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Scattering amplitude and Poisson relation in obstacle scattering

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1 Introduction

The scattering operator $S(\lambda)$ constitutes a mathematical model for the data obtained in a scattering experiment. The kernel of $S(\lambda) - I$ is given by the *scattering amplitude* $a(\lambda, \theta, \omega)$ which roughly speaking is the leading term of the asymptotic of an outgoing solution $v_s(r\theta, \lambda)$ as $|x| = r \to \infty$. In the obstacle scattering the perturbation is a bounded obstacle K with a connected complement Ω and $S(\lambda)$ is related to the boundary problems for the wave equation in $\Omega \times \mathbb{R}$ (see [4]). For the inverse scattering problems it is rather important to describe a scattering data which we can observe. In this direction the *sojourn times* of the so called (ω, θ)-rays present a natural observable data. In the last twenty years the progress of the microlocal analysis led to many new ideas and results concerning the asymptotics of the scattering amplitude and the singularity of the Fourier transform $s(t, \theta, \omega)$ of $s(\lambda, \theta, \omega)$ called *scattering kernel*. The singularities of $s(t, \theta, \omega)$ are included in the set of sojourn times of the (ω, θ)-rays. The purpose of this article is to present a brief description of some results in this direction and to motivate the study of the inverse scattering problems.

2 Scattering amplitude for strictly convex obstacles

Let $K \subset \mathbb{R}^3$ be a bounded domain with C^∞ boundary ∂K and connected complement $\Omega = \mathbb{R}^3 \setminus \overline{K}$. We will consider the Dirichlet problem for the Laplacian.

To introduce the scattering amplitude $a(\lambda, \theta, \omega)$, $\theta, \omega \in S^2$, consider the ***outgoing solution*** $v_s = v_s(x, \lambda)$ of the problem

$$\begin{cases} (\Delta + \lambda^2)v_s = 0 \text{ in } \Omega, \\ v_s + e^{-i\lambda\langle x,\omega\rangle} = 0 \text{ on } \partial K \end{cases} \tag{2.1}$$

satisfying the so called $(i\lambda)$ - outgoing Sommerfeld radiation condition. This condition means that as $|x| = r \longrightarrow \infty$ we have

$$v_s(r\theta, \lambda) = \frac{e^{-i\lambda r}}{r}\Big(a(\lambda, \theta, \omega) + \mathcal{O}\big(\frac{1}{r}\big)\Big), \; x = r\theta .$$

We can interpret $v_i = e^{-i\lambda<x,\omega>}$ as an ***incoming plane wave***, while $v_s(x, \lambda)$ is the ***outgoing wave*** obtained after the impact of v_i on ∂K. To obtain a formula for the leading term $a(\lambda, \theta, \omega)$ we apply the Green formula combined with the outgoing condition and deduce the following representation

$$v_s(x, \lambda) = \int_{\partial K}\Big[E_\lambda(x - y)\frac{\partial v_s}{\partial \nu}(y, \lambda) - \frac{\partial E_\lambda}{\partial \nu}(x - y)v_s(y, \lambda)\Big]dS_y , \tag{2.2}$$

where $E_\lambda(x)$ is the outgoing Green function

$$E_\lambda(x) = -\frac{1}{4\pi}\frac{e^{-i\lambda r}}{r}$$

and $\nu(x)$ is the unit normal to $x \in \partial K$ pointing into Ω. Next, we multiply (2.2) by $e^{i\lambda r}r$, put $x = r\theta$ and taking the limit $r \to \infty$ we get

$$a(\lambda, \theta, \omega) = -\frac{1}{4\pi}\int_{\partial K}\Big(i\lambda < \nu(x), \theta > e^{i\lambda<x,\theta-\omega>} + e^{i\lambda\langle x,\theta\rangle}\frac{\partial v_s}{\partial \nu}(x, \lambda)\Big)dS_x , \tag{2.3}$$

where $< \bullet, \bullet >$ denotes the scalar product in $\mathbb{R}^3$.

