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PERIODICITY IN NONLINEAR DIFFERENCE EQUATIONS

E.A. Grove and G. Ladas
Department of Mathematics
University of Rhode Island
Kingston, Rhode Island, 02881 USA

1 Introduction

Our aim in this expository article is to bring to the attention of the greater mathematical community a wealth of examples of nonlinear difference equations having the property that every solution of each equation is **periodic with the same period**, a wealth of other equations having the property that all their solutions are **eventually periodic with prescribed periods**, and a large number of equations having the property that **all of their solutions converge to periodic solutions with the same period**.

We shall also ask some difficult questions and pose several thought provoking **open problems and conjectures** which may require no particular special training in the area of difference equations other than a clever analytical mind and a desire to understand the fascinating world of periodic solutions of difference equations.

A *difference equation of order* $(k+1)$ is an equation of the form

$$x_{n+1} = f(x_n, x_{n-1}, \ldots, x_{n-k}) \quad , \quad n = 0, 1, \ldots \tag{1}$$

where f is a function which maps some set I^{k+1} into I. I is usually an interval of real numbers, or a union of intervals, but it may even be a discrete set such as the set of *integers* $\mathbf{Z} = \{\ldots, -1, 0, 1, \ldots\}$.

A *solution* of Eq.(1) is a sequence $\{x_n\}_{n=-k}^{\infty}$ which satisfies Eq.(1) for all $n \geq 0$. If we prescribe a set of $(k+1)$ *initial conditions* $x_{-k}, x_{-k+1}, \ldots, x_0 \in I$ then

$$\begin{aligned} x_1 &= f(x_0, x_{-1}, \ldots, x_{-k}) \\ x_2 &= f(x_1, x_0, \ldots, x_{-k+1}) \\ &\vdots \end{aligned}$$

and so the solution $\{x_n\}_{n=-k}^{\infty}$ of Eq.(1) is uniquely determined by the initial conditions.

We say that the point $\bar{x} \in I$ is an equilibrium point of Eq.(1) if

$$\bar{x} = f(\bar{x}, \bar{x}, \ldots \bar{x}).$$

That is, the constant sequence $x_n = \bar{x}$ for all $n \geq -k$ is a solution of Eq.(1).

A solution $\{x_n\}_{n=-k}^{\infty}$ of Eq.(1) is called *periodic with period p* if there exists an integer $p \geq 1$ such that

$$x_{n+p} = x_n \quad \text{for all} \quad n \geq -k. \tag{2}$$

We say that the solution is periodic with *prime period* p if p is the smallest positive integer for which Eq.(2) holds. In this case, a p-tuple $(x_{n+1}, x_{n+2}, \ldots, x_{n+p})$ of any p consecutive values of the solution is called a *p-cycle* of Eq.(2).

A solution $\{x_n\}_{n=-k}^{\infty}$ of Eq.(1) is called *eventually periodic with period p* if there exists an integer $N \geq -k$ such that $\{x_n\}_{n=N}^{\infty}$ is periodic with period p; that is,

$$x_{n+p} = x_n \quad \text{for all} \quad n \geq N.$$

2 What do the following equations have in common?

$$x_{n+1} = \frac{1}{x_n x_{n-1}} \quad , \quad n = 0, 1, \ldots \tag{3}$$

$$x_{n+1} = \frac{1}{x_{n-1}} \quad , \quad n = 0, 1, \ldots \tag{4}$$

$$x_{n+1} = \frac{x_n}{x_{n-1}} \quad , \quad n = 0, 1, \ldots . \tag{5}$$

The answer is that every solution of each of the above equations is periodic with the same period.

Every solution of Eq.(3) is **periodic with period 3.**

Every solution of Eq.(4) is **periodic with period 4.**

Every solution of Eq.(5) is **periodic with period 6.**

Indeed, if the initial conditions are nonzero real numbers denoted by

$$x_{-1} = \alpha \quad \text{and} \quad x_0 = \beta$$

then the solution of Eq.(3) is the **3-cycle**

$$\left(\alpha,\ \beta,\ \frac{1}{\alpha\beta}\right).$$

The solution of Eq.(4) is the **4-cycle**

$$\left(\alpha,\ \beta,\ \frac{1}{\alpha},\ \frac{1}{\beta}\right)$$

and the solution of Eq.(5) is the **6-cycle**

$$\left(\alpha,\ \beta,\ \frac{\beta}{\alpha},\ \frac{1}{\alpha},\ \frac{1}{\beta},\ \frac{\alpha}{\beta}\right).$$

What is it that makes every solution of a difference equation periodic with the same period?

Is there an easily verifiable test that we can apply to determine whether or not this is true?

