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Basic Clifford Analysis

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Abstract

The basic concepts of Clifford analysis are introduced both in euclidean space and over the sphere. Cauchy integral formulae and links to conformal structure are described.

1 Introduction

Clifford analysis initially arose out of attempts to extend aspects of one variable complex analysis to other algebras. Though generalizations of Cauchy-Riemann equations can be described over arbitrary algebras one also needs an analogue of Cauchy's integral formula. A number of authors noted that a suitable generalization of the Cauchy integral formula can be introduced over the quaternion algebra. Fundamental use appeared to be made of the fact that the quaternion algebra is a division algebra. Early progress in this direction was made by A. C. Dixon [Di]. By the 1930's more consolidated attempts to develop quaternionic analysis were developed by the Swiss mathematician Rudolph Fueter [F] and his students and also by the Romanian mathematicians Moisil and Teodorescu [MTh].

Later it was noted that the full structure of the quaternionic division algebra was not being used to develop quaternionic analysis. In fact to set up the Cauchy integral formula from quaternionic analysis one only relies on the fact that each non-zero vector has a multiplicative inverse. In fact one only needed a subspace of the algebra to poses the property that each non-zero vector in the subspace has a multiplicative inverse in order to set up such an integral formula. This allowed one to open the door a bit wider and note that one could introduce aspects of one variable complex analysis over R^n for arbitrary n using Clifford algebras. Ultimately it was understood that almost every aspect of quaternionic analysis extended to all dimensions using Clifford algebras. The earlier aspects of this study was developed by amongst others Richard Delanghe [D], Viorel Iftimie [I] and David Hestenes [H]. The subject that has grown from these works is now called Clifford analysis.

In more recent times Clifford analysis has found a wealth of unexpected applications in a number of branches of mathematical analysis particularly classical harmonic analysis, see for instance the work of Alan McIntosh and his collaborators, for instance [LMcQ, LMcS], Marius Mitrea [M1, M2] and papers in [R4]. Links to representation theory and several complex variables may be found in [GiMu, R1-R3].

The purpose of this paper is to present a review of many of the basic aspects of Clifford analysis. We begin by trying to motivate Clifford algebras via geometric considerations. After developing these ideas we move on to introduce the Dirac operator, which plays the role of a generalized Cauchy-Riemann operator, and to introduce the Cauchy integral formula. We also introduce other analogues of basic results from one variable complex analysis. We also describe the function associated to the equation $D^k f = 0$ where D is the Dirac operator. After this we go back to Clifford algebras to introduce the so called Vahlen matrices. These matrices were introduced by K. Th. Vahlen in 1902 in [V]. Except for an occasional reference they were promptly forgotten until their properties were rediscovered and developed in a sequence of papers by Ahlfors [A] in the 1980's. These matrices give an elegant way of describing conformal transformations over $R^n \cup \{\infty\}$.

We move on to describe the link between Vahlen matrices and Clifford analysis. We use them as a link to a suitable euclidean analogue of a cross ratio and Schwarzian derivative. We describe the invariance of solutions to our generalized Cauchy Riemann equations under Möbius transformations. Via a Cayley transformation we conclude by introducing analogues of these results over domains on the sphere.

Alternative accounts of much of this work together with other related results can be found in [BDSo, DSoSou, GSp, GiMu, KSh, O, R4].

2 Algebraic and Geometric Preliminaries

Consider the unit circle S^1 lying in the xy-plane and a point X lying on either the positive or the negative x axis. If we draw the line connecting this point to the north pole of S^1, namely the co-ordinate $(0, 1)$, then this line cuts the circle at precisely one point. On drawing the line from the south pole, $(0, -1)$, to this point the new line cuts the x-axis at a second point X'. We would like to know the relationship between the numbers X and X'. Simple arguments for triangles involving elementary trigonometry tell us that $X' = \frac{1}{X}$ or X^{-1}.

The lines that we drew in this calculation help describe stereographic projections of first $S^1\backslash\{(0,1)\}$ and then $S^1\backslash\{(0,-1)\}$ onto the x axis.

We may use similar constructions for two dimensional space. First consider the unit sphere $S^2 = \{x \in R^3 : \|x\| = 1\}$ lying in R^3. Let us consider the horizontal plane to be the xy plane. The north pole of the sphere will be the co-ordinate $(0,0,1)$ while the south pole is the co-ordinate $(0,0,-1)$. Taking any non-zero vector (x,y) in the xy-plane we may consider the plane containing this point and the north and south poles of S^2. We may identify S^1 with the restriction of the sphere to this plane and repeat the geometric construction we developed earlier. The point we end up with will be the co-ordinate $(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2})$. On identifying the xy plane with the complex plane in the usual way, so that (x,y) is identified with $x+iy = z$, then the co-ordinate $(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2})$ is identified with the complex number $\frac{z}{\|z\|^2}$.

The geometric argument we have developed in one and two dimensional space using stereographic projections from a sphere in one higher dimension clearly can be extended to all dimensions. If we consider a non-zero vector $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$ the analogous argument will give the point $\frac{x}{\|x\|^2}$. This vector is often called the Kelvin inverse of the non-zero vector x. The transformation $Inv : \mathbf{R}^n\backslash\{0\} \to \mathbf{R}^n\backslash\{0\}$: $Inv(x) = \frac{x}{\|x\|^2}$ will turn the interior of the closed unit disc in $\mathbf{R}^n$ into the complement of the same closed unit disc.

In the one and two dimensional settings we saw that Kelvin inversion is intimately related to the algebra of the real and complex number systems. We would like to relate Kelvin inversion in higher dimensions to some similar algebraic structure. We begin in three dimensions. For each vector $(t,x,y) \in \mathbf{R}^3$ we shall write this vector as $t+ix+jy$ and denote it as q. The Kelvin inverse is $\frac{1}{\|q\|^2}(t+ix+jy)$. Following the complex setting where the Kelvin inverse may be expressed as $\frac{\bar{z}}{\|z\|^2}$. We will write the Kelvin inverse of q as $\frac{\bar{q}}{\|q\|^2}$. In the complex number system $z\bar{z} = \|z\|^2$. We would like to consider q as an element of an algebra $\mathbf{H}$ such that $q\bar{q} = \|q\|^2$ where $\bar{q} = t - ix - jy$. In terms of the elements 1, i and j this means that $i^2 = j^2 = -1$ and $ij = -ji$. It follows that $(ij)^2 = -1$. On placing $k = ij$ we get that $k^2 = ijk = -1$. These identities are the usual identities for the generators of the quaternion algebra. So $\mathbf{H}$ is the quaternion algebra. Let us now consider a general element $q = t + ix + jy + kw$ of $\mathbf{H}$. We denote its conjugate by $\bar{q}$ where $\bar{q} = t - ix - jy - kw$. Using the identities governing the generators $1, i, j, k$ of $\mathbf{H}$ it may be seen that $q\bar{q} = \|q\|^2$. Consequently $\mathbf{H}$ is a division algebra with each non-zero element q of $\mathbf{H}$ having multiplicative inverse $q^{-1} = \frac{\bar{q}}{\|q\|^2}$. It should be mentioned that this algebra was introduced by the Irish mathematician William Hamilton in 18... Let us also observe that the relations the generators of $\mathbf{H}$ satisfy show us that the quaternion algebra is non-commutative.

Let us move on to the general n-dimensional case. First in order to simplify notation slightly let us note that if $q = ix + jy + kw$ then $\bar{q} = -q$. If we write a vector $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$ as $x_1e_1 + \ldots + x_ne_n$ where $e_1, \ldots, e_n$ is the standard orthonormal basis for $\mathbf{R}^n$ then we might consider $\mathbf{R}^n$ as embedded in an algebra Cl_n in such a way that $-xx = \|x\|^2$ or $x^2 = -\|x\|^2$ for each $x \in \mathbf{R}^n$. The algebra Cl_n so far has not been properly introduced. It has mearly been vaguely mentioned as an algebraic structure that enables one to do a desirable algebraic manipulation. No attempt has so far been made here to formally introduce this algebra. We shall now at least partially correct that omission. We will set up the algebra in terms of generators. As we are interested in the identity $x^2 = -\|x\|^2$ in terms of the vectors $e_1, \ldots, e_n$ we get that $e_i^2 = -1$ for $1 \leq i \leq n$ and $e_ie_j = -e_je_i$ for $i \neq j$. As we need no other relations on the e_j's the Clifford algebra Cl_n will be defined as the real algebra generated from $\mathbf{R}^n$ via the relationship

$$e_ie_j + e_je_i = -2\delta_{ij}.$$

Cl_n will have as basis the elements

$$1, e_1, \ldots, e_n, e_1e_2, \ldots, e_{n-1}e_n, \ldots, e_{j_1} \ldots e_{j_r}, \ldots, e_1 \ldots e_n$$

where $j_1 < \ldots < j_r$ and $1 \leq r \leq n$.

As the collection of elements of the type $e_{j_1} \ldots e_{j_r}$ is in one to one correspondence with the number of subsets of $\{e_1, \ldots, e_n\}$ containing precisely r elements the binomial theorem tells us there are precisely $\binom{n}{r}$ elements of this type. So the dimension of Cl_n is 2^n.

We have set up the Clifford algebra Cl_n in a basis dependent way. A more general approach can be found in [AtBS] and elsewhere.

One interesting feature of the algebra Cl_n is that each non-zero vector $x \in \mathbf{R}^n \subset Cl_n$ has a multiplicative inverse $x^{-1} = \frac{-x}{\|x\|^2}$. Up to the minus sign this inverse corresponds to the Kelvin inverse of x. However Cl_n is not a division algebra for $n > 2$. When $n = 3$ the elements $E_\pm = \frac{1}{2}(1 \pm e_1e_2e_3)$ belong to Cl_3 and $E_+E_- = 0$.

Another point worth mentioning is that it is clear from the relationship on the generators of Cl_n that this algebra is non-commutative for $n > 1$.

When $n = 1$ the basis of the algebra is $1, e_1$ and $e_1^2 = -1$. So Cl_1 is the complex number system. When $n = 2$ a simple calculation shows that Cl_2 is the quaternion algebra.

Later we shall need an analogue of the conjugation operator we saw over the complex number system and the quaternions. The conjugate operator is the linear transform given by

$$- : Cl_n \to Cl_n : e_{j_1} \ldots e_{j_r} \to (-1)^r e_{j_r} \ldots e_{j_1}.$$

It is easy to check in the cases $n = 1$ and $n = 2$ that this operator corresponds to the operator of conjugation over the complex and quaternionic algebras. Instead of writing $-(X)$ we shall write $\overline{X}$ for each $X \in Cl_n$. Moreover the real part or identity component of $X\overline{X}$ is equal to $\|X\|^2 = x_0^2 + \ldots + x_{1\ldots n}^2$. It is relatively easy to note for each pair $X, Y \in Cl_n$ that $\overline{XY} = \overline{Y}\,\overline{X}$. So conjugation reverses the order of multiplication and so is an example of an anti-automorphism on Cl_n. Closely associated to conjugation is the operator

$$\sim: Cl_n \to Cl_n : e_{j_1} \dots e_{j_r} \to e_{j_r} \dots e_{j_1}.$$

Again we will write $\tilde{X}$ instead of $\sim X$ and it may be deduced that $\tilde{XY} = \tilde{Y}\tilde{X}$.