In the physical literature $a(\lambda, \theta, \omega)$ is called ***scattering amplitude*** and the analysis of the leading term of the asymptotic of $a(\lambda, \theta, \omega)$ as $\lambda \to +\infty$ in the mathematical physics has a long tradition. The simplest case to deal with is that when $\theta \neq \omega$ and K is a strictly convex obstacle. First consider the integral

$$I(\lambda) = -\frac{i\lambda}{4\pi}\int_{\partial K} < \nu(x), \theta > e^{i\lambda<x,\theta-\omega>}dS_x .$$

The phase function $< x, \theta - \omega > |_{x\in\partial K}$ has two critical points $x_\pm$ with

$$< x_+, \theta - \omega >= \min_{y\in\partial K} < y, \theta - \omega >,$$

$$< x_-, \theta - \omega >= \max_{y\in\partial K} < y, \theta - \omega >,$$

$$\nu(x_{\pm}) = \pm \frac{\theta - \omega}{|\theta - \omega|}.$$

Here x^- denotes the point in the *illuminated region* (see Figure 1)

$$\partial K_+(\omega) = \{y \in \partial K :< \nu(y), \omega >< 0\}$$

related to ω, while x^- lies in the *shadow region*

$$\partial K_-(\omega) = \{y \in \partial K :< \nu(y), \omega >> 0\}.$$

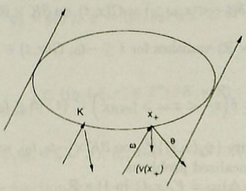

Figure 1

Applying a stationary phase argument we obtain

$$I(\lambda) = \tfrac{1}{2} e^{i\lambda < x_+, \theta - \omega >} \mathcal{K}(x_+)^{-1/2} < \nu(x_+), \theta > \tag{2.4}$$

$$-\frac{1}{2} e^{i\lambda < x_-, \theta - \omega >} \mathcal{K}(x_-)^{-1/2} < \nu(x_-), \theta > + \mathcal{O}(|\lambda|^{-1}),$$

$\mathcal{K}(y) > 0$ being the Gauss curvature at $y \in \partial K$.

The analysis of the term involving $\frac{\partial v_s}{\partial \nu}$ is more complicated. In the mathematical physics many efforts have been concerned with the construction of an approximative outgoing solution $w_0(x, \lambda)$ of the problem

$$\begin{cases} (\Delta + \lambda^2) w_0 = f(x, \lambda) \text{ in } \Omega, \\ w_0 + e^{-i\lambda \langle x, \omega \rangle} = g(x, \lambda) \text{ on } \partial K \end{cases}$$

with $f(x, \lambda) \in C^\infty(\Omega)$, $g(x, \lambda) \in C^\infty(\partial K)$. This leads to considerable difficulties when we must describe the form of the solution w_0 in a domain close to the grazing submanifold

$$G(\omega) = \{y \in \partial K : < \nu(y), \omega >= 0\}.$$

In the seventies the progress of the microlocal analysis led to the investigation of the above problem without a precise information for w_0 in a neighbourhood of $G(\omega)$. This was given by Majda [5] exploiting the works of Hörmander [3], Taylor [17] and Melrose [7] for the propagation of the singularities. We will expose briefly the approach of Majda and refer to [5] for more details.

Consider the boundary problem

$$\begin{cases} (\partial_t^2 - \Delta)u_0 = F(x,t) \text{ in } \Omega \times \mathbb{R}, \\ u_0 + \delta(t - < x, \omega >) = G(x,t) \text{ on } \partial K \times \mathbb{R}, \end{cases}$$

where $F(x,t) \in C^\infty(\Omega \times \mathbb{R})$ vanishes for $t \leq -t_0$, $G(x,t) \in C_0^\infty(\partial K \times \mathbb{R})$ and t_0 is chosen so that

$$\text{supp}_t\ \delta\Big(t - < x.\omega > |_{x \in \partial K}\Big) \subset \{t : |t| \leq t_0\}.$$