3 What do the following equations have in common

$$x_{n+1} = \frac{\max\{x_n, 1\}}{x_{n-1}} \quad , \quad n = 0, 1, \ldots \tag{6}$$

$$x_{n+1} = \frac{\max\{x_n, 1\}}{x_n x_{n-1}} \quad , \quad n = 0, 1, \ldots \tag{7}$$

$$x_{n+1} = \frac{\max\{x_n, 1\}}{x_n^2 x_{n-1}} \quad , \quad n = 0, 1, \ldots \tag{8}$$

$$x_{n+1} = \frac{\max\{x_n^2, 1\}}{x_n x_{n-1}} \quad , \quad n = 0, 1, \ldots \tag{9}$$

$$x_{n+1} = \frac{\max\{x_n^2, 1\}}{x_n^3 x_{n-1}} \quad , \quad n = 0, 1, \ldots . \tag{10}$$

The answer is that every positive solution of each of the above equations is periodic with the same period.

Every positive solution of Eq.(6) is periodic with period 5.

Every positive solution of Eq.(7) is periodic with period 7.

Every positive solution of Eq.(8) is periodic with period 8.

Every positive solution of Eq.(9) is periodic with period 9.

Every positive solution of Eq.(10) is periodic with period 12.

The proof that every positive solution of Eq.(7) is periodic with period 7 is evident from the following table.

Case 1	Case 2	Case 3	Case 4
$x_{-1} = \alpha \leq 1$ $x_0 = \beta \leq 1$	$x_{-1} = \alpha \geq 1$ $x_0 = \beta \leq 1$	$x_{-1} = \alpha \leq 1$ $x_0 = \beta \geq 1$	$x_{-1} = \alpha \geq 1$ $x_0 = \beta \geq 1$
$x_1 = \frac{1}{\alpha\beta}$	$x_1 = \frac{1}{\alpha\beta}$	$x_1 = \frac{1}{\alpha}$	$x_1 = \frac{1}{\alpha}$
$x_2 = \frac{1}{\beta}$	$x_2 = \max\left\{\alpha, \frac{1}{\beta}\right\}$	$x_2 = \frac{1}{\beta}$	$x_2 = \frac{\alpha}{\beta}$
$x_3 = \alpha\beta$	$x_3 = \alpha\beta$	$x_3 = \alpha\beta$	$x_3 = \max\{\alpha, \beta\}$
$x_4 = \frac{1}{\alpha}$	$x_4 = \frac{1}{\alpha}$	$x_4 = \max\left\{\beta, \frac{1}{\alpha}\right\}$	$x_4 = \frac{\beta}{\alpha}$
$x_5 = \frac{1}{\alpha\beta}$	$x_5 = \frac{1}{\beta}$	$x_5 = \frac{1}{\alpha\beta}$	$x_5 = \frac{1}{\beta}$
$x_6 = \alpha$	$x_6 = \alpha$	$x_6 = \alpha$	$x_6 = \alpha$
$x_7 = \beta$	$x_7 = \beta$	$x_7 = \beta$	$x_7 = \beta$

Are there any other values of k and l for which every positive solution of the difference equation

$$x_{n+1} = \frac{\max\{x_n^k, 1\}}{x_n^l x_{n-1}} \quad , \quad n = 0, 1, \ldots$$

is periodic with the same period? What are they?

Is there an easily applicable test to determine this? What is it?

4 Lyness' Equation

This is the equation

$$x_{n+1} = \frac{1 + x_n}{x_{n-1}} \quad , \quad n = 0, 1, \ldots \tag{11}$$

which was introduced by Lyness in 1942 (see [30]) while he was working on a problem in Number Theory. See also [13], [18], and [36].

Every positive solution of Eq.(11) is periodic with period 5. Indeed if

$$x_{-1} = \alpha \quad \text{and} \quad x_0 = \beta$$

are positive initial conditions, then the solution $\{x_n\}_{n=-1}^{\infty}$ is the **5-cycle**

$$\left(\alpha,\ \beta,\ \frac{1+\beta}{\alpha},\ \frac{1+\alpha+\beta}{\alpha\beta},\ \frac{1+\alpha}{\beta}\right). \tag{12}$$

Lyness' equation arises in *frieze patterns* (see [9]). An example of a frieze pattern is the following.

$$\begin{array}{ccccccccccc}
\cdots & 1 & & 1 & & 1 & & 1 & & 1 & \cdots \\
\cdots & & 1 & & 3 & & 1 & & 3 & & \cdots \\
\cdots & 2 & & 2 & & 2 & & 2 & & 2 & \cdots \\
\cdots & & 3 & & 1 & & 3 & & 1 & & \cdots
\end{array}$$

The property which defines a frieze pattern is that except for possible borders of zeros and ones, every four adjacent numbers forming a rhombus

$$\begin{array}{ccc}
 & q & \\
p & & r \\
 & s &
\end{array}$$

are positive and satisfy the unimodular equation $pr - qs = 1$.