3 Some Clifford Analysis

We start by replacing the vector $x = x_1e_1 + \dots + x_ne_n$ by the differential operator $D = \Sigma_{j=1}^n e_j \frac{\partial}{\partial x_j}$. One basic but interesting property of D is that $D^2 = -\Delta_n$, the Laplacian $\Sigma_{j=1}^n \frac{\partial^2}{\partial x_j^2}$ in $\mathbf{R}^n$. The differential operator D will be called a Dirac operator. This is because the classical Dirac operator constructed over four dimensional Minkowski space squares to give the wave operator.

Definition 1 *Suppose that U is a domain in $\mathbf{R}^n$ and f and g are C^1 functions defined on U and taking values in Cl_n. Then f is called a left monogenic function if $Df = 0$ on U while g is called a right monogenic function on U if $gD = 0$ where $gD = \Sigma_{j=1}^n \frac{\partial g}{\partial x_j} e_j$.*

Examples of such functions include the gradients of real valued harmonic functions on U. So if h is harmonic on U and it is also real valued then Dh is a vector valued left monogenic function. It is also a right monogenic function. Such a function is more commonly referred to as a conjugate harmonic function or a harmonic 1-form. See for instance [StW]. An example of such a function is $G(x) = \frac{x}{\|x\|^n}$.

To introduce other possible examples of left monogenic functions suppose that μ is a Cl_n valued measure with compact support $[\mu]$ in $\mathbf{R}^n$. Then the convolution $\int_{[\mu]} G(x-y)d\mu(y)$ defines a left monogenic function on the maximal domain lying in $\mathbf{R}^n \backslash [\mu]$.

Another way to construct examples of left monogenic functions was introduced by Littlewood and Gay in [LiG] for the case $n = 3$ and independently re-introduced for all n by Sommen [S2]. Suppose U' is a domain in $\mathbf{R}^{n-1}$, the span of $e_2, \dots, e_n$. Suppose also that $f'(x')$ is a Cl_n valued function such that at each point $x' \in U'$ there is a multiple series expansion in $x_2, \dots, x_n$ that converges uniformly on some neighbourhood of x' in U' to f'. Such a function is called a real analytic function. The series

$$\Sigma_{k=0}^\infty \frac{1}{k!} x_1^k (-e_1 D' f'(x') = \exp(-x_1 e_1 D') f'(x')$$

where $D' = \Sigma_{j=2}^n e_j \frac{\partial}{\partial x_j}$, defines a left monogenic function f in some neighbourhood $U(f')$ in $\mathbf{R}^n$ of U'. The left monogenic function f is the Cauchy-Kowalewska extension of f'.

It should be noted that if f is a left monogenic function then $\overline{f}$ and $\tilde{f}$ are both right monogenic functions.

We now turn to analogues of Cauchy's Theorem and Cauchy's integral formula.

Theorem 1 (The Clifford-Cauchy Theorem): *Suppose that f is a left monogenic function on U and g is a right monogenic function on U. Suppose also that V is a bounded subdomain of U with piecewise differentiable boundary S lying in U. Then*

$$\int_S g(x)n(x)f(x)d\sigma(x) = 0 \qquad (1)$$

where $n(x)$ is the outward pointing normal vector to S at x and σ is the Lebesgue measure on S.

The proof follows directly from Stokes' Theorem. One important point to keep in mind though is that as Cl_n is not a commutative algebra then it is important to place the vector $n(x)$ between f and g. One then has that

$$\int_S g(x)n(x)f(x)d\sigma(x) = \int_V ((g(x)D)f(x) + g(x)(Df(x)))dx^n = 0.$$

Suppose that g is the gradient of a real valued harmonic function and $f = 1$. Then the real part of Equation 1 gives the following well known integral formula.

$$\int < gradg(x), n(x) > d\sigma(x) = 0.$$

We now turn to the analogue of a Cauchy integral formula.

Theorem 2 (Clifford-Cauchy Integral Formula): *Suppose that U, V, S, f and g are all as in Theorem 1 and that $y \in V$. Then*

$$f(y) = \frac{1}{\omega_n}\int_S G(x-y)n(x)f(x)d\sigma(x)$$

and

$$g(y) = \frac{1}{\omega_n}\int_S g(x)n(x)G(x-y)d\sigma(x)$$

where ω_n is the surface area of the unit sphere in $\mathbf{R}^n$.

Proof: The proof follows very similar lines to the argument in one variable complex analysis. We shall establish the formula for $f(y)$ the proof being similar for $g(y)$. First let us take a sphere $S^{n-1}(y,r)$ centered at y and of radius r. The radius r is chosen sufficently small so that the closed disc with boundary $S^{n-1}(y,r)$ lies in V. Then by the Clifford-Cauchy theorem

$$\int_S G(x-y)n(x)f(x)d\sigma(x) = \int_{S^{n-1}(y,r)} G(x-y)n(x)f(x)d\sigma(x).$$

However on $S^{n-1}(y,r)$ the vector $n(x) = \frac{y-x}{\|x-y\|}$. So $G(x-y)n(x) = \frac{1}{r^{n-1}}$. So

$$\int_{S^{n-1}(y,r)} G(x-y)n(x)f(x)d\sigma(x) = \int_{S^{n-1}(y,r)} \frac{1}{r^{n-1}}(f(x)-f(y))d\sigma(x)$$

$$+\int_{S^{n-1}(y,r)} \frac{f(y)}{r^{n-1}} d\sigma(x).$$

The right side of this previous expression reduces to

$$\int_{S^{n-1}(y,r)} \frac{(f(x)-f(y))}{r^{n-1}} d\sigma(x) + f(y)\int_{S^{n-1}} d\sigma(x).$$

Now $\int_{S^{n-1}} d\sigma(x) = \omega_n$ and by continuity $\lim_{r\to 0} \int_{S^{n-1}(y,r)} \frac{(f(x)-f(y))}{r^{n-1}} d\sigma(x) = 0$. The result follows. □

One important feature is to note that Kelvin inversion plays a fundamental role in this proof. Moreover the proof is almost exactly the same as the proof of Cauchy's Integral Formula for piecewise C^1 curves in one variable complex analysis.

Having obtained a Cauchy Integral Formula in $\mathbf{R}^n$ a number of basic results that one might see in a first course in one variable complex analysis carry over more or less automatically to the context described here. This includes a Liouville Theorem and Weierstrass Convergence Theorem. We leave it as an exercise to the interested reader to set up and establish the Clifford analysis analogues of these results. Their statements and proofs can be found in [BDSo].

Theorems 1 and 2 show us that the individual components of the equations $Df = 0$ and $gD = 0$ comprise generalized Cauchy-Riemann equations. In the particular case where f is just vector valued so $f = \Sigma_{j=1}^n f_j e_j$ then the generalized Cauchy-Riemann equations become $\frac{\partial f_j}{\partial x_i} = \frac{\partial f_i}{\partial x_j}$ whenever $i \neq j$ and $\Sigma_{j=1}^n \frac{\partial f_j}{\partial x_j} = 0$. This system of equations is often referred to as the Riesz system.

Having obtained an analogue of Cauchy's integral formula in euclidean space we shall now exploit this result to show how many consequences of the classical Cauchy integral carry over to the context described here. We begin with the Mean Value Theorem.

Theorem 3 *Suppose that $D(y,R)$ is a closed disc centered at y, of radius R and lying in U. Then for each monogenic function f on U*

$$f(y) = \frac{1}{R\omega_n}\int_{D(y,R)} \frac{f(x)}{\|x-y\|^{n-1}} dx^n.$$

Proof: We have already seen that for each $r \in (0,R)$

$$f(y) = \frac{1}{\omega_n}\int_{S^{n-1}(y,r)} \frac{f(x)}{\|x-y\|^{n-1}} d\sigma(x),$$

where $S^{n-1}(y,r)$ is the $(n-1)$-dimensional sphere centered at y and of radius r. We obtain the result by integrating both sides of this expression with respect to the variable r and dividing throughout by R. $\square$

Let us now turn to explore the real analyticity properties of monogenic functions. First it may be noted that when n is even $G(x-y) = (-1)^{\frac{n-2}{2}}(x-y)^{-n+1}$. Also $(x-y)^{-1} = x^{-1}(1-yx^{-1})^{-1} = (1-x^{-1}y)^{-1}x^{-1}$, and $\|x^{-1}y\| = \|yx^{-1}\| = \frac{\|y\|}{\|x\|}$. So for $\|y\| < \|x\|$

$$(x-y)^{-1} = x^{-1}(1+yx^{-1}+\ldots+yx^{-1}\ldots yx^{-1}+\ldots)$$
$$= (1+x^{-1}y+\ldots+x^{-1}y\ldots x^{-1}y+\ldots)x^{-1}.$$

Hence these two sequences converge uniformly to $(x-y)^{-1}$ provided $\|y\| \le r < \|x\|$ and they converge pointwise to $(x-y)^{-1}$ provided $\|y\| < \|x\|$. One can now take $(-1)^{\frac{n-2}{2}}$ times the $(n-1)$-fold product of the series expansions of $(x-y)^{-1}$ with itself to obtain a series expansion for $G(x-y)$. In this process of multiplying series together in order to maintain the same radius of convergence one needs to group together all linear combinations of monomials in $y_1,\ldots,y_n$ that are of the same order. Thus we have deduced that when n is even the multiple Taylor series expansion

$$\Sigma_{j=0}^{\infty}(\Sigma_{\substack{j_1,\ldots,j_n\\ j_1+\ldots+j_n=j}}\frac{y_1^{j_1}\cdots y_n^{j_n}}{j_1!\cdots j_n!}\frac{\partial^j G(x)}{\partial x_1^{j_1}\ldots\partial x_n^{j_n}})$$

converges uniformly to $G(x-y)$ provided $\|y\| < r < \|x\|$ and converges pointwise to $G(x-y)$ provided $\|y\| < \|x\|$.

A similar argument may be developed when n is odd.

