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FOURIER INTEGRAL OPERATORS: ORIGIN AND USEFULNESS

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1 Introduction

The theory of Fourier integral operators, which arose at the end of the 1960's, is presently undergoing rapid growth. Dozens of works which have appeared in the world mathematical literature expound, generalize and utilize the theory of Fourier integral operators. And this is natural. After the fascination with the elliptic theory, which caused the theory of pseudodifferential operators to flourish, there appeared at first a timid, and then stronger and stronger interest in the non-elliptic theory equations with real characteristics. However, the technique of pseudodifferential operators, which works well in the elliptic theory, turned out to be unsuitable for the solution of this new group of problems.

The essential novelty in the theory of equations with real characteristics lies in the fact that, as opposed to the elliptic case, here the almost inverse operator is not a pseudodifferential operator. At first there were attempts to correct or somehow to add on to the old techniques, in such a way as to make them applicable to the new situation. Next, several generalizations of pseudodifferential operators appeared in a series of works. The linear phase in the Fourier integral was replaced by an arbitrary homogeneous function. This was the first step in the right direction. Gradually experience with this sort of operator accumulated. A whole series of problems repeatedly indicated the existence of some sort of general technical apparatus. The future Fourier integral operators appeared in seemingly very unexpected situations, for example in the study of the transformations of pseudodifferential operators induced by a canonical diffeomorphism of phase space, etc. The general principles of the new technique began to show through more and more clearly. Only one step remained to be taken. At this point, finally, in 1971 appeared the publication of the Swedish mathematician L. Hörmander [3], in which the mathematical apparatus which made it possible to solve the necessary problems, and which he called the method of Fourier integral operators, was presented. The new technique developed rapidly, and soon Fourier integral operators won wide popularity among specialists. A stream of articles on the application and generalization of the method of Fourier integrals sprang forth. For details and further studies, we recommend the book by F. Tréves [6], from which we have borrowed most of the ideas (and the style) to write this expository article.

2 The Cauchy problem for hyperbolic equations

We shall consider here differential operators in which one of the variables, called the *time* variable and usually denoted by t, plays a privileged role. Because of this this role, we shall assume that the dimension of the space will be equal to $n+1$. The n variables will be denoted by $x = (x^1, \ldots, x_n)$; we shall often refer to them as the *space* variables. That the time plays a privileged role is implicit in our assumption that the differential operator under study is of the form

$$P(x, t, D_x, \partial_t) = \partial_t^m + \sum_{j=1}^{m} P_j(x, t, D_x)\partial_t^{m-j}. \tag{2.1}$$

We have used, and shall systematically use the notation $\partial_t = \partial/\partial t$. For each j, $P_j(x,t,D_x)$ is a differential operator of order $\leq j$ in the x-variables, whose coefficients we assume to be C^∞ functions of t in the *closed* interval $[-T_0, T_0]$, $T_0 > 0$, valued in $C^\infty(\Omega)$, Ω being an open subset of R^n. In other words,

$$P_j(x,t,D_x) = \sum_{|\alpha|\leq j} c_{\alpha,j}(x,t) D^\alpha,$$

where $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$, $D_i = -\sqrt{-1}\partial/\partial x^i$, $i = 1,\ldots,n$, $\alpha = (\alpha_1,\ldots,\alpha_n)$, $D = (D_1,\ldots,D_n)$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $c_{\alpha,j} \in C^\infty(\Omega \times [-T_0,T_0])$.

We recall the statement of the *Cauchy problem.* We are given a function or a distribution $f(x,t)$ in $\Omega\times]-T_0,T_0[$, and m functions or distributions of x alone, defined in Ω, $u_0(x),\ldots,u_{m-1}(x)$. We seek a function or a distribution $u(x,t)$ in $\Omega\times]-T_0,T_0[$ satisfying the following set of conditions:

$$P(x,t,D_x,\partial_t)u = f \quad \text{in} \quad \Omega\times]-T_0,T_0[, \tag{2.2}$$

$$\partial_t^j u|_{t=0} = u_j \quad \text{in} \quad \Omega, j = 0,\ldots,m-1. \tag{2.3}$$

Of course this is a purely *formal* statement: for instance, the solution $u(x,t)$ should be a distribution such that the traces on the hyperplane $t = 0$ of its t-derivatives of order $< m$ are well defined. In general, the Cauchy problem does not have solutions at all: the classical example is provided by

$$P(x,t,D_x,\partial_t) = \partial_t - D_x \quad (\text{here } n = 1), \tag{2.4}$$

with $f(x,t) \equiv 0$ and $u_0(x)$ *nonanalytic* in Ω.

In order to get some feeling about the problem (2.2)-(2.3), we shall consider a relatively simple case: that where $\Omega = R^n$, the coefficients of the operator (2.1) are constant (we shall then denote by $P(D_x,\partial_t)$ the operator (2.1)) and where the right-hand side f and the Cauchy data u_j are C^∞ functions with compact support (in $\Omega\times]-T_0,T_0[$ and in Ω, respectively). The natural idea is to perform a Fourier transformation with respect to x and thus transform the problem (2.2)-(2.3) into:

$$P(\xi,\partial_t)\hat{u} = \hat{f}\ , \quad -T_0 < t < T_0, \tag{2.5}$$

$$\partial_t^j \hat{u}|_{t=0} = \hat{u}_j, \tag{2.6}$$

where the "hats" denote the Fourier transforms with respect to the space variables (the variable on the side of the Fourier transforms is denoted by ξ).