Coxeter (see [10]) has shown that every frieze pattern is periodic. For example, the frieze pattern shown below is **periodic with period five**.

$$\begin{array}{cccccccccccc}
\cdots & 1 & & 1 & & 1 & & 1 & & 1 & & \cdots \\
\cdots & & x_1 & & x_3 & & x_5 & & x_2 & & x_4 & \cdots \\
\cdots & x_5 & & x_2 & & x_4 & & x_1 & & x_3 & & \cdots \\
\cdots & & 1 & & 1 & & 1 & & 1 & & 1 & \cdots
\end{array}$$

If $x_1 = \alpha$ and $x_2 = \beta$ are arbitrary positive numbers, then from the definition of frieze patterns it follows that

$$x_3 = \frac{1+\beta}{\alpha}, \quad x_4 = \frac{1+\alpha+\beta}{\alpha\beta}, \quad \text{and} \quad x_5 = \frac{1+\alpha}{\beta}.$$

Therefore the above pattern is generated by Lyness' equation.

For a given positive number a, what are all positive periodic solutions that the difference equation

$$x_{n+1} = \frac{a + x_n}{x_{n-1}} \quad , \quad n = 0, 1, \ldots$$

possesses?

5 Todd's Equation

Todd's equation is the equation

$$x_{n+1} = \frac{1 + x_n + x_{n-1}}{x_{n-2}} \quad , \quad n = 0, 1, \ldots . \tag{13}$$

Every positive solution of Eq.(13) is periodic with period 8. (See [18].) Indeed if

$$x_{-2} = \alpha, \; x_{-1} = \beta, \text{ and } x_0 = \gamma$$

are given initial conditions, then it follows by a simple computation that the solution $\{x_n\}_{n=-2}^{\infty}$ of Eq.(13) is the 8-cycle

$$\left(\alpha, \; \beta, \; \gamma, \; \frac{1+\beta+\gamma}{\alpha}, \; \frac{1+\alpha+\beta+\gamma+\alpha\gamma}{\alpha\beta}, \right.$$

$$\left. \frac{(1+\alpha+\beta)(1+\beta+\gamma)}{\alpha\beta\gamma}, \; \frac{1+\alpha+\beta+\gamma+\alpha\gamma}{\beta\gamma}, \; \frac{1+\alpha+\beta}{\gamma} \right).$$

Are there values of a, other than $a = 1$, such that every positive solution of the equation

$$x_{n+1} = \frac{a + x_n + x_{n-1}}{x_{n-2}} \quad , \quad n = 0, 1, \ldots \tag{14}$$

is periodic? Does Eq.(14) possess a positive nonperiodic solution?

Remark 5.1 Note that every solution of

$$x_{n+1} = \frac{1}{x_n} \quad , \quad n = 0, 1, \ldots$$

is **periodic with period 2**, every solution of

$$x_{n+1} = \frac{1 + x_n}{x_{n-1}} \quad , \quad n = 0, 1, \ldots$$

is **periodic with period 5**, and every solution of

$$x_{n+1} = \frac{1 + x_n + x_{n-1}}{x_{n-2}} \quad , \quad n = 0, 1, \ldots$$

is **periodic with period 8**. Is there a pattern here?

Unfortunately, **this pattern does not continue in any obvious way!** For example, if $\{x_n\}_{n=-3}^{\infty}$ is the solution of the difference equation

$$x_{n+1} = \frac{1 + x_n + x_{n-1} + x_{n-2}}{x_{n-3}} \quad , \quad n = 0, 1, \ldots$$

with initial conditions $x_{-3} = x_{-2} = x_{-1} = 1$, then the first twelve terms of $\{x_n\}_{n=-3}^{\infty}$ are

$$1,\ 1,\ 1,\ 1,\ 4,\ 7,\ 13,\ 28,\ \frac{49}{4},\ \frac{31}{4},\ \frac{49}{13},\ \frac{23}{26}$$

and so $\{x_n\}_{n=-3}^{\infty}$ **is not periodic with period 11**.

Remark 5.2 It is interesting to note the great similarities between Eq.(6) and Eq.(11). The solution of Eq.(6) with initial conditions

$$x_{-1} = \alpha \quad \text{and} \quad x_0 = \beta$$

is the **5-cycle**

$$\left(\alpha,\ \beta,\ \frac{\max\{1, \beta\}}{\alpha},\ \frac{\max\{1, \alpha, \beta\}}{\alpha\beta},\ \frac{\max\{1, \alpha\}}{\beta}\right).$$

Compare this with the **5-cycle** in (12).