Returning to Cauchy's integral formula let us suppose that f is a left monogenic function defined in a neighbourhood of the closure of some ball $B(0,R)$. Then

$$f(y) = \frac{1}{\omega_n}\int_{\partial B(0,R)} G(x-y)n(x)f(x)d\sigma(x) =$$

$$\frac{1}{\omega_n}\int_{\partial B(0,R)}\Sigma_{j=0}^{\infty}(\Sigma_{\substack{j_1\ldots j_n\\ j_1+\ldots+j_n=j}}\frac{y_1^{j_1}\cdots y_n^{j_n}}{j_1!\cdots j_n!}\frac{\partial^j G(x)}{\partial x_1^{j_1}\ldots\partial x_n^{j_n}})n(x)f(x)d\sigma(x)$$

provided $\|y\| < \|x\|$. As this series converges uniformly on each ball $B(0,r)$ for each $r < R$ then this last integral can be re-written as

$$\frac{1}{\omega_n}\Sigma_{j=0}^{\infty}\int_{\partial B(0,R)}(\Sigma_{\substack{j_1\ldots j_n\\ j_1+\ldots+j_n=j}}\frac{y_1^{j_1}\cdots y_n^{j_n}}{j_1!\cdots j_n!}\frac{\partial^j G(x)}{\partial x_1^{j_1}\ldots\partial x_n^{j_n}}n(x)f(x))d\sigma(x).$$

As the summation within the parentheses is a finite summation this last expression easily reduces to

$$\frac{1}{\omega_n}\Sigma_{j=0}^{\infty}(\Sigma_{\substack{j_1\ldots j_n\\ j_1+\ldots+j_n=j}}\frac{y_1^{j_1}\cdots y_n^{j_n}}{j_1!\cdots j_n!}\int_{\partial B(0,R)}\frac{\partial^j G(x)}{\partial x_1^{j_1}\ldots\partial x_n^{j_n}}n(x)f(x))d\sigma(x).$$

On placing

$$\frac{1}{\omega_n}\int_{\partial B(0,R)} \frac{\partial^j G(x)}{\partial x_1^{j_1}\dots\partial x_n^{j_n}} n(x)f(x)d\sigma(x) = a_{j_1\dots j_n}$$

it may be seen that on $B(0, R)$ the series

$$\Sigma_{j=0}^{\infty}(\Sigma_{\substack{j_1\dots j_n \\ j_1+\dots+j_n=j}} \frac{x_1^{j_1}\dots x_n^{j_n}}{j_1!\dots j_n!} a_{j_1\dots j_n})$$

converges pointwise to $f(y)$. Convergence is uniform on each ball $B(0, r)$ provided $r < R$.

Similarly if g is a right monogenic function defined in a neighbourhood of the closure of $B(0, R)$ then the series

$$\Sigma_{j=0}^{\infty}(\Sigma_{\substack{j_1\dots j_n \\ j_1+\dots+j_n=j}} b_{j_1\dots j_n} \frac{y_1^{j_1}\dots y_n^{j_n}}{j_1!\dots j_n!})$$

converges pointwise on $B(0, R)$ to $g(y)$ and converges uniformly on $B(0, r)$ for $r < R$, where

$$b_{j_1\dots j_n} = \frac{1}{\omega_n}\int_{\partial B(0,R)} g(x)n(x)\frac{\partial^j G(x)}{\partial x_1^{j_1}\dots\partial x_n^{j_n}} d\sigma(x).$$

By translating the ball $B(0, R)$ to the ball $B(w, R)$ where $w = w_1e_1+\dots+w_ne_n$ one may readily observe that for any left monogenic function f defined in a neighbourhood of the closure of $B(w, R)$ the series

$$\Sigma_{j=0}^{\infty}(\Sigma_{\substack{j_1\dots j_n \\ j_1+\dots+j_n=j}} \frac{(y_1-w_1)^{j_1}\dots(y_n-w_n)^{j_n}}{j_1!\dots j_n!} a'_{j_1\dots j_n})$$

converges pointwise on $B(w, R)$ to $f(y)$, where

$$a'_{j_1\dots j_n} = \frac{1}{\omega_n}\int_{\partial B(w,R)} \frac{\partial^j G(x-w)}{\partial x_1^{j_1}\dots x_n^{j_n}} n(x)f(x)d\sigma(x).$$

Again the series converges uniformly on $B(w, r)$ for each $r < R$. A similar series may be readily obtained for any right monogenic function defined in a neighbourhood of the closure of $B(w, R)$.

The types of power series that we have developed for left monogenic functions are not entirely satisfactory. In particular, unlike their complex analogues the homogeneous polynomials

$$\Sigma_{\substack{j_1\dots j_n \\ j_1+\dots+j_n=j}} \frac{x_1^{j_1}\dots x_n^{j_n}}{j_1!\dots j_n!} a_{j_1\dots j_n}$$

are not expressed as a linear combination of left monogenic polynomials. To rectify this situation let us first take a closer look at the Taylor expansion for the Cauchy kernel $G(x-y)$ where all the Taylor coefficients are real. Let us first look at the first order terms in the Taylor expansion. This is the expression

$$y_1\frac{\partial G(x)}{\partial x_1}+\ldots+y_n\frac{\partial G(x)}{\partial x_n}.$$

As G is a monogenic function then $\frac{\partial G(x)}{\partial x_1}=-\Sigma_{j=2}^n e_1^{-1}e_j\frac{\partial G(x)}{\partial x_j}$. Therefore the first order terms of the Taylor expansion for $G(x-y)$ can be re-expressed as

$$\Sigma_{j=2}^n(y_j-e_1^{-1}e_jy_1)\frac{\partial G(x)}{\partial x_j}.$$

Moreover, for $2\leq j\leq n$ the first order polynomial $y_j-e_1^{-1}e_jy_1$ is a left monogenic polynomial. Let us now go to second order terms. Again we will replace the operator $\frac{\partial}{\partial x_1}$ by the operator $-\Sigma_{j=2}^n e_1^{-1}e_j\frac{\partial}{\partial x_j}$ whenever it arises. Let us consider the term $\frac{\partial^2 G(x)}{\partial x_i\partial x_j}$ where $i\neq j\neq 1$. We end up with the polynomial $y_iy_j-y_iy_1e_1^{-1}e_j-y_jy_1e_1^{-1}e_i=\frac{1}{2}((y_i-y_1e_1^{-1}e_i)(y_j-y_1e_1^{-1}e_j)+(y_j-y_1e_1^{-1}e_j)(y_i-y_1e_1^{-1}e_i))$. Similarly the polynomial attached to the term $\frac{\partial^2 G(x)}{\partial x_i^2}$ is $(y_i-y_1e_1^{-1}e_i)^2$. Using the Clifford algebra anti-commutation relationship $e_ie_j+e_je_i=-2\delta_{ij}$ and on replacing the differential operator $\frac{\partial}{\partial x_1}$ by the operator $-\Sigma_{j=2}^n e_1^{-1}e_j\frac{\partial}{\partial x_j}$ it may be determined that the power series we previously obtained for $G(x-y)$ can be replaced by the series $\Sigma_{j=0}^\infty(\Sigma_{\substack{j_2\cdots j_n\\ j_2+\ldots+j_n=j}} P_{j_2\ldots j_n}(y)\frac{\partial^j G(x)}{\partial x_2^{j_2}\ldots\partial x_n^{j_n}})$, where $\|y\|<\|x\|$ and

$$P_{j_2\ldots j_n}(y)=\frac{1}{j!}\Sigma(y_{\sigma(1)}-y_1e_1^{-1}e_{\sigma(1)})\ldots(y_{\sigma(j)}-y_1e_1^{-1}e_{\sigma(j)}).$$

Here $\sigma(i)\in\{2,\ldots,n\}$ and the previous summation is taken over all permutations of the monomials $(y_{\sigma(i)}-y_1e_1^{-1}e_{\sigma(i)})$ without repetition. The quaternionic monogenic analogues for these polynomials were introduced by Fueter [F] while the Clifford analogues, $P_{j_2\ldots j_n}$, described here were introduced by Delanghe in [D]. It should be noted that each polynomial $P_{j_2\ldots j_n}(y)$ takes its values in the space spanned by 1, $e_1e_2,\ldots,e_1e_n$. Also each such polynomial is homogeneous of degree j. Similar arguments to those just outlined give that $G(x-y)=\Sigma_{j=0}^\infty(\Sigma_{\substack{j_2\cdots j_n\\ j_2+\ldots+j_n=j}}\frac{\partial^j G(x)}{\partial x_2^{j_2}\ldots x_n^{j_n}}\widetilde{P_{j_2\ldots j_n}}(y))$ provided $\|y\|<\|x\|$.

Proposition 1 *Each of the polynomials $P_{j_2\ldots j_n}(y)$ is a left monogenic polynomial.*

Proof: As $DP_{j_2\ldots j_n}(y)=e_1(\frac{\partial}{\partial y_1}+e_1^{-1}\Sigma_{j=2}^n e_j\frac{\partial}{\partial y_j}P_{j_2\ldots j_n}(y))$ then we shall consider the expression $(\frac{\partial}{\partial y_1}+\Sigma_{j=2}^n e_1^{-1}e_j\frac{\partial}{\partial y_j})P_{j_2\ldots j_n}(y)$. This term is equal to

$$(\frac{\partial}{\partial y_j}+\Sigma_{j=2}^n e_1^{-1}e_j\frac{\partial}{\partial y_j})\Sigma(y_{\sigma(1)}-e_1^{-1}e_{\sigma(1)}y_1)\ldots$$

$$\dots(y_{\sigma(i-1)} - e_1^{-1}e_{\sigma(i-1)}y_1)(y_{\sigma(i)} - e_1^{-1}e_{\sigma(i)}y_1)(y_{\sigma(i+1)} - e_1^{-1}e_{\sigma(i+1)}y_1)\dots$$
$$\dots(y_{\sigma(j)} - e_1^{-1}e_{\sigma(j)}y_1).$$

This is equal to

$$\Sigma(y_{\sigma(1)} - e_1^{-1}e_{\sigma(1)}y_1)\dots(y_{\sigma(i-1)} - e_1^{-1}e_{\sigma(i-1)}y_1)(-e_1^{-1}e_{\sigma(i)})(y_{\sigma(i+1)} - e_1^{-1}e_{\sigma(i+1)}y_1)$$
$$\dots(y_{\sigma(j)} - e_1^{-1}e_{\sigma(j)}y_1) + \Sigma e_1^{-1}e_{\sigma(i)}(y_{\sigma(1)} - e_1^{-1}e_{\sigma(1)}y_1)\dots(y_{\sigma(i-1)} - e_1^{-1}e_{\sigma(i-1)}y_1)$$
$$(y_{\sigma(i+1)} - e_1^{-1}e_{\sigma(i+1)}y_1)\dots(y_{\sigma(j)} - e_1^{-1}e_{\sigma(j)}y_1).$$

If we multiply the previous term by y_1 and add to it the following term, which is equal to zero,

$$\Sigma(y_{\sigma(1)} - e_1^{-1}e_{\sigma(1)}y_1)\dots(y_{\sigma(i-1)} - e_1^{-1}e_{\sigma(i-1)}y_1)(y_{\sigma(i)} - y_{\sigma(i)})$$
$$(y_{\sigma(i+1)} - e_1^{-1}e_{\sigma(i+1)}y_1)\dots(y_{\sigma(j)} - e_1^{-1}e_{\sigma(i)}y_1)$$

we get, after regrouping terms,

$$\Sigma(y_{\sigma(1)} - e_1^{-1}e_{\sigma(1)}y_1)\dots(y_{\sigma(i-1)} - e_1^{-1}e_{\sigma(i-1)}y_1)(y_{\sigma(i)} - e_1^{-1}e_{\sigma(i)}y_1)$$
$$(y_{\sigma(i+1)} - e_1^{-1}e_{\sigma(i+1)}y_1)\dots(y_{\sigma(j)} - e_1^{-1}e_{\sigma(j)}y_1)$$
$$-\Sigma(y_{\sigma(i)} - e_1^{-1}e_{\sigma(i)}y_1)(y_{\sigma(1)} - e_1^{-1}e_{\sigma(1)}y_1)\dots(y_{\sigma(i-1)} - e_1^{-1}e_{\sigma(i-1)}y_1)$$
$$(y_{\sigma(i+1)} - e_1^{-1}e_{\sigma(i+1)}y_1)\dots(y_{\sigma(j)} - e_1^{-1}e_{\sigma(j)}y_j).$$

As summation is taken over all possible permutations without repetition this last term vanishes. □

Using Proposition 1 and the results we previously obtained on series expansions we can obtain the following generalization of Taylor expansions from one variable complex analysis.