Of course, Eq. (2.5) is an ordinary differential equation, linear and with constant coefficients which happen to depend on the parameter ξ. Of course, the solution to (2.5)-(2.6) exists and is unique. It also depends smoothly on ξ: more precisely, it can be extended to the complex values of ξ in $\mathbb{C}^n$ as an entire function of exponential type (of course, depending smoothly on t, $-To < t < T_0$). This means that, if $\hat{u}(\xi, t)$ is the Fourier transform with respect to x of a distribution, the latter must have compact support. But whether $\hat{u}(\xi, t)$ is such a Fourier transform depends on its behaviour for real ξ: by the Paley-Wiener-Schwartz theorem, we know that $\hat{u}(\xi, t)$ is the Fourier transform with respect to x of an element of $\mathcal{E}(R^n)$ (then denoted by u) if and only if it can be extended to $\mathbb{C}^n$ as an entire function of exponential type and, moreover, its absolute value is bounded by a Polynomial $\varpi(\xi)$ on R^n.

That this might not be the case is clear on inspection of the example (2.4). The corresponding transformed Cauchy Problem reads

$$\partial_t \hat{u} = \xi \hat{u} + \hat{f}\ , \quad -T_0 < t < T_0, \tag{2.7}$$

$$\hat{u}|_{t=0} = \hat{u}_0. \tag{2.8}$$

Let us take $\hat{f} \equiv 0$; then the unique solution of (2.7)-(2.8) is given by

$$\hat{u}(\xi, t) = \hat{u}_0(\xi)\ e^{\xi t},$$

which is *never* tempered with respect to real ξ, at least when the Cauchy datum $u_0 \in C_c^\infty(R^n)$ is not identically zero. Indeed, if it were of slow growth for $|\xi| \to +\infty$ for at least one positive value of t and one negative value of t, we would conclude that, for some $\varepsilon > 0$,

$$|\hat{u}_0(\xi)| \leq \text{const.}\ e^{-\varepsilon|\xi|}, \quad \forall \xi \in R^n \tag{2.10}$$

which implies easily (by the Fourier inversion formula) that $u_0(x)$ can be extended to the complex values of x, provided that $|\mathrm{Im}\ x|$ remains $< \varepsilon$, as a holomorphic function.

It is clear that the trouble,when there is trouble, when there is trouble, lies with the roots of the polynomial $P(\xi, \tau)$ in the τ variable.

We wish to take a closer look at the behaviour of the solution $\hat{u}(\xi, t)$ of (2.5)-(2.6) when $\xi \in R^n$. It is checked at once that we can write:

$$\hat{u}(\xi, t) = \sum_{j=0}^{m-1} \varepsilon_j(\xi, t)\hat{u}_j(\xi) + \int_0^t \varepsilon_{m-1}(\xi, t - t')\hat{f}(\xi, t')dt', \tag{2.11}$$

where, for each $j = 0, \dots, m-1$, $\varepsilon_j(\xi, t)$ is the unique solution of the following problem:

$$P(\xi, \partial_t)\varepsilon_j = 0 \quad (\text{for all } t \in R^1), \tag{2.12}$$

$$\partial_t^k \varepsilon_j|_{t=0} = 1 \quad \text{if} \quad j = k, \ = 0 \quad \text{if} \quad j \neq k. \tag{2.13}$$

If we want $\hat{u}(\xi, t)$ to be tempered for $\xi \in R^n$ whatever the data f and u_j it is necessary and sufficient that $\varepsilon_j(\xi, t)$ be tempered for $\xi \in R^n$ whatever $j = 0, \dots, m-1$.

Let us fix ξ arbitrarily and denote by $\tau_1, \dots, \tau_m$ the m roots of the polynomial in $\tau, P(\xi, \tau)$. We observe that, for every $k = 1, \dots, m$,

$$h_k(\xi, t) = \exp\{t\tau_k(\xi)\} \tag{2.14}$$

is the solution of Problem (2.5)-(2.6) where we take $\hat{f} \equiv 0$ and $\hat{u}_j = \tau_k^j$, $j = 0, \dots, m-1$. By (2.11) we must therefore have:

$$h_k(\xi, t) = \sum_{j=0}^{m-1} \varepsilon_j(\xi, t)\tau_k(\xi)^j \tag{2.15}$$

When the roots τ_k are distinct, it is easy to solve (2.15) in term of the ε_j. Let $V(\tau_1, \dots, \tau_m)$ denote the Vandermonde determinant in the $\tau_k' s$ and $V_j(\tau_i, \dots, \tau_m; t)$ the same but where the row $(\tau_1^j, \dots, \tau_m^j)$ has been replaced by $(e^{t\tau_1}, \dots, e^{t\tau_m})$. Then:

$$\varepsilon_j(\xi, t) = V_j(\tau_1, \dots, \tau; t)/V(\tau_j, \dots, \tau_m).$$

But notice that the right-hand side can be regarded as an *entire* function of the variable $(\tau_1, \dots, \tau_m)$ in C^m (depending analytically on the real variable t). It must therefore represent $\varepsilon_j(\xi, t)$ even when the roots are not distinct. Note also that it is a symmetric function of the τ_k — and therefore an entire analytic function of the coefficients of $P(\xi,\tau)$, and by way of consequence also of ξ — as we have already pointed out.

