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DISCRETE SYSTEMS WITH ADVANCED ARGUMENT

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Abstract

The existence and uniqueness of solutions to the difference equation with advanced argument $\Delta x(n) = f(n, x(n), x(g(n)))$, $g(n) \geq n+1$, are discussed.

1 Introduction

In this paper, where we denote $\mathrm{N} = \{0, 1, 2, \ldots\}$, we treat the problem of existence and uniqueness of the solutions to the initial value problem (IVP)

$$\begin{cases} \Delta x(n) &= f(n, x(n), x(g(n))), \; n \in \mathrm{N}, \\ x(0) &= \xi, \end{cases} \tag{1.1}$$

(where the difference operator Δ is defined by $\Delta x(n) = x(n+1) - x(n)$) corresponding to the difference equation with advanced argument

$$\Delta x(n) = f(n, x(n), x(g(n))). \tag{1.2}$$

The sequence $\{g(n)\}$ satisfies $g(n) \geq n+1$, $\forall n \in \mathrm{N}$. For equations with continuous argument, this problem has been analyzed in [9], and the present paper can be considered, with appropriate modifications, as the discrete version of those results.

By a solution of Eq. (1.2) we will understand a sequence $x : \mathrm{N} \to R^n$ that satisfies this equation on all of N. Thus, we are treating a non local problem. This implies that the known methods for a certain class of equation with advanced argument appearing in the theory of equations with delay [6] are not applicable to Eq. (1.2).

Although, a solution of Eq. (1.2) has not a clear physical meaning, from a simple inspection of problem (1.1) we may observe that the present state $x(n)$ is conditioned to the future understanding of the sequence $x(k)$, $k \geq n$. The difficulties arising in the study of equations with advanced argument remind the problem of backward prolongation of delay differential equations [7].

The antecedents of this study are the following: The beautiful paper of Sugiyama [9], who, by simple examples, shows that, in general, the uniqueness of the IVP fails if we do not restrict the analyze of these equations to a specific functional space; the paper of Popenda and E. Schmeidel [8], who study the problem of existence of convergent solutions of scalar equations with advanced argument; our own research on this subject, mainly dedicated to linear problems [1, 2, 3, 4, 5].

2 Existence and uniqueness

Throughout we will use a sequence $\{h(n)\}$ satisfying

(H0) $$h(0) = 1,\ 0 \leq h(n) \leq h(n+1), \forall n \in \mathrm{N},\ \sum_{n=0}^{\infty} h(n)^{-1} < \infty.$$

We define

$$\ell_h^\infty = \{f : \mathrm{N} \to R^n,\ \ ||f||_h < \infty\},$$

where

$$||f||_h = \sup\{|h(n)^{-1}f(n)|, \ \ n \in \mathrm{N}\}.$$

If $||f||_h < \infty$, then we will say that f is an h-bounded sequence.

The Eq. (1.2) is defined by the function $f : \mathrm{N} \times R^n \times R^n \to R^n$, which is assumed to be continuous with respect to $(x, y) \in R^n \times R^n$ for any fixed n.

Let us consider the following conditions on the IVP (1.1):

(H1) For any point $\xi \in R^n$, the following sequence converges

$$\sum_{m=0}^{n-1} h(m)^{-1}|f(m, \xi, \xi)|.$$

(H2) Let $w(n, \lambda, \mu)$ be a nonnegative and nondecreasing function with respect to λ and μ for any fixed n, $w : \mathrm{N} \times [0, +\infty) \times [0, +\infty) \to [0, +\infty)$, $w(n, 0, 0) = 0$, such that the series

$$\Omega(\gamma) = \sum_{m=0}^{\infty} w(m, \gamma, \alpha(m)\gamma)$$

is convergent, where the sequence $\{\alpha(n)\}$ is defined by

$$\alpha(n) = \frac{h(g(n))}{h(n)}, \quad \forall n \in \mathrm{N}.$$

(H3) For a nonnegative constant γ, we define the sequence $\{M_k(\gamma)\}$ by

$$M_0(\gamma) = \gamma, \ \ M_{k+1}(\gamma) = \Omega(M_k(\gamma)), \ \ k = 0, 1, 2, \ldots$$

We will assume that for any γ, the series $\sum_{k=0}^{\infty} M_k(\gamma)$ converges.