What is it that these two equations have in common? Are there other pairs of equations with similar behavior?

6 The Gingerbreadman difference equation

The **gingerbreadman difference equation** is the piecewise linear difference equation

$$x_{n+1} = |x_n| - x_{n-1} + 1 \quad , \quad n = 0, 1, \ldots \tag{15}$$

which was investigated by Devaney (see [12]) and was shown to be chaotic in certain regions and stable in others. The name of this equation is due to the fact that the orbits of certain points in the plane fill a region that looks like a "gingerbreadman."

If you use a computer to plot the orbit of the solution $\{x_n\}_{n=-1}^{\infty}$ of Eq.(15) with initial conditions

$$(x_{-1},\ x_0) = \left(-\frac{1}{10},\ 0\right)$$

the computer may predict that after 100,000 iterations, **the solution is still not periodic**. Although a computer may be fooled due to round-off and truncation errors, one can show that the orbit of the solution of Eq.(15) with initial condition

$$(x_{-1},\ x_0) = \left(-\frac{1}{10},\ 0\right)$$

is **periodic with period 126**. An easy way to see this is to make the substitution

$$x_n = \frac{1}{10} y_n.$$

Then Eq.(15) is transformed into the difference equation

$$y_{n+1} = |y_n| - y_{n-1} + 10 \quad , \quad n = 0, 1, \ldots \tag{16}$$

and the initial conditions $(x_{-1},\ x_0) = \left(-\frac{1}{10},\ 0\right)$ of the solution $\{x_n\}_{n=-1}^{\infty}$ of Eq.(15)

are transformed into

$$(y_{-1},\ y_0) = (-1,\ 0).$$

Let $\{y_n\}_{n=-1}^{\infty}$ be the solution of Eq.(16) with initial conditions $(y_{-1},\ y_0) = (-1,\ 0)$ Then the values of y_n for $-1 \le n \le 126$ are given as follows:

$y_{-1} = -1$	$y_0 = 0$					
$y_1 = 11$	$y_2 = 21$	$y_3 = 20$	$y_4 = 9$	$y_5 = -1$	$y_6 = 2$	$y_7 = 13$
$y_8 = 21$	$y_9 = 18$	$y_{10} = 7$	$y_{11} = -1$	$y_{12} = 4$	$y_{13} = 15$	$y_{14} = 21$
$y_{15} = 16$	$y_{16} = 5$	$y_{17} = -1$	$y_{18} = 6$	$y_{19} = 17$	$y_{20} = 21$	$y_{21} = 14$
$y_{22} = 3$	$y_{23} = -1$	$y_{24} = 8$	$y_{25} = 19$	$y_{26} = 21$	$y_{27} = 12$	$y_{28} = 1$
$y_{29} = -1$	$y_{30} = 10$	$y_{31} = 21$	$y_{32} = 21$	$y_{33} = 10$	$y_{34} = -1$	$y_{35} = 1$
$y_{36} = 12$	$y_{37} = 21$	$y_{38} = 19$	$y_{39} = 8$	$y_{40} = -1$	$y_{41} = 3$	$y_{42} = 14$
$y_{43} = 21$	$y_{44} = 17$	$y_{45} = 6$	$y_{46} = -1$	$y_{47} = 5$	$y_{48} = 16$	$y_{49} = 21$
$y_{50} = 15$	$y_{51} = 4$	$y_{52} = -1$	$y_{53} = 7$	$y_{54} = 18$	$y_{55} = 21$	$y_{56} = 13$
$y_{57} = 2$	$y_{58} = -1$	$y_{59} = 9$	$y_{60} = 20$	$y_{61} = 21$	$y_{62} = 11$	$y_{63} = 0$
$y_{64} = -1$	$y_{65} = 11$	$y_{66} = 22$	$y_{67} = 21$	$y_{68} = 9$	$y_{69} = -2$	$y_{70} = 3$
$y_{71} = 15$	$y_{72} = 22$	$y_{73} = 17$	$y_{74} = 5$	$y_{75} = -2$	$y_{76} = 7$	$y_{77} = 19$
$y_{78} = 22$	$y_{79} = 13$	$y_{80} = 1$	$y_{81} = -2$	$y_{82} = 11$	$y_{83} = 23$	$y_{84} = 22$
$y_{85} = 9$	$y_{86} = -3$	$y_{87} = 4$	$y_{88} = 17$	$y_{89} = 23$	$y_{90} = 16$	$y_{91} = 3$
$y_{92} = -3$	$y_{93} = 10$	$y_{94} = 23$	$y_{95} = 23$	$y_{96} = 10$	$y_{97} = -3$	$y_{98} = 3$
$y_{99} = 16$	$y_{100} = 23$	$y_{101} = 17$	$y_{102} = 4$	$y_{103} = -3$	$y_{104} = 9$	$y_{105} = 22$
$y_{106} = 23$	$y_{107} = 11$	$y_{108} = -2$	$y_{109} = 1$	$y_{110} = 13$	$y_{111} = 22$	$y_{112} = 19$
$y_{113} = 7$	$y_{114} = -2$	$y_{115} = 5$	$y_{116} = 17$	$y_{117} = 22$	$y_{118} = 15$	$y_{119} = 3$
$y_{120} = -2$	$y_{121} = 9$	$y_{122} = 21$	$y_{123} = 22$	$y_{124} = 11$		
$y_{125} = -1$	$y_{126} = 0.$					