Taking a partition of unity $\{\psi_j(x,t)\}_{j=1}^M$ on $\partial K \times [-t_0, t_0]$, we pass to the analysis of the solutions of the localized problems

$$\begin{cases} (\partial_t^2 - \Delta)u_j = F_j(x,t) \text{ in } \Omega \times \mathbb{R}, \\ u_j + \psi_j\delta(t - < x, \omega >) = G_j(x,t) \text{ on } \partial K \times \mathbb{R}, \end{cases} \quad (2.5)$$

with $F_j(x,t) \in C^\infty(\Omega \times \mathbb{R})$, $G_j(x,t) \in C_0^\infty(\partial K \times \mathbb{R})$ and $F_j = 0$ for $t \leq t_0$. Then using the decay of local energy for strictly convex obstacles we get

$$\frac{\partial v_s}{\partial \nu}\Big|_{\partial K} = \sum_{j=1}^{M} \int e^{-i\lambda t} \frac{\partial u_j(x,t)}{\partial \nu}\Big|_{\partial K \times \mathbb{R}} dt + \mathcal{O}(|\lambda|^{-N}),\ \forall N.$$

The results for the propagation of the wave front set $WF(u_j)$ of the solutions of (2.5) (see [17], [7]) say that

$$WF\Big(\frac{\partial u_j}{\partial \nu}\Big|_{\partial K \times \mathbb{R}}\Big) \subset WF\Big(\psi_j\delta(t - < x, \omega >)|_{\partial K \times \mathbb{R}}\Big). \quad (2.6)$$

In the case supp $\psi_j \cap \Big(G(\omega) \times \mathbb{R}\Big) = \emptyset$ the above relation follows from the pseudo-local property of pseudo-differential operators [3] since we have modulo smooth terms the representation

$$\frac{\partial u_j}{\partial \nu}\Big|_{\partial K \times \mathbb{R}} = B_j\Big[\psi_j\delta(t - < x, \omega >)|_{\partial K \times \mathbb{R}}\Big], \quad (2.7)$$

B_j being a first order pseudo-differential operator, while for supp $\psi_j \cap \Big(G(\omega) \times \mathbb{R}\Big) \neq \emptyset$ we apply the results of Taylor [17] and Melrose [7]. Thus we are going to study the expression

$$\sum_j \int \int_{\partial K} e^{-i\lambda(t - <x,\theta>)} \frac{\partial u_j}{\partial \nu} dt dS_x. \quad (2.8)$$

where the integral is interpreted in the sense of distributions. By using the definition of the wave front it is easy to see that the condition

$$(y', t, d'_y\Phi, d_t\Phi) \cap WF(u) = \emptyset, \ y' \in D \subset \mathbb{R}^2$$

implies

$$\int_{\mathbb{R}}\int_D e^{-i\lambda\Phi(y',t)} u(y',t)dtdy' = \mathcal{O}(|\lambda|^{-N}), \ \forall N .$$

In order to apply this assume that in local coordinates $U_j \cap \partial K$ is given by

$$y_3 = g(y'), \ y' = (y_1, y_2) \in D \subset \mathbb{R}^2 .$$

Then (2.6) yields

$$WF\Big(\frac{\partial u_j}{\partial \nu}\Big|_{\partial K\times\mathbb{R}}\Big) \subset \{(y,t,\xi,\tau) \in T^*(\partial K \times \mathbb{R}) : \ t = < y, \omega >,$$

$$y \in \text{supp } \psi_j(y, < y, \omega >), \ (\xi, \tau) = \pm(-\omega' - \nabla g(y')\omega_3, 1)\} .$$

Clearly, for the phase function $\Phi = t - < y, \omega > |_{y\in U_j \cap \partial K}$ we have $d\Phi = (-\theta' - \nabla g(y')\theta_3, 1)$ which coincides with the directions of the wave front of $\frac{\partial u_j}{\partial \nu}\Big|_{\partial K\times\mathbb{R}}$ only in the case

$$-\omega' - \nabla g(y')\omega_3 = -\theta' - \nabla g(y')\theta_3 .$$

Thus we deduce immediately

$$\frac{\theta - \omega}{|\theta - \omega|} = \pm\nu(y', g(y')) .$$

The assumption $\theta \neq \omega$ says that for $y \in G(\omega)$ the last condition is impossible. Moreover, the same argument shows that supp $\psi_j(y, < y, \omega >)$ must be included in small neighbourhood $U_\pm$ of $x_\pm$ with $\psi_j(y, < y, \omega >) = 1$ in a neighbourhood of $x_\pm$.