Theorem 4 (Taylor Series) *Suppose that f is a left monogenic function defined in an open neighbourhood of the closure of the ball $B(w, R)$. Then*

$$f(y) = \Sigma_{j=0}^{\infty}(\Sigma_{\substack{j_2\dots j_n \\ j_2+\dots+j_n=j}} P_{j_2\dots j_n}(y-w)a_{j_2\dots j_n}),$$

where $a_{j_2\dots j_n} = \frac{1}{\omega_n}\int_{\partial B(w,R)} \frac{\partial^j G(x-w)}{\partial x_2^{j_2}\dots\partial x_n^{j_n}} n(x)f(x)d\sigma(x)$ *and* $\|y-w\| < R$. *Convergence is uniform provided* $\|x-w\| < r < R$.

A simple application of Cauchy's theorem now tells us that the Taylor series that we obtained for f in the previous theorem remains valid on the largest open ball on which f is defined and the largest open ball on which g is defined. Also the previous identities immediately yield the mutual linear independence of the collection of the left monogenic polynomials $\{P_{j_2\dots j_n} : j_2 + \dots + j_n = j \text{ and } 0 \le j < \infty\}$.

4 The Equation $D^k f = 0$

Here we will examine some of the basic properties of solutions to the equation $D^k f = 0$ for an integer $k > 1$. We start with the case $k = 2$. In this case we get Laplace's equation $\triangle_n f = -D^2 f = 0$, and the solutions are harmonic functions.

It is reasonably well known that if h is a real valued harmonic function defined on a domain $U \subset R^n$ then for each $y \in U$ and each compact, piecewise C^1 surface S lying in U such that S bounds a subdomain V of S and $y \in V$, then

$$h(y) = \frac{1}{\omega_n}\int_S (H(x-y) < n(x), grad\ h(x) > - < G(x-y), n(x) > h(x))d\sigma(x),$$

where $H(x-y) = \frac{1}{(n-2)\|x-y\|^{n-2}}$. This formula is Green's formula for a harmonic function, and it heavily relies on the standard inner product on R^n. Introducing the Clifford algebra Cl_n the right side of Green's formula is the real part of

$$\frac{1}{\omega_n}\int_S (G(x-y)n(x)h(x) - H(x-y)n(x)Dh(x))d\sigma(x).$$

Assuming that the function h is C^2 then on applying Stokes' theorem the previous integral becomes

$$\frac{1}{\omega_n}\int_{S^{n-1}(y,r(y))} (G(x-y)n(x)h(x) - H(x-y)n(x)Dh(x))d\sigma(x),$$

where $S^{n-1}(y, r(y))$ is a sphere centered at y, of radius $r(y)$ and lying in V. On letting the radius $r(y)$ tend to zero the first term of the integral tends to $h(y)$ while the second term tends to zero. Consequently the Clifford analysis version of Green's formula is

$$h(y) = \frac{1}{\omega_n}\int_S (G(x-y)n(x)h(x) - H(x-y)n(x)Dh(x))d\sigma(x).$$

This formula was obtained under the assumption that h is real valued and C^2. The fact that we have assumed h to be real valued can easily be observed to be irrelevant, and so we can assume that h is Cl_n valued. From now on we shall assume that all harmonic functions take their values in Cl_n. If h is also a left monogenic function then the Clifford analysis version of Green's formula becomes Cauchy's integral formula.

The assumption that h is C^2 can also be dropped as we shall see shortly. First we establish the following result.

Proposition 2 *Suppose that f is a monogenic function on some domain U. Then $xf(x)$ is harmonic.*

Proof: $Dxf(x) = -nf(x) - \Sigma_{j=1}^{n} x_j \frac{\partial f(x)}{\partial x_j} - \Sigma_{\substack{j,k \\ j \neq k}} x_k e_k e_j \frac{\partial f(x)}{\partial x_j}$. Now

$$\Sigma_{\substack{j,k \\ j \neq k}} x_k e_k e_j \frac{\partial f(x)}{\partial x_j} = \Sigma_{k=1}^{n} \Sigma_{j \neq k} x_k e_k e_j \frac{\partial f(x)}{\partial x_j}.$$

As f is left monogenic this last expression simplifies to $\Sigma_{k=1}^{n} x_k \frac{\partial f(x)}{\partial x_k}$. Moreover $D\Sigma_{j=1}^{n} x_j \frac{\partial f(x)}{\partial x_j} = 0$. Consequently $D^2 x f(x) = 0$. □

The previous proof is a generalization of the statement- "if $h(x)$ is a real valued harmonic function then so is $< x, grad\ h(x) >$".

In fact in the previous proof we determine that $Dxf(x) = -nf(x) - 2\Sigma_{j=1}^{n} x_j \frac{\partial f(x)}{\partial x_j}$. In the special case where $f(x) = P_k(x)$, a left monogenic polynomial of order k, this equation simplifies to $DxP_k(x) = -(n+2k)P_k(x)$. Suppose now that $h(x)$ is a harmonic function defined in a neighbourhood of the ball $B(0, R)$. Now Dh is a left monogenic function so we know that there is a series $\Sigma_{l=0}^{\infty} P_l(x)$ of left monogenic polynomials with each P_l homogeneous of degree l and such that the series converges locally uniformly on $B(0, R)$ to $Dh(x)$. Now consider the series $\Sigma_{l=0}^{\infty} \frac{-1}{n+2l} P_l(x)$. As $\frac{1}{n+2l} \|P_l(x)\| < \|P_l(x)\|$ then this new series converges locally uniformly on $B(0, R)$ to a left monogenic function $f_1(x)$. Moreover, $Dxf_1(x) = Dh(x)$ on $B(0, R)$. Consequently $h(x) - xf_1(x)$ is equal to a left monogenic function $f_2(x)$ on $B(0, R)$. Thus we have established:

Proposition 3 *Suppose that h is a harmonic function defined in a neighbourhood of $B(0, R)$ then there are left monogenic functions f_1 and f_2 defined on $B(0, R)$ such that $h(x) = xf_1(x) + f_2(x)$ for each $x \in B(0, R)$.*

This result remains invariant under translation. As a consequence it shows us that all harmonic functions are real analytic functions. So there is no need to specify whether or not a harmonic function is C^2. The result also provides an Almansi type decomposition of harmonic functions in terms of monogenic functions over any ball in R^n.

It should be noted that Proposition 3 remains true if h is only real valued.

Proposition 3 gives rise to an alternative proof of the Mean Value Theorem for harmonic functions.

Theorem 5 *For any harmonic function h defined in a neighbourhood of a ball $B(a, R)$*

$$h(a) = \frac{1}{\omega_n} \int_{\partial B(a,r)} h(x) d\sigma(x)$$

for any $r < R$.

Proof: Proposition 3 tells us that there is a pair of left monogenic functions f_1 and f_2 such that $h(x) = (x - a)f_1(x) + f_2(x)$ on $B(a, R)$. So $h(a) = f_2(a)$, and we have previously shown that $\frac{1}{\omega_n}\int_{\partial B(a,r)} f_2(x)d\sigma(x) = f_2(a)$. Now $\int_{\partial B(a,r)}(x-a)f_1(x)d\sigma(x) = r\int_{\partial B(a,r)} n(x)f_1(x)d\sigma(x) = 0$. □

The following is an immediate consequence of Proposition 3.

Proposition 4 *If $h_l(x)$ is a harmonic polynomial homogeneous of degree l then*

$$h_l(x) = p_l(x) + xp_{l-1}(x)$$

where p_l is a left monogenic polynomial homogeneous of degree l while p_{l-1} is a left monogenic polynomial which is homogeneous of degree $l - 1$.

It is well known that pairs of homogeneous harmonic polynomials of differing degrees of homogeneity are orthogonal with respect to the usual inner product over the unit sphere. Proposition 4 offers a further refinement to this. Suppose that f and g are Cl_n valued functions defined on S^{n-1} and each component of f and g is square integrable. If we define the Cl_n inner product of f and g to be

$$< f, g >= \frac{1}{\omega_n}\int_{S^{n-1}} \overline{f(x)}g(x)d\sigma(x)$$

then if f and g are both real valued this inner product is equal to

$$\frac{1}{\omega_n}\int_{S^{n-1}} f(x)g(x)d\sigma(x)$$

which is the usual inner product for real valued square integrable functions defined on S^{n-1}. Now

$$< xp_{l-1}(x), p_l(x) >= -\frac{1}{\omega_n}\int_{S^{n-1}} \bar{p}_{l-1}(x)xp_l(x)d\sigma(x)$$

$$= -\frac{1}{\omega_n}\int_{S^{n-1}} \bar{p}_{l-1}(x)n(x)p_l(x)d\sigma(x) = 0.$$

The evaluation of the last integral is an application of Cauchy's theorem.

Let us denote the space of Cl_n valued functions defined on S^{n-1} and such that each component is square integrable by $L^2(S^{n-1}, Cl_n)$. Clearly the space of real valued square integrable functions defined on S^{n-1} is a subset of $L^2(S^{n-1}, Cl_{n-1})$. The space $L^2(S^{n-1}, Cl_n)$ is a Cl_n module.

We have shown that by introducing the module $L^2(S^{n-1}, Cl_n)$ Proposition 4 provides a further orthogonal decomposition of harmonic polynomials using left monogenic polynomials. We shall return to this theme later. This decomposition was introduced for the case $n = 4$ by Sudbery [Su] and independently extended for all n by Sommen [So2].

Let us now consider higher order iterates of the Dirac operator D. In the same way as we have that $DH(x) = G(x)$ there is a function $G_3(x)$ defined on $R^n\backslash\{0\}$ such that $DG_3(x) = H(x)$. Specifically $G_3(x) = C(n,3)\frac{x}{\|x\|^{n-2}}$ for some dimensional constant $C(n,3)$. Continuing inductively we may find a function $G_k(x)$ on $R^n\backslash\{0\}$ such that $DG_k(x) = G_{k-1}(x)$. Specifically

$$G_k(x) = C(n,k)\frac{x}{\|x\|^{n-k+1}}$$

when n is odd and so is k.

$$G_k(x) = C(n,k)\frac{1}{\|x\|^{n-k}}$$

when n is odd and k is even

$$G_k(x) = C(n,k)\frac{x}{\|x\|^{n-k+1}}$$

when n is even, k is odd and $k < n$

$$G_k(x) = C(n,k)\frac{1}{\|x\|^{n-k}}$$

when n is even, k is even and $k < n$

$$G_k(x) = C(n,k)(x^{k-n}\log\|x\| + A(n,k)x^{k-n})$$

when n is even and $k \geq n$. In the last expression $A(n,k)$ is a real constant dependent on n and k. $C(n,k)$ is a constant dependent on n and k throughout.