Therefore the sequence $\{y_n\}_{n=-1}^{\infty}$ (and hence also $\{x_n\}_{n=-1}^{\infty}$) is periodic with period 126.

It is interesting to note that the gingerbreadman difference equation is a special case of the **max difference equation**

$$x_{n+1} = \frac{\max\{x_n^2, A\}}{x_n x_{n-1}}\ , \quad n = 0, 1, \dots . \tag{17}$$

Indeed the change of variables (see [22])

$$x_n = \begin{cases} A^{\frac{1+y_n}{2}} & \text{if} \quad A > 1 \\ e^{\frac{y_n}{2}} & \text{if} \quad A = 1 \\ A^{\frac{-1+y_n}{2}} & \text{if} \quad 0 < A < 1 \end{cases}$$

reduces Eq.(17) to the difference equation

$$y_{n+1} = |y_n| - y_{n-1} + \delta \quad , \quad n = 0, 1, \ldots$$

where

$$\delta = \begin{cases} -1 & \text{if} \quad A > 1 \\ 0 & \text{if} \quad A = 1 \\ 1 & \text{if} \quad A < 1. \end{cases}$$

Note that Eq.(17) with $A \in (0,\ 1)$ reduces to the gingerbreadman difference equation (15).

When $A = 1$ Eq.(17) reduces to Eq.(9) which by the above change of variables is transformed into the difference equation

$$y_{n+1} = |y_n| - y_{n-1} \quad , \quad n = 0, 1, \ldots . \tag{18}$$

Hence **every solution of Eq.(18) is periodic with period 9.**

What is the set of initial conditions $(x_{-1},\ x_0) \in (0, \infty) \times (0, \infty)$ through which the solutions of Eq.(15) are periodic?

Are there values of A, other than $A = 1$, for which every solution of Eq.(17) is periodic with the same period? What do the solutions of Eq.(17) do for values of A not equal to 1?

7 The $(3x + 1)$ conjecture

This is the well known and famous, but still not confirmed, conjecture that every solution of the difference equation

$$x_{n+1} = \begin{cases} \dfrac{3x_n+1}{2} & \text{if } \ x_n \ \text{ is odd} \\[2ex] \dfrac{x_n}{2} & \text{if } \ x_n \ \text{ is even} \end{cases}, \qquad n = 0, 1, \ldots \tag{19}$$

with initial condition

$$x_0 \in \{1, 2, \ldots\}$$

is eventually the **two-cycle** $(1,\ 2)$.

On the other hand, it is conjectured that every solution $\{x_n\}_{n=0}^{\infty}$ of Eq.(19) with initial condition

$$x_0 \in \{0,\ -1,\ -2, \ldots\},$$

is eventually either the **one-cycle** (0), the **one-cycle** (-1), the **three-cycle** $(-5,\ -7,$ or the **eleven-cycle**

$$(-17,\ -25,\ -37,\ -55,\ -82,\ -41,\ -61,\ -91,\ -136,\ -68,\ -34).$$

The $(3x + 1)$ conjecture is also known as the *Collatz Problem*, the *Syracuse Problem*, *Kakutani's Problem*, *Ulam's Problem*, and *Hasse's Algorithm*. According to Paul Erdös, **mathematics is not yet ready for such problems.**

See the interesting paper [26] by J.C. Lagarias for the history of the $(3x + 1)$ conjecture, and for a survey on the literature of this problem. In fact, the following beautiful excerpt comes from [26].

"*Is the* (3x+1) *problem intractably hard*? The problem of settling the (3x+1) problem seems connected to the fact that it is a deterministic process that simulates "random" behavior. We face this dilemma: On the one hand, to the extent that the problem has structure, we can analyze it—yet it is precisely this structure that seems to prevent us from proving that it behaves "randomly". On the other hand, to the extent that the problem is structureless and "random", we have nothing to analyze and consequently cannot rigorously prove anything. Of course there remains the possibility that someone will find some hidden regularity in the (3x+1) problem that allows some of the conjectures about it to be settled.