Since x_- lies in the shadow region we have $< \nu(x_-), \omega >> 0$ and the solution of the wave equation which is smooth for $t < 0$ in a small neighbourhood of $(x_-, < x_-, \omega >)$ has the form $u_- = -\delta(t - < x, \omega >)$. Thus we obtain

$$\frac{\partial v_s}{\partial \nu}\Big|_{U_-\cap\partial K} = i\lambda < \nu, \omega > e^{-i\lambda< x,\omega>}|_{U_-\cap\partial K}$$

and replacing $\frac{\partial r_+}{\partial \nu}|_{U_-\cap\partial K}$ in the expression (2.3) we see that the shadow region gives no contribution to $a(\lambda, \theta, \omega)$ since

$$< \nu(x_-), \theta + \omega >= 0 .$$

Passing to the illuminated region, denote by ψ_+ and B_+ the cut-off function and the pseudo-differential operator related to U_+. Then for the formally adjoint operators B_+^* we obtain

$$\frac{1}{4\pi}\int\int_{U_\pm} B_+^*\Big(e^{-i\lambda(t-<y',\theta'>-g(y')\theta_3)}\Big)\psi_+\delta\Big(t-<y',\omega'>-g(y')\omega_3\Big)$$

$$\times\Big(1+|\nabla g(y')|^2\Big)^{1/2} dt dy'$$

$$= \frac{\lambda}{4\pi}\int_{U_+} e^{i\lambda(<y',\theta'-\omega'>+g(y')(\theta_3-\omega_3))} b_+(y',\theta)dy' + \mathcal{O}(1)$$

with

$$b_+(y',\theta) = -i\beta_+(y',\theta'+\nabla g(y')\theta_3,-1)\Big(1+|\nabla g(y')|^2\Big)^{1/2},$$

$i\beta_+$ being the principal symbol of B_+. Thus we are reduced to study an integral having the same form as $I(\lambda)$.

Without loss of the generality we can assume that $\nabla g(x'_+) = 0$. From the construction of the asymptotic solution in a neighbourhood of x_+ we obtain

$$\beta_+(x'_+, \theta', -1) =< \nu(x_+), \theta >> 0$$

and we conclude that

$$\frac{\lambda}{4\pi}\int_{U_+} e^{i\lambda(<y',\theta'-\omega'>+g(y')(\theta_3-\omega_3))} b_+(y',\theta)dy'$$

$$= \frac{1}{2}e^{i\lambda<x_+,\theta-\omega>}\mathcal{K}(x_+)^{-1/2} < \nu(x_+), \theta > .$$

Combined this with (2.4) we get

$$a(\lambda,\theta,\omega) = e^{i\lambda<x_+,\theta-\omega>}\mathcal{K}(x_+)^{-1/2} < \nu(x_+), \theta > +\mathcal{O}(|\lambda|^{-1}) .$$

Finally, in the illuminated region we have

$$< \nu(x_+), \theta >= \frac{< \theta - \omega, \theta >}{|\theta - \omega|^2} = \frac{1}{2}$$

and

$$a(\lambda, \theta, \omega) = \tfrac{1}{2} e^{i\lambda < x_+, \theta - \omega >} \mathcal{K}(x_+)^{-1/2} + \mathcal{O}(|\lambda|^{-1}). \tag{2.9}$$