From (2.15) it follows that if the $\varepsilon_j(\xi,t)$ are tempered so are the $h_k(\xi,t)$. It should be underlined, however, that in general the latter cannot be regarded as functions of ξ, because of possible ramifications. But one can say that

$$\sum_{k=1}^{m} |h_k(\xi,t)| \leq \text{const.}(1+|\xi|)^J, \quad \forall \xi \in R^n, \tag{2.17}$$

for some J, and therefore, for a suitable $C > O$,

$$\sum_{k=1}^{m} |Re\ \tau_k(\xi)| \leq C \log(1+|\xi|), \quad \forall \xi \in R^n. \tag{2.18}$$

However, by virtue of the fact that the τ_k are roots of a polynomial whose coefficients are themselves polynomials of the variable ξ, and in view of the Seidenberg-Tarski theorem, we cannot have (2.18), unless we have, for another suitable constant $C' > 0$,

$$\sum_{k=1}^{m} |Re\ \tau_k(\xi)| \leq C', \quad \forall \xi \in R^n. \tag{2.19}$$

One can then also prove that, if (2.19) holds, the $\varepsilon_j(\xi,\tau)$ are tempered, as functions of ξ in R^n. Thus (2.19) turns out to be the necessary and sufficient condition for the Cauchy problem (2.2)-(2.3) to be *well posed* — when the coefficients of the differential operator are constant and when $\Omega = R^n$. This is a classical theorem, due to L. Gårding.

We shall restrict ourselves to a particular case of the one just described. We introduce the principal part $P_m(\xi,\tau)$ of $P(\xi,\tau)$: $P_m(\xi,\tau)$ is a homogeneous polynomial of degree m with respect to (ξ,τ) and $P(\xi,\tau) - P_m(\xi,\tau)$ is a polynomial of degree not exceeding $m-1$.

Definition 2.1 *The differential operator $P(D_x, \partial_t)$ is said to be strongly (or strickly) hyperbolic if, given any $\xi \in R^n$, $\xi \neq 0$, the roots of the polynomial in $\tau, P_m(\xi,\tau)$, are purely imaginary and distinct.*

Our hypothesis in the remainder of this section is that $P(D_x, \partial_t)$ is strongly hyperbolic. Let us denote by $i\lambda_1, \ldots, i\lambda_m$ (λ_j real) the roots of $P_m(\xi, \tau)$; since the λ_j are distinct, we may order them so as to have $\lambda_1 < \cdots < \lambda_m$. It is clear that each one of them is an analytic function of ξ in $R^n \backslash \{0\}$, homogeneous of degree one.

Thus we have, for some $c_0 > 0$,

$$c_0|\xi| \le |\lambda_j(\xi) - \lambda_k(\xi)| \quad \text{if} \quad j \neq k, \quad \forall \xi \in R^n. \tag{2.20}$$

As for the roots τ_k of $P(\xi, \tau)$ they are distinct as soon as $|\xi|$ is large enough. We may number them in such a way as to have, for $|\xi| \sim +\infty$,

$$\tau_k(\xi) = i\lambda_k(\xi) + \sum_{\ell=0}^{+\infty} \tau_{k,\ell}(\xi), \tag{2.21}$$

where, for each $\ell, \tau_{k,\ell}$ is homogeneous of degree $-\ell$ with respect to ξ. The series, at the right in (2.21), converges uniformly in sets of the kind $|\xi| > M$ for M sufficiently close to $+\infty$. The way to prove (2.21) is by substituting the right-hand side for τ in $P(\xi, \tau)$ and determine each $\tau_{k,\ell}$ in terms of $\tau_{k,\ell'}$ with $\ell' < \ell$, which is possible because of (2.20).

Let us now return to (2.16) and take advantage of (2.20) and (2.21). For $|\xi| > M$ we may write:

$$\varepsilon_j(\xi, t) = \sum_{k=1}^{m} c_{j,k}(\xi) e^{t\{\tau_k(\xi) - i\lambda_k(\xi)\}} e^{it\lambda_k(\xi)}, \tag{2.22}$$

where every $c_{j,k}$ is a homogeneous function of ξ of degree $d_{j,k} \le 0$. But in view of (2.21) we may write

$$e^{t\{\tau_k(\xi) - i\lambda_k(\xi)\}} = \sum_{\ell=0}^{+\infty} e^{t\tau_{k,0}(\xi)} \varpi_{k,\ell}(\xi, t), \tag{2.23}$$

where each $\varpi_{k,\ell}(\xi, t)$ is a polynomial in t (of degree $\le \ell$) whose coefficients are homogeneous functions of degree $-\ell$ with respect to ξ.

Finally we see that we may write, for $|\xi| > M$,

$$\varepsilon_j(\xi, t) = \sum_{k=1}^{m} a_{j,k}(\xi, t) e^{it\lambda_k(\xi)}, \tag{2.24}$$

where each $a_{j,k}$ can be expressed as a series of the following kind

$$a_{j,k}(\xi,t) = \sum_{\ell=0}^{+\infty} a_{j,k,\ell}(\xi,t). \tag{2.25}$$

Each $a_{j,k,\ell}(\xi,t)$ is a homogeneous function of degree $-\ell$ with respect to ξ, and is analytic with respect to (ξ,t) in $(R^n\backslash\{0\})\times R^1$. We set

$$E_j(t)u(x) = (2\pi)^{-n}\int e^{ix\cdot\xi}\varepsilon_j(\xi,t)\hat{u}(\xi)d\xi, \quad u\in C_c^\infty(R^n). \tag{2.26}$$

We have just shown that

$$\begin{aligned} E_j(t)u(x) &= \sum_{k=1}^{m}(2\pi)^{-n}\int_{|\xi|>M} e^{i\{x\cdot\xi+t\lambda_k(\xi)\}}a_{j,k}(\xi,t)\hat{u}(\xi)d\xi \\ &\quad +(2\pi)^{-n}\int_{|\xi|\le M} e^{ix\cdot\xi}\varepsilon_j(\xi,t)\hat{u}(\xi)d\xi. \end{aligned} \tag{2.27}$$