If the (3x+1) *problem is intractable, why should one bother to study it?* One answer is provided by the following aphorism: "No problem is so intractable that something interesting cannot be said about it." Study of the (3x+1) problem has uncovered a number of interesting phenomena; I believe further study of it may be rewarded by the discovery of other new phenomena."

8 Some conjectures in the spirit of the $(3x+1)$ conjecture

In this section we give some **known results, open problems, and conjectures** about difference equations of the form

$$x_{n+1} = \begin{cases} \dfrac{\alpha x_n + \beta x_{n-1}}{2} & \text{if} \quad x_n + x_{n-1} \quad \text{is even} \\ \\ \gamma x_n + \delta x_{n-1} & \text{if} \quad x_n + x_{n-1} \quad \text{is odd} \end{cases}, \quad n = 0, 1, \ldots . \tag{20}$$

where

$$\alpha, \beta, \gamma, \delta \in \{-1, 1\} \text{ and } x_{-1}, x_0 \in \mathbf{Z} = \{\ldots, -1, 0, 1, \ldots\}.$$

For some references about these problems, see [1], [8], [24], and [25].

Note that if $\{x_n\}_{n=-1}^{\infty}$ is a solution of Eq.(20), then $\{-x_n\}_{n=-1}^{\infty}$ is also a solution of Eq.(20). The change of variables

$$x_{2n-1} = y_{2n-1} \quad \text{and} \quad x_{2n} = -y_{2n}$$

reduces the 16 possible cases of Eq.(20) to 8, because the study of the solutions of Eq.(20) in the case of a given set of parameters $(\alpha, \beta, \gamma, \delta)$ is similar to that of the case of $(-\alpha, \beta, -\gamma, \delta)$.

Known Results, Open Problems, And Conjectures.

Given integers $p, q \in Z$, let $gcod(p, q)$ denote the greatest common odd divisor of p and q.

1. **The case**

$$(\alpha,\ \beta,\ \gamma,\ \delta) = (1,\ 1,\ 1,\ 1).$$

Theorem 8.1 (See [1].) *The following statements are true.*

(a) There exist solutions of Eq.(20) which are eventually constant, and there exist solutions of Eq.(20) which are not eventually constant.

(b) Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq.(20) which is not eventually constant. Then either $\lim_{n\to\infty} x_n = -\infty$ or $\lim_{n\to\infty} x_n = \infty$.

Open Problem 8.1 Find all points $(x_{-1},\ x_0) \in \mathbf{Z} \times \mathbf{Z}$ through which the solution $\{x_n\}_{n=-1}^{\infty}$ of Eq.(20) is eventually constant.

2. **The case**

$$(\alpha,\ \beta,\ \gamma,\ \delta) = (1,\ 1,\ 1,\ -1).$$

Theorem 8.2 (D. Clark and J.T. Lewis (See [8].)) *Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq.(20). Suppose that $x_{-1} \neq x_0$ and that $gcod(x_{-1}, x_0) = 1$. Then $\{x_n\}_{n=-1}^{\infty}$ is either eventually the constant* 1, *the constant* -1, *or the six-cycle* $(1,\ 3,\ 2,\ -1,\ -3,\ -2)$.

3. **The case**

$$(\alpha,\ \beta,\ \gamma,\ \delta) = (1,\ 1,\ -1,\ 1).$$

Theorem 8.3 (See [1].) *Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq.(20). Suppose that $x_{-1} \neq x_0$ and that $gcod(x_{-1}, x_0) = 1$. Then $\{x_n\}_{n=-1}^{\infty}$ is either eventually the constant* 1, *the constant* -1, *the four-cycle* $(1, 2, -1, 3)$, *the four-cycle* $(-1, -2, 1, -3)$, *or the six-cycle* $(1, 0, 1, -1, 0, -1)$.

4. **The case**

$$(\alpha,\ \beta,\ \gamma,\ \delta) = (1,\ -1,\ 1,\ 1).$$

Conjecture 8.1 *Let* $\{x_n\}_{n=-1}^{\infty}$ *be a solution of Eq.(20). Suppose that* $gcod(x_{-1}, x_0) = 1$*. Then* $\{x_n\}_{n=-1}^{\infty}$ *is either eventually the three-cycle* $(0,\ 1,\ 1)$*, the three-cycle* $(0, -1, -1)$*, or the ten-cycle* $(3, 2, 5, 7, 1, -3, -2, -5, -7, -1)$.