(H4) The function $f(n, x, y)$ satisfies the inequality

$$|h(n)^{-1}(f(n, x_1, y_1) - f(n, x_2, y_2))| \leq w(n, h(n)^{-1}|x_1 - x_2|, h(n)^{-1}|y_1 - y_2|)$$

for any $n \in \mathrm{N}$ and x_1, x_2, y_1, y_2 in R^n.

Theorem 1. *The hypotheses* (H0, H1, H2, H3, H4) *imply the existence of a unique solution* $\{\varphi(n)\}$ *in the space* ℓ_h^∞ *of the IVP* (1.1) *for every* $\xi \in R^n$.

Proof. We will use the successive approximations method to demonstrate the existence of solution of the problem (1.1). Let us define the recurrence

$$\begin{aligned} x_0(n) &= \xi, \\ x_{k+1}(n) &= \xi + \sum_{m=0}^{n-1} f(m, x_k(m), x_k(g(m))),\ k = 0, 1, 2, \dots \end{aligned}$$

We will prove that the estimate

$$|h(n)^{-1}(x_{k+1}(n) - x_k(n))| \le M_k(b), \quad k = 0, 1, 2, \dots$$

is valid for any $n \in \mathbf{N}$. For $k = 1$ we have

$$|h(n)^{-1}(x_1(n) - \xi)| \le \sum_{m=0}^{n-1} |h(n)^{-1} f(m, x_0(m), x_0(g(m)))|.$$

From **(H1)** and the notations introduced in **(H4)** we write

$$|h(n)^{-1}(x_1(n) - \xi)| \le \sum_{m=0}^{\infty} |h(m)^{-1} f(m, \xi, \xi)| = b = M_0(b).$$

Let us suppose that

$$|h(n)^{-1}(x_k(n) - x_{k-1}(n))| \le M_{k-1}(b), \quad k = 1, 2, \dots$$

Since $|h(n)^{-1}(x_{k+1}(n) - x_k(n))|$ is majorated by

$$\sum_{m=0}^{n-1} w(m, h(m)^{-1}|x_k(m) - x_{k-1}(m)|, h(m)^{-1}|x_k(g(m)) - x_{k-1}(g(m))|),$$

then it follows the inequality

$$|h(n)^{-1}(x_k(n) - x_{k-1}(n))| \le \sum_{m=0}^{\infty} w(m, M_{k-1}(b), \alpha(m) M_{k-1}(b)) = M_k(b).$$

From the telescope identity

$$x_{k+1}(n) = x_{k+1}(n) - x_k(n) + x_k(n) - x_{k-1}(n) + \ldots + x_1(n) - \xi + \xi,$$

we conclude that the convergence of sequence $\{x_k\}$ is equivalent to the convergence of the series

$$\xi + \sum_{k=0}^{\infty}(x_{k+1}(n) - x_k(n)).$$

In ℓ_h^∞, the partial sums of this series are majorated by:

$$|h(n)^{-1}\xi| + |h(n)^{-1}(x_1(n) - \xi)| + \ldots + |h(n)^{-1}(x_{k+1}(n) - x_k(n))| \leq$$

$$||\xi||_h + M_0(b) + M_1(b) + \ldots + M_k(b).$$

The condition **(H3)** implies the convergence of the series $\sum_{k=0}^{\infty} M_k(b)$ assuring, by the Weierstrass criterion, the uniform convergence of $\{x_k\}_{k=0}^{\infty}$, on all **N**, to a sequence $\varphi(n)$ belonging to ℓ_h^∞. Since $\{x_k(n)\}_{k=0}^{\infty}$ converges coordinate by coordinate to $\varphi(n)$, then $\varphi(n)$ is a solution of the IVP (1.1).

Now, we will prove the uniqueness of the solution φ of IVP (1.1) in ℓ_h^∞. Suppose that $\{x(n)\}$, $\{y(n)\}$ are two h-bounded solutions of problem (1.1). From **(H4)** it follows

$$\begin{aligned}|h(n)^{-1}(x(n) - y(n))| &\leq \sum_{m=0}^{n-1} |h(n)^{-1}(f(m, x(m), x(g(m))) \\ &\quad -f(m, y(m), y(g(m))))| \\ &\leq \sum_{m=0}^{n-1} w(m, |h(m)^{-1}(x(m) - y(m))|, |h(m)^{-1}(x(g(m)) \\ &\quad -y(g(m)))|).\end{aligned}$$