In conclusion we see that, modulo a pseudodifferential operator of order $-\infty$ (essentially represented by the integral over the ball $|\xi|\le M$ in the right-hand side of (2.27)), $E_j(t)$ is equal to a sum of m operators of the following kind:

$$F(t)u(x) = (2\pi)^{-n}\int e^{i\phi(x,t,\xi)}a(\xi,t)\hat{u}(\xi)d\xi, \tag{2.28}$$

where $a(\xi,t)$ is a C^∞ function of (ξ,t) in $R^n\times R^1$ which has a series representation of the kind (2.25),

$$a(\xi,t) = \sum_{\ell=0}^{+\infty} a_\ell(\xi,t), \tag{2.29}$$

where the $a_\ell(\xi,t)$ are homogeneous of degree $-\ell$ with respect to ξ for $|\xi|$ large, say for $|\xi|>2/3$. We have, moreover,

$$\phi(x,t,\xi) = x\cdot\xi + t\lambda(\xi), \tag{2.30}$$

where $\lambda(\xi)$ is analytic and homogeneous of degree one in $R^n\backslash\{0\}$. We know that $i\lambda(\xi)$ is a root of the polynomial in $\xi, P_m(\xi,\tau)$, which means that the function ϕ is a solution of the *characteristic equation*

$$p_m(\partial_x\phi, i\partial_t\phi) = 0, \tag{2.31}$$

where $\partial_x\phi = \mathrm{grad}_x\phi$. Furthermore,the function ϕ satisfies the Cauchy condition:

$$\phi|_{t=0} = x \cdot \xi. \tag{2.32}$$

Eq. (2.31) is first-order but nonlinear; it splits into the m equations:

$$\partial_t\phi = \lambda_k(\partial_x\phi), \tag{2.33}$$

for $k = 1, \ldots, m$. Each problem (2.32)-(2.33) has a unique smooth solution (which is real-valued), and (2.30) is the solution of one of these problems.

It is not difficult to extend to differential operators with variable coefficients some of the methods devised above. Thus we return to the operator (2.1) but we shall make right away the hypothesis that it is strongly hyperbolic:

Definition 2.2 *We say that the operator $P(x, t, D_x, \partial_t)$ is strongly hyperbolic in $\Omega\times]-T_0, T_0[$ if, for every point (x_0, t_0) of this open set, the constant coefficient operator $P(x_0, t_0, D_x, \partial_t)$ is strongly hyperbolic in R^{n+1} (Def. 1.1).*

This means that the polynomial in $\tau, P_m(x, t, \xi, \tau)$ [(+)], has m distinct, purely imaginary roots whatever (x, t) in $\Omega\times]-T_0, T_0[$ and $\xi \in R^n\backslash\{0\}$. We may denote them by $i\lambda_j(x, t, \xi)$, $j = l, \ldots, m$, with the agreement that $\lambda_1 < \cdots < \lambda_m$. It is easy to check that the $\lambda_j(x, t, \xi)$ are C^∞ functions in $\Omega\times]-T_0, T_0[\times(R^n\backslash\{0\})$ homogeneous of degree one with respect to ξ. Needless to say, they are all real-valued.

In principle we should be able to express the solution u of Problem (2.2)-(2.3) (which is unique, by the general theory of hyperbolic equations), in the following form:

$$u(x, t) = \sum_{j=0}^{m-1} E_j(t)u_j(x) + \int_0^t E_{m-1}(t, t')f(x, t')dt'. \tag{2.34}$$

We shall continue to assume that f and the u_j, are test-functions, in $\Omega\times]-T_0, T_0[$ and in Ω respectively. Our problem is to find a "good" integral representation for the operators $E_j(t)(0 \le j \le m-1)$ and $E_{m-1}(t, t')$ generalizing (2.27). Let us define

(+) $P_m(x, t, \xi, \tau)$ is not quite the *principal symbol* of $P(X, t, D_x, \partial_t)$ — the latter is equal to $P_m(x, t, \xi, i\tau)$.

$E_j(t,t')$ as the solution of the following problem:

$$P(x,t,D_x,\partial_t)E_j(t,t') = 0, \quad -T_0 < t < T_0, \tag{2.35}$$

$$\partial_t^k E_j(t,t')|_{t=t'} = 0 \quad \text{if} \quad k \neq j, \; = I \quad \text{if} \quad k = j(k = 0,\dots,m-1), \tag{2.36}$$

where I stands for the identity mapping on functions and distributions in the space variables (defined in Ω). The problem (2.35) is to be understood in the operators sense: $E_j(t,t')$ is a smooth function of t in $[-T_0,T_0]$ (it also depends smoothly on t' in $[-T_0,T_0]$) with values in the space of linear operators on, say, test-functions of x in Ω. The coefficients $P_j(x,t,D_x)$ of $P(x,t,D_x,\partial_t)$ (see (2.1)), which are also such operator-valued functions of t, act on $E_j(t,t')$ by composition. Also $E_j(t) = E_j(t,0)$, $j = 0,\dots,m-1$, and since the problem (2.35)-(2.36) is not much different whether t' is equal to zero or not, we shall content ourselves with studying the operators $E_j(t)$. Also, we shall not overly worry about the existence and uniqueness of the operators $E_j(t)$. There are general results about hyperbolic equations which assert both properties under reasonable hypotheses on the coefficients of $P(x,t,D_x,D_t)$.

In the present situation, however there is not much hope to get (in general) an *exact* representation, and we shall only seek an approximate one, modulo regularizing operators. This is what is referred to as "parametrices for the Cauchy problem".