It should be noted that $G_1(x) = G(x)$ and $G_2(x) = H(x)$. It should also be noted that $D^kG_k(x) = 0$.

Here is a simple technique for constructing solutions to the equation $D^kg = 0$ from left monogenic functions. The special case $k = 2$ was illustrated in Proposition 2.

Proposition 5 *Suppose that f is a left monogenic function on U then* $D^kx^{k-1}f(x) = 0$.

Proof The proof is by induction. We have already seen the result to be true in the case $k = 2$ in Proposition 2. If k is odd then $Dx^{k-1}f(x) = (k-1)x^{k-2}f(x)$. If k is even then

$$Dx^{k-1}f(x) = -n(k-1)x^{k-2}f(x) + x^{k-2}\Sigma_{j=1}^{n}e_jx\frac{\partial f(x)}{\partial x_j}.$$

By arguments presented in Proposition 5 this expression is equal to

$$-n(k-1)x^{k-2}f(x) + x^{k-2}\Sigma_{j=1}^{n}x_j\frac{\partial f(x)}{\partial x_j}.$$

The induction hypothesis tells us that the only term we need consider is $x^{k-2}\Sigma_{j=1}^{n} x_j \frac{\partial f(x)}{\partial x_j}$. However $\Sigma_{j=1}^{n} x_j \frac{\partial f(x)}{\partial x_j}$ is a left monogenic function. So proof by induction is now complete. □

In future we shall refer to a function $g : U \to Cl_n$ which satisfies the equation $D^k g = 0$ as a left k-monogenic function. Similarly if $h : U \to Cl_n$ satisfies the equation $hD^k = 0$ then h is a right k-monogenic function. In the case where $k = 1$ we return to the setting of left, or right, monogenic functions and when $k = 2$ we return to the setting of harmonic functions. When $k = 4$ the equations $D^4 g = 0$ and $gD^4 = 0$ correspond to the equations $\triangle_n^2 g = 0$ and $\triangle_n^2 h = 0$. So left or right 4-monogenic functions are in fact biharmonic functions. In greater generality if k is even then a left or right k-monogenic function f automatically satisfies the equation $\triangle_n^{\frac{k}{2}} f = 0$.

Proposition 6 *Suppose that p is a left k-monogenic polynomial homogeneous of degree q then there are left monogenic polynomials $f_0, \ldots, f_{k-1}$ such that*

$$p(x) = f_0(x) + \ldots + x^{k-1} f_{k-1}(x)$$

and each polynomial f_j is homogeneous of degree $q - j$ whenever $q - j \geq 0$ and is identically zero otherwise.

Proof: The proof is via induction on k. The case $k = 2$ is established immediately after the proof of Proposition 2. Let us now consider $Dp(x)$. This is a left $k-1$-monogenic polynomial homogeneous of degree $q - 1$. So by the induction hypothesis $Dp(x) = g_1(x) + \ldots + x^{k-2} g_{k-1}(x)$ where each g_j is a left monogenic polynomial homogeneous of degree $q - j$ whenever $q - j \geq 0$ and is equal to zero otherwise. Using Euler's lemma and the observations made after the proof of Proposition 5 one may now find left monogenic polynomials $f_1(x), \ldots, f_{k-1}(x)$ such that $D(xf_1(x) + \ldots + x^{k-1} f_{k-1}(x)) = Dp(x)$ and $f_j(x) = c_j g_j(x)$ for some $c_j \in R$ and for $1 \leq j \leq k-1$. It follows that $p(x) - \Sigma_{j=1}^{k-1} x^j f_j(x)$ is a left monogenic polynomial f_0 homogeneous of degree q. □

One may now use Proposition 6 and the arguments used to establish Proposition 3 to deduce:

Theorem 6 *Suppose that f is a left k-monogenic function defined in a neighbourhood of the ball $B(0, R)$ then there are left monogenic functions $f_0, \ldots, f_{k-1}$ defined on $B(0, R)$ such that $f(x) = f_0(x) + \ldots + x^{k-1} f_{k-1}(x)$ on $B(0, R)$.*

Theorem 6 establishes an Almansi decomposition for left k-monogenic functions in terms of left monogenic functions over any open ball. It also follows from this theorem that each left k-monogenic function is a real analytic function. It is also reasonably well known that if h is a biharmonic function defined in a neighbourhood of $B(0, R)$ then there are harmonic functions h_1 and h_2 defined on $B(0, R)$ and such that $h(x) = h_1(x) + \|x\|^2 h_2(x)$. In the special case where $k = 4$ Theorem 6 both establishes this result and refines it.

As each left k-monogenic function is a real analytic function then we can immediately use Stokes' theorem to deduce the following Cauchy-Green type formula.

Theorem 7 *Suppose that f is a left k-monogenic function defined on some domain U and suppose that S is a piecewise C^1 compact surface lying in U and bounding a bounded subdomain V of U. Then for each $y \in V$*

$$f(y) = \frac{1}{\omega_n} \int_S (\Sigma_{j=1}^k (-1)^{j-1} G_j(x-y) n(x) D^{j-1} f(x)) d\sigma(x).$$

5 Vahlen Matrices and Clifford Analysis

Here we will examine the role played by the conformal group within parts of Clifford analysis. Our starting point is to ask what type of diffeomorphisms acting on subdomains of R^n preserve monogenic functions. If a diffeomorphism ϕ can transform the class of left monogenic functions on one domain U to a class of left monogenic functions on the domain $\phi(U)$ and do the same for the class of right monogenic functions on U then it must preserve Cauchy's theorem. So if f is left monogenic on U and g is right monogenic on U and these functions are transformed to f' and g' respectively left and right monogenic functions on $\phi(U)$ then

$$\int_S g(x) n(x) f(x) d\sigma(x) = 0 = \int_{\phi(S)} g'(y) n(y) f'(y) d\sigma(y)$$

where S is a piecewise C^1 compact surface lying in U and $y = \phi(x)$. An important point to note here is that we need to assume that ϕ preserves vectors orthogonal to the tangent spaces at x and $\phi(x)$. As the choice of x and S is arbitrary it follows that the diffeomorphism ϕ is angle preserving. In other words ϕ is a conformal transformation. A theorem of Liouville [Lio] tells us that for dimensions 3 and greater the only conformal transformations on domains are Möbius transformations.

In order to deal with Möbius transformations using Clifford algebras we need to introduce some more algebra.

Consider the action $e_1 x e_1$. This gives rise to a reflection along the line spanned by e_1. More specifically on placing $x = x_1 e_1 + x_2 e_2 + \ldots + x_n e_n$ and multiplying out $e_1(x_1 e_1 + x_2 e_2 + \ldots + x_n e_n) e_1$ we get $-x_1 e_1 + x_2 e_2 + \ldots + x_n e_n$, and this describes the desired reflection. In greater generality we may take a vector $y \in S^{n-1}$, the unit sphere in R^n, and consider the triple product yxy where as before $x \in R^n$. One may rewrite x as $\lambda(x) y + y_x^\perp$, where $\lambda(x) \in R$ and $y_x^\perp$ is a vector in R^n that is perpendicular to y. In this case $yxy = \lambda(x) y^3 + y y_x^\perp y$. This expression simplifies to $-\lambda(x) y + y_x^\perp$. This shows that the triple product yxy gives rise to a reflection in R^n along the line spanned by the vector y.

One may now consider a finite sequence of vectors $y_1, \ldots, y_p \in S^{n-1}$. On placing $a = y_1 \ldots y_p$ and proceeding inductively we may observe that the triple product $ax\tilde{a} = y_1 \ldots y_p x y_p \ldots y_1$ gives rise to a sequence of consecutive reflections along the lines spanned by the vectors $y_1, \ldots, y_p$.

Definition 2 *A linear transformation $O : R^n \to R^n$ is called an orthogonal transformation if for each pair of vectors $x, y \in R^n$ the inner product $< x, y >$ is equal to $< O(x), O(y) >$.*

It is a simple exercise to deduce that any reflection is an example of an orthogonal transformation, and that the set, $O(n)$, of all orthogonal transformations over R^n forms a group under composition. Therefore the triple product $ax\tilde{a}$ describes an orthogonal transformation. Moreover the set $\{a \in Cl_n : a = y_1 \ldots y_p$ with $y_j \in S^n \subset R^n$ and $1 \leq j \leq p$ for p an arbitrary positive integer$\}$ is a group lying inside the Clifford algebra Cl_n. Traditionally this group is called the pin group and is denoted by $Pin(n)$. For each $a \in Pin(n)$ as $a = y_1 \ldots y_p$ and each $y_j \in S^{n-1}$ then $a\overline{a} = 1$. Thus $Pin(n)$ is a subset of the unit sphere in Cl_n.

Our previous construction shows that there is a group homomorphism

$$\theta : Pin(n) \to O(n) : \theta(a) = O_a.$$

where $O_a(x) = ax\tilde{a}$. We would like to show that the group homomorphism θ is surjective. This follows automatically from the the fact that each orthogonal transformation $O \in O(n)$ can be expressed as the composition of at most n reflections.

So we have seen that Clifford algebras are well equipped to describe orthogonal transformations and inversion of vectors. Both of these types of functions can be extended to homeomorphisms over the one point compactification $R^n \cup \{\infty\}$ of R^n. These are special examples of Möbius transformations. Also for each $v \in R^n$ the translation map $T_v : R^n \to R^n : T_v(x) = x + v$ can be extended to a homeomorphism over $R^n \cup \{\infty\}$ by setting $T_v(\infty) = \infty$. Furthermore for $\lambda \in R^+$ we can extend the dilation map $D_\lambda : R^n \to R^n : D_\lambda(x) = \lambda x$ to a homeomorphism over $R^n \cup \{\infty\}$ by setting $D_\lambda(\infty) = \infty$. These are all examples of Möbius transformations.

Definition 3 *A Möbius transformation is a function $M : R^n \cup \{\infty\} \to R^n \cup \{\infty\}$ which can be expressed as a finite composition of translations, dilations, orthogonal transformations and inversions.*

Each Möbius transformation is a homeomorphism from $R^n \cup \{\infty\}$ to itself. The set of all Möbius transformations of $R^n \cup \{\infty\}$ forms a group and we denote it by $M(n)$.

In general it is rather messy to write out an arbitrary Möbius transformations as the definition is expressed in terms of generators of the group $M(n)$. In two dimensions one can identify R^2 with the complex plane C in the usual way. Then on restricting to only using the special orthogonal group rather than the full orthogonal group a Möbius transformation can be expressed as $\frac{az+b}{cz+d}$ where a, b, c and $d \in C$ and $det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1$.

Over the quaternion algebra we can write out Möbius transformations of the type $(aq+b)(cq+d)^{-1}$ where a, b, c, d and $q \in \mathbf{H}$ and the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible in the algebra $\mathbf{H}(2)$ of 2×2 matrices with quaternionic coefficents. This certainly mimics the two dimensional situation and is easier than working with generators of a group. It would be nice to find conditions on the quaternionic coefficents a, b, c and d such that when $q = x \in R^3$ then $(ax+b)(cx+d)^{-1}$ is a Möbius transformation on $R^3 \cup \{\infty\}$ where R^3 is the span of i, j and k. We shall now try to determine such conditions.