Since $\{x(n)\}$, $\{y(n)\}$ are h-bounded, we can define $\delta = ||x - y||_h$. Therefore

$$|h(n)^{-1}(x(n) - y(n))| \leq \sum_{m=0}^{n-1} w(m, \delta, \alpha(m)\delta) \leq \Omega(\delta),$$

from whence

$$\delta \leq \Omega(\delta).$$

We will see that the unique nonnegative number satisfying the above inequality is $\delta = 0$. First, we prove that

$$\delta \leq M_k(\delta), \quad k = 1, 2, \ldots$$

The definition of Ω leads to the estimate $\Omega(\delta) = M_1(\delta)$, implying $\delta \leq M_1(\delta)$. If $\delta \leq M_{k-1}(\delta)$, then

$$\begin{aligned} M_k(\delta) &= \Omega(M_{k-1}(\delta)) = \sum_{m=0}^{\infty} w(m, M_{k-1}(\delta), \alpha(m)M_{k-1}(\delta)) \\ &\geq \sum_{m=0}^{\infty} w(m, \delta, \alpha(m)\delta) = \Omega(\delta) \geq \delta. \end{aligned}$$

The series $\sum_{k=0}^{\infty} M_k(\delta)$ converges, what implies $\lim_{k\to\infty} M_k(\delta) = 0$. From $\delta \leq M_k(\delta)$ we get $\delta = 0$. ∎

Theorem 2. *Under conditions* (**H0, H1, H2, H3, H4**), *the solution* $\{\varphi(n)\}$ *of IVP* (1.1) *in the space* ℓ_h^∞ *has a limit as* $n \to \infty$ *and both, the solution and its limit, continuously depend on the initial value* ξ.

Proof of the existence of the limit. Let $\varphi(n)$ an h-bounded solution of IVP (1.1). We shall prove the existence of the limit $\lim_{n\to\infty} h(n)^{-1}\varphi(n)$. Since

$$h(n)^{-1}\varphi(n) = h(n)^{-1}\xi + \sum_{m=0}^{n-1} h(n)^{-1} f(m, \varphi(m), \varphi(g(m))),$$

it is sufficient to prove the convergence of the sequence

$$\sum_{m=0}^{n-1} h(n)^{-1} f(m, \varphi(m), \varphi(g(m))).$$

From the property

$$\lim_{n\to\infty} h(n)^{-1} = 0$$

and the Lebesgue dominate convergence theorem we obtain

$$\lim_{n\to\infty} \sum_{m=0}^{n-1} h(n)^{-1}(f(m, \varphi(m), \varphi(g(m))) - f(m, \xi, \xi)) = 0.$$

Therefore, the identity

$$\sum_{m=0}^{n-1} h(n)^{-1} f(m, \varphi(m), \varphi(g(m))) =$$

$$\sum_{m=0}^{n-1} h(n)^{-1}(f(m, \varphi(m), \varphi(g(m))) - f(m, \xi, \xi)) + \sum_{m=0}^{n-1} h(n)^{-1} f(m, \xi, \xi)$$

implies

$$\lim_{n\to\infty} \sum_{m=0}^{n-1} h(n)^{-1} f(m, \varphi(m), \varphi(g(m))) = \lim_{n\to\infty} \sum_{m=0}^{n-1} h(n)^{-1} f(m, \xi, \xi).$$

This last limit exists due to the condition **(H1)**.

Proof of the continuous dependence on the initial values. Let $\varphi_1(n)$, $\varphi_2(n)$ be h-bounded solutions of the IVP (1.1) with initial values $\varphi_1(0) = \xi_1$, $\varphi_2(0) = \xi_2$. Hence

$$|h(n)^{-1}(\varphi_1(n) - \varphi_2(n))| \le |h(n)^{-1}(\xi_1 - \xi_2)|$$

$$+ \sum_{m=0}^{n-1} |h(m)^{-1}(f(m, \varphi_1(m), \varphi_1(g(m))) - f(m, \varphi_2(m), \varphi_2(g(m))))|$$

$$\le |h(n)^{-1}(\xi_1 - \xi_2)|$$

$$+ \sum_{m=0}^{n-1} w(m, |h(m)^{-1}(\varphi_1(m) - \varphi_2(m))|, |h(m)^{-1}(\varphi_1(g(m)) - \varphi_2(g(m)))|)$$

