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Resonances in the Euclidean Scattering

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0 Introduction

The purpose of this article is to review the most significant results about the resonances associated to selfadjoint second order differential operators (with absolutely continuous spectrum only) which coincide with the Euclidean Laplacian outside a compact domain. Typical examples of such operators are the Schrödinger operator with a compactly supported potential, or the Laplace operator in the exterior of a bounded domain (with Dirichlet or Neumann boundary conditions). The latter describes the propagation of acoustic waves outside an obstacle, which reflect from the boundary but do not enter inside. The resonances are important objects in the scattering theory of such operators. They are complex numbers usually defined as the poles of the meromorphic continuation of the resolvent (acting on suitable spaces) through the real axis. Physically, a resonance $\lambda \in \mathbf{C}$ (with $\mathrm{Re}\,\lambda > 0$, $\mathrm{Im}\,\lambda > 0$) describes a nonstable quantum state oscilating with a frequency $\mathrm{Re}\,\lambda$, whose life-time is proportional to $1/\mathrm{Im}\,\lambda$. Therefore, the closer a resonance is to the real axis (that is, the smaller its imaginary part is), the longer it lives, and hence the more interesting it is from physical point of view. In the physical experiments the real parts of the resonances are observed as the points at which the first derivative, $s'(\lambda)$, of the phase $s(\lambda)$, $\lambda \in \mathbf{R}$, of the scattering matrix has peaks. Therefore, it is important to study the relationship between the behaviour of $s(\lambda)$ (or of $s'(\lambda)$) and the resonances. The knowledge of the resonances near the real axis enables also to deduce important information about the decay of the local energy of the solutions of the corresponding wave equation.

In Section 1 I consider mainly the case of scattering by both an arbitrary obstacle and a perturbed metric (not necessairily Riemannian). Among other things, the problems of existence of resonances, lower and upper bounds on their counting functions are discussed. In Section 2 I consider operators satisfying the so-called generalized Huyghens principle (see (2.1.1)). Usually this principle is satisfied in the cases when there are no trapped rays, and that is why such perturbations are called nontrapping. Typical examples are the scattering by a strictly convex obstacle or the Schrödinger operator with a smooth compactly supported potential. In Section 3 I discuss the scattering by several strictly convex obstacles, which is a typical example of trapping perturbations. One sees that in this case the resonances near the real axis are distributed differently compared with the case of nontrapping perturbations, namely, the existence of infinitely many resonances in a strip is an indication of presence of trapped rays. In Section 4 I discuss the Neumann problem in linear elasticity. This problem is interesting (and different from the problems discussed in the previous sections) because of the existence of surface waves (called Rayleigh waves) moving on the boundary of the obstacle. As a consequence, the generalized Huyghens principle is never fulfilled in this case. Note that there are no such surface waves in the Dirichlet problem in linear elasticity. Finally, in Section 5 I discuss the transmission problem which describes the propagation of acoustic waves which penetrate into the obstacle and move in its interior with a different speed. Because of the existence of periodic broken rays in the obstacle, the generalized Huyghens principle is not fulfilled for this problem neither.

I am not going to discuss resonances for semi-classical problems nor for operators on spaces with negative curvature. For other review articles on the resonances I refer the readers to Zworski's articles [92], [95].

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1 Distribution and density of the resonances

1.1 Properties of the outgoing resolvent of the Laplace operator on $\mathbf{R}^n$

Let $\Delta = \sum_{j=1}^n \partial_{x_j}^2$ be the Laplace operator in $\mathbf{R}^n$, $n \geq 2$. Denote by P_0 the self-adjoint realization of $-\Delta$ on the Hilbert space $H_0 = L^2(\mathbf{R}^n)$. Denote by $\mathcal{F}$ the Fourier transform, that is,

$$(\mathcal{F}u)(\xi) = (2\pi)^{-n/2} \int_{\mathbf{R}^n} e^{-i\langle x,\xi\rangle} u(x)dx.$$

Clearly, $\mathcal{F}$ is an unitary operator on H_0, and

$$P_0 = \mathcal{F}^{-1}|\xi|^2\mathcal{F}. \tag{1.1.1}$$

Hence P_0 is a positive self-adjoint operator with absolutely continuous spectrum only with domain of definition, $D(P_0)$, which coincides with the Sobolev space $H^2(\mathbf{R}^n)$. The representation (1.1.1) allows to find an explicit formulae for the kernel of the operator $f(\sqrt{P_0}) = \mathcal{F}^{-1}f(|\xi|)\mathcal{F}$ in terms of oscilatory integrals:

$$[f(\sqrt{P_0})](x,y) = (2\pi)^{-n} \int_{\mathbf{S}^{n-1}} \int_0^\infty e^{-ir\langle x-y,w\rangle} r^{n-1} f(r) dr dw, \tag{1.1.2}$$

where $\mathbf{S}^{n-1}$ denotes the unit sphere in $\mathbf{R}^n$. Let $h(s) = (s^2 - \lambda^2)^{-1}$, $s \in \mathbf{R}$, $\lambda \in \mathbf{C}$, $\operatorname{Im}\lambda < 0$. The outgoing resolvent, $R_0(\lambda)$, of P_0 is defined as follows:

$$R_0(\lambda) := (P_0 - \lambda^2)^{-1} := h(\sqrt{P_0}) \quad \text{for} \quad \operatorname{Im}\lambda < 0.$$

By the spectral theorem we have

$$\|R_0(\lambda)\|_{\mathcal{L}(H_0)} \leq \sup_{s\in\mathbf{R}} |h(s)| \leq \frac{1}{|\lambda||\operatorname{Im}\lambda|} \quad \text{for} \quad \operatorname{Im}\lambda < 0.$$

The kernel of $R_0(\lambda)$ is given by

$$[R_0(\lambda)](x,y) = E_\lambda(x-y),$$

where $E_\lambda(x)$ is the outgoing fundamental solution of the operator $-\Delta - \lambda^2$, that is, $(-\Delta - \lambda^2)E_\lambda(x) = \delta(x)$, $\delta(x)$ being the Dirac function. The function $E_\lambda(x)$ can be expressed in terms of the Henkel functions of first type by the formulae

$$E_\lambda(x) = \frac{i}{4}\left(\frac{\lambda}{2\pi|x|}\right)^p H_p^{(1)}(\lambda|x|), \quad p = (n-2)/2.$$

Theorem 1.1.1. *For every $\rho \in C_0^\infty(\mathbf{R}^n)$ the operator-valued function $\rho R_0(\lambda)\rho : H_0 \to H^2$ admits an analytic extension on the complex plane* $\mathbf{C}$ *if $n \geq 3$ is odd, and on the Riemann surface of $\log z$, $\Lambda := \{-\infty < \arg z < +\infty\}$, if $n \geq 2$ is even. Moreover,*

$$\rho R_0(\lambda)\rho = A(\lambda) + B(\lambda)\lambda^{n-2}\log\lambda, \tag{1.1.3}$$

where $A(\lambda)$ and $B(\lambda)$ are entire operator-valued functions ($B(\lambda) \equiv 0$ if n is odd), while $\log\lambda$ takes its principal branch on $-i\mathbf{R}^+$, that is, $\log\lambda = \log|\lambda| + i(\arg\lambda + \frac{\pi}{2})$. Moreover, the following estimate holds

$$\|\lambda\rho R_0(\lambda)\rho\|_{\mathcal{L}(H_0)} \leq C \quad \text{for} \quad \operatorname{Im}\lambda \leq 0. \tag{1.1.4}$$

This theorem can be proved by using the formulae

$$-i\lambda R_0(\lambda) = \int_0^\infty e^{-it\lambda}\cos\left(t\sqrt{P_0}\right)dt \quad \text{for} \quad \operatorname{Im}\lambda < 0, \tag{1.1.5}$$

and the following

Proposition 1.1.2. *There exists a constant $T > 0$ (depending on $\operatorname{supp}\rho$) so that*

$$\rho\cos\left(t\sqrt{P_0}\right)\rho = 0 \quad \text{for} \quad t \geq T \text{ if } n \geq 3 \text{ is odd}, \tag{1.1.6}$$

and $\rho\cos\left(t\sqrt{P_0}\right)\rho$ is analytic in t for $t \geq T$ if n is even, and has the following expansion

$$\rho\cos\left(t\sqrt{P_0}\right)\rho = \sum_{j=0}^{\infty} t^{-n-j}Q_j, \quad t \geq T, \tag{1.1.7}$$

where Q_j are finite rank operators.

The property (1.1.6) is known as Huyghens principle. This proposition can be easily proved by using the formulae (1.1.2). Indeed, the kernel $K(x,y,t)$ of $\rho\cos\left(t\sqrt{P_0}\right)\rho$ is of the form

$$K(x,y,t) = (2\pi)^{-n}t^{-n}\rho(x)\psi\left(\frac{x-y}{t}\right)\rho(y),$$

where

$$\psi(z) = \int_{\mathbf{S}^{n-1}} \varphi(\langle z, w\rangle)dw,$$

with a function $\varphi(k)$ defined by the oscilatory integral

$$\varphi(k) = \int_0^\infty r^{n-1}\cos r e^{-ikr}dr = \mathcal{F}_{r\to k}\left(r^{n-1}g(r)\cos r\right)$$

$$= (-i\partial_k)^{n-1}\,\mathcal{F}_{r\to k}\left(g(r)\cos r\right),$$

where $g(r) = 1$ for $r \geq 0$, $g(r) = 0$ for $r < 0$. It is clear from these representations that to prove the above proposition it suffices to show that $\varphi(k)$ is analytic at $k = 0$ and

$$\varphi(k) = \begin{cases} \sum_{j=0}^{\infty} a_j k^{2j+1} & \text{if } n \text{ is odd,} \\ \sum_{j=0}^{\infty} b_j k^{2j} & \text{if } n \text{ is even.} \end{cases} \tag{1.1.8}$$

We write

$$\mathcal{F}_{r\to k}\left(g(r)\cos r\right) = \frac{1}{2}\lim_{\varepsilon\to 0^+}\left(\int_0^\infty e^{ir(1-k+i\varepsilon)}dr + \int_0^\infty e^{ir(-1-k+i\varepsilon)}dr\right)$$

$$= \frac{i}{2}\lim_{\varepsilon\to 0^+}\left(\frac{1}{1-k+i\varepsilon} - \frac{1}{1+k-i\varepsilon}\right) = \frac{i}{2}\left(\frac{1}{1-k} - \frac{1}{1+k}\right) = i\sum_{j=0}^{\infty} k^{2j+1}$$

in a neighbourhood of $k = 0$, which clearly implies (1.1.8).

By (1.1.5)-(1.1.7) we can write

$$-i\lambda\rho R_0(\lambda)\rho = \int_0^T e^{-it\lambda}\rho\cos\left(t\sqrt{P_0}\right)\rho dt + \sum_{j=0}^{\infty} Q_j \int_T^\infty t^{-n-j}e^{it\lambda}dt, \tag{1.1.9}$$

where $Q_j \equiv 0$ if n is odd. Put $\lambda = -iz$, $z > 0$ real, and set

$$q_m(z) = \int_T^\infty t^{-n-j}e^{-tz}dt, \quad m = 1, 2, \ldots.$$

It is easy to check by induction in m that

$$q_m(z) = \frac{(-1)^m}{(m-1)!}z^{m-1}\log z + \text{entire function},$$

which clearly extends for all values of $z \in \Lambda$. Thus (1.1.3) and (1.1.4) follow from (1.1.9).