Let us first assume that $c \neq 0$. In this case $(ax+b)(cx+d)^{-1}$ factors as $(ac^{-1}(cx+d) + (b - ac^{-1}d))(cx+d)^{-1}$ which in turn factors to $ac^{-1} + (b - ac^{-1}d)(cx+d)^{-1}$. It would be reasonable to assume that this expression represents a translation through a vector $ac^{-1} \in R^3$ followed by a orthogonal transformation and dilation represented by $cx\tilde{c}$ and a further translation and one inversion. In order for $cx\tilde{c}$ to represent a dilation and special orthogonal transformation we should consider c to be a product of non-zero vectors from R^3. As we are assuming that $ac^{-1} \in R^3$ it would follow that $a \in R^3$. Also we should place $b - ac^{-1}d = \lambda\tilde{c}^{-1}$, where $\lambda \in R\backslash\{0\}$. It would then follow that $d\tilde{c} \in R^3$, and so d is also a product of vectors from R^3. As $d\tilde{c}$ is assumed to belong to R^3 then $d\tilde{c} = \widetilde{d\tilde{c}} = c\tilde{d}$ and the equation $b - ac^{-1}d = \lambda\tilde{c}^{-1}$ becomes $b\tilde{c} - a\tilde{d} = \lambda$.

Let us now turn to the case where $c = 0$. In order to get a Möbius transformation we must assume that $d \neq 0$. In order that the expression $(ax+b)d^{-1}$ remains in $R^3 \cup \{\infty\}$ we need to assume $bd^{-1} \in R^3$. Consequently we need to assume that b is also a product of vectors from R^3. We also need to assume that $d^{-1} = \lambda\tilde{a}$ for some $\lambda \in R\backslash\{0\}$.

In summary if the quaternions a, b, c and d are all products of vectors from R^3 and satisfy the following criteria:

(i) $ac^{-1}, d\tilde{c} \in R^3$, provided $c \neq 0$

(ii) $a\tilde{d} - b\tilde{c} = \pm 1$

(iii) $bd^{-1} \in R^3$ when $c = 0$

then the expression $(ax+b)(cx+d)^{-1}$ describes a Möbius transformation on $R^3 \cup \{\infty\}$.

One remarkable feature of the conditions that we worked out for the coefficents a, b, c and d is that they readily extend to all dimensions. This extension was apparently first worked out by Karl Theodor Vahlen [V] in 1902, and was re-introduced by Ahlfors [A] over eighty years later.

The trick to extend to all Clifford algebras Cl_n, which are no longer division algebras, is to consider coefficents a, b, c and d in Cl_n that are all products of vectors from R^n. Then in this case each element a, b, c and d is either invertible in Cl_n or the 0 vector. Let us also impose the following three conditions on a, b, c and d.

(i) $ac^{-1}, d\tilde{c} \in R^n$ when $c \neq 0$

(ii) $a\tilde{\ }d - b\tilde{c} = \pm 1$
(iii) $bd^{-1} \in R^n$.

It is now an easy exercise to repeat our earlier calculations over the quaternions in the new context and see that when $\underline{x} \in R^n$ the expression $(a\underline{x}+b)(c\underline{x}+d)^{-1}$ factorizes to equal $ac^{-1} \pm (c\underline{x}\tilde{x}+d\tilde{c})^{-1}$ when $c \neq 0$ and it simplifies to $\pm a\underline{x}\tilde{a} + bd^{-1}$ when $c = 0$. In both cases these are examples of Möbius transformations.

Definition 4 *A matrix* $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *with coefficents belonging to* Cl_n *is called a Vahlen matrix if*
(i) The coefficents a, b, c, *and* d *are all products of vectors from* R^n.
(ii) $a\tilde{c}$, $c\tilde{d}$, $d\tilde{b}$ *and* $d\tilde{a} \in R^n$
and
(iii) $a\tilde{d} - b\tilde{c} = \pm 1$.

The conditions (ii) and (iii) in the last definition are not quite as rigid as might be first thought. For instance if $a\tilde{c} \in R^n$ then $a\tilde{c} = \widetilde{a\tilde{c}} = c\tilde{a}$. Consequently $\tilde{a}c \in R^n$. Also $\tilde{c}xc \in R^n$ for each $x \in R^n$. In particular $\tilde{c}(a\tilde{c})c \in R^n$. But $\tilde{c}c$ is a scalar, and so $\tilde{c}a \in R^n$. On letting $\sim$ act on this vector we see that $\tilde{a}c$ belongs to R^n too. Similar arguments may be made for the other vectors arising in condition (ii) of the previous definition.

Similarly $\widetilde{a\tilde{d} - b\tilde{c}} = \widetilde{\pm 1}$. So $d\tilde{a} - c\tilde{b} = \pm 1$. Also consider $\tilde{c}(a\tilde{d} - b\tilde{c})c$. This is equal to $\pm c\tilde{c}$. But $\tilde{c}(a\tilde{d} - b\tilde{c})c = \tilde{c}a\tilde{d}c - \tilde{c}b\tilde{c}c$. As $\tilde{c}a = \tilde{a}c$ and $\tilde{d}c = \tilde{c}d$ then this last expression simplifies to $(\tilde{a}d - \tilde{c}b)(c\tilde{c})$. So provided $c \neq 0$ then $\tilde{a}d - \tilde{c}b = \pm 1$. If $c = 0$ then up to a sign a and d are inverses of each other. Similarly $\tilde{d}a - \tilde{b}c = \pm 1$.

A main point to remember is that Cl_n is not a commutative algebra. The lack of commutativity necessitates all the calculations in the previous two paragraphs.

Let us now assume that we have a Vahlen matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Let us suppose also that $c \neq 0$. Then $\tilde{c}(cx\tilde{c} + d\tilde{c})^{-1}$ is well defined provided $cx\tilde{c} \neq -d\tilde{c}$. But this expression factors to $(cx+d)^{-1}$. Hence the term $(ax+b)(cx+d)^{-1}$ is well defined within the algebra Cl_n and for each $x \in R^n$ provided $cx\tilde{c} \neq -d\tilde{c}$. In this case the expression $(ax+b)(cx+d)^{-1}$ is equal to $ac^{-1} \pm (cx\tilde{c} + d\tilde{c})^{-1}$. As $c^{-1} = \frac{\tilde{c}}{c\tilde{c}}$ then $ac^{-1} \in R^n$ and $(ax+b)(cx+d)^{-1}$ is a Möbius transformation over $R^n \cup \{\infty\}$. When $c = 0$ the expression $(ax+b)(cx+d)^{-1}$ is equal to $\pm(ax\tilde{a} + bd^{-1})$ which again can be seen to be a Möbius transformation over $R^n \cup \{\infty\}$. So each Vahlen matrix can easily be used to describe a Möbius transformation.

Examples of Vahlen matrices include $\begin{pmatrix} \lambda^{\frac{1}{2}} & 0 \\ 0 & \lambda^{-\frac{1}{2}} \end{pmatrix}$ where $\lambda \in R^+$, $\begin{pmatrix} a & 0 \\ 0 & \tilde{a}^{-1} \end{pmatrix}$ where $a \in Pin(n)$, $\begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix}$ where $v \in R^n$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. These are the Vahlen matrices that correspond to dilation by λ, orthogonal transformation by $\theta(a)$, translation by v and inversion. So each generator of the Möbius group $M(n)$ has a corresponding Vahlen matrix.

We can in fact easily combine dilation and orthogonal transformation by considering the group generated by $R^n \backslash \{0\}$ within the algebra Cl_n. We shall call this group the Clifford group and we denote it by Γ_n. In fact $\Gamma_n = Pin(n) \times R^+$. Hence $\begin{pmatrix} a & 0 \\ 0 & \tilde{a}^{-1} \end{pmatrix}$ is the Vahlen matrix corresponding to the Möbius transformation $ax\tilde{a}$ whenever $a \in \Gamma_n$. Note that for each $a \in \Gamma_n$ then $\|a\|^2 = a\bar{a}$, also if $a = y_1 \ldots y_r$ with each $y_j \in R^n \backslash \{0\}$ for $1 \leq j \leq r$ then $\|a\| = \|y_1\| \ldots \|y_r\|$.

The Vahlen matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ corresponds to the Möbius transformation $-x^{-1} = \frac{x}{\|x\|^2}$ and this transformation gives what is commonly called the Kelvin inverse of a non-zero vector $x \in R^n$.

From the expression $ac^{-1} \pm (cx\tilde{c} + d\tilde{c})^{-1}$ it may be assumed that whenever $c \neq 0$ the Vahlen matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is equal to

$$\begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c & 0 \\ 0 & \tilde{c}^{-1} \end{pmatrix} \begin{pmatrix} 1 & c^{-1}d \\ 0 & 1 \end{pmatrix}. \qquad (2)$$

One can verify this assumption by multiplying out these matrices. Similarly one can verify that

$$\begin{pmatrix} a & b \\ 0 & \tilde{a}^{-1} \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & \tilde{a}^{-1} \end{pmatrix} \begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix}. \qquad (3)$$

Consequently each Vahlen matrix can be expressed as a finite product of the Vahlen matrices linked to the generators of the Möbius group acting over $R^n \cup \{\infty\}$. Let us denote the set of all Vahlen matrices with coefficents in $\Gamma_n \cup \{0\}$ by $V(n)$.

Theorem 8 *The set $V(n)$ is a group under matrix multiplication.*

Proof: The identity matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ can easily be verified to be a Vahlen matrix. Let us now consider an arbitrary Vahlen matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and its product with the following special Vahlen matrices:

(i) $\begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix}$ where $v \in R^n$

(ii) $\begin{pmatrix} a & 0 \\ 0 & \tilde{a}^{-1} \end{pmatrix}$ where $a \in \Gamma_n$

(iii) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

On multiplying A on the right by either the matrix appearing in (ii) or (iii) it is easy to verify that the resulting matrix is indeed a Vahlen matrix.

It remains to show that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & av + b \\ c & cv + d \end{pmatrix}$$

is a Vahlen matrix. The main problem arises in verifying that $(av + b)(\widetilde{cv + d})$ is an element of R^n. However, up to a scalar $(av+b)(\widetilde{cv + d}) = (av+b)(cv+d)^{-1}$ whenever $cv\tilde{c} \neq -d\tilde{c}$. Consequently $(av + b)(\widetilde{cv + d}) \in R^n$ whenever $cv\tilde{c} \neq d\tilde{c}$. When $cv\tilde{c} = d\tilde{c}$ then $(av + b)(\widetilde{cv + d}) = 0$. It now follows from expressions (1) and (2) that $V(n)$ is a group. □

From expressions (2) and (3) and the proof of the previous theorem it may be observed that the matrices $\begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix}$ where $v \in R^n$, $\begin{pmatrix} a & 0 \\ 0 & \tilde{a}^{-1} \end{pmatrix}$ where $a \in \Gamma_n$, and $\begin{pmatrix} 0 & \pm 1 \\ 1 & 0 \end{pmatrix}$ are generators of the group $V(n)$. Also our calculations on Vahlen matrices and Möbius transformations provide a surjective group homomorphism

$$\vartheta : V(n) \to M(n) : \vartheta(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = (ax+b)(cx+d)^{-1}.$$

It is easy to check that both $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ belong to the kernel of ϑ. One can also see that the Vahlen matrices $\begin{pmatrix} \pm e_1 \ldots e_n & 0 \\ 0 & \mp(e_n \ldots e_1)^{-1} \end{pmatrix}$ belong to the kernel of ϑ. It is left as an exercise to verify that these are the only matrices in the kernel of ϑ.