Thus from the limit

$$|a(\omega, \theta)| = \lim_{\lambda \to \infty} |a(\lambda, \omega, \theta)|$$

we may determine the Gauss curvature $\mathcal{K}(x_+)$ at x_+. If (ω, θ) run over a set $V \in S^2 \times S^2 \setminus \{(\omega, \omega) : \omega \in S^2\}$ we may recover the Gauss curvature $\mathcal{K}(y)$ at every point $y \in \partial K$ if the map

$$V \ni (\omega, \theta) \longrightarrow \frac{\theta - \omega}{|\theta - \omega|} \in S^2$$

is onto. On the other hand, the knowledge of the Gauss curvature at all points of ∂K determines uniquely ∂K (see [5] for more details).

The case $\omega = \theta$ is more complicated since the singularities associated to diffractive rays must be taken into account. We refer to [9] and [18] for the results in this direction.

3 Singularities of the scattering kernel

Throughout this section we assume that $\theta \neq \omega$. To study the general case of non-convex obstacles it is more convenient to consider the scattering kernel $s(t, \theta, \omega)$ defined as the Fourier transform

$$s(t, \theta, \omega) = \mathcal{F}_{\lambda \to t}\Big(i\lambda\, \overline{a(\lambda, \theta, \omega)}\Big),$$

where $\mathcal{F}_{\lambda \to t}\varphi(\lambda) = (2\pi)^{-1} \int e^{it\lambda}\varphi(\lambda) d\lambda$ for functions $\varphi \in \mathcal{S}'(\mathbb{R}^n)$.

Let $V(x, t) = \mathcal{F}_{\lambda \to t}\overline{v_s(x, \lambda)}$ be the solution of the problem

$$\begin{cases} (\partial_t^2 - \Delta)V = 0 \text{ in } \Omega \times \mathbb{R}, \\ V + \delta(t + < x, \omega >) = 0 \text{ on } \partial K \times \mathbb{R}, \\ V|_{t < -t_0} = 0. \end{cases}$$

Then we have

$$s(t, \theta, \omega) = -\frac{1}{4\pi} \int_{\partial K} \Big(\frac{\partial V}{\partial \nu} + < \nu, \theta > \frac{\partial V}{\partial t}\Big)(x, t - < x, \theta >) dS_x,$$

where the integral is interpreted in the sense of distributions. Our aim will be to examine the singularities of $s(t, \theta, \omega)$ with respect to t.

First we will define the so called (ω, θ)-rays. Given two directions θ, $\omega \in S^2$, consider a curve $\gamma \in \Omega$ having the form

$$\gamma = \cup_{i=0}^{m} l_i, \quad m \geq 1,$$

where $l_i = [x_i, x_{i+1}]$ are finite segments for $i = 1, \ldots, m-1$, $x_i \in \partial K$, and l_0 (resp. l_m) is the infinite segment starting at x_1 (resp. at x_m) and having direction $-\omega$ (resp. θ). The curve γ will be called *reflecting* (ω, θ)-ray in Ω if for $i = 0, 1, \ldots, m-1$ the segments l_i and l_{i+1} satisfy the law of reflection at x_{i+1} with respect to ∂K. The points $x_1, \ldots, x_m$ will be called *reflection points* of γ and this ray is called *ordinary reflecting* if γ has no segments tangent to ∂K.

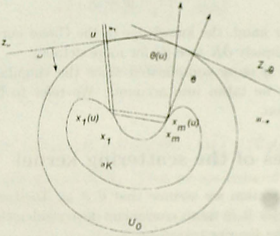

Figure 2

Now we will define two important notions related to scattering rays. Fix an arbitrary open ball U_0 with radius $a > 0$ containing K. For $\xi \in S^2$ introduce the plane Z_ξ orthogonal to ξ and such that ξ is pointing into the interior of the open half space H_ξ with boundary Z_ξ containing U_0. Let $\pi_\xi : \mathbb{R}^3 \longrightarrow Z_\xi$ be the orthogonal projection. For a reflecting (ω, θ)-ray γ in Ω with successive reflecting points $x_1, \ldots, x_m$ the *sojourn time* T_γ of γ is defined by

$$T_\gamma = \|\pi_\omega(x_1) - x_1\| + \sum_{i=1}^{m-1} \|x_i - x_{i+1}\| + \|x_m - \pi_{-\theta}(x_m)\| - 2a\,.$$