1.2 Definition of the resonances

Let $\mathcal{O} \subset \mathbf{R}^n$ be a compact domain with a C^∞-smooth boundary, Γ, and a connected complement $\Omega = \mathbf{R}^n \setminus \mathcal{O}$. Consider in Ω the Laplace-Beltrami operator Δ_g defined by

$$\Delta_g = c(x)^2\sum_{i,j=1}^{n} \partial_{x_i}(g_{ij}(x)\partial_{x_j}),$$

where $c(x), g_{ij}(x) \in C^\infty(\Omega)$ are real-valued functions such that $c(x) = 1$, $g_{ij}(x) = \delta_{ij}$ for $|x| \geq \rho_0 \gg 1$, δ_{ij} being the Kronecker symbol. In other words, Δ_g coincides with the Euclidean Laplace operator Δ outside some compact. We suppose that

$$g(x,\xi) := c(x)^2 \sum_{i,j=1}^{n} g_{ij}(x)\xi_i\xi_j \geq 0, \quad \forall (x,\xi) \in T^*\Omega,$$

where $T^*\Omega$ denotes the cotangential bundle of Ω. Note that Δ_g is not necessarily an elliptic operator. We suppose that $-\Delta_g$ has a positive, self-adjoint Dirichlet realization (which will be denoted by P) on the Hilbert space $H = L^2(\Omega; c(x)^2 dx)$ with absolutely continuous spectrum only. Here "Dirichlet" means that the functions belonging to the domain of definition, $D(P)$, of P vanish on Γ. We also suppose that P is a subelliptic operator with loss of $2 - 2\varepsilon$ derivatives, $0 < \varepsilon \leq 1$, that is,

$$\|u\|_{H^{2\varepsilon}(\Omega)} \leq C\left(\|Pu\|_H + \|u\|_H\right), \quad \forall u \in D(P), \tag{1.2.1}$$

where $H^s(\Omega)$ denotes the usual Sobolev space. Clearly, for elliptic operators (1.2.1) holds with $\varepsilon = 1$. In the same way as in the previous section we define the outgoing resolvent of P by

$$R(\lambda) := (P - \lambda^2)^{-1} := h(\sqrt{P}) \quad \text{for} \quad \operatorname{Im}\lambda < 0.$$

Fix a $\lambda_0 \in \mathbf{C}$, $\operatorname{Im}\lambda_0 < 0$ and let $\chi \in C_0^\infty(\mathbf{R}^n)$, $\chi = 1$ for $|x| \leq \rho_0 + 1$. The estimate (1.2.1) guarantees that

$$\text{the operator} \quad \chi R(\lambda_0) : H \to H \quad \text{is compact.} \tag{1.2.2}$$

Under these assumptions we have the following

Theorem 1.2.1. *The operator-valued function*

$$R_\chi(\lambda) := \chi R(\lambda)\chi : H \to H$$

admits a meromorphic continuation to $\mathbf{C}$ *if* $n \geq 3$ *is odd, and to the Riemann surface* Λ *if* n *is even. Moreover, the coefficients in the Laurent expansion at each pole are of finite rank.*

This theorem follows from the Fredholm theory and the following representation (for example, see [80]):

$$R_\chi(\lambda)(1 - K(\lambda)) = K_1(\lambda), \tag{1.2.3}$$

where $K(\lambda)$ and $K_1(\lambda)$ are of the form

$$K(\lambda) = ([\chi_1, \Delta]R_0(\lambda)\eta - [\chi_1, \Delta]R_0(\lambda_0)\eta)\, A + (\lambda^2 - \lambda_0^2)\chi_2 R_\chi(\lambda_0),$$

$$K_1(\lambda) = (1 - \chi_1)\,(\chi R_0(\lambda)\eta - \chi R_0(\lambda_0)\eta)\, A + R_\chi(\lambda_0).$$

Here $\lambda_0 \in \mathbf{C}$, $\operatorname{Im}\lambda_0 < 0$, is fixed, $\eta, \chi_j \in C_0^\infty(\mathbf{R}^n)$, $j = 1, 2$, $\chi_1 = 1$ for $|x| \le \rho_0$, $\chi_2 = 1$ on $\operatorname{supp}\chi_1$, $\chi = 1$ on $\operatorname{supp}\chi_2$, $\eta = 0$ on $\operatorname{supp}\chi_1$ and $\eta = 1$ on $\operatorname{supp}\chi(1 - \chi_2)$, A is a bounded operator independent of λ. By Theorem 1.1.1, the operator-valued functions $[\chi_1, \Delta]R_0(\lambda)\eta$ and $\chi R_0(\lambda)\eta$ are analytic on $\mathbf{C}$ if n is odd, and on Λ if n is even, with values in the compact operators on H. Hence, so are $K(\lambda)$ and $K_1(\lambda)$. On the other hand, since $K(\lambda_0) = 0$, $1 - K(\lambda)$ is invertible at $\lambda = \lambda_0$. Therefore, $(1 - K(\lambda))^{-1}$ forms a meromorphic family on $\mathbf{C}$ if n is odd, and on Λ if n is even, with finite rank coefficients in the Laurent expansion at each pole, and by (1.2.3) so is true for $R_\chi(\lambda)$.

The resonances associated to the operator P are defined as being the poles of the meromorphic continuation of $R_\chi(\lambda)$ and they do not depend on the choice of the cutoff function χ provided $\chi = 1$ on the support of the perturbation. To each resonance $\lambda \neq 0$ we associate a multiplicity as follows:

$$\operatorname{mult}(\lambda) := \operatorname{rank} \int_{\gamma(\lambda)} z R_\chi(z)\, dz,$$

where $\gamma(\lambda) = \{z = \lambda + \varepsilon\zeta, \zeta \in \mathbf{C}, |\zeta| = 1\}$, $\varepsilon > 0$ being such that there are no other resonances in the interior of $\gamma(\lambda)$. Denote by $\mathcal{R}$ the set of all resonances repeated according to their multiplicities. Clearly, $\mathcal{R}$ is a discrete set in $\mathbf{C}$ if n is odd, and in Λ if n is even.

The resonances can be also defined (for example, see [26]) as being the set of all complex numbers λ for which the Helmholtz equation has a nontrivial solution $u \in H^2_{loc}(\Omega)$:

$$\begin{cases} (\Delta_g + \lambda^2)u &= 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \Gamma, \\ u &- \lambda - \text{outgoing}. \end{cases} \tag{1.2.4}$$

Here "λ-outgoing" means that u satisfies the Sommerfeld radiation condition at infinity, that is,

$$u(r\theta) = r^{-\frac{n-1}{2}} e^{-i\lambda r}(w(\theta) + o(1)), \quad \partial_r u + i\lambda u = o(1)u, \quad r \to +\infty,$$

uniformly in $\theta \in \mathbf{S}^{n-1}$ with some function $w \in C^\infty(\mathbf{S}^{n-1})$, or what is equivalent,

$$u|_{|x| \ge \tilde\rho_0} = R_0(\lambda) v|_{|x| \ge \tilde\rho_0}$$

for some $\rho_0 \gg 1$ and a compactly supported (in a compact independent of λ) function v.

Note that one can define in the same way the resonances associated to the Neumann realization of $-\Delta_g$ as well as to the Schrödinger operator $-\Delta + V(x)$ on $\mathbf{R}^n$ with a potential $V \in L^\infty_{comp}(\mathbf{R}^n)$. Moreover, all the results concerning the Dirichlet problem discussed in Section 1 are valid for the Neumann problem as well.

1.3 Sharp upper bounds on the number of the resonances

An important quantity which gives a valuable information on the distribution of the resonances on $\mathbf{C}$ (resp. Λ) is their counting function. The most natural counting functions are the following:

$$N(r) := \sharp\{\lambda \in \mathcal{R} \subset \mathbf{C} : |\lambda| \le r\} \quad \text{if } n \text{ is odd},$$

$$N(r,a) := \sharp\{\lambda \in \mathcal{R} \subset \Lambda : |\lambda| \le r, |\arg \lambda| \le a\} \quad \text{if } n \text{ is even},$$

where $r, a \gg 1$. It turns out that the behaviour in r of these counting functions is closely related with the behaviour of the counting function of the eigenvalues of the so-called reference operator, $\widetilde{P}$, obtained by restricting P in a neighbourhood of the support of the perturbation. More precisely, $\widetilde{P}$ is the Dirichlet self-adjoint realization of $-\Delta_g$ on the Hilbert space $\widetilde{H} = L^2(\widetilde{\Omega}; c(x)^2 dx)$, where $\widetilde{\Omega} := \{x \in \Omega : |x| \le \rho_0\}$. Under (1.2.1), $\widetilde{P}$ is of compact resolvent and hence the spectrum of $\widetilde{P}$ consists of eigenvalues only. Denote the set of these eigenvalues, repeated according to multiplicity, by $\widetilde{\mathcal{R}} \subset [0, +\infty)$, and introduce the counting function

$$\widetilde{N}(r) := \sharp\{z \in \widetilde{\mathcal{R}} : z \le r^2\}.$$

Suppose that $\widetilde{N}(r)$ satisfies the bound

$$\widetilde{N}(r) \le \varphi(r), \quad r > 1, \tag{1.3.1}$$

where $\varphi(r) \in C^\infty(1, +\infty)$ is an increasing function such that $\varphi(r) \ge \widetilde{C} r^n$ with some constant $\widetilde{C} > 0$. Clearly, if P is elliptic, (1.3.1) is fulfilled with $\varphi(r) = C' r^n$, $C' > 0$. More generally, under (1.2.1) we have (1.3.1) with $\varphi(r) = C'' r^{n/\varepsilon}$, $C'' > 0$. It is worth noticing that it might happend that for hypoelliptic operators satisfying (1.2.1) the bound (1.3.1) holds with a function $\varphi(r) \ll r^{n/\varepsilon}$. Examples of such operators can be found in [40], [41], [59].

Theorem 1.3.1. *Under the assumption (1.3.1), there exists a constant $C > 0$ so that the following bounds hold:*

$$N(r) \le C\varphi(r), \tag{1.3.2}$$

$$N(r,a) \le Ca(\varphi(r) + (\log a)^n). \tag{1.3.3}$$

In particular, if P is elliptic, we have

$$N(r) \le Cr^n, \tag{1.3.4}$$

$$N(r,a) \le Ca(r^n + (\log a)^n). \tag{1.3.5}$$

Sharp bounds on the number of the resonances have been first obtained by Melrose [34], where he proved (1.3.4) in the case when $\Delta_g \equiv \Delta$. Then Zworski [91] proved (1.3.4) in the case of the Schrödinger operator $-\Delta + V(x)$ on $\mathbf{R}^n$ with a compactly supported potential. Later on the bound (1.3.4) was proved in [79] in the case when $\mathcal{O} = \emptyset$ and Δ_g is elliptic. In the greater generality, the bound (1.3.2) was first proved by Sjöstrand-Zworski [63] by using the so-called complex scaling method. Another proof of (1.3.2) based on the approach originating from Melrose's works [33], [34] is presented in [80]. In [82], [83] this approach was adapted to the case of even dimensions in order to prove (1.3.3). Note that if $|a| < \frac{\pi}{2}$ the bound (1.3.3) can be also obtained by the complex scaling method developed by Sjöstrand-Zworski [63].

Melrose's method is based on the representation (1.2.3) and the observation that, under (1.2.1), there exists an integer $p \ge 1$ so that the operator $K(\lambda)^p$ is trace class, so the following determinant is well defined:

$$h(\lambda) := \det(1 - K(\lambda)^p).$$

Then the following proposition allows to conclude that the poles of $R_\chi(\lambda)$ (that is, the resonances) are among the zeros of $h(\lambda)$, counting the multiplicity (for example, see the appendix of [82]).

Proposition 1.3.2. *Let $\Theta \subset \mathbf{C}$ be an open neighbourhood of 0 and let $\mathcal{K}(z)$ be analytic in Θ with values in the trace class operators on a Hilbert space H. Suppose that there exists a function f holomorphic in Θ, $f(0) \neq 0$, such that*

$$\det(1 - \mathcal{K}(z)) = z^l f(z).$$

Then for every $B(z), C(z) \in \mathcal{L}(H,H)$, holomorphic in Θ, we have

$$\operatorname{rank} \int_\gamma B(z)(1 - \mathcal{K}(z))^{-1} C(z)\,dz \le l, \tag{1.3.6}$$

where γ is a circle centered at $z = 0$ of a sufficiently small radius.

Thus the problem of obtaining upper bounds on the number of the resonances is reduced to the problem of obtaining upper bounds on the number of the zeros of entire functions (resp. functions holomorphic on Λ). In the first case this can be done by using the well known Jensen's inequality, while in the second case one can use the classical Carleman theorem. This latter theorem also alows to get upper bounds on the number of the resonances outside a conic neighbourhood of the real axis which does not depend on the behaviour of $\varphi(r)$. More precisely, we have the following (see [81], [46], [47])

Theorem 1.3.3. *Let $n \geq 3$ be odd. Then there exists a constant $C > 0$ so that*

$$\sum_{\lambda_j \in \mathcal{R}:|\lambda_j| \leq r} \frac{\operatorname{Im} \lambda_j}{|\lambda_j|^2} \leq C r^{n-1}. \tag{1.3.7}$$

In particular, $\forall \delta > 0 \, \exists C_\delta > 0$ so that

$$\sharp\{\lambda \in \mathcal{R} : |\lambda| \leq r, \operatorname{Im} \lambda \geq \delta |\operatorname{Re} \lambda|\} \leq C_\delta r^n. \tag{1.3.8}$$

The bound (1.3.8) combined with the upper bounds on the number of the resonances in small conic neighbourhoods of the real axis obtained by Sjöstrand-Zworski [64] lead to a more precise upper bound on the counting function $N(r)$ in the case when the function $\varphi(r)$ dominates r^n.