Having used Clifford algebras to describe Möbius transformations in such an elegant way we can now use this approach to introduce some properties of Möbius transformations that cannot be described without the use of Clifford algebras. These are all generalizations of known properties of Möbius transformations in the complex plane. Let us begin with the cross ratio.

For an ordered quadruple of points $\{w_1, w_2, w_3, w_4\}$ lying in R^n, no three of which coincide, we define the cross ratio $[w_1, w_2, w_3, w_4]$ of these points to be

$$(w_1 - w_3)(w_1 - w_4)^{-1}(w_2 - w_4)(w_3 - w_4)^{-1}.$$

If $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in V(n)$ and $\psi(x) = (ax+b)(cx+d)^{-1}$ then for each distinct pair x and $y \in R^n$

$$\psi(x) - \psi(y) = (\widetilde{cy+d})^{-1}(x-y)(cx+d)^{-1}.$$

This equality is easy to verify when $c = 0$. When $c \neq 0$ then $\psi(x) - \psi(y) = (cx\tilde{c} + d\tilde{c})^{-1} - (cy\tilde{c} + d\tilde{c})^{-1}$. In turn this last expression is equal to $(cx\tilde{c}+d\tilde{c})^{-1}(cy\tilde{c} - cx\tilde{c})(cy\tilde{c} + d\tilde{c})^{-1}$. As $d\tilde{c} = c\tilde{d}$ then equation 3 follows. Thus it may be determined that

$$[\psi(w_1), \psi(w_2), \psi(w_3), \psi(w_4)] = (\widetilde{cw_3+d})^{-1}[w_1, w_2, w_3, w_4](\widetilde{cw_3+d}).$$

This formula is also the formula one would get in the complex plane setting. However, in that context the algebra is commutative and so the terms $(\widetilde{cw_3+d})^{-1}$ and $(\widetilde{cw_3+d})$ would cancel.

Closely related to the cross ratio is the Schwarzian derivative. Before introducing a generalization of the Schwarzian derivative let us first look at $\frac{\partial \psi(x)}{\partial x_j}$. When $c \neq 0$ then $\psi(x) = ac^{-1} \pm \tilde{c}^{-1}(x + c^{-1}d)^{-1}c^{-1}$. Before differentiating this expression let us consider the special case $\psi(x) = x^{-1}$. On considering $\frac{\partial x x^{-1}}{\partial x_j}$ it is easy to verify that $\frac{\partial x^{-1}}{\partial x_j} = -x^{-1}e_j x^{-1}$. It follows that when $c \neq 0$ then

$$\frac{\partial \psi(x)}{\partial x_j} = \mp \tilde{c}^{-1}(x + c^{-1}d)^{-1}e_j(x + c^{-1}d)^{-1}c^{-1}.$$

On noting that $c^{-1}d = \tilde{d}\tilde{c}^{-1}$ this equation simplifies to

$$\frac{\partial\psi(x)}{\partial x_j} = \mp(\widetilde{cx+d})^{-1}e_j(cx+d)^{-1}.$$

In greater generality, if y is a point on the unit sphere S^{n-1} then the partial derivative of $\psi(x)$ along the y direction is

$$\mp(\widetilde{cx+d})^{-1}y(cx+d)^{-1}.$$

It follows that for each x, $w \in R^n$ the derivative, $D\psi_x(w)$, of ψ at x and acting on w is $-(\widetilde{cx+d})^{-1}w(cx+d)^{-1}$. Whenever $x \neq c^{-1}d$ the term $cx+d$ belongs to the Clifford group Γ_n. So for fixed $x \in R^n\backslash\{c^{-1}d\}$ the derivative $D\psi_x(w)$ describes an orthogonal transformation combined with a dilation. Consequently for each fixed x the linear transformation $D\psi_x$ preserves angles between vectors. A differentiable function defined over a domain in R^n taking values in R^n and whose derivative at each point preserves the angle between vectors is called a conformal map. This definition for a conformal map readily extends to the setting of manifolds.

Let $\psi(x)_{yy}$ denote the second order partial derivative of $\psi(x)$ in the direction of y. So when $y = e_j$ then $\psi(x)_{yy} = \frac{\partial^2\psi(x)}{\partial x_j^2}$. Let also $\psi(x)_{yyy}$ denote the third order partial derivative of $\psi(x)$ in the y direction. Then

$$\psi(x)_{yy} = \mp 2\tilde{c}^{-1}(x+c^{-1}d)^{-1}y(x+c^{-1}d)^{-1}y(x+c^{-1}d)^{-1}c^{-1}$$

while

$$\psi(x)_{yyy} = \pm 6\tilde{c}^{-1}(x+c^{-1}d)^{-1}y(x+c^{-1}d)^{-1}y(x+c^{-1}d)^{-1}y(x+c^{-1}d)^{-1}c^{-1}.$$

A simple computation now reveals that

$$\psi(x)_{yyy}\psi(x)_y^{-1} - \frac{3}{2}(\psi(x)_{yy}\psi(x)_y^{-1})^2 = 0.$$

This is a direct analogue of the Schwarzian derivative arising in complex analysis. For Ω a domain in the complex plane and $f(z)$ a non-constant holomorphic function defined on Ω the Schwarzian derivative of f is defined to be $\{S,f\} = \frac{f'''}{f'} - \frac{3}{2}(\frac{f''}{f'})^2$, where f', f'' and f''' denote the first, second and third derivatives of f respectively.

Definition 5 *Suppose that U is a domain in R^n, so U is an open, connected non-empty subset of R^n, and suppose also that $F: U \to R^n$ is an invertible C^3 map. Then the Schwarzian derivative of F in the direction of y is defined to be*

$$\{S,F\}_y = F_{yyy}F_y^{-1} - \frac{3}{2}(F_{yy}F_y^{-1})^2,$$

where F_y, F_{yy} and F_{yyy} denote the first, second and third order partial derivatives of F in the direction of y respectively.

This Schwarzian derivative was introduced in [R6].

One main difference between this Schwarzian derivative and the classical one arising in the complex plane is that the Schwarzian we have introduced in Euclidean space depends on the direction $y \in S^{n-1}$. This problem is overcome by Wada [W].

As F is an invertible and differentiable map then F_y^{-1}, the multiplicative inverse in Cl_n of F_y, exists everywhere on U. As F_y, F_{yy} and F_{yyy} all belong to R^n then $\{S, F\}_y$ takes its values in the subspace of Cl_n spanned by 1, e_ie_j where $1 \leq i < j \leq n$ and $e_ie_je_ke_l$ where $1 \leq i < j < k < l \leq n$. This last component of quadruple products of the basis vectors $e_1 \ldots e_n$ only arises if $n \geq 4$, and then only arises in the second half of the formula for $\{S, F\}_y$. We shall denote the identity component of $\{S, F\}_y$ by $\{S, F\}_{y,0}$. We shall denote the component of $\{S, F\}$ spanned by the bivectors e_ie_j where $1 \leq i < j \leq n$ by $\{S, F\}_{y,i,j}$, while we shall denote the last component of $\{S, F\}_y$ by $\{S, F\}_{y,i,j,k,l}$.

One main feature of the Schwarzian derivative in the complex plane is that $\{S, f\} = \{S, M(f)\}$ for any Möbius transformation $\frac{az+b}{cz+d}$. Let us now see what happens in the Euclidean setting. On taking the composition $\psi(F)$ a straightforward mimic of previous arguments shows that the partial derivative of $\psi(F)$ in the direction of y is

$$\mp(\widetilde{cF(x) + d})^{-1}F(x)_y(cF(x) + d)^{-1},$$

or alternatively when $c \neq 0$

$$\mp\tilde{c}^{-1}(F(x) + c^{-1}d)^{-1}F(x)_y(F(x) + c^{-1}d)^{-1}.$$

One may now deduce that

$$\{S, \psi(F)\}_y = (cF + d)^{-1}\{S, F\}_y(cF + d).$$

As $ae_{j_1} \ldots e_{j_r}\tilde{a} = \pm ae_{j_1}\tilde{a}a \ldots \tilde{a}ae_{j_r}\tilde{a}$ for each $a \in Pin(n)$ and as the vectors $e_1, \ldots, e_n$ are an orthonormal basis for R^n the element $ae_{j_1} \ldots e_{j_r}\tilde{a}$ will be a linear combination of r-fold products of orthonormal vectors from R^n. Hence

$$\{S, \psi(F)\}_{y,0} = (cF + d)^{-1}\{S, F\}_{y,0}(cF + d) = \{S, F\}_{y,0}$$

and

$$\{S, \psi(F)\}_{y,i,j} = (cF + d)^{-1}\{S, F\}_{y,i,j}(cF + d),$$

and

$$\{S, \psi(F)\}_{y,i,j,k,l} = (cF + d)^{-1}\{S, F\}_{y,i,j,k,l}(cF + d).$$

So the scalar part of the Schwarzian derivative is left invariant under Möbius transformations while there is a simple and elegant formula to describe the conformal covariance of the other two components.

We shall now proceed to show that each Möbius transformation preserves monogenicity. Sudbery [Su] and also Bojarski [B] have established this fact. We will need the following two lemmata.

Lemma 1 *Suppose that $\phi(x) = (ax+b)(cx+d)^{-1}$ is a Möbius transformation then*

$$G(u-v) = J(\phi, x)^{-1} G(x-y) \tilde{J}(\phi, y)^{-1}$$

where $u = \phi(x)$, $v = \phi(y)$ and $J(\phi, x) = \frac{\widetilde{(cx+d)}}{\|cx+d\|^n}$.

Proof The proof essentially follows from the fact that

$$(x^{-1} - y^{-1}) = x^{-1}(y-x)y^{-1}.$$

Consequently $\|x^{-1} - y^{-1}\| = \|x\|^{-1}\|x-y\|\|y\|^{-1}$. Also $ax\tilde{a} - ay\tilde{a} = a(x-y)\tilde{a}$.

If one breaks the transformation down into terms arising from the generators of the Möbius group and use the previous set of equations then one will readily arrive at the result. □

Lemma 2 *Suppose that $y = \phi(x) = (ax+b)(cx+d)^{-1}$ is a Möbius transformation and for domains U and V we have $\phi(U) = V$ then*

$$\int_S f(u)n(u)g(u)d\sigma(u) = \int_{\psi^{-1}(S)} f(\psi(x))\tilde{J}(\psi, x)n(x)J(\psi, x)g(\psi(x))d\sigma(x)$$

where $u = \psi(x)$, S is a orientable hypersurface lying in U and $J(\psi, x) = \frac{\widetilde{cx+d}}{\|cx+d\|^n}$.