We shall try to represent each operator $E_j(t)$ as a sum of m operators of the kind $F(t)$ in (2.28). But we cannot expect that the symbol a will here also be independent of x. Thus we shall write

$$E_j(t)u(x) = \sum_{k=1}^{m}(2\pi)^{-n}\int e^{i\phi_k(x,t,\xi)}a_{jk}(x,t,\xi)\hat{u}(\xi)d\xi + R_j(t)u(x), \tag{2.37}$$

where $R_j(t)$ is regularizing (see, [2]). Our task, of course, is to determine the symbols ϕ_k and a_{jk}, translating the conditions (2.35) and (2.36) in terms of these symbols.

In analogy with (2.30) we shall take the functions $\phi_k(x,t,\xi)$ to be homogeneous of degree one with respect to ξ (in fact, this choice will soon be justified) and in analogy with (2.29):

$$a_{jk}(x,t,\xi) = \sum_{\ell=0} a_{jkl}(x,t,\xi), \tag{2.38}$$

where each term is homogeneous with respect to ξ; the homogeneity degree of $a_{jk\ell}$ with respect to ξ will be equal to $-j-\ell$, as we are now going to determine. It should be said that when we speak of functions which are homogeneous with respect to ξ, this must as usually be understood for $|\xi|$ large, say for $|\xi| > 2/3$; what happens near the origin can only contribute regularizing terms in the expression of $E_j(t)$ and can therefore be neglected, from the viewpoint which we have adopted here.

We must now write that the homogeneous terms (homogeneous always means with respect to ξ) in the left-hand side of (2.35) all vanish. The term with maximum homogeneity degree is $P_m(x, t, \partial_x\phi_k, i\partial_t\phi_k)a_{jk0}(x, t, \xi)$ and we shall require

$$P_m(x, t, \partial_x\phi_k, i\partial_t\phi_k) = 0, \tag{2.39}$$

which is the generalization of (2.31).
Because of our hypothesis of strong hyperbolicity, Eq. (2.39) splits into m equations of the type (2.33) — provided that the gradient of ϕ_k have a sufficiently large norm. As we shall see $|\text{grad } \phi_k|$ will be of the order of $|\xi|$ (at least for small $|t|$), and shall assume that

$$\partial_t\phi_k = \lambda_k(x, t, \partial_x\phi_k), \tag{2.40}$$

which in general is a *nonlinear* equation which always have *local* solutions (in (x, t) space) but not always global ones. We shall add to this equation the "initial" condition (which implies our contention about the growth of $|\text{grad } \phi|$ with $|\xi|$):

$$\phi_k|_{t=0} = x \cdot \xi. \tag{2.41}$$

To be more precise, given any relatively compact open subset Ω' of Ω, there is a number $\delta > 0$ (which tends to zero as Ω' expands to Ω) such that (2.40)-(2.41) has a unique solution in the "cylinder" $\{(x, t);\ x \in \Omega',\ |t| < \delta\}$; we may therefore use the representation (2.37) when $x \in \Omega'$ and $|t| < \delta$. The determination of the homogeneous terms $a_{jk\ell}$ can be made likewise (see [6]) adjoining appropriate conditions at time $t = 0$ in order to take into consideration (2.36).

A remark about the representation (2.37) is that the $a_{jk}(x, t, \xi)$ in the right-hand side should be given the meaning of *symbols*, as defined in [2]. Thus the series representation (2.38) is formal (and in particular we are free to regard its terms as homogeneous functions of ξ in $R^n\backslash\{0\}$, as in [2]. In order to form a representative of

$a_{jk}(x,t,\xi)$ one can multiply each homogeneous term $a_{jk\ell}(x,t,\xi)$ by a cut-off function $\chi_\ell(x,t,\xi)$ in the fashion explained in [2].

In this way, we succeed in establishing the representation (2.37). Indeed, if we set

$$F_j(t)u(x) = \sum_{k=1}^{m}(2\pi)^{-n}\int e^{i\phi_k(x,t,\xi)}a_{jk}(x,t,\xi)\hat{u}(\xi)d\xi, \tag{2.42}$$

it is not difficult to check that we can prove the following:

$$P(x,t,D_x,\partial_t)F_j(t) = S(t), \tag{2.43}$$

$$\partial_t^{j'}F_j|_{t=0} = \begin{cases} I + T_{j'} & \text{if } \; j = j', \\ & \qquad\qquad 0 \le j \le m-1, \\ T_{j'} & \text{if } \; j \ne j', \end{cases} \tag{2.44}$$

where $S(t)$ is a regularizing operator (acting on distribution in Ω with support contained in K and transforming them into C^∞ functions in Ω'), depending smoothly on t, $|t| < \delta$, and where the $T_{j'}$, $0 \le j' \le m-1$, are also regularizing operators, in the same sense as $S(t)$. Here K is some compact subset of Ω (independent of t) which contains the x-supports of the data $f(x,t)$ and $u_j(x)$ in (2.2)-(2.3). These hypothesis are aimed at preventing the difficulties which might arise from the behaviour of the coefficients of $P(x,t,D_x,D_t)$ near the boundary of Ω. But in view of (2.34) we may then write:

$$F_j(t) - E_j(t) = \sum_{j'=0}^{m-1}E_{j'}(t)T_{j'} + \int_0^t E_{m-1}(t,t')S(t')dt', \tag{2.45}$$

and the right-hand side is obviously regularizing (if we admit that the operators $E_j(t,t')$ transform C^∞ functions into C^∞ functions, which they do according to the general theory of hyperbolic equations).