Theorem 1.3.4. *Under the condition (1.3.1), for $0 < \theta \ll 1$, we have*

$$\sharp\{\lambda \in \mathcal{R} : 0 < \arg \lambda \leq \theta ; |\lambda| \leq r\} \leq \varphi(r)(1 + \varepsilon(\theta)) + O_\theta(r^n), \tag{1.3.9}$$

where $\varepsilon(\theta) \to 0$ as $\theta \to 0$ does not depend on r. In particular, if $n \geq 3$ is odd and if (1.3.1) is fulfilled with a function $\varphi(r)$ satisfying

$$\lim_{r \to +\infty} \frac{r^n}{\varphi(r)} = 0,$$

then

$$N(r) \leq (2 + o(1))\varphi(r). \tag{1.3.10}$$

1.4 Poisson formulae for resonances

Let B_0 be a ball containing the obstacle $\mathcal{O}$ and denote by χ_0 its characteristic function. Define the distribution $u(t)$ as follows

$$\langle u, \phi \rangle = 2\operatorname{tr} \int \phi(t) \left(\cos\left(t\sqrt{P}\right) - (1 - \chi_0) \cos\left(t\sqrt{P_0}\right)(1 - \chi_0)\right) dt$$

$$+2\mathrm{tr}\int \phi(t)\chi_0\cos\left(t\sqrt{P_0}\right)\chi_0 dt, \quad \phi\in C_0^\infty(\mathbf{R}),$$

where the first trace is taken on the Hilbert space H, while the second one is taken on H_0. When $\mathcal{O}=\emptyset$ we take $\chi_0\equiv 0$. It is not hard to see that $u(t)$ is an even distribution belonging to the Schwartz class $\mathcal{S}'(\mathbf{R})$. Therefore, the Fourier transform $\widehat{u}(\lambda)$ of $u(t)$ is again a distribution belonging to $\mathcal{S}'(\mathbf{R})$. In fact, $\widehat{u}(\lambda)$ is a polynomially bounded C^∞ function which is nothing else but the first derivative, $s'(\lambda)$, of the scattering phase, $s(\lambda)$, which in turn can be defined in terms of the scattering matrix, $S(\lambda)$, as follows

$$s(\lambda)=(2\pi i)^{-1}\log\det S(\lambda), \quad s(0)=0, \quad s(\lambda)=-s(-\lambda).$$

There is a close relationship between $u(t)$ and the resonances of P, which is expressed by the Poisson formulae (which is an analogue of the Poisson formulae for eigenvalues). To simplify the exposition of the Poisson formulae we will make the following assumption:

$$\|\lambda R_\chi(-i\lambda)\|_{\mathcal{L}(H)}<+\infty \quad \text{as} \quad \lambda\to 0,\ \lambda>0. \tag{1.4.1}$$

This assumption guarantees that 0 is not neither an eigenvalue nor a resonance of P. In fact, (1.4.1) is always fulfilled in the setting we discuss provided that the operator Δ_g is elliptic, that is, when $g(x,\xi)\ge C|\xi|^2>0$, $\forall(x,\xi)\in T^*\Omega$ (for example, see Appendix B.2 of [3]).

Theorem 1.4.1. *The following identities hold in sense of distributions for* $t>0$,

$$u(t)=\sum_{\lambda_j\in\mathcal{R}} e^{it\lambda_j}, \quad \textit{if } n \textit{ is odd}, \tag{1.4.2}$$

$$u(t)=2\sum_{\lambda_j\in\mathcal{R}:0<\arg\lambda_j\le\rho} e^{it\lambda_j}+2\int_0^\infty \psi(\lambda)\widehat{u}(\lambda)\cos(t\lambda)d\lambda+v_{\rho,\psi}(t), \quad \textit{if } n \textit{ is even}, \tag{1.4.3}$$

where $0<\rho<\frac{\pi}{2}$, $\psi\in C_0^\infty(\mathbf{R})$, $\psi=1$ *in a neighbourhood of* 0, *and* $v_{\rho,\psi}\in C^\infty(0,+\infty)$ *satisfies*

$$\partial_t^k v_{\rho,\psi}(t)=\mathcal{O}(t^{-N}), \quad \forall k, N,\ t\gg 1.$$

The Possion formulae (1.4.2) was first obtained by Bardos-Guillot-Ralston [1] for t large enough, and then extended by Melrose [32] (see also [66]) for every $t>0$, using the Lax-Phillips theory (see [26]). A simpler proof has been later found by Zworski [93] based on a previous work by Guillopé-Zworski [13]. His proof avoids the use of the Lax-Phillips theory and allows to treat more general situations as well as the case of even dimension. The identity (1.4.3) can be considered as an analogue of the Poisson formulae for even dimensional spaces and was proved by Zworski [94]. Note that a local semi-classical trace formulae involving the resonances, for a very large class of perturbations, has been obtained by Sjöstrand [60]. This formulae, however, does not imply (1.4.3).

Let $\Theta := [1/2, 3/2] + i[0, 1/2]$. It follows easily from (1.4.2) and (1.4.3) that for every $\phi \in C_0^\infty(0, +\infty)$,

$$\widehat{\phi u}(\lambda) = \sum_{\lambda_j \in \mathcal{R}: \lambda_j/\lambda \in \Theta} \widehat{\phi}(\lambda - \lambda_j) + O(\lambda^{-\infty}), \quad \lambda \to +\infty. \tag{1.4.4}$$

Note that (1.4.4) also follows from Sjöstrand's local trace formulae ([60]). This identity is very useful for studying the relationship between the nonzero singularities of $u(t)$ and the resonances.

Given any $\gamma > 0$ denote by $u_\gamma(t)$ the distribution defined by the sum

$$u_\gamma(t) := \sum_{\lambda_j \in \mathcal{R}_\gamma} e^{it\lambda_j},$$

where $\mathcal{R}_\gamma := \{\lambda \in \mathcal{R} : 0 < \arg \lambda < \pi/2, \operatorname{Im} \lambda \le \gamma \log |\lambda|\}$. As a consequence of the above theorem we have the following (see [94])

Theorem 1.4.2. *Let $n \ge 3$. Then for every integer $k \ge 0$, we have $u(t) - u_\gamma(t) \in C^k(t_k, +\infty)$, $t_k = (n+k)/\gamma$, and for $t > t_k$,*

$$|\partial_t^k(u(t) - u_\gamma(t))| \le \begin{cases} C_k e^{-\alpha_k t} & \text{if } n \text{ is odd}, \\ C_k t^{-n+2-k} & \text{if } n \text{ is even}. \end{cases} \tag{1.4.5}$$

Note that the polynomial decay in the even dimensional case comes from the contribution of the integral in the RHS of (1.4.3). To compute this contribution one needs to know the behaviour of $\widehat{u}(\lambda)$ as $\lambda \to 0$, that is, the behaviour of the scattering matrix $S(\lambda)$ at zero. Since $S(\lambda)$ can be expressed in terms of $R_\chi(\lambda)$, one needs to know the behaviour of $R_\chi(\lambda)$ at zero. The following proposition gives the leading singularity of $R_\chi(\lambda)$ at $\lambda = 0$ (see [86]):

Proposition 1.4.3. *Let (1.4.1) be fulfilled. If $n \ge 3$ is odd, we have*

$$R_\chi(\lambda) = \lambda^{-1}\mathcal{P}_n + \mathcal{E}_n(\lambda), \tag{1.4.6}$$

where $\mathcal{E}_n(\lambda)$ is analytic at $\lambda = 0$, and $\mathcal{P}_n \equiv 0$ if $n \ge 5$, $\operatorname{rank} \mathcal{P}_3 \le 1$.

If $n \geq 2$ is even, we have

$$R_\chi(\lambda) = \mathcal{M}_n \lambda^{n-2} \log \lambda + \mathcal{F}_n(\lambda) + O(\lambda^{n-2}), \quad \lambda \to 0, |\arg \lambda + \pi/2| \leq \pi, \quad (1.4.7)$$

where rank $\mathcal{M}_n = 1$ *and* $\mathcal{F}_n(\lambda)$ *is a polynomial of degree* $\leq n-3$ *if* $n \geq 4$, $\mathcal{F}_2(\lambda) \equiv 0$.

1.5 Lower bounds on the number of the resonances

It follows from Theorem 1.4.2 that if the distribution $u(t)$ does not belong to $C^\infty(0,+\infty)$, there exists a $\gamma > 0$ so that the set $\mathcal{R}_\gamma$ contains infinitely many resonances, that is, the nonzero singularities of $u(t)$ produce infinitely many resonances in a logarithmic neighbourhood of the real axis. So, it is natural to expect that the knoweledge of such a singularity would yield a lower bound for the counting function

$$N_\gamma(r) := \sharp\{\lambda \in \mathcal{R}_\gamma : |\lambda| \leq r\}.$$

Indeed, such lower bounds were obtained by Sjöstrand-Zworski [65]:

Theorem 1.5.1. *Suppose that there exists a $d > 0$ and a function $\phi_d \in C_0^\infty(0,+\infty)$, $\phi_d = 1$ in a neighbourhood of d, such that*

$$\widehat{\phi_d u}(\lambda) \geq b(1-o(1))\lambda^k, \quad \lambda \gg 1, \quad b > 0. \quad (1.5.1)$$

Then there exists a $\gamma > 0$ so that if $k \geq 0$ we have

$$N_\gamma(r) \geq \frac{b(1-o(1))}{2\pi(k+1)} r^{k+1}, \quad r \gg 1, \quad (1.5.2)$$

while if $k < 0$, then $\forall \delta > 0$, $\exists r_0(\delta) > 1$ so that

$$N_\gamma(r) \geq r^{1-\delta}, \quad r \geq r_0(\delta). \quad (1.5.3)$$

Moreover, if (1.5.1) holds for a sequence $d_j \to +\infty$ (uniformly in d_j), then the above lower bounds hold for every $\gamma > 0$.

Let $\Pi(T) \subset \Sigma := \{(x,\xi) \in T^*\Omega : |\xi|_x = 1\}$ be the union of all periodic trajectories with period $T \neq 0$ and let $d\vartheta$ be the Liouville measure on Σ. Using the above theorem Popov [50] proved the following sharp lower bound on the number of the resonances (which generalizes a previous result obtained by Sjöstrand-Zworski [65]):

Theorem 1.5.2. *Suppose that $\vartheta(\Pi(T_0)) > 0$ for some $T_0 > 0$. Then for every $\gamma > 0$, we have*

$$N_\gamma(r) \geq \frac{\vartheta(\Pi(T_0))}{n(2\pi)^n}(1 - o(1))r^n, \quad r \gg 1. \tag{1.5.4}$$

Under the same assumptions as in the above theorem, Petkov-Zworski [47] proved the existence of accumulation of resonances at a sequence of real numbers (similar to the clustering properties of the eigenvalues). As a consequence, they obtained a lower bound of the counting function of the resonances in the strip $\operatorname{Im}\lambda \leq \varepsilon$, $\forall\varepsilon > 0$, of the form $Cr^n(1 - o_\varepsilon(1))$ with a constant $C > 0$ independent of ε, which however is smaller than the constant in (1.5.4). In particular, there exists an infinite sequence of resonances $\mu_j \in \mathcal{R}$ such that $\operatorname{Im}\mu_j \to 0$.

It is much more complicated to derive lower bounds on the number of the resonances from the singularity of $u(t)$ at zero. One of the reasons for this is that the Poisson formulae is not valid at $t = 0$. Nevertheless, some information about the resonances has been recently obtained in the case of the Schrödinger operator using the lower singularities of $u(t)$ at $t = 0$.