Obviously, $T_\gamma + 2a$ coincides with the length of this part of γ which lies in $H_\omega \cap H_{-\theta}$ (see Figure 2). In fact, the sojourn time T_γ does not depend on the choice of the ball U_0 since it follows easily that

$$\|\pi_\omega(x_1) - x_1\| = a + < x_1, \omega >, \ \|x_m - \pi_{-\theta}(x_m)\| = a - < x_m, \theta >$$

and we obtain

$$T_\gamma = < x_1, \omega > + \sum_{i=1}^{m-1} \|x_i - x_{i+1}\| - < x_m, \theta >$$

which proves the independence of T_γ of the choice of U_0.

Next for an ordinary reflecting (ω, θ)-ray γ set $u_\gamma = \pi_\omega(x_1)$. There exists a small neighbourhood W_γ of u_γ in Z_ω such that for every $u \in W_\gamma$ there exists an unique direction $\theta(u) \in S^2$ and points $x_1(u), ..., x_m(u)$ which are the successive reflection points of a reflecting $(u, \theta(u))$-ray in Ω with $\pi_\omega(x_1(u)) = u$. We define a smooth map

$$J_\gamma : W_\gamma \ni u \longrightarrow \theta(u) \in S^2$$

and $dJ_\gamma(u_\gamma)$ is called *differential cross section*. We say that the ray γ is non-degenerate if

$$\det dJ_\gamma(u_\gamma) \neq 0 .$$

The notion of sojourn time as well as that of differential cross section are well known in the physical literature. The definitions given above are due to Guillemin [1].

For strictly convex obstacles all reflecting rays have only one reflection point x_1 and the corresponding sojourn time is equal to $< x_1, \omega - \theta >$. Moreover, the stationary phase argument of the previous section implies that $\overline{a(\lambda, \omega, \theta)}$ has a complete asymptotic expansion

$$\overline{a(\lambda, \omega, \theta)} = e^{i<x_+, \omega - \theta>} \sum_{j=0}^{N} c_j \lambda^{-j} + \mathcal{O}(|\lambda|^{-N})$$

and we deduce

$$\text{sing supp } s(t, \theta, \omega) = \{-T_+\},$$

$T_+ = < x_+, \omega - \theta >$ being the sojourn time of the (ω, θ)-ray γ_+ reflecting at x_+. A simple geometric argument implies that

$$|\det dJ_{\gamma_+}(u_{\gamma_+})| = 4\mathcal{K}(x_+)$$

and for t close to $-T_+$ we have

$$s(t,\theta,\omega) = \left|dJ_{\gamma_+}(u_{\gamma_+})\right|^{-1/2} \delta'(t+T_+) + \text{lower order singularities.} \quad (3.1)$$

For strictly convex obstacles T_+ is an isolated singularity of $s(t,\theta,\omega)$ related to an ordinary reflecting ray. Our purpose is to generalize this result for arbitrary obstacles treating multiple reflecting rays leading to isolated singularities. Roughly speaking the singularities of the scattering kernel are included in the set of sojourn times of (ω,θ)-rays but we must consider all rays incoming with direction ω and outgoing with direction θ (see [12], Chapter 9 and [8]). In general, there exist (ω,θ)-rays with grazing or gliding segments (see Figure 3).

The precise definition of a (ω,θ)-ray is based on the notion of generalized bicharacteristics of the operator $\Box = \partial_t^2 - \Delta_x$ given as trajectories of the generalized Hamilton flow $\mathcal{F}_t$ in Ω generated by the symbol $\sum_{i=1}^3 \xi_i^2 - \tau^2$ of $\Box$ (see [10] for a precise definition). In general, $\mathcal{F}_t$ is not smooth and in some cases there may exist two different integral curves issued from the same point in the phase space (see [17] for an example). To avoid this situation we assume the following generic condition.