What is important is that we have constructed a parametrix for the Cauchy problem (2.2)-(2.3) by means of operators of the kind

$$F(t)u(x) = (2\pi)^{-n}\int e^{i\phi(x,t,\xi)}a(x,t,\xi)\hat{u}(\xi)d\xi \tag{2.46}$$

where ϕ and a are symbols (depending on the time t) in the open set Ω, as we have defined them in [2]. It is well known that many interesting properties of the olution $u(x,t)$ of (2.2)-(2.3) can be extracted from the study of the relevant ϕ's and 's — for instance, information about the *propagation of singularities*. This kind of nformation is useful in problem other than the Cauchy problem and for PDEs which re not necessarily hyperbolic. Integral operators given by formulas such as (2.46) re a natural generalization of the pseudodifferential operators, given by formulas uch as (2.36), [2]. They enable us to construct parametrices for certain non-elliptic inear PDEs and might give us new insight in the theory of such equations, in articular by allowing us to "lift" their study in the cotangent bundle (as we have lready done, to some extent, by means of pseudodifferential operators, [2]).

3 Usefulness of Fourier Integral Operators in the Study of the Local Solvability of Linear Partial Differential Equations

n [2], we have shown that the theory of pseudodifferential operator enables us to educe the problem of local solvability of a linear partial differential (or pseudodfferential) equation with simple real characteristics, of order $m > 0$, in an open ubset Ω of $\mathbb{R}^N$,

$$P(x,D)u = f \tag{3.1}$$

o that of the first-order pseudodifferential equation

$$L(x,D)v = g(x,D)w, \tag{3.2}$$

vhere $g(x,\xi)$ is a certain C^∞ cut-off function in an open cone in $T^*(\Omega)$, positively omogeneous of degree zero, and w is an approximate solution of some elliptic pseuodifferential equation of order $m-1$. A further reduction, of crucial significance, vill now be made possible by the theory of Fourier Integral Operators.

It is convenient, and instructive to view equations such as (3.2) as *evolution quations*. Let us change variables, and set $t = x^N - x_0^N$, $y^j = x^j$ for $j \leq N-1 = n$ the covariables ξ_j, $j < N$, will be denoted by η_j and ξ_N by τ). In this notation we are dealing with equations of the general type

$$D_t u - \lambda(y,t,D_y)u - c(y,t,D_y,D_t)u = f, \tag{3.3}$$

where $\lambda(y,t,n)$ is homogeneous of degree 1 in η and $c(y,t,\eta,\tau)$ is a symbol of degree < 0. Both these functions can be assumed to be C^∞ with respect to all arguments in $\mathbb{R}^{n+1} \times (R^{n+1}\backslash\{0\})$ (this tacitely assumes that we have extended them to the whole space and multiplied them by suitable cut-off functions; this does not affect the preceding argument, it merely introduces here and there a few more error terms, expressed by pseudodifferential operators of order $-\infty$). We may even assume that the (y,t)-projections of the supports of $\lambda(y,t,\eta)$ and $c(y,t,\eta,\tau)$ are compact. Then we know that the pseudodifferential operator $c(y,t,D_y,D_t)$ defines a bounded linear operator of each Sobolev space $H^s(\mathbb{R}^{n+1})$ into itself, and because of this can easily be eliminated from the picture. We may concentrate our attention upon the equation

$$Lu = D_t u - \lambda(y,t,D_y)u = f. \tag{3.4}$$

We shall write $\lambda = a + \sqrt{-1}b$, where a and b are *real-valued* C^∞ functions of (y,t,η), $\eta \neq 0$, homogeneous of degree one in η. Observe that the pseudodifferential operator $a(y,t,D_y)$ is an operator on distributions in the y-variables, depending smoothly on t. When so viewed, let us denote it by $A(t)$; similarly, denote $b(y,t,D_y)$ by $B(t)$. If we replace f by $-\sqrt{-1}f$, the equation (3.4) reads:

$$\partial_t u - \sqrt{-1}A(t)u + B(t)u = f. \tag{3.5}$$

The theory of Fourier integral operators will now enable us to get rid of the therm $\sqrt{-1}A(t)$. Observe that since both a and b are *real*, the operators $A(t)$ and $B(t)$, defined on the dense subset $H^1(\mathbb{R}^n_y)$ of $H^0(\mathbb{R}^n_y)$, are equal to two self-adjoint (unbounded) linear operators on $H^0(\mathbb{R}^n_y)$, at least modulo bounded linear operators. Suppose for a moment that A and B are independent of t. The preceding observation allows us to solve the operator equation:

$$\partial_t U = iAU, \tag{3.6}$$

with initial condition:

$$U|_{t=0} = I, \quad \text{the identity operator.} \tag{3.7}$$

As a matter of fact, the solution is well known: $U(t) = \exp(iAt)$, the group of unitary operators on $L_2(\mathbb{R}^n)$ with infinitesimal generator A (when A is self-adjoint; when A is merely self-adjoint modulo bounded linear operators, $U(t)$ is "almost unitary".)

Set then $u = U(t)v$, $f = U(t)g$ in Eq. (3.5), where A and B are independent of t. It gets transformed into

$$\partial_t v + U(t)^{-1}BU(t)v = g. \tag{3.8}$$

It is checked at once that, up to a bounded linear operator (depending smoothly on t),

$$B^{\#}(t) = U(t)^{-1}BU(t) \tag{3.9}$$

is self-adjoint. If we set, as usually done in Lie groups and Lie algebras theory,

$$(Ad\ A)(B) = [A, B] = AB - BA, \tag{3.10}$$

we see easily that

$$B^{\#}(t) = e^{-iAt}Be^{iAt} = e^{-it(Ad\ A)}B = \sum_{j=0}^{+\infty} \frac{1}{j!}(-it)^j (Ad\ A)^j B. \tag{3.11}$$