Outline Proof On breaking ψ up into the generators of the Möbius group the result follows from noting that

$$\frac{\partial x^{-1}}{\partial x_j} = -x^{-1}e_j x^{-1}.$$ □

It follows from Cauchy's Theorem that if $g(u)$ is a left monogenic function in the variable u then $J(\psi, x)f(\psi(x))$ is left monogenic in the variable x.

When $\phi(x)$ is the Cayley transformation $y = (e_n x + 1)(x + e_n)^{-1}$ we can use this transformation to establish a Cauchy-Kowalewska extension in a neighbourhood of the sphere. If $f(x)$ is a real analytic function defined on an open subset U of $S^{n-1}\backslash\{e_n\}$ then $l(y) = J(\phi^{-1}, y)^{-1}f(\phi(y))$ is a real analytic function on the open set $V = \phi^{-1}(U)$. This function has a Cauchy-Kowalewska extension to a left monogenic function $L(y)$ defined on an open neighbourhood $V(g) \subset R^n$ of V. Consequently $F(x) = J(\phi^{-1}, x)L(\phi^{-1}(x))$ is a left monogenic defined on an open neighbourhood $U(f) = \phi^{-1}(V(g))$ of U. Moreover $F_{|U} = f$. Combing with similar arguments for the other Cayley transformation $y = (-e_n x + 1)(x - e_n)^{-1}$ one can deduce:

Theorem 9 (Cauchy-Kowalewska Theorem) *Suppose that f is a Cl_n valued real analytic function defined on S^{n-1}. Then there is a unique left monogenic function F defined on an open neighbourhood $U(f)$ of S^{n-1} such that $F_{|S^{n-1}} = f$.*

In fact if $f(u)$ is defined on some domain and satisfies the equation $D^k f = 0$ then the function $J_k(\psi, x)f(\psi(x))$ satisfies the same equation, where $J_k(\psi, x) = \frac{\widetilde{cx+d}}{\|cx+d\|^{n-k+1}}$.

Theorem 10 (Sce's Theorem) *Suppose that $f = u + iv$ is a holomorphic function on a domain $\Omega \subset C$ and that $\Omega = \overline{\Omega}$ and $f(\overline{z}) = \overline{f(z)}$. Then the function $F(\underline{x}) = u(x_1, \|x'\|) + e_1^{-1}\frac{x'}{\|x'\|}v(x_1, \|x'\|)$ is a unital left $n-1$-monogenic function on the domain $\{\underline{x} : x_1 + i\|x'\| \in \Omega\}$ whenever n is even. Here $x' = x_2e_2 + \ldots + x_ne_n$.*

Proof: First let us note that $x^{-1}e_1$ is left $n-1$ monogenic whenever n is even. It follows that $\frac{\partial^k}{\partial x_1^k}x^{-1}e_1 = c_kx^{-k-1}e_1$ is $n-1$ left monogenic for each positive integer k. Here c_k is some non-zero real number. Using Kelvin inversion it follows that x^ke_1 is left $n-1$ monogenic for each positive integer k. By taking translations and Taylor series expansions for the function f the result follows. □

This result was first established for the case $n = 4$ by Fueter, [F], see also Sudbery [Su]. It was extended to all even dimensions by Sce [Sc], though the methods used do not make use of the conformal group.

6 Dirac Operators on Spheres

We can use Vahlen matrices to describe a Cayley transformation from R^n to the punctured sphere lying in R^{n+1} and then set up a Dirac operator with a Cauchy integral formula on the sphere. It should be mentioned that Dirac operators over general manifolds have been utilised in a number of contexts, see for instance [GiMu, LaMi, B-BW]. In [C, M2] attempts are made to draw Clifford analysis closer to this more general setting.

While one can regard the sphere, minus one point, to be topologically equivalent to euclidean space via a stereographic projection, and in one variable complex analysis one often interchanges domains in the complex plane for their stereographic images on the 2-sphere, this does not agree with the setting described here. While the stereographic projection does correspond to the Cayley transformation used here in the case of the sphere, in order to set up a Dirac operator on the sphere and push through the associated function theory we need to also use a multiplier operator from the Clifford algebra.

Let us now consider the algebra Cl_{n+1} and the Cayley transformation

$$C_1 : R^n \to S^n : x \to (x - e_{n+1})(-e_{n+1}x + 1)^{-1},$$

where $x = x_1e_1 + \ldots + x_ne_n \in R^n$, and e_{n+1} is a unit vector in R^{n+1} which is orthogonal to R^n. Now $C_1(R^n) = S^n \backslash \{e_{n+1}\}$.

For f a left monogenic function and g a right monogenic function defined on a domain $U \subseteq R^n$ we have that

$$0 = \int_S g(y)n(y)f(y)d\sigma(y)$$

$$= \int_{C_1^{-1}(S)} g(C_1^{-1}(x)) \tilde{J}(C_1^{-1}, x) n(x) J(C_1^{-1}, x) f(C_1^{-1}(x)) d\pi(x),$$

where S is a piecewise smooth surface lying in U and bounding a bounded subregion of U, and $C_1(y) = x$. Moreover, S^n has the Riemannian structure inherited as a submanifold of R^{n+1}. So $n(x)$ is a unit vector lying in the tangent space TS_x^n and is normal to $TC_1^{-1}(S)_x$. Also, π is the Lebesgue measure on the surface $C_1(S)$.

On applying Stokes' theorem to the integral

$$\int_{C_1^{-1}(S)} g(C_1^{-1}(x)) \tilde{J}(C_1^{-1}, x) n(x) J(C_1^{-1}, x) f(C_1^{-1}(x)) d\pi(x)$$

we obtain

$$\int_{C_1^{-1}(R)} (g(C_1^{-1}(x)) \tilde{J}(C_1^{-1}, x) D_{S^n}) J(C_1^{-1}, x) f(C_1^{-1}(x)) d\eta(x)$$

$$+ \int_{C_1^{-1}(R)} g(C_1^{-1}(x)) \tilde{J}(C_1^{-1}, x)(D_{S^n} J(C_1^{-1}, x) f(C_1^{-1}(x))) d\eta(x),$$

where R is the region bounded by S, D_{S^n} is the Dirac operator on S^n arising from the application of Stokes' theorem, and η is the Lebesgue surface measure on S^n.

As S is arbitrary it follows that

$$g(C_1^{-1}(x)) \tilde{J}(C_1^{-1}, x) D_{S^n} = 0$$

and

$$D_{S^n} J(C_1^{-1}, x) f(C_1^{-1}(x)) = 0.$$

Definition 6 *Suppose that V is a domain on the sphere S^n then a pointwise differentiable function $f : V \to Cl_{n+1}$ is said to be spherical left monogenic if $D_{S^n} f = 0$ on V.*

A similar definition may be given for spherical right monogenic functions.

From the previous calculations it is relatively easy to deduce:

Theorem 11 *A function $h(y)$ is spherical left monogenic if and only if*

$$J(C_1, x) h(C_1(x))$$

is left monogenic, where $C_1(x) = y$.

A similar result may be deduced for right monogenicity.

It follows from theorem 11 and [R5] that any spherical left monogenic function is also a real analytic function, and similarly any right spherical monogenic function is a real analytic function.

We would like a more explicit representation for the spherical Dirac operator D_{S^n}. We follow an argument presented by Cnops and Malonek in [CnM]. Suppose first that f and g are continuously differentiable functions defined on a domain U lying on the sphere S^n and f and q are Cl_{n+1} valued. We extend these functions to functions F and Q defined on a domain in R^{n+1} by placing $F(x) = r^{\frac{1-n}{2}}f(\eta)$ and $Q(x) = r^{\frac{1-n}{2}}q(\eta)$ where $x = r\eta$ and $\eta \in U$. Now consider the domain V in R^{n+1} where $V = \{x = r\eta : 0 < r_1 < r < r_2 \text{ and } \eta \in U\}$. Stokes' theorem tells us that if S is a piecewise smooth surface lying in U and bounding a subdomain W then

$$\int_{\partial W'} F(x)n(x)Q(x)d\sigma(x) = \int_{W'}((F(x)D)Q(x) + F(x)(DQ(x)))dx^{n+1}$$

where here D is the Dirac operator in R^{n+1} and $W' = \{r\eta : r_1 < r_3 \leq r \leq r_4 < r_2$ and $\eta \in W\}$.

The integral over $\partial W'$ can be divided into three parts. First there is the integral over the part where $r = r_3$ then there is the integral over the part where $r = r_4$ and last there is the integral over the part where $\eta \in S$. The first two integrals cancel each other. This is because the vectors $n(x)$ are opposite on these surfaces and on re-writing the integral to an integral over W the extensions of f and q combine to give the same values. The last integral can be rewritten as

$$\int_S f(\eta)n'(\eta)q(\eta)(\int_{r_3}^{r_4} r^{n-2}r^{1-n}dr)d\sigma(\eta)$$

where $d\sigma$ here represents the volume element of S.

Before evaluating the integral over W' let us express the Dirac operator D in terms of spherical co-ordinates. Now $D = -\eta\eta D$ and $\eta D = \frac{\partial}{\partial r} + \Gamma$ where Γ is purely a spherical operator. So $D = -\eta(\frac{\partial}{\partial r} + \Gamma)$ and $DQ(r\eta) = r^{\frac{-n-1}{2}}\eta(\Gamma - \frac{n-1}{2})q(\eta)$. A similar formula holds for $F(r\eta)D$. So

$$\int_{W'}((F(x)D)Q(x) + F(x)DQ(x))dx^{n+1} =$$

$$\int_W (f(\eta)(\Gamma - \frac{n-1}{2})\eta q(\eta) + f(\eta)\eta(\Gamma + \frac{n-1}{2})q(\eta))d\sigma(\eta)\int_{r_3}^{r_4} r^{-1}dr.$$

As S and W are arbitrary it now follows that $D_{S^n} = \eta(\Gamma - \frac{n-1}{2})$.

Now for any Möbius transformation $y = \psi(x) = (ax+b)(cx+d)^{-1}$ we have that

$$G(\psi(x) - \psi(u)) = \tilde{J}(\psi,x)^{-1}G(x-u)J(\psi,u)^{-1}. \quad (4)$$

This follows from the identity $(x^{-1} - u^{-1}) = u^{-1}(u-x)x^{-1}$, see for instance [PQ]. Consequently, from theorems 2 and 3 we obtain:

Theorem 12 *Suppose that V is a domain in $S^n\backslash\{e_{n+1}\}$ and R is a subdomain of U, with $cl(R) \subset U$. Moreover, $C_1^{-1}(S)$ is a piecewise smooth surface, where $S = \partial U$, the boundary of U. Then for each $y \in R$ and each spherical left monogenic function f defined on U we have:*

$$f(y) = \frac{1}{\omega_n} \int_S G(x-y)n(x)f(x)d\pi(x).$$

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