Theorem 1.5.3. *The resonances associated to the Schrödinger operator $-\Delta + V(x)$ on $\mathbf{R}^n$, $V \in C_0^\infty(\mathbf{R}^n)$, not identically zero, satisfy*

$$\limsup_{r\to+\infty} \frac{N(r)}{r(\log r)^{-p}} = +\infty, \quad \forall p > 1, \quad n \geq 3 \text{ odd}, \tag{1.5.5}$$

$$\sum_{\lambda\in\mathcal{R}} \left((\log|\lambda|)^2 + (\arg\lambda)^2\right)^{-1/2} = +\infty, \quad n \geq 4 \text{ even}. \tag{1.5.6}$$

In particular,

$$\limsup_{r\to+\infty} \frac{\sharp\{\lambda \in \mathcal{R} : |\arg\lambda| \leq \log r,\ r^{-1} \leq |\lambda| \leq r\}}{\log r(\log\log r)^{-p}} = +\infty, \quad \forall p > 1, \quad n \geq 4 \text{ even}. \tag{1.5.7}$$

The property (1.5.5) is proved by Christiansen [10], while (1.5.6) and (1.5.7) are proved by Sá Barreto [56]. The existence of infinitely many resonances for the Schrödinger operator with a nonidenticaly zero compactly supported smooth potential was first established by Melrose [37] in the case of $n = 3$. Using Melrose's argument Sá Barreto-Zworski [58] extended this result to any $n \geq 3$ odd and to super-exponentially decaying potentials. In [57] they proved analogous results for metric perturbations in $\mathbf{R}^3$.

For some perturbations in odd dimensional spaces it is possible to obtain lower bounds on the counting function of the resonances lying on the imaginary axis:

$$N_I(r) := \sharp\{\lambda \in \mathcal{R} : \operatorname{Re}\lambda = 0, \operatorname{Im}\lambda \leq r\}.$$

Theorem 1.5.4. *Let $n \geq 3$ be odd. Then for the resonances in the obstacle scattering (which corresponds to $\Delta_g \equiv \Delta$) or for those associated to the Schrödinger operator $-\Delta + V(x)$, $V \in C_0^\infty(\mathbf{R}^n)$ being a nonidentically zero, real-valued function which does not change the signe (that is, either $V(x) \geq 0, \forall x \in \mathbf{R}^n$, or $V(x) \leq 0, \forall x \in \mathbf{R}^n$), the following lower bound holds:*

$$N_I(r) \geq Cr^{n-1}, \quad r \gg 1, \quad C > 0. \tag{1.5.8}$$

In the case of obstacle scattering this theorem was proved by Lax-Phillips [27]. They also showed that if the obstacle is star-shaped, there is an upper bound of the same order, that is, $N_I(r) = O(r^{n-1})$. In the case of potential scattering (1.5.8) is proved by Vasy [78].

1.6 Relationship between quasimodes and resonances

Finer lower bounds on the number of the resonances close to the real axis can be obtained in the cases when one can construct quasimodes. Such a construction is usually possible when there is a strong trapping, for example elliptic periodic trajectories.

Suppose that there exist an infinite sequence of real numbers $\{k_j\}$, $k_j \to +\infty$, and functions $u_j \in D(P)$, $\operatorname{supp} u_j \subset K$ (K being a compact independent of j), so that

$$\left\{ \begin{array}{rcl} \|(P - k_j^2)u_j\|_H & \leq & \widetilde{F}(k_j), \\ |\langle u_i, u_j\rangle_H - \delta_{ij}| & \leq & \widetilde{F}(k_j), \end{array} \right. \tag{1.6.1}$$

where δ_{ij} denotes the Kronecker symbol and $\widetilde{F} \in C(1, +\infty)$, $\widetilde{F}(t) = O(t^{-\infty})$, $t \gg 1$. Then we have the following

Theorem 1.6.1. *Under the above assumptions, there exists an infinite sequence $\{\mu_j\} \in \mathcal{R}$ such that $\operatorname{Im}\mu_j \leq F(|\mu_j|)$ with a function $F \in C(1, +\infty)$, $F(t) = O(t^{-\infty})$. Moreover, the following lower bound holds*

$$\begin{aligned} N_F(r) &:= \sharp\{\lambda \in \mathcal{R} : |\arg\lambda| < \pi/2, \operatorname{Im}\lambda \leq F(|\lambda|), |\lambda| \leq r\} \\ &\geq N_{quasi}(r - r^{-k}) - O_k(1), \quad r \gg 1, \forall k \geq 1, \end{aligned} \tag{1.6.2}$$

where $N_{quasi}(r) := \sharp\{k_j : k_j \leq r\}$.

The fact that the existence of quasimodes implies existence of resonances close to the real axis was first observed by Stefanov-Vodev [73], where the first part of the above theorem was proved in the case of odd dimensional spaces. This has been later extended to even dimensional spaces and noncompactly supported perturbations by Tang-Zworski [74]. The bound (1.6.2) has been proved by Stefanov [68]. He also proved in [69] that the existence of infinitly many resonances $\{\mu_j\} \subset \mathcal{R}$ with $\operatorname{Im}\mu_j = O(|\mu_j|^{-\infty})$ implies the existence of quasimodes (u_j, k_j) satisfying (1.6.1).

As a consequence of (1.6.2) one gets a sharp lower bound on $N_F(r)$ if there exists an elliptic periodic broken ray. Suppose that $\Delta_g \equiv \Delta$ and let γ be a periodic broken ray in Ω with vertices $\rho_j \in \Gamma$, $j = 0, 1, ..., j_0$. Denote by $\mathcal{P}$ the Poincaré map associated to γ. The periodic broken ray γ is said to be elliptic if all eigenvalues of $D\mathcal{P}(\rho_0)$ lie on the unit cercle and are different from ± 1. Let $e^{i\alpha_j}$, $e^{-i\alpha_j}$, $0 < |\alpha_j| < \pi$, $j = 1, ..., n-1$, be these eigenvalues. We make the following assumptions:

$$\text{The Poincaré map } \mathcal{P} \text{ is 5-elementary, that is, } m_1\alpha_1 + ... + m_{n-1}\alpha_{n-1} \neq 0$$

$$\text{for all integers } m_1, ..., m_{n-1} \text{ such that } 1 \leq |m_1| + ... + |m_{n-1}| \leq 5. \tag{1.6.3}$$

$$\text{The Birkhoff normal form of } \mathcal{P} \text{ is nongenerate.} \tag{1.6.4}$$

Under these assumptions Popov [49] (see also [4]) constructed quasimodes (u_j, k_j) satisfying (1.6.1) whose counting function satisfies the asymptotics

$$N_{quasi}(r) = \frac{\text{mes}(G_E)}{n(2\pi)^n}(1 + o(1))r^n, \tag{1.6.5}$$

where G_E is a Cantor set with nonzero measure associated with the invariant tori of the Poincaré map $\mathcal{P}$. Combining (1.6.2) and (1.6.5) leads to the following

Theorem 1.6.2. *Suppose that there exists an elliptic periodic broken ray satisfying the above conditions. Then there exists a function $F(t) = O(t^{-\infty})$ such that the counting function N_F of the resonances satisfies the lower bound*

$$N_F(r) \geq \frac{\text{mes}(G_E)}{n(2\pi)^n}(1 - o(1))r^n. \tag{1.6.6}$$

It is quite possible that if the boundary Γ is analytic at ρ_j, $j = 0, ..., j_0$, one could construct quasimodes satisfying (1.6.1) with $\tilde{F}(t) = e^{-c_1 t}$, $c_1 > 0$, and hence Theorem 1.6.2 would hold with $F(t) = e^{-c_2 t}$, $c_2 > 0$, in this case. In what follows in this section we will consider an example of metric perturbations for which we have such a lower bound on the number of the resonances in an exponentially small neighbourhood of the real axis.

Suppose that there exist $0 < \rho_1 < \rho_2 < \rho_0$ such that $\mathcal{O} \subset \{|x| \le \rho_1\}$, and in $\rho_1 < |x| < \rho_2$ the operator Δ_g is of the form $c(|x|)^2\Delta$ with a positive function $c(r) \in C^\infty(\rho_1, \rho_2)$. Set $f(r) = \frac{c(r)}{r}$ and suppose that there exists $r_0 \in (\rho_1, \rho_2)$ such that

$$f'(r_0) = 0, \quad f''(r_0) > 0, \tag{1.6.7}$$

where f' and f'' denote the first and the second derivative, respectively. We also suppose that $c(r)$ is analytic at $r = r_0$.

Theorem 1.6.3. *Under the above assumptions, there exist constants $C, \beta > 0$ so that*

$$N_F(r) \ge Cr^n, \quad r \gg 1, \tag{1.6.8}$$

with a function $F(t) = e^{-\beta t}$.

Note that the existence of infinitely many resonances converging to the real axis exponentially fast for a class of operators of the form $c(|x|)^2\Delta$ on $\mathbf{R}^n$ was obtained by Ralston [54]. Ralston's method however does not allow to get lower bounds on the density of these resonances.

The above theorem can be derived from the standard quasimode construction for the semiclassical Schrödinger operator with a positive potential having a stable stationary point in the following way. The operator $Q = -c(|x|)^2\Delta$ can be written in the polar coordinates $(r, \theta) \in \mathbf{R}^+ \times \mathbf{S}^{n-1}$ as follows:

$$Q = -c(r)^2\partial_r^2 - \frac{n-1}{r}c(r)^2\partial_r - \frac{c(r)^2}{r^2}\Delta_{\mathbf{S}^{n-1}},$$

where $\Delta_{\mathbf{S}^{n-1}}$ denotes the Laplace-Beltrami operator on $\mathbf{S}^{n-1}$. Let w_μ be an eigenfunction of $-\Delta_{\mathbf{S}^{n-1}}$ with an eigenvalue μ^2, $\mu > 0$. Then $Q(v(r)w_\mu) = (Q_\mu v)w_\mu$, where

$$Q_\mu = -c(r)^2\partial_r^2 - \frac{n-1}{r}c(r)^2\partial_r + \mu^2\frac{c(r)^2}{r^2}.$$

Introduce a new variable $t = t(r)$ defined by

$$t = \int_{r_0}^{r} \frac{d\sigma}{c(\sigma)}.$$

In the new coordinates the operator Q_μ takes the form

$$Q_\mu = -\partial_t^2 + p(t)\partial_t + \mu^2 q(t),$$

where $q(t) = \frac{c(r(t))}{r(t)}$ satisfies

$$q(0) > 0, \quad q'(0) = 0, \quad q''(0) > 0, \tag{1.6.9}$$

and $p(t)$ and $q(t)$ are analytic at $t = 0$. Using the results of Helffer-Sjöstrand [15], [16] one can see that for every integer $k \geq 0$, there exists a C^∞ function $v_{k,\mu}$ supported in a neighbourhood of $t = 0$ such that

$$\|(Q_\mu - \nu_{k,\mu}^2)v_{k,\mu}\|_{L^2(\mathbf{R})} + |\langle v_{k,\mu}, v_{k',\mu}\rangle_{L^2(\mathbf{R})} - \delta_{k,k'}| \leq e^{-\beta_0 \mu},$$

where $\nu_{k,\mu}^2 = q(0)^2\mu^2 + \sqrt{q''(0)}(2k+1)\mu + O(1)$ and $\beta_0 > 0$ is a constant independent of k and μ. Then the functions $u_{k,\mu} = v_{k,\mu}w_\mu$ satisfy (1.6.1) with $k_j = \nu_{k,\mu}$, provided $k = O(\mu)$, and a function $\widetilde{F}(t) = e^{-\beta_1 t}$, $\beta_1 > 0$. Counting μ with the multiplicity (which is $\sim C_n\mu^{n-2}$, $\mu \gg 1$), it is easy to see that
$\sharp\{(k, \mu) : k$ is an integer,

$$0 \leq k \leq \mu,\ \mu \in \operatorname{spec}\sqrt{-\Delta_{\mathbf{S}^{n-1}}},\ \nu_{k,\mu}^2 \leq r^2\} \geq C'r^n,\ r \gg 1, C' > 0,$$

which implies (1.6.8) in view of the bound (1.6.2).

1.7 Asymptotics of the number of the resonances

In contrast to the counting function of the eigenvalues, there are very few examples of perturbations for which the counting function of the resonances is known to have an asymptotic behaviour. The most typical one is the case of degenerate perturbations, that is, when the counting function $\widetilde{N}(r)$ admits asymptotics of the form

$$\widetilde{N}(r) = \varphi(r)(1 + o(1)) \tag{1.7.1}$$

with a smooth increasing function φ satisfying

$$\lim_{r \to +\infty} \frac{r^n}{\varphi(r)} = 0. \tag{1.7.2}$$

Namely, we have the following improvement of Theorem 1.3.4.