These formulae have a geometrical "substratum", as we well know. Suppose for instance that both A and B are first-order differential operators with real coefficients (i.e., real vector fields) multiplied by $-\sqrt{-1}$

$$A = \sum_{k=1}^{n} a^k(y)D_{y^k}, \qquad B = \sum_{k=1}^{n} b^k(y)D_{y^k}.$$

Consider then the solution $z = z(y,t)$ of the system differential equations:

$$\frac{dz^k}{dt} = -a^k(z), \quad 1 \le k \le n, \tag{3.12}$$

with initial conditions:

$$z^k|_{t=0} = y^k, \quad 1 \le k \le n, \tag{3.13}$$

For small values of $|t|$,

$$z = z(y,t) \tag{3.14}$$

defines a C^∞ change of variables in the neighborhood of any given point. Let

$$y = \mathcal{Y}(z,t), \tag{3.15}$$

denote the inverse change of variables. We note that $t \longrightarrow (\mathcal{Y}(z,t),t)$ is the (piece of) integral curve of the vector field $\partial_t - iA$ through the point z. The change of variables $(y,t) \longrightarrow (z,t)$ transforms that vector field into ∂_t. An easy computation shows that

$$B^{\#}(t) = \sum_{k,\ell=1}^{n} b^k(\mathcal{Y}(z,t))\frac{dz^\ell}{dy^k}D_{z^\ell}. \tag{3.16}$$

Observe that $b^k(\mathcal{Y}(z,t))$ is the value of b^k at the point reached at time t when we move along the curve $t \longmapsto \mathcal{Y}(z,t)$ (z is the point of this curve obtained at $t=0$). Note also that the symbol of $B^{\#}(t)$ is

$$b\left(\mathcal{Y}(z,t)\ ,\ {}^t\left(\frac{\partial z}{\partial y}\right)\zeta\right), \tag{3.17}$$

where $\frac{\partial z}{\partial y}$ stands for the Jacobian matrix of the $z's$ with respect to the $y's$. The symbol (3.17) is nothing else but the transform of the symbol $b(y,\eta)$ of B under the mapping $(z,\zeta) \longmapsto (y,\eta)$, where

$$y = \mathcal{Y}(z,t), \quad \eta = {}^t\left(\frac{\partial z}{\partial y}\right)(y,t). \tag{3.18}$$

Now, (3.18) is the transformation in the cotangent bundle (over $\mathbb{R}^n$ or over an open subset of $\mathbb{R}^n$) associated with the transformation (3.15) in the base. The curve $t \longmapsto \mathcal{Y}(z,t)$ in the base is the projection of the curve $t \longmapsto (\mathcal{Y}(z,t),\ {}^t(\partial z/\partial y)\zeta)$ in the cotangent bundle, and the symbol of $B^{\#}(t)$ is obtained by displacing that of B along the latter curve. This describes completely the transformation of B into $B^{\#}(t)$, and shows that introducing the cotangent bundle was not just a dressing-up in fancy language of otherwise plain material, but had to do with really deep aspects of the problem.

In the more general situation, when A and B are not vector fields on an open set of $\mathbb{R}^n$, they can still be regarded as vector fields on an appropriate Lie group (they can be regarded as elements of its Lie algebra) and $B^{\#}(t)$ can once more be interpreted as a "displacement" of B. However, when, as it is the case in our problem, they are defined by pseudodifferential operators of order one, a more concrete interpretation of the whole operation is possible, thanks to Fourier integral operators.

Let us therefore go back to the case where

$$A(t) = a(y, t, D_y), \quad B(t) = b(y, t, D_y)$$

are essentially self-adjoint pseudodifferential operators of order one. Formally the process is the same as that which lead us from Eq. (3.5) to Eq. (3.8): we solve the Cauchy problem (3.6)-(3.7) and introduce the transform $B^{\#}(t)$ (note only that $B = B(t)$ now). It is not difficult to prove that $U(t)$ is essentially unitary (on $L^2(\mathbb{R}^n)$), and that $B^{\#}(t)$, like $B(t)$, is essentially self-adjoint. But the question for us is whether $B^{\#}(t)$ is also a pseudodifferential operator of order one, to which some appropriate analysis can be applied. The beauty of the approach is that this is indeed so and that, moreover, $B^{\#}(t)$ can be related to $B(t)$ in a simple and elegant manner, generalizing what happens when A and B are vector fields. This is due to the fact that $U(t)$ can be approximated, modulo regularizing operators, by a Fourier integral operator:

$$K(t)u(y) = (2\pi)^{-n} \int e^{i\phi(y,t,\eta)} k(y, t, \eta)\hat{u}(\eta)d\eta, \tag{3.19}$$

where $k(y, t, \eta)$, the *amplitude-function*, is a symbol of degree zero (depending smoothly on t) and the *phase-function* ϕ is C^∞ with respect to (y, t, η), $\eta \neq 0$, homogeneous of degree one with respect to η, and *real-valued*. In fact, it is the unique solution of the following (nonlinear) first-order Cauchy problem:

$$\phi_t = a(y, t, \phi_y), \qquad \phi|_{t=0} = y \cdot \eta. \tag{3.20}$$

Assuming that we have added a zero-order term to $a(y, t, D_y)$ so as to make it self-adjoint, we can obtain that $U(t)^{-1}$ be equal to the adjoint $K(t)^*$ of $K(t)$ modulo regularizing operators, and thus (again modulo regularizing operators)

$$B^{\#}(t) \equiv K(t)^* B(t) K(t). \tag{3.21}$$

At this stage the important *Egorov's theorem* comes to our rescue: it states that K^*BK is a pseudodifferential operator of same order as B, here *one*, and that modulo symbols of order strictly less, its symbol can be computed out of that of B by a formula similar, and generalizing the one which yielded (3.17). The relevant curves in the cotangent bundle are now the *bicharacteristic strips* of $\tau - a(y, t, \eta)$, i.e., of the symbol of $\frac{1}{i}(\partial_t - iA)$. These strips are the integral curves of the *Hamiltonian* vector field of $\tau - a(y, t, \eta)$, which is the vector field

$$\mathcal{H} = \partial_t - a_\eta(y, t, \eta) \cdot \partial_y + a_y(y, t, \eta) \cdot \partial_\eta$$