Since both $\varphi_1(n)$ and $\varphi_2(n)$ are h-bounded, we may define the nondecreasing function

$$\mu(\varepsilon) = \sup\{|h(n)^{-1}(\varphi_1(n) - \varphi_2(n))| : \ |h(n)^{-1}(\xi_1 - \xi_2)| < \varepsilon, \ \varepsilon > 0\}.$$

From **(H2)** we have

$$|h(n)^{-1}(\varphi_1(n) - \varphi_2(n))| \le |h(n)^{-1}(\xi_1 - \xi_2)| + \sum_{m=0}^{\infty} w(m, \mu(\varepsilon), \alpha(m)\mu(\varepsilon)).$$

If we compute the supremum on all n such that $|h(n)^{-1}(\xi_1-\xi_2)|<\varepsilon$, then we obtain

$$\mu(\varepsilon) \leq \varepsilon + \Omega(\mu(\varepsilon)).$$

Taking into account the existence of the limit

$$\mu_0 = \lim_{\varepsilon\to 0^+} \mu(\varepsilon),$$

it is follows that $\mu_0 \leq \Omega(\mu_0)$.

The same tokens used in the proof of Theorem 1 show that the last inequality is a contradiction unless $\mu_0 = 0$. This proves the continuous dependence, in the space ℓ_h^∞, of the bounded solutions of system (1.1) with respect to the initial values as well as the continuous dependence of the limits at $n=\infty$ of these solutions. ∎

3 Linear equations

How does the theory developed in section 2 work for the linear system

$$x(n+1) = A(n)x(n) + B(n)x(g(n))?, \tag{3.3}$$

where $\{A(n)\}$, $\{B(n)\}$ are sequences of $r\times r$ matrices, that are not required to be invertible. Let us assume the following set of conditions:

(C1) $$n+1 \leq g(n) \leq n+N,$$

where N is a constant natural number.

(C2) $$\frac{h(m)}{h(n)} \leq H,\ \forall n+1 \leq m \leq n+N,\ \forall n,$$

where $\{h(n)\}$ is an increasing sequence satifying **(H0)**. Defining

$$f(n,x,y) = A(n)x + B(n)y,$$

the condition **(H1)** will be accomplished if

(C3) $$\rho = \sum_{n=0}^{\infty}(|A(n)| + H|B(n)|) < 1.$$

The function $w(n,\lambda,\mu) = |A(n)|\lambda + |B(n)|\mu$ satisfies the condition **(H4)** and condition **(H2)**, since

$$\Omega(\gamma) = \sum_{m=0}^{\infty} w(m,\gamma,H\gamma) = \gamma \sum_{m=0}^{\infty}(|A(m)| + H|B(m)|) < \infty$$

The sequence defined in **(H3)** turns to be $M_k(\gamma) = \rho^k\gamma$. Thus, we may ennounce the following

Theorem 3. *Under conditions* **(H0, C1, C2, C3)**, *the IVP*

$$\begin{cases} x(n+1) &= A(n)x(n) + B(n)x(g(n)), \\ x(0) &= \xi, \end{cases} \tag{3.4}$$

has a unique solution in the space ℓ_h^∞. *This solution, in the metric of space* ℓ_h^∞, *depends continuously on the initial data* ξ. *Moreover this solution converges as* $n \to \infty$.

The conditions **(C1)**-**(C3)** are stringent for the linear system (3.3). For example, Theorem 3 cannot be applied to solve the IVP (3.4) if $A(n)$ is constant. Linear systems with advanced argument have been studied in [1, 2, 3, 4, 5], where conditions of existence and uniqueness, more general than those given by Theorem 3, are given.

4 Another class of problems

Let us consider the equation

$$\Delta x(n) = A(n)x(n) + B(n)x(g(n)) + \sum_{s=0}^{\hat{g}(n)} K(\hat{g}(n),s)x(s),$$

where $g(n) \geq n+1$ and $\hat{g}(n) \geq n$ for all $n \in \mathbf{N}$. The sequence $\{h(n)\}$ satisfies conditions **(H0, C2)**. Also assume

$$\textbf{(C4)} \qquad \lim_{n\to\infty} \sum_{s=0}^{n-1} \left(|A(s)| + \alpha(s)|B(s)| + H \sum_{m=0}^{\hat{g}(s)} |K(\hat{g}(s),m)| \right) = \rho < 1,$$

where the sequence $\{\alpha(n)\}$ was defined in **(H2)**.