Theorem 1.7.1. *Under the assumptions (1.7.1) and (1.7.2), for* $0 < \theta \ll 1$,

$$\sharp\{\lambda \in \mathcal{R} : 0 < \arg\lambda \leq \theta, |\lambda| \leq r\} = \varphi(r)(1 + o(1)). \tag{1.7.3}$$

In particular, if $n \geq 3$ *is odd,*

$$N(r) = 2\varphi(r)(1 + o(1)). \tag{1.7.4}$$

The asymptotics (1.7.3) are proved by Sjöstrand [61]. The asymptotics (1.7.4) follow from (1.7.3) combinned with (1.3.8) and the fact that in the odd dimensional case the resonances are symmetric with respect to the imaginary axis. Note that (1.7.4) has been previously proved in [84] under a slightly stronger assumption than (1.7.2) by using the Poisson formulae (1.4.2) and the bound (1.3.7).

It is much more complicated to get asymptotics of the number of the resonances for elliptic perturbations. Such asymptotics are known in the case of the Schrödinger operator with radial potentials.

Theorem 1.7.2. *Let $V(x)$ be a real-valued function of the form $V(x) = q(|x|)$, where $q \in C^2[0,a]$, $q(a) \neq 0$, $q(t) = 0$ for $t > a$. If $n \geq 1$ is odd, the counting function of the resonances associated to the Schrödinger operator $-\Delta + V(x)$ on $\mathbf{R}^n$ satisfies*

$$N(r) = C_n a^n r^n (1 + o(1)), \tag{1.7.5}$$

where $C_n > 0$ is a constant depending on n only, $C_1 = 2/\pi$.

This theorem is established by Zworski [89], [90]. It is worth noticing that Zworski's proof allows to get the same type of asymptotics in the case of scattering by a sphere of radius a in odd dimensional spaces.

1.8 Decay of the local energy and distribution of the resonances near the real axis

Throughout this section we will suppose that the operator Δ_g is elliptic. Let $u(t,x)$ be the solution to the wave equation

$$\begin{cases} (\partial_t^2 - \Delta_g)u(t,x) &= 0 \quad \text{in } \Omega \times \mathbf{R}, \\ u(t,x) &= 0 \quad \text{on } \Gamma \times \mathbf{R}, \\ u(0,x) &= f_1(x), \\ \partial_t u(0,x) &= f_2(x). \end{cases} \tag{1.8.1}$$

The energy of $u(t,x)$ in $\Omega_{R_0} := \Omega \cap \{|x| \leq R_0\}$, $R_0 > \rho_0$, is given by

$$E_{R_0}(u) = \frac{1}{2}\int_{\Omega_{R_0}} \left(\sum_{i,j=1}^n g_{ij}(x)\partial_{x_i} u \partial_{x_j}\overline{u} + c(x)^{-2}|\partial_t u|^2 \right) dx.$$

Given a $m \geq 0$, set

$$p_m(t) = \sup\left\{ \frac{\|\nabla_x u\|_{L^2(\Omega_{R_0})} + \|\partial_t u\|_{L^2(\Omega_{R_0})}}{\|\nabla_x f_1\|_{H^m(\Omega_{R_0})} + \|f_2\|_{H^m(\Omega_{R_0})}}, (0,0) \neq (f_1, f_2) \in [C^\infty(\overline{\Omega})]^2, \operatorname{supp} f_j \subset \Omega_{R_0} \right\}.$$

Clearly, $\forall m \geq 0$,

$$E_{R_0}(u) \leq C(m, R_0)p_m(t)^2 \left(\|\nabla_x f_1\|^2_{H^m(\Omega)} + \|f_2\|^2_{H^m(\Omega)}\right), \tag{1.8.2}$$

for all functions (f_1, f_2) such that $\nabla_x f_1 \in H^m(\Omega)$, $f_2 \in H^m(\Omega)$, $\operatorname{supp} f_j \subset \Omega_{R_0}$. In other words, $p_m(t)$ measures the rate of decay of the local energy of the solutions to (1.8.1) as $t \to +\infty$. It turns out that there is a close relationship between the behaviour of $p_m(t)$, $t \gg 1$, and the behaviour of the norm of $R_\chi(\lambda)$, $|\lambda| \to +\infty$, on the real axis. Suppose that

$$\|\lambda R_\chi(\lambda)\|_{\mathcal{L}(H)} \leq CM(|\lambda|), \quad \lambda \in \mathbf{R}, \tag{1.8.3}$$

where $C > 0$ is independent of λ and $M(t) \in C^\infty(\mathbf{R})$. The best possible bound in (1.8.3) is that one we have for the free resolvent (see (1.1.4)), that is, when $M = 1$. So, we may suppose that $M \geq 1$. Using the representation (1.2.3) it is not hard to prove (for example, see [86]) the following

Proposition 1.8.1. *Under the assumption (1.8.3), there exists a constant $C_1 > 0$ so that $R_\chi(\lambda)$ extends holomorphically to the region $\operatorname{Im}\lambda \leq C_1/M(|\lambda|)$, and satisfies there the bound (1.8.3) with possibly a new constant $C > 0$.*

Suppose that the function $M(t)$, $t \geq 1$, is increasing and denote by $f(t)$ its inverse, that is, $M(f(t)) = t$. We suppose that $f(t) = O(t^{k_0})$, $k_0 > 0$, and that for every constant $C > 0$ there exists a constant $\widetilde{C} = \widetilde{C}(C) > 0$ such that $f(Ct) \leq \widetilde{C} f(t)$, $\forall t \gg 1$. Using the above proposition one can prove (for example, see Section 3 of [51]) the following

Theorem 1.8.2. *For every $m \geq 0$ there exists a $C_m > 0$ so that*

$$p_m(t) \leq C_m \left(f(t/\log t)^{-m} + \varepsilon_n t^{-n}\right), \quad t \gg 1, \tag{1.8.4}$$

where $\varepsilon_n = 0$ if n is odd, $\varepsilon_n = 1$ if n is even.

It turns out that in the situation we discuss the norm of the cutoff resolvent is always exponentially bounded on the real axis.

Theorem 1.8.3. *There exist constants $C, \gamma > 0$ so that*

$$\|\lambda R_\chi(\lambda)\|_{\mathcal{L}(H)} \leq Ce^{\gamma|\lambda|}, \quad \lambda \in \mathbf{R}. \tag{1.8.5}$$

As a consequence,

$$p_m(t) \leq C_m(\log t)^{-m}, \quad t \gg 1. \tag{1.8.6}$$

Clearly, (1.8.6) follows from combining (1.8.5) and (1.8.4). Note also that it follows from (1.8.5) and Proposition 1.8.1 that there are no resonances in a region of the form $\mathrm{Im}\,\lambda \leq C_1 e^{-\gamma|\lambda|}$, $|\lambda| \geq C_2$, for some constants $C_1, C_2 > 0$. The above theorem is proved by Burq [3] using the Carleman estimates previously obtained by Lebeau-Robbiano [30], [31]. Another proof of Burq's result is presented in [87]. The method of [87] has been extended in [88] to prove an analogue of (1.8.5) for a class of metrics which differ from the Euclidean one by functions of order $O(e^{-C|x|^p})$, $|x| \gg 1$, where $C > 0$ and $p > 2$.

Clearly, the estimates (1.8.4) and (1.8.6) are trivial if $m = 0$. It turns out that to be able to derive some information about $p_0(t)$ from (1.8.3), one needs to have this latter bound with $M = 1$. More precisely, we have the following

Theorem 1.8.4. *The following three statements are equivalent:*

$$\lim_{t \to +\infty} p_0(t) = 0. \tag{1.8.7}$$

$$\|\lambda R_\chi(\lambda)\|_{\mathcal{L}(H)} \leq C_1, \quad \lambda \in \mathbf{R},\ |\lambda| \geq C_2, \tag{1.8.8}$$

with some constants $C_1, C_2 > 0$.

$$p_0(t) \leq \begin{cases} Ce^{-\beta t}, & n \text{ odd}, \\ Ct^{-n}, & n \text{ even}, \end{cases} \tag{1.8.9}$$

with some constants $C, \beta > 0$.

Note that the implication (1.8.7)⇒(1.8.9) was first proved by Morawetz [42] (see also [25], [43]) in the case of odd dimension using the Lax-Phillips theory. In the case of even dimension this implication is proved by Kawashita [23]. In [86] a proof of the above theorem is presented based on Proposition 1.8.1 and the following

Proposition 1.8.5. *Suppose that* $R_\chi(\lambda)$ *admits analytic continuations in the regions* $\{\lambda \in \mathbf{C} : 0 \leq \mathrm{Im}\,\lambda \leq C_1, \pm\mathrm{Re}\,\lambda > 0\}$, $C_1 > 0$, *such that*

$$\|\lambda R_\chi(\lambda)\|_{\mathcal{L}(H)} \leq C_2|\lambda|^k, \quad \mathrm{Im}\,\lambda \leq C_1,\ |\lambda| \geq C_3, \tag{1.8.10}$$

with some constants $C_2, C_3 > 0$ *and* $k \geq 0$. *Then there exist constants* $C, \beta > 0$ *so that*

$$p_k(t) \leq \begin{cases} Ce^{-\beta t}, & n \text{ odd}, \\ Ct^{-n}, & n \text{ even}. \end{cases} \tag{1.8.11}$$

1.9 Properties of the scattering phase

The scattering phase, $s(\lambda)$, can be considered as an analogue of the counting function of the eigenvalues of the Lapalce-Beltrami operator on compact manifolds. So, it is natural to expect that $s(\lambda)$ has an asymptotic behaviour as $\lambda \to +\infty$. In contrast to the counting function of the eigenvalues, however, obtaining asymptotics for $s(\lambda)$ is much more complicated because in general $s(\lambda)$ is not an increasing function and hence it is not possible to apply Tauberian arguments to it. Nevertheless, such asymptotics have been proved in the most interesting cases. Let us consider first the case when Δ_g is elliptic. We have the following

Theorem 1.9.1. *The following asymptotics hold*

$$s(\lambda) = a_0\lambda^n + O(\lambda^{n-1}), \tag{1.9.1}$$

where

$$a_0 = \tau_n \mathrm{Vol}(\mathcal{O}) + \tau_n \int_\Omega \left(\left(\det(c(x)^2 g_{ij}(x))\right)^{-1/2} - 1 \right) dx$$

$$\tau_n = (2\pi)^{-n}\mathrm{Vol}(\{x \in \mathbf{R}^n : |x| \le 1\}).$$

In this generality, (1.9.1) has been first proved by Melrose [36] in the case of odd dimensional spaces using the Poisson formulae (1.4.2) and the sharp upper bound (1.3.4). His method consists of decomposing the scattering phase as a sum of an increasing function and a symbol of order n, and then applying a Tauberian argument to the monotone part. It seems that this approach works in the case of even dimensional spaces as well using (1.4.3) instead of (1.4.2) and (1.3.5) instead of (1.3.4). Melrose's approach is based on the following result of Ivrii [21] (see also [35]) concerning the singularity of $u(t)$ at $t = 0$:

Theorem 1.9.2. *There exists an* $\varepsilon_0 > 0$ *so that if* $\phi \in C_0^\infty(\mathbf{R})$, $\phi(t) = 1$ *for* $|t| \le \varepsilon_0$, $\phi(t) = 0$ *for* $|t| \ge 2\varepsilon_0$, *then*

$$\widehat{\phi u}(\lambda) \sim \sum_{j=0}^{\infty} \alpha_j \lambda^{n-1-j}, \tag{1.9.2}$$

where $\alpha_0 = na_0$.

If we assume that the set of the closed transversally reflected geodesics has measure zero, then we have (1.9.1) with a second term of the form $a_1\lambda^{n-1}$ and a remainder $o(\lambda^{n-1})$ (see [21]).

Robert [55] found another proof of Theorem 1.9.1 which does not exploit the Poisson formulae and the resonances, and which works equally well in both cases of odd and even dimensions. Note that this theorem also follows from the quite general results obtained by Christiansen [9].

In the case of degenerate perturbations the scattering phase behaves very much like the counting function of the resonances.

Theorem 1.9.3. *Under the assumptions (1.7.1) and (1.7.2), as* $\lambda \to +\infty$,

$$s(\lambda) = \varphi(\lambda)(1 + o(1)). \tag{1.9.3}$$

Let ϕ be as in Theorem 1.9.2 and define the function s_ϕ so that $s'_\phi(\lambda) = \widehat{\phi u}(\lambda)$, $s_\phi(0) = 0$. It is not hard to see that under the assumptions (1.7.1) and (1.7.2), we have $s_\phi(\lambda) = \varphi(\lambda)(1 + o(1))$ and $s'_\phi(\lambda) = o(1)\varphi(\lambda)$. Therefore, the asymptotics (1.9.3) can be proved by using Melrose's approach (see [84] for more details). Another proof is given by Christiansen [9].