They are the curves described by the point $(z(y, t, \eta), t, \zeta(y, t, \eta))$, where

$$\frac{dz}{dt} = -a_\eta(z, t, \zeta), \quad \frac{d\zeta}{dt} = a_y(z, t, \zeta), \tag{3.22}$$

$$z = y, \qquad \zeta = \eta \qquad \text{at} \qquad t = 0. \tag{3.23}$$

For small values of $|t|$, the mapping $(y, \eta) \longmapsto (z(y, t, \eta), \zeta(y, t, \eta))$ is a diffeomorphism; let us denote by

$$y = \mathcal{Y}(z, t, \zeta), \qquad \eta = \eta(z, t, \zeta), \tag{3.24}$$

the inverse transformation. If $b^{\#}(y, t, \eta)$ denotes the symbol of $B^{\#}(t)$, we have:

$$b^{\#}(z, t, \zeta) \equiv b(\mathcal{Y}(z, t, \zeta), t, \eta(z, t, \zeta)) \quad \text{modulo symbols of degree zero.} \tag{3.25}$$

The reader will check without too much difficulty that this generalizes the formula (3.16).

In such a manner have we reduced our original problem to the solvability of the evolution equation:

$$\partial_t u + b^{\#}(y, t, D_y)u = f, \tag{3.26}$$

where we have the right to assume that the *principal symbol* $b_0^{\#}(y,t,\eta)$ of $b^{\#}(y,t,D_y)$ is *real* (and is defined by property (3.25)).

With the pseudodifferential equation (3.26) we associate the following ordinary differential equation, depending on the parameters (y,η)

$$\partial_t w + b^{\#}(y,t,\eta)w = \hat{f}(y,t,\eta). \tag{3.27}$$

Eq. (3.27) is first-order and linear. All solutions are known. They can be written in the form:

$$w(y,t,\eta) = \int_{T_0}^{t} e^{B(y,t,t'\eta)} \hat{f}(y,t',\eta)dt', \tag{3.28}$$

where

$$B(y,t,t',\eta) = \int_{t}^{t'} b^{\#}(y,t'',\eta)dt''. \tag{3.29}$$

We shall consider the right-hand sides $\hat{f}$ having compact support with respect to (y,t), contained in a small slab $|t| < T$, and tempered with respect to η. Then we ask whether we can always (i.e., given any such right-hand side $\hat{f}$) find a solution w of (3.27) which is *tempered* with respect to η at infinity. A simple argument shows that this property is equivalent to the following one:

(ψ) *There is a number* $T_0, |T_0| \leq T$, *such that, for every* t, $t' \in]-T,T[$ *such that* t' *lies in the segment joining* t *to* T_0, *for all* y,η,

$$B(y,t,t',\eta) \leq 0 \quad \text{(mod symbols of degree zero).} \tag{3.30}$$

The existence of the number T_0 is, in turn, equivalent to the following condition:

($\tilde{\psi}$) *whatever* y,η, *if* $b_0^{\#}(y,t,\eta) < 0$ *for some* t, $|t| < T$, *then* $b_0^{\#}(y,t',\eta) \leq 0$ *for every* t', $t < t' < T$.

We may translate ($\tilde{\psi}$) in terms of $b(y,t,\eta)$ (which, we recall, is homogeneous of degree one with respect to η):

(Ψ) *along every bicharacteristic strip of* $\tau - a(y,t,\eta)$ (in a neighborhood of a point $(y_0,0,\eta^0,\tau^0)$), *if* $b(y,t,\eta)$ *is* < 0 *at some point, it remains* ≤ 0 *at every later point* (bicharacteristic strips are oriented curves).

But suppose that $b(y,t,\eta)$, restricted to a bicharacteristic strip Γ of $\tau - a(y,t,\eta)$, changes sign at some point $(y_1, t_1, \eta^1, \tau^1)$ of Γ. If (Ψ) holds, it must necessarily change sign from $+$ to $-$. If we now make the "symmetry" $(y,t,\eta,\tau) \longrightarrow (y,t,-\eta,-\tau)$ and look at the behavior of b along the bicharacteristic strip of $\tau - a(y,t,\eta)$ through $(y,t_1,-\eta^1,-\tau^1)$, we see that b changes sign there from $-$ to $+$, and therefore violates (Ψ)! Thus, due to its homogeneity of degree one with respect to η, we see that (Ψ) is equivalent with:

($\mathcal{P}$) *along every bicharacteristic strip $\tau - a(y_0,t,\eta)$, in a neighborhood of $(y_0, 0, \eta^0, \tau^0)$ $b(y,t,\eta)$ does not change sign.*

This is in essence the solvability condition $(\mathcal{P})$ which, as R. Beals and C. Fefferman have shown, implies the local solvability of the original equation (3.1) (of course, for this it must be satisfied everywhere in the cotangent bundle over a neighborhood of the point $x_0 = (y_0, 0)$ under consideration). We have seen that it means that the ordinary differential equation (3.27) has a tempered solution (tempered with respect to η) whenever the right-hand side is tempered.

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