5. **The case**

$$(\alpha,\ \beta,\ \gamma,\ \delta) = (1,\ 1,\ -1,\ -1).$$

Theorem 8.4 (See [1].) *Let* $\{x_n\}_{n=-1}^{\infty}$ *be a solution of Eq.(20). Suppose that* $x_{-1} \neq x_0$ *and that* $gcod(x_{-1}, x_0) = 1$*. Then* $\{x_n\}_{n=-1}^{\infty}$ *is either eventually the constant* 1*, the constant* -1*, the three-cycle* $(1,\ 0,\ -1)$*, or the three-cycle* $(-1,\ 0,\ 1)$.

6. **The case**

$$(\alpha,\ \beta,\ \gamma,\ \delta) = (1,\ -1,\ 1,\ -1).$$

Theorem 8.5 (See [1].) *Let* $\{x_n\}_{n=-1}^{\infty}$ *be a solution of Eq.(20). Suppose that* $x_{-1} \neq x_0$ *and that* $gcod(x_{-1}, x_0) = 1$*. Then* $\{x_n\}_{n=-1}^{\infty}$ *is eventually the six-cycle* $(1,\ 1,\ 0,\ -1,\ -1,\ 0)$.

7. **The case**

$$(\alpha,\ \beta,\ \gamma,\ \delta) = (1,\ -1,\ -1,\ 1).$$

Theorem 8.6 (See [1].) *Let* $\{x_n\}_{n=-1}^{\infty}$ *be a solution of Eq.(20). Suppose that* $x_{-1} \neq x_0$ *and that* $gcod(x_{-1}, x_0) = 1$ *and* $x_{-1} \neq x_0$*. Then* $\{x_n\}_{n=-1}^{\infty}$ *is eventually the eight-cycle* $(1,\ 1,\ 0,\ 1,\ -1,\ -1,\ 0,\ -1)$.

8. **The case**

$$(\alpha,\ \beta,\ \gamma,\ \delta) = (1,\ -1,\ -1,\ -1).$$

Conjecture 8.2 *Let* $\{x_n\}_{n=-1}^{\infty}$ *be a solution of Eq.(20). Suppose that* $gcod(x_{-1}, x_0) = 1$ *and* $x_{-1} \neq x_0$*. Then* $\{x_n\}_{n=-1}^{\infty}$ *is eventually either the three-cycle* $(1,\ 2,\ -3)$*, the three cycle* $(-1,\ -2,\ 3)$*, the four-cycle* $(1,\ 0,\ -1,\ 1)$*, or the four-cycle* $(-1,\ 0,\ 1,\ -1)$.

Similar problems are of interest for the equation

$$x_{n+1} = \begin{cases} \dfrac{x_n + x_{n-1}}{3} & \text{if 3 divides } x_n + x_{n-1} \\ \\ x_n + x_{n-1} & \text{otherwise} \end{cases} \quad , \quad n = 0, 1, \ldots . \tag{21}$$

where $x_{-1}, x_0 \in \mathbf{Z}$.

Conjecture 8.3 *The following statements are true.*

(a) Every positive solution of Eq.(21) which is not eventually a three-cycle converges to ∞.

(b) Eq.(21) has an unbounded solution.

9 The generalized Lozi's equation

Lozi's map is the system of difference equations

$$\begin{cases} x_{n+1} &= 1 - a|x_n| + y_n \\ \\ y_{n+1} &= bx_n \end{cases} \quad , \quad n = 0, 1, \ldots$$

introduced by Lozi (see [29]) in 1978 as a piecewise linear analogue of the *Hénon map*

$$\begin{cases} x_{n+1} &= 1 - ax_n^2 + y_n \\ \\ y_{n+1} &= bx_n \end{cases} \quad , \quad n = 0, 1, \ldots .$$

The Hénon map was introduced by the theoretical astronomer Hénon (see [16]) in 1976 to illuminate the strange attractor which was observed by the meteorologist Lorenz (see [28]) in 1963 in the simple-looking nonlinear system of differential equations

$$\begin{cases} \dfrac{dx}{dt} &= 10(y-x) \\ \dfrac{dy}{dt} &= x(28-z)-y \\ \dfrac{dz}{dt} &= xy-\dfrac{8}{3}z \end{cases}$$

which Lorenz used in his research to model weather patterns.

When Lorenz used Euler's method to integrate this system numerically in his Royal-McBee LGP-30 computer, the solutions of this system exhibited **extremely complicated** behavior. The solutions exhibited **sensitive dependence upon initial conditions** about which Lorenz coined the phrase **butterfly effect**. If a butterfly flaps its wings in Temuco, Chile, this may cause it to rain in Kingston, Rhode Island 4 days later. This is bad news for numerical methods, and means that we should be suspicious of what we "see in the computer" until we establish it by a rigorous proof.