In contrast to the scattering phase, its first derivative $s'(\lambda)$ has a much more complicated behaviour and is much more sensible to the distribution of the resonances near the real axis.

Theorem 1.9.4. *Suppose that there exists a non-decreasing positive continuous function* $\widetilde{M}(t)$ *satisfying* $\widetilde{M}(t+\delta) \le C\widetilde{M}(t)$ *for* $0 < \delta \le 1$, *and such that there are no resonances in the region* $\{\operatorname{Im}\lambda \le C'\widetilde{M}(|\lambda|)^{-1}\}$. *Then,*

$$|s'(\lambda)| \le C_0|\lambda|^{n-1}\widetilde{M}(|\lambda|), \quad \lambda \in \mathbf{R}. \tag{1.9.4}$$

In particular, $s'(\lambda) = O(e^{\gamma_0|\lambda|})$, $\gamma_0 > 0$.

This theorem is proved by Petkov-Zworski [47]. The exponential bound of $s'(\lambda)$ follows from (1.9.4) combinned with Proposition 1.8.1 and Theorem 1.8.3.

2 Nontrapping perturbations

2.1 Distribution of the resonances and uniform decay of the local energy

The operator P will be said to be a nontrapping perturbation if the kernel $U(t,x,y)$ of the operator $\cos\left(t\sqrt{P}\right)$ satisfies the generalized Huyghens principle:

$$\forall \chi \in C_0^\infty(\mathbf{R}^n),\ \exists T_0 > 0, \text{ so that } \chi(x)U(t,x,y)\chi(y) \in C^\infty([T_0,+\infty) \times \overline{\Omega} \times \overline{\Omega}). \tag{2.1.1}$$

According to the results of Melrose-Sjöstrand [38], [39] on the propagation of the C^∞ singularities, in the situation we discuss the above condition is fulfilled if every generalized bicharacteristic (the definition being quite technical, we refer to [38], [39] for the details) leaves every compact in a finite time. In other words, the generalized Huyghens principle is fulfilled when there are no trapped rays. A typical example of a nontrapping perturbation is the scattering by a strictly convex obstacle, that is, when $\Delta_g \equiv \Delta$ and $\mathcal{O}$ is strictly convex.

Theorem 2.1.1. *Suppose (2.1.1) fulfilled. Then,* $\forall N > 0, \exists C_N > 0$ *so that there are no resonances in the region* $\operatorname{Im}\lambda \leq N\log|\lambda| - C_N$. *Moreover, there exists a constant* $C > 0$ *so that*

$$\|\lambda R_\chi(\lambda)\|_{\mathcal{L}(H)} \leq C, \quad \lambda \in \mathbf{R}. \tag{2.1.2}$$

As a consequence,

$$p_0(t) \leq \begin{cases} Ce^{-\beta t}, & n \text{ odd}, \\ Ct^{-n}, & n \text{ even}, \end{cases} \tag{2.1.3}$$

with some constants $C, \beta > 0$.

In this generality, the above theorem is proved by Vainberg [76], [77] (see also [28]). It turns out that in some cases of nontrapping perturbations one can have a better free of resonances region than that one given by the above theorem. Suppose that $\Delta_g \equiv \Delta$ and that the boundary Γ is analytic. Then according to Lebeau's result [29] on the propagation of the analytic singularities, we have an analogue of (2.1.1) with C^∞ replaced by the Gevrey class G^3, provided that every generalized bicharacteristic leaves every compact in a finite time. In this case Bardos-Lebeau-Rauch [2] proved the following

Theorem 2.1.2. *If* $\mathcal{O}$ *is a nontrapping obstacle with analytic boundary* Γ, $\Delta_g \equiv \Delta$, *then there exist positive constants* C_1 *and* C_2 *such that there are no resonances in the region* $\operatorname{Im}\lambda \leq C_1|\lambda|^{1/3} - C_2$.

Another proof of this theorem, based on Lebeau's result [29] and Vainberg's method [76], [77], is presented in [48]. Note that, if the obstacle is strictly convex, the optimal value of the constant C_1 can be calculated explicitly in terms of the first zero of the Airy function and the second fundamental form of Γ (see [2]). It turns out that for strictly convex obstacles one can get a similar free of resonances region without assuming analyticity of the boundary. The following result is established by Hargé-Lebeau [14] using the complex scaling up to the boundary Γ instead of the propagation of the analytic singularities (see also [67]):

Theorem 2.1.3. *If $\mathcal{O}$ is a strictly convex obstacle with C^∞ boundary Γ, $\Delta_g \equiv \Delta$, then there exist positive constants C_1' and C_2' such that there are no resonances in the region* $\operatorname{Im}\lambda \leq C_1'|\lambda|^{1/3} - C_2'$.

The constant C_1' can be calculated explicitly, too, and in fact we have $C_1' \leq C_1$. In other words, the more regular the boundary is, the larger free of resonances region we have.

2.2 Asymptotic expansion of the scattering phase

It follows from Theorem 2.1.1 and (1.9.4) that for nontrapping perturbations the first derivative of the scattering phase satisfies the bound $s'(\lambda) = O(\lambda^{n-1})$. In fact, much more is true.

Theorem 2.2.1. *Under the assumption (2.1.1), the following asymptotics hold:*

$$s'(\lambda) \sim \sum_{j=0}^{\infty} \alpha_j \lambda^{n-1-j}, \tag{2.2.1}$$

where α_j are the same as in Theorem 1.9.2.

This theorem is established by Petkov-Popov [45]. Clearly, to prove (2.2.1) it suffices to show that under the condition (2.1.1),

$$s'(\lambda) = \widehat{\phi u}(\lambda) + O(\lambda^{-\infty}). \tag{2.2.2}$$

When $n \geq 3$, (2.2.2) can be derived from the free of resonances region established in Theorem 2.1.1 combinned with Theorem 1.4.2. Indeed, by Theorem 2.1.1 the set $\mathcal{R}_\gamma$ contains finitely many resonances for any γ, hence $u_\gamma \in C^\infty(\mathbf{R})$, $\partial_t^k u_\gamma(t) = O(e^{-c_0 t})$, $c_0 > 0$, $\forall k \geq 0$. Thus, by Theorem 1.4.2 one concludes that $(1-\phi)u \in C^\infty(\mathbf{R})$ and $\partial_t^k((1-\phi)u) \in L^1(\mathbf{R})$ for every integer $k \geq 0$. Hence, $\lambda^k\widehat{(1-\phi)u}(\lambda) = O(1)$ for every integer $k \geq 0$, which implies (2.2.2).

3 Scattering by several strictly convex bodies

3.1 Distribution of the resonances for two strictly convex bodies

In this section we will discuss the Dirichlet problem in the case when $\Delta_g \equiv \Delta$ and $\mathcal{O} = \mathcal{O}_1 \cup \mathcal{O}_2$, where $\mathcal{O}_1$ and $\mathcal{O}_2$ are strictly convex bounded domains in

$\mathbf{R}^n$ with C^∞-smooth boundaries Γ_1 and Γ_2, $\Gamma_1 \cap \Gamma_2 = \emptyset$. Let $d > 0$ denote the distance between Γ_1 and Γ_2, and let $a_j \in \Gamma_j$, $j = 1, 2$ be the unique points such that $\mathrm{dist}(a_1, a_2) = d$. Then the ray connecting a_1 and a_2 is the only periodic (and hence trapped) ray in $\Omega = \mathbf{R}^n \setminus \mathcal{O}$, so the generalized Huyghens principle (2.1.1) is never fulfilled in this case. That is way one should expect a different distribution of the resonances near the real axis.

Denote by $\mathcal{P}$ the Poincaré map associated to the periodic ray $[a_1, a_2]$. It follows from the strictly convexity that $\mathcal{P}$ is a hyperbolic ray, that is, the eigenvalues of $D\mathcal{P}(a_1)$, which are real-valued in this case, are different from 1. Let e^{μ_j}, $j = 1, ..., n-1$, be the eigenvalues bigger than 1, i.e. $\mu_j > 0$. Denote $z_j = \frac{\mu_1 + ... + \mu_{n-1}}{4d} i + \frac{\pi}{d} j$, $j = 1, 2, ...$. The following result is due to Ikawa [17], [18]:

Theorem 3.1.1. *There exist positive constants C_1, C_2 and j_0 such that the only resonances, $\{\lambda_j\}$, in $\{\mathrm{Im}\,\lambda \le C_1, \mathrm{Re}\,\lambda \ge C_2\}$, are of multiplicity one and satisfy the expansion*

$$\lambda_j = z_j + \sum_{k=1}^{\infty} \beta_k j^{-k}, \quad j = j_0, j_0 + 1, ... \tag{3.1.1}$$

In particular, the above theorem implies that in the case of two strictly convex bodies there exists a strip of the form

$$\mathrm{Im}\,\lambda \le \frac{\mu_1 + ... + \mu_{n-1}}{4d} - \varepsilon, \quad \mathrm{Re}\,\lambda \ge C_\varepsilon > 0, \ \forall 0 < \varepsilon \ll 1,$$

free of resonances. The above theorem is extended by Gérard [11] who described all resonances below some logarithmic curves. To state Gérard's result, introduce the pseudo-resonances (which were first introduced in [1])

$$z_{j,\alpha} = \frac{\mu_1 \alpha_1 + ... + \mu_{n-1}\alpha_{n-1}}{4d} i + \frac{\pi}{d} j, \quad (j, \alpha) \in \mathbf{N} \times \mathbf{N}^{n-1}.$$

To each $z_{j,\alpha}$, associate a multiplicity $\mathrm{mult}(z_{j,\alpha})$ which is the number of all $\beta \in \mathbf{N}^{n-1}$ such that

$$\mu_1 \beta_1 + ... + \mu_{n-1}\beta_{n-1} = \mu_1 \alpha_1 + ... + \mu_{n-1}\alpha_{n-1}.$$

The following result is due to Gérard [11]:

Theorem 3.1.2. *For every $C \gg 1$ there exists a $C_1 = C_1(C) > 0$ so that in a neighbourhood of each $z_{j,\alpha} \in \{\mathrm{Im}\,\lambda \le C, \mathrm{Re}\,\lambda \ge C_1\}$ there is a resonance $\lambda_{j,\alpha}$ of multiplicity* $\mathrm{mult}(z_{j,\alpha})$, *and these are all resonances in* $\{\mathrm{Im}\,\lambda \le C, \mathrm{Re}\,\lambda \ge C_1\}$.

Gérard also obtained an asymptotic expansion of $\lambda_{j,\alpha} - z_{j,\alpha}$ as a fractional power series in $z_{j,\alpha}^{-1}$.

3.2 Distribution of the resonances for many strictly convex bodies

Let $\mathcal{O} = \cup_{j=1}^{J} \mathcal{O}_j$, $J \geq 3$, where each $\mathcal{O}_j$ is a strictly convex bounded domain in $\mathbf{R}^n$ with C^∞-smooth boundary. The obstacle $\mathcal{O}$ is a trapping one with infinitely many periodic rays in $\Omega = \mathbf{R}^n \setminus \mathcal{O}$, which makes the study of the resonances generated by these trapped rays much more complicated compared with the case $J = 2$. Nevertheless, there are some results in this direction essentially due to Ikawa. Consider the Dirichlet problem in Ω, $\Delta_g \equiv \Delta$, and suppose that $n = 3$. Let γ be a periodic ray in Ω. Denote by d_γ the length of γ and let $\beta_\gamma, \beta'_\gamma$ be the eigenvalues of the linear Poincaré map $P_\gamma = D\mathcal{P}_\gamma$ (as above $\mathcal{P}_\gamma$ denotes the Poincaré map associated to γ) such that $|\beta_\gamma|, |\beta'_\gamma| < 1$. Set $\lambda_\gamma = |\beta_\gamma \beta'_\gamma|^{1/2}$. We make the following assumptions:

The convex hull of every two connected components of $\mathcal{O}$ does not have common points

with any other connected component of $\mathcal{O}$. (3.2.1)

There exists a constant $\alpha > 0$ such that

$$\sum \lambda_\gamma d_\gamma e^{\alpha d_\gamma} < +\infty, \tag{3.2.2}$$

where the sum is taken over all primitive periodic rays γ in Ω.