(G) If for $(x,\xi) \in T^*(\partial K)$ the normal curvature of ∂K vanishes of infinite order in direction ξ, then ∂K is convex at x in direction ξ.

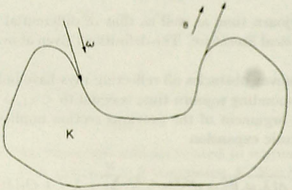

Figure 3

In the following we assume (G) fulfilled. A curve $\gamma = \{x(t) \in \Omega : t \in \mathbb{R}\}$ is called (ω,θ)-ray if there exist real numbers $t_1 < t_2$ so that

$$\tilde{\gamma}(t) = \{t, x(t), -1, \xi(t)\} \in T^*(\mathbb{R} \times \Omega)$$

is a *generalized bicharacteristic* of $\Box$ and

$$\xi(t) = \omega \text{ for } t \le t_1, \; \xi(t) = \theta \text{ for } t \ge t_2.$$

provided that the time t increases when we move along $\hat{\gamma}$. Introduce the set

$$\mathcal{L}_{\omega,\theta}(\Omega) = \bigcup_{(\omega,\theta)-\text{rays}} \{\gamma\},$$

where the union is taken over all (ω,θ)-rays in Ω. The sojourn time of $\delta \in \mathcal{L}_{\omega,\theta}(\Omega)$ is defined as the length of the part of δ lying in $H_\omega \cap H_{-\theta}$.

Passing to the problem of the behaviour of $s(t,\theta,\omega)$, assume that γ is a fixed *non-degenerate ordinary reflecting* (ω,θ)-ray such that

$$T_\gamma \neq T_\delta \text{ for every } \delta \in \mathcal{L}_{\omega,\theta}(\Omega) \setminus \{\gamma\}. \tag{3.2}$$

By using the continuity of the generalized Hamilton flow it is easy to show that

$$(-T_\gamma - \epsilon, -T_\gamma + \epsilon) \cap \text{sing supp } s(t,\theta,\omega) = \{-T_\gamma\}$$

for $\epsilon > 0$ sufficiently small. The analysis of the singularity of $s(t,\theta,\omega)$ for t close to $-T_\gamma$ is based on a global construction of an asymptotic solution as a Fourier integral operator ([2], [11], [12], Chapter 9). It was proved in [11] that under the assumption (3.2) we have

$$-T_\gamma \in \text{singsupp} s(t,\theta,\omega) \tag{3.3}$$

and for t close to $-T_\gamma$ the scattering kernel has the form

$$s(t,\theta,\omega) = i(-1)^{m_\gamma - 1} \exp\left(i\tfrac{\pi}{2}\beta_\gamma\right) \tag{3.4}$$

$$\times \left| \frac{\det dJ_\gamma(u_\gamma) < \nu(q_1), \omega >}{< \nu(q_m), \theta >} \right|^{-1/2} \delta'(t + T_\gamma) + \text{lower order singularities.}$$

Here m_γ is the number of reflections of γ, q_1 (resp. q_m) is the first (resp. the last) reflection point of γ and $\beta \in \mathbb{N}$. For strictly convex obstacles we have $\beta = -1$, $q_1 = q_m$ and $< \nu(q_1), \omega + \theta >= 0$.

4 Poisson relation for the scattering kernel

In this section we assume $\omega \neq \theta$ and the assumption (G) fulfilled. First we have the following relation

$$\text{sing supp } s(t,\theta,\omega) \subset \{-T_\gamma : \gamma \in \mathcal{L}_{\omega,\theta}(\Omega)\} \tag{4.1}$$

called *Poisson relation* for the scattering kernel. The proof is based on the results of propagation of singularities and we refer to [12], Chapter 9 and [8] for more details.