Theorem 4. *For any $\xi \in R^n$, there exists a unique h-bounded solution $x(n)$ of the IVP*

$$\begin{cases} \Delta x(n) &= A(n)x(n) + B(n)x(g(n)) + \sum_{s=0}^{\hat{g}(n)} K(\hat{g}(n), s)x(s) \\ x(0) &= \xi, \end{cases}$$

provided the conditions (**H0, C2, C4**) *are fulfilled. Moreover, this solution continuously depends on the initial values.*

Proof. Let us define the recurrence

$$\begin{aligned} x_0(n) &= \xi, \\ x_{k+1}(n) &= \xi + \sum_{s=0}^{n-1}(A(s)x_k(s) + B(s)x_k(g(s)) + \\ & \sum_{m=0}^{\hat{g}(s)} K(\hat{g}(s), m)x_k(m)), \ k = 0, 1, 2, \ldots \end{aligned}$$

We will prove that the estimate

$$|h(n)^{-1}(x_{k+1}(n) - x_k(n))| \leq |\xi|\rho^{k+1}, \quad k = 0, 1, 2, \ldots \tag{4.5}$$

is valid for any $n \in \mathbf{N}$. Taking into account condition (**H0**), for $k = 1$, we have

$$|h(n)^{-1}(x_1(n) - \xi)| \leq \sum_{s=0}^{n-1}(|A(s)| + |B(s)| + \sum_{m=0}^{\hat{g}(s)} |K(\hat{g}(s), m)|)|\xi|.$$

From condition (**C4**) we obtain

$$|h(n)^{-1}(x_1(n) - \xi)| \leq |\xi|\rho.$$

Suppose that

$$|h(n)^{-1}(x_k(n) - x_{k-1}(n))| \leq |\xi|\rho^k, \ k = 0, 1, 2, \ldots$$

Then

$$|h(n)^{-1}(x_{k+1}(n)-x_k(n))| \le \sum_{s=0}^{n-1}(|A(s)||h(s)^{-1}(x_k(s)-x_{k-1}(s))|$$
$$+\alpha(s)|B(s)||h(g(s))^{-1}(x_k(g(s))-x_{k-1}(g(s)))|$$
$$+\sum_{m=0}^{\hat{g}(s)} H|K(\hat{g}(s),m)||h(m)^{-1}(x_k(m)-x_{k-1}(m))|)$$

then it follows (4.5). The convergence of sequence $\{x_k\}$ is equivalent to the convergence of the series

$$\xi+\sum_{k=0}^{\infty}(x_{k+1}(n)-x_k(n))$$

on the space ℓ_h^{∞}. From condition **(C4)**, the series $\sum_{k=0}^{\infty}\rho^{k+1}$ is convergent. Hence the sequence $\{x_k(n)\}$ converges to a bounded function $x(n)$ belonging to the space ℓ_h^{∞}. Moreover,

$$|h(n)^{-1}x(n)| \le \sum_{s=0}^{n-1}\left(|A(s)|+\alpha(s)|B(s)|+H\sum_{m=0}^{\hat{g}(s)}|K(\hat{g}(s),m)|\right)||x||_h+||\xi||_h,$$

that is

$$|h(n)^{-1}x(n)| \le \rho||x||_h+||\xi||_h.$$

Thus

$$||x||_h \le \frac{||\xi||_h}{1-\rho},$$

from whence we obtain the continuous dependence of the solution x on the initial data ξ. The solution $x(n)$ is unique, because if there were two bounded solutions $x(n)$, $y(n)$, then for $z(n)=x(n)-y(n)$, we would have

$$\Delta z(n)=A(n)z(n)+B(n)z(g(n))+\sum_{m=0}^{\hat{g}(n)}K(\hat{g}(n),m)z(m)$$

which leads us to

$$z(n)=\sum_{m=0}^{n-1}\left[A(m)z(m)+B(m)z(g(m))+\sum_{s=0}^{\hat{g}(m)}K(\hat{g}(m),s)z(s)\right].$$

Thus,

$$
\begin{aligned}
|h(n)^{-1}z(n)| &\leq ||z||_h \sum_{m=0}^{n-1}\left[|A(m)| + \alpha(m)|B(m)| + H\sum_{s=0}^{\hat{g}(m)}|K(\hat{g}(m),s)|\right], \\
&\leq ||z||_h\rho,
\end{aligned}
$$

implying $||z||_h \leq ||z||_h\rho$, from whence $||z||_h = 0$, because $\rho < 1$. ∎

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