Ikawa [19] proved the following

Theorem 3.2.1. *Under the assumptions (3.2.1) and (3.2.2), there exists a constant $c_0 > 0$ so that there are no resonances in $\operatorname{Im}\lambda \leq c_0$. Moreover, the following estimate holds*

$$p_2(t) \leq e^{-Ct}, \quad t \gg 1, \tag{3.2.3}$$

with some constant $C > 0$.

It seems that Ikawa's proof works also in the case of Neumann boundary conditions. It is also natural to expect that an analogue of the above theorem still holds for any space dimension $n \geq 2$ (with bound $O(t^{-n})$ in (3.2.3) if n is even).

The class of obstacles satisfying the assumptions (3.2.1) and (3.2.2) provides quite rich examples of trapping obstacles for which there are no infinitely many resonances tending to the real axis. It is natural to ask, however, if for this class of trapping obstacles there is a strip containing infinitely many resonances (which is the case if $J = 2$). It turns out that the answer is yes for the Neumann problem, while in the case of Dirichlet boundary conditions the problem is of a quite great complexity and to my best knowledge it is still open. The following theorem is due to Ikawa [20]:

Theorem 3.2.2. *Under the assumption (3.2.1), there exists a constant* $C > 0$ *so that there are infinitely many Neumann resonances in* $\mathrm{Im}\,\lambda \leq C$.

Petkov [44] obtained a lower bound of the form $O_\delta(r^{1-\delta})$, $\forall\delta > 0$, for the counting function of these resonances under weaker assumptions than (3.2.1). Ikawa proved the above theorem when the space dimension $n \geq 3$ is odd using the Poisson formulae (1.4.2) (note that in the case of even space dimension the proof goes in the same way using (1.4.3) instead of (1.4.2)) and the observation that in the case of Neumann boundary conditions the leading singular part of the distribution $u(t)$ is of the form (see [12]):

$$\sum T_\gamma \left|\det(Id - P_\gamma)\right|^{-1/2} \delta(t - d_\gamma),$$

where T_γ denotes the primitive period of γ, and the sum is taken over all periodic rays in Ω. Hence no cancelation of singularities is possible in this case. In the case of Dirichlet boundary conditions the leading singular part of the distribution $u(t)$ is of the form

$$\sum(-1)^{k_\gamma} T_\gamma \left|\det(Id - P_\gamma)\right|^{-1/2} \delta(t - d_\gamma),$$

where k_γ denotes the Maslov index of γ. Thus in this case the singularities may cancel, and in particular Ikawa's argument does not work anymore.

4 The Neumann problem in linear elasticity

4.1 Free of resonances regions

In this section we are going to discuss the resonances for a class of matrix-valued second order differential operators. Let $\mathcal{O} \subset \mathbf{R}^n$, $n \geq 2$, be a compact domain with C^∞-smooth boundary Γ and connected complement $\Omega = \mathbf{R}^n \setminus \mathcal{O}$. Denote by Δ_e the elasticity operator, which is a $n \times n$ matrix-valued differential operator defined by

$$\Delta_e u = \mu_0 \Delta u + (\lambda_0 + \mu_0)\mathrm{grad}(\mathrm{div}\, u),$$

where $u = {}^t(u_1, ..., u_n)$. Here λ_0 and μ_0 are the Lamé constants which are supposed to satisfy

$$\mu_0 > 0, \quad n\lambda_0 + 2\mu_0 > 0. \tag{4.1.1}$$

The operator Δ_e describes the propagation of two waves moving with speeds $c_1 = \sqrt{\mu_0}$ and $c_2 = \sqrt{\lambda_0 + 2\mu_0}$. Denote by Δ_e^N the selfadjoint realization of Δ_e in Ω with Neumann boundary conditions:

$$(Bu)_i := \sum_{j=1}^{n} \sigma_{ij}(u)\nu_j|_\Gamma = 0, \quad i = 1, ..., n,$$

where $\sigma_{ij}(u) = \lambda_0\delta_{ij}\mathrm{div}\, u + \mu_0(\partial_{x_j} u_i + \partial_{x_i} u_j)$ is the stress tensor and $\nu = (\nu_1, ..., \nu_n)$ is the outer unit normal to Γ. The resonances associated to Δ_e^N are defined in the same way as in Section 1.2, that is, as the poles of the meromorphic continuation of the cutoff resolvent

$$R_\chi(\lambda) := \chi(\Delta_e^N + \lambda^2)^{-1}\chi : L^2(\Omega; \mathbf{C}^n) \to L^2(\Omega; \mathbf{C}^n)$$

from $\mathrm{Im}\,\lambda < 0$ to the complex plane $\mathbf{C}$ if n is odd, and to the Riemann surface, Λ, of the logarithme if n is even. At first glance the operator Δ_e^N looks very much like the operators discussed in the previous sections and has similar properties. Indeed, most of the properties and the results discussed in Section 1 are valid for Δ_e^N as well. The only difference is that the Neumann boundary conditions for Δ_e do not satisfy the so-called Lopatinski-Shapiro condition, which has for consequence that the generalized Huyghens principle (2.1.1) is never fulfilled for this problem. In other words, every obstacle (even the ball) is a trapping obstacle for the operator Δ_e^N. The fact that the Lopatinski-Shapiro condition is violated is expressed by the fact that the formal parametrix (which is a $n \times n$ matrix-valued semiclassical pseudodifferential operator on Γ) of the Dirichlet-to-Neumann map is not elliptic in the elliptic region $\mathcal{E} = \{(x, \xi) \in T^*\Gamma : c_1|\xi|_x > 1\}$. Explicit calculations show that its principal symbol has one eigenvalue which vanishes on a variety $\Sigma = \{(x, \xi) \in T^*\Gamma : c_R|\xi|_x = 1\} \subset \mathcal{E}$, while the other $n - 1$ eigenvalues are nonvanishing. The existence of such a characteristic variety is interpreted as existence of surface waves moving on Γ with a speed $c_R < c_1$ (see [75]), called Rayleigh waves after Lord Rayleigh who was first to study these waves in the case of a half space (see [53]). Since in our case Γ is compact, these surface waves stay trapped in a compact set, and hence every compact obstacle is trapping for this problem. It is worth noticing that there are no such surface waves for the Dirichlet realization of Δ_e. For example, the strictly convex obstacles are nontrapping (satisfy (2.1.1)) for the elasticity equation with Dirichlet boundary conditions.

Since the generalized Huyghens principle (2.1.1) is not satisfied for Δ_e^N, it is not natural to expect that the resonances of Δ_e^N have a distribution like in Theorem 2.1.1. However, the fact that the characteristic variety Σ is included in the elliptic region $\mathcal{E}$ enables to obtain quite large regions free of resonances if $\mathcal{O}$ is strictly convex. More precisely, we have the following

Theorem 4.1.1. *Let $\mathcal{O}$ be a bounded strictly convex domain with C^∞-smooth boundary Γ. Then for every $M, N > 1$ there exist constants $C_M, C_N > 0$ so that there are no resonances of Δ_e^N in the region $C_N|\lambda|^{-N} \leq \operatorname{Im}\lambda \leq M\log|\lambda| - C_M$. Moreover, if Γ is analytic, then the free of resonances region is of the form $Ce^{-\gamma|\lambda|} \leq \operatorname{Im}\lambda \leq M\log|\lambda| - C_M$, where $C, \gamma > 0$.*

In the C^∞ case this theorem is proved in [72], while the case of analytic boundary is treated in [85]. Kawashita [24] obtained the same type of free of resonances regions as in the C^∞ case for more general obstacles having no trapped rays in Ω (such obstacles are in fact nontrapping for Δ_e in Ω with Dirichlet boundary conditions on Γ, and hence the only trapping in the case of Neumann boundary conditions comes from the Rayleigh waves on Γ). It is worth noticing that in analogy with the Laplacian and in view of Theorem 2.1.3 it is natural to expect that when $\mathcal{O}$ is strictly convex there should not be resonances of Δ_e^N in a region of the form $\{1 \leq \operatorname{Im}\lambda \leq C_1|\lambda|^{1/3} - C_2\}$ with some positive constants C_1 and C_2, certainly different from those in Theorem 2.1.3. This is proved in [71] when $\mathcal{O}$ is a ball.

4.2 Existence of Rayleigh resonances

It is natural to expect that the Rayleigh waves generate infinitely many resonances (which in view of Theorem 4.1.1 should be very close to the real axis) as otherwise the distribution of the resonances of Δ_e^N for strictly convex obstacles would be the same as in the case of nontrapping perturbations. In fact, we have the following

Theorem 4.2.1. *For any obstacle $\mathcal{O}$ with C^∞-smooth boundary there exists an infinite sequence $\{\lambda_j\}$ of different resonances of Δ_e^N such that*

$$0 < \operatorname{Im}\lambda_j \leq C_N|\lambda_j|^{-N}, \quad \forall N \geq 1. \tag{4.2.1}$$

Moreover, if at least one of the connected components of $\mathcal{O}$ is of analytic boundary, then there exists an infinite sequence $\{\lambda_j\}$ of different resonances of Δ_e^N such that

$$0 < \operatorname{Im}\lambda_j \leq Ce^{-\gamma|\lambda_j|} \tag{4.2.2}$$

with some positive constants C and γ.

In the C^∞ case this theorem is proved in [73] (see also [72]) , while the case of analytic boundary is treated in [85]. The proof is based on Theorem 1.6.1 and the observation that the existence of Rayleigh waves in the elliptic region $\mathcal{E}$ allows to construct quasimodes supported in a neighbourhood of the boundary. Moreover, if one has more information about the counting function of these quasimodes one could obtain lower bounds of the counting function of the resonances near the real axis. Such lower bounds are obtained by Stefanov [70]:

Theorem 4.2.2. *For any obstacle $\mathcal{O}$ with C^∞-smooth boundary there exists a function $F \in C(1, +\infty)$, $F(t) = O(t^{-\infty})$, $t \gg 1$, such that*

$$\sharp\{\lambda_j - \textit{resonance of } \Delta_e^N : 0 < \operatorname{Im}\lambda_j \leq F(|\lambda_j|), |\lambda_j| \leq r\}$$

$$\geq \tau_{n-1} c_R^{-n+1} \operatorname{Vol}(\Gamma) r^{n-1} - O(r^{n-2}), \tag{4.2.3}$$

where the constant τ_{n-1} is defined in Theorem 1.9.1 (with n replaced by $n-1$).

For obstacles for which Theorem 4.1.1 holds there are near the real axis only resonances generated by the Rayleigh waves, so it is natural to expect that these resonances are close to the corresponding quasimodes and that their counting function has the same behaviour as the counting function of the quasimodes. In other words, one should expect asymptotic behaviour of the counting function of the Rayleigh resonances. The following theorem is due to Sjöstrand-Vodev [62]:

Theorem 4.2.3. *Let $\mathcal{O}$ be a strictly convex obstacle with C^∞-smooth boundary Γ. Then*

$$N_R(r) := \sharp\{\lambda_j - \textit{resonance of } \Delta_e^N : 0 < \operatorname{Im}\lambda_j \leq 1, |\lambda_j| \leq r\}$$

$$= \tau_{n-1} c_R^{-n+1} \operatorname{Vol}(\Gamma) r^{n-1} - O(r^{n-2}). \tag{4.2.4}$$

4.3 Behaviour of the local energy

In the same way as in the case of the Laplace operator one can define the quantities $p_m(t)$ (see Section 1.8) in the case of the Neumann problem in linear elasticity, which measure the rate of decay of the local energy of the solutions to the corresponding mixed problem. Because of the existence of resonances close to the real axis due to the Rayleigh surface waves, one should not expect that $p_m(t)$ would decay very fast to zero as $t \to +\infty$. In the next theorem $\mathcal{O}$ is an arbitrary bounded obstacle with C^∞-smooth boundary.

Theorem 4.3.1. *There exists a constant $\alpha > 0$ such that*

$$p_0(t) \geq \alpha, \quad t \gg 1. \tag{4.3.1}$$

For every $m > 0$ and every $\delta > 0$ we have

$$\limsup_{t \to +\infty} t^\delta p_m(t) > 0. \tag{4.3.2}$$

Moreover, if at least one of the connected components of $\mathcal{O}$ is of analytic boundary, then for every $m > 0$ we have

$$\limsup_{t \to +\infty} (\log t)^m p_m(t) > 0. \tag{4.3.3}$$

The inequality (4.3.1) is proved by Kawashita [22]. It also follows from the fact that there exist infinitely many resonances converging to real axis (see Theorem 4.2.1). The inequality (4.3.2) was first proved by Kawashita [24] for a restricted class of obstacles, and then extended in [85] to an arbitrary obstacle. The inequality (4.3.3) is proved in [85]. In particular, it shows that the bound (1.8.6) obtained by Burq [3] is sharp.