For the inverse scattering problems it is natural to obtain some geometrical information from the scattering data given by the knowledge of singsupp$s(t,\theta,\omega)$. In this direction it is very important to know whenever (14) becomes an equality, that is if the singularities of $s(t, \theta, \omega)$ determine all sojourn times T_γ, $\gamma \in \mathcal{L}_{\omega,\theta}(\Omega)$. Recently, Stoyanov [14] established an impressive result saying that there exists a subset $\mathcal{R} \subset S^2 \times S^2$ of full Lebesgue measure in $S^2 \times S^2$ such that for all $(\omega, \theta) \in \mathcal{R}$ we have

$$\text{sing supp } s(t, \theta, \omega) = \{-T_\gamma : \gamma \in \mathcal{L}_{\omega,\theta}(\Omega)\}. \tag{4.2}$$

For this purpose first one proves that for a subset $\mathcal{R}'$ of full Lebesgue measure in $S^2 \times S^2$ every reflecting (ω, θ)-ray in Ω is a non-degenerate ordinary reflecting one and $T_\gamma \neq T_\delta$ whenever γ and δ are different reflecting (ω, θ)-rays (see [13]). Now to apply the results of Section 3 we need the condition (3.2) since the contributions related to different (ω, θ)-rays with equal sojourn times could cancel. On the other hand, the description of the singularity associated to a ray having gliding or grazing segments is a very hard problem, especially when we have a higher order tangent segments. Intuitively it is clear that (ω, θ)-rays with grazing or gliding segments could exist only for some very specially chosen directions ω and θ and we may expect to avoid such rays for almost all directions.

Stoyanov [14] proved that there exists a subset $\mathcal{R}'' \subset S^2 \times S^2$ of full Lebesgue measure in $S^2 \times S^2$ such that every (ω, θ)-ray with $(\omega, \theta) \in \mathcal{R}''$ is a reflecting one. This is a global result and the proof is based on a fine analysis of some regularity properties of the generalized Hamilton flow $\mathcal{F}_t$. Now taking $\mathcal{R} = \mathcal{R}' \cap \mathcal{R}''$ we are in position to apply (3.3) for (ω, θ)-rays with $(\omega, \theta) \in \mathcal{R}$.

The *scattering length spectrum* (SLS) of an obstacle K is defined as the set of real numbers

$$\mathcal{SL}_K = \bigcup_{(\omega,\theta) \in S^2 \times S^2} \mathcal{SL}_K(\omega, \theta),$$

where

$$\mathcal{SL}_K(\omega, \theta) = \{T_\gamma : \gamma \in \mathcal{L}_{(\omega,\theta)}\}, \ (\omega, \theta) \in S^2 \times S^2.$$

The equality (4.2) says that for almost (ω, θ) we have

$$\mathcal{SL}_K(\omega, \theta) = \text{sing supp } s(t, \omega, \theta).$$

From the results of Majda [6] it follows that for arbitrary obstacles we have

$$\max_t \text{sing supp } s(t, -\omega, \omega) = -T_0(\omega),$$

where $T_0(\omega) = 2r(\omega) = 2\min_{y \in \partial K} < x, \omega >$ and this is the smallest sojourn time of the (ω, θ)- rays incoming with direction ω and outgoing with direction $-\omega$. Since

$$\text{convex hull } \partial K = \bigcap_{\omega \in S^2} \{y : < y, \omega > \geq r(\omega)\},$$

we can recover the *convex hull* of an obstacle from $S\mathcal{L}_K$.

The case of non-convex obstacles is much more complicated and in the general, as an example of M. Livshits shows (see [8], Chapter 5), $S\mathcal{L}_K$ does not determine K uniquely. On the other side, Stoyanov [15] proved that if two obstacles K and L have almost the same SLS then the corresponding flows $\mathcal{K}_t^{(K)}$ and $\mathcal{K}_t^{(L)}$ are conjugated on their phase spaces minus the set of so called trapping points. This makes possible to prove the uniqueness of the inverse scattering problem related to SLS for some class of obstacles. We refer to [15] and [16] for more details.

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