4.4 Asymptotic behaviour of the scattering phase

The scattering phase, $s_N(\lambda)$, associated to the operator Δ_e^N can be defined in the same way as in Section 1.4. Because of the existence of resonances exponentially close to the real axis one can easily see that one may have bounds of the form $|s'_N(k_j)| \geq e^{\beta k_j}$, $\beta > 0$, for an infinite sequence of real numbers $k_j \to +\infty$. So, as one may expect, the asymptotics (2.2.1) are far from being true for $s'_N(\lambda)$. On the other hand, we have an analogue of the asymptotics (1.9.1) for $s_N(\lambda)$ (with a different constant a_0). It turns out that for strictly convex obstacles $s_N(\lambda)$ has two term asymptotics with a remainder of right order. The following theorem is due to Cardoso-Vodev [7]:

Theorem 4.4.1. *If $\mathcal{O}$ is strictly convex with C^∞-smooth boundary Γ, then*

$$s_N(\lambda) = a_0\lambda^n + a_1\lambda^{n-1} + r(\lambda), \tag{4.4.1}$$

where $r(\lambda) = O(\lambda^{n-2})$ if $n > 2$, $r(\lambda) = a_2 \log\lambda + O(1)$ if $n = 2$, and $a_0 = \tau_n((n-1)c_1^{-n} + c_2^{-n})\mathrm{Vol}(\mathcal{O})$.

The constant a_1 is of the form $b_1 \mathrm{Vol}(\Gamma)$, where b_1 depends on the Lamé constants and the dimension. One could expect that the above asymptotics hold for every obstacle which is nontrapping for the Dirichlet realization of Δ_e. Another natural expectation is that (4.4.1) holds with $r(\lambda) = o(\lambda^{n-1})$ under the assumption that the measure of the set of the periodic transversally reflected rays in $T^*\Omega$ is zero. Furthermore, if $\mathcal{O}$ is strictly convex such that the measure of the set of the periodic geodesics in $T^*\Gamma$ is zero, one should expect that (4.4.1) holds with $r(\lambda) = a_2\lambda^{n-2} + o(\lambda^{n-2})$ if $n > 2$. Such a behaviour is suggested by the following

Theorem 4.4.2. *If $\mathcal{O}$ is strictly convex with C^∞-smooth boundary, then there exists a function of the form*

$$g(\lambda) = \sum_{k=0}^{n-1} b_k\lambda^{n-k} + b_n \log\lambda,$$

where $b_0 = a_0$, such that for every $p \gg 1$, $0 < \delta \ll 1$, we have

$$N_R(\lambda - \lambda^{-p}) - O_{p,\delta}(1) - O(\lambda^\delta) \le s_N(\lambda) - g(\lambda) \le N_R(\lambda + \lambda^{-p}) + O_p(1), \quad (4.4.2)$$

where $N_R(\lambda)$ is the counting function of the Rayleigh resonances introduced in Theorem 4.2.3.

This theorem is proved for odd $n \ge 3$ in [8] by using the Poisson formulae (1.4.2) (in the case of even n one should use (1.4.3) instead) and the free of resonances region established in Theorem 4.1.1. Note that the asymptotics (4.4.1) also follow from combining (4.4.2) and (4.2.4). Moreover, obtaining a third term in (4.4.1) is equivalent to obtaining second term asymptotics for $N_R(\lambda)$, that is, for the counting function of the quasimodes generated by the Rayleigh waves. Since these latter objects can be viewed as eigenvalues of a pseudodifferential operator on Γ, such two terms asymptotics quite probably hold under the assumption mentioned just before Theorem 4.4.2.

5 The transmission problem

5.1 The case of interior totally reflected rays

Let $\mathcal{O}$ be a bounded strictly convex domain with C^∞-smooth boundary Γ and denote $\Omega = \mathbf{R}^n \setminus \mathcal{O}$. Let $c \neq 1$ and α be two positive constants. The complex number λ will be said to be a resonance for the transmission problem associated to the obstacle $\mathcal{O}$ if the following problem has a nontrivial solution:

$$\left\{\begin{array}{rcl} (c^2\Delta + \lambda^2)u_1 & = & 0 \quad \text{in } \mathcal{O}, \\ (\Delta + \lambda^2)u_2 & = & 0 \quad \text{in } \Omega, \\ u_1 - u_2 & = & 0 \quad \text{on } \Gamma, \\ \partial_{\nu'}u_1 + \alpha\partial_\nu u_2 & = & 0 \quad \text{on } \Gamma, \\ u_2 & - & \lambda - \text{outgoing}, \end{array}\right. \tag{5.1.1}$$

where $\Delta = \sum_{j=1}^n \partial_{x_j}^2$, ν denotes the outer unit normal to Γ and $\nu' = -\nu$ is the inner one. The resonances of the transmission problem can be also defined as the poles of the meromorphic continuation of the cutoff resolvent, $R_\chi(\lambda) := \chi(G + \lambda^2)^{-1}\chi$, of the operator

$$Gu := (c^2\Delta u_1, \Delta u_2), \quad u = (u_1, u_2) \in D(G),$$

on the Hilbert space $H = L^2(\mathcal{O}; \alpha^{-1}c^{-2}dx) \oplus L^2(\Omega)$, where the domain of definition of G is given by

$$D(G) := \{(u_1, u_2) \in H, u_1 \in H^2(\mathcal{O}), u_2 \in H^2(\Omega), u_1|_\Gamma = u_2|_\Gamma, \partial_{\nu'}u_1|_\Gamma = -\alpha\partial_\nu u_2|_\Gamma\}.$$

The problem (5.1.1) describes the propagation of acoustic waves moving in two different media, with speed 1 in Ω and with speed c in $\mathcal{O}$. When a wave moving in Ω (resp. in $\mathcal{O}$) reaches the boundary Γ there is one part which enters in $\mathcal{O}$ (resp. in Ω) and another part which reflects from the boundary. In other words, there is a transmission of energy through the boundary. Consequently, there is a propagation of C^∞ singularities along the periodic broken rays in $\mathcal{O}$, and hence the generalized Huyghens principle (2.1.1) is not fulfilled for the transmission problem. Moreover, if $c < 1$, the broken rays in $\mathcal{O}$ close enough to the glancing manifold $\mathcal{K} = \{(x, \xi) \in T^*\Gamma : c|\xi|_x = 1\}$ lie in the elliptic region $\mathcal{E} = \{(x, \xi) \in T^*\Gamma : |\xi|_x > 1\}$ for the exterior problem, so such rays stay trapped in $\mathcal{O}$ for a long time. More precisely, if a broken ray $\gamma(t)$ is ε close to $\mathcal{K}$ for $t = 0$ (in a suitable metric) it remains 2ε close to $\mathcal{K}$ for $|t| \le C_N\varepsilon^{-N}$, $\forall N \gg 1$. This property is known as "effective stability of the billiard flow in $\mathcal{O}$ near the glancing manifold" and such broken rays are called "interior totally reflected rays". The existence of such rays suggests that quite a lot of energy should stay trapped in the obstacle $\mathcal{O}$ for a long time, which could produce infinitely many resonances converging to the real axis. This was proved by Popov-Vodev [52]:

Theorem 5.1.1. *Let $\mathcal{O}$ be strictly convex with C^∞-smooth boundary. Then, if $c < 1$, there exists an infinite sequence of different resonances $\{\lambda_j\}$ of G such that*

$$0 < \operatorname{Im}\lambda_j \le C_N|\lambda_j|^{-N}, \quad \forall N \ge 1. \tag{5.1.2}$$

In view of Theorem 1.6.1, to prove the above theorem it suffices to construct compactly supported quasimodes for the transmission problem (1.5.1). Such quasimodes, concentrated in a neighbourhood of the boundary Γ (microlocally supported in $\mathcal{K}$) are constructed in [52] using the fact that when $c < 1$, the glancing manifold $\mathcal{K}$ is included in the elliptic region $\mathcal{E}$.

5.2 The case of exterior totally reflected rays

When $c > 1$ the glancing manifold $\mathcal{K}$ is included in the hyperbolic region $\mathcal{H} = \{(x,\xi) \in T^*\Gamma : |\xi|_x < 1\}$ of the exterior problem. That is why there are no interior totally reflected rays in $\mathcal{O}$ and the construction of quasimodes like those in the case discussed in Section 5.1 is no longer possible. Instead, there are exterior totally reflected rays in Ω which do not enter into the obstacle. This suggests that in the case when $c > 1$ the distribution of the resonances near the real axis should look much more like the distribution of the resonances for nontrapping perturbations than like that one in the case $c < 1$. The following theorem is established in [5] (in a more general setting):

Theorem 5.2.1. *Let $\mathcal{O}$ be strictly convex with C^∞-smooth boundary. Then, if $c > 1$, the cutoff resolvent satisfies the estimate*

$$\|\lambda R_\chi(\lambda)\|_{\mathcal{L}(H)} \le C, \quad \lambda \in \mathbf{R}. \tag{5.2.1}$$

As a consequence, there are no resonances of G in a strip $0 < \operatorname{Im}\lambda \le \gamma$, $\gamma > 0$, and the following estimate holds, for $t \ge 1$,

$$p_0(t) \le \begin{cases} C'e^{-\beta t}, & n \text{ odd}, \\ C't^{-n}, & n \text{ even}, \end{cases} \tag{5.2.2}$$

with constants $C', \beta > 0$.

Note that (5.2.2) and the existence of a free of resonances strip follow from (5.2.1) combinned with Theorem 1.8.4 and Proposition 1.8.1. The above theorem shows in particular that for the transmission problem with $c > 1$ we have the same type of uniform decay of the local energy as in the case of nontrapping perturbations although this problem does not satisfy the generalized Huyghens principle (2.1.1). Moreover, when $\mathcal{O}$ is a ball it is not hard to see that there is a sequence of resonances $\{\mu_j\}$ such that $\mu_j \to \gamma_0 > 0$. Hence it is not possible to have a better free of resonances region (as for example in Theorem 2.1.1) than a strip.

5.3 Asymptotics of the number of the resonances near the real axis

As in the case of the Neumann problem in linear elasticity (see Theorem 4.1.1) one could expect that in the case of the transmission problem there should be also large free of resonances regions far from the real axis and that the counting function of the resonances lying below such regions admits an asymptotic behaviour. Indeed, such results are obtained in [6] under some natural assumptions on the parameter α appearing in the boundary conditions:

Theorem 5.3.1. *Let $\mathcal{O}$ be strictly convex with C^∞-smooth boundary and let $c < 1$. Then there exists a constant $\alpha_0 > 0$ so that if $\alpha \le \alpha_0$, there are no resonances of G in a region of the form $\{C\alpha \le \operatorname{Im}\lambda \le C_1|\lambda|^{1/3} - C_2\}$, where C, C_1 and C_2 are positive constants independent of α. Moreover, the following asymptotics hold*

$$\sharp\{\lambda_j - \text{resonance of } G : 0 < \operatorname{Im}\lambda_j \le C\alpha, |\lambda_j| \le r\}$$

$$= \tau_n c^{-n}\operatorname{Vol}(\mathcal{O})r^n + O_\varepsilon(r^{n-1/3+\varepsilon}), \quad \forall\varepsilon > 0. \tag{5.3.1}$$

Theorem 5.3.2. *Let $\mathcal{O}$ be strictly convex with C^∞-smooth boundary and let $c > 1$. Then there exists a constant $A_0 > 0$ so that if $\alpha \ge A_0$, there are no resonances of G in a region of the form $\{C/\alpha \le \operatorname{Im}\lambda \le C_1|\lambda|^{1/3} - C_2\}$, where C, C_1 and C_2 are positive constants independent of α. Moreover, the following asymptotics hold*

$$\sharp\{\lambda_j - \text{resonance of } G : 0 < \operatorname{Im}\lambda_j \le C/\alpha, |\lambda_j| \le r\}$$

$$= \tau_n c^{-n}\operatorname{Vol}(\mathcal{O})r^n + O_\varepsilon(r^{n-1/3+\varepsilon}), \quad \forall\varepsilon > 0. \tag{5.3.2}$$

Note that the constant C_1 above can be taken the same as the constant C_1' in Theorem 2.1.3.

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