The solutions oscillate irregularly, never exactly repeating but always remaining in a bounded region in the (x, y, z) space, and they settle onto a complicated set resembling an owl's mask or a pair of butterfly wings, which we now call a *strange attractor*, *strange* because its boundary is a fractal (with dimension between 2 and 3). All solutions approach the attractor quite rapidly, and there are no periodic or asymptotically periodic solutions. The term *strange attractor* was coined by Ruelle and Takens (see [35]) in 1971.

The Lozi map is the first system for which it was established (by Misiurewicz (see [31]) in 1980) that it possesses a strange attractor. For the Hénon map, the existence of a strange attractor was established by Benedicks and Carleson (see [4]) in 1991. The existence of a strange attractor for the Lorenz equations was recently announced by Warwick Tucker at the Berlin Equadiff last August, 1999 (see [38]). Here is an excerpt from Warwick's homepage (http://www.math.uu.se/~warwick/).

Warwick's Homepage

I have just completed my Ph.D. studies at the department of mathematics in

Uppsala. In my thesis, I prove that the Lorenz attractor really is a strange attractor as conjectured for more than 35 years ago. During my studies I also spent some time at the Royal Institute of Technology in Stockholm. My field of interest is dynamical systems, and I am currently doing a post-doc at IMPA, Rio de Janeiro.

By eliminating the variable y_n, Lozi's map reduces to the second order piecewise linear difference equation

$$x_{n+1} = 1 - a|x_n| + bx_{n-1} \quad , \quad n = 0, 1, \ldots \tag{22}$$

where a and b are real numbers.

Several of the equations which we have recently investigated, and which exhibit an interesting periodic character, are of the form

$$x_{n+1} = \frac{\max\{x_n^k, A\}}{x_n^l x_{n-1}^m} \quad , \quad n = 0, 1, \ldots \tag{23}$$

where

$$k, l, m \in \mathbf{Z} \quad \text{and} \quad A, x_{-1}, x_0 \in (0, \infty).$$

Some special cases of this equation were investigated in [3], [17], [20], and [22] and were found to have very interesting dynamics.

When $A = 1$ and $m = 1$, every solution of Eq.(23) is **periodic with period**

$$\begin{array}{rlcll}
3 & \text{if} & k=0 & \text{and} & l=1 \\
4 & \text{if} & k=0 & \text{and} & l=0 \\
5 & \text{if} & k=1 & \text{and} & l=0 \\
6 & \text{if} & k=0 & \text{and} & l=-1 \\
7 & \text{if} & k=1 & \text{and} & l=1 \\
8 & \text{if} & k=1 & \text{and} & l=2 \\
9 & \text{if} & k=2 & \text{and} & l=2 \\
12 & \text{if} & k=2 & \text{and} & l=3.
\end{array}$$

It follows easily that the change of variables (see [22])

$$x_n = \begin{cases} A^{\frac{1+y_n}{k}} & \text{if } A > 1 \\ e^{\frac{y_n}{k}} & \text{if } A = 1 \\ A^{\frac{-1+y_n}{k}} & \text{if } A < 1 \end{cases}$$

together with the observation that

$$\max\{\alpha, \beta\} = \frac{1}{2}[(\alpha + \beta) + |\alpha - \beta|]$$

transforms Eq.(23) into the piecewise linear equation

$$y_{n+1} = \frac{k}{2}|y_n| + \left(\frac{k}{2} - l\right) y_n - m y_{n-1} + \delta \quad , \quad n = 0, 1, \ldots \tag{24}$$

where

$$\delta = \begin{cases} k - 1 - l - m & \text{if } \; A > 1 \\ 0 & \text{if } \; A = 1 \\ -(k - 1 - l - m) & \text{if } \; A < 1. \end{cases}$$

We call Eq.(24) the **generalized Lozi's equation.** See [22].

Are there other values of k and l for which every solution of Eq.(24) with $m = 1$ and $\delta = 0$ is periodic with the same period?

10 When is every solution of $\mathbf{x_{n+1} = \dfrac{\alpha + \beta x_n + \gamma x_{n-1}}{A + Bx_n + Cx_{n-1}}}$ periodic?

Consider the nonlinear second order rational difference equation

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1}}{A + Bx_n + Cx_{n-1}} \quad , \quad n = 0, 1, \ldots \tag{25}$$

where the parameters $\alpha, \beta, \gamma, A, B, C$ are nonnegative real numbers with $B+C > 0$, and where the initial conditions x_{-1} and x_0 are nonnegative real numbers such that the right hand side of Eq.(25) is well defined for all $n \geq 0$.

The following four special examples of Eq.(25)

$$x_{n+1} = \frac{1}{x_n}, \quad n = 0, 1, \ldots \tag{26}$$

$$x_{n+1} = \frac{1}{x_{n-1}}, \quad n = 0, 1, \ldots \tag{27}$$

$$x_{n+1} = \frac{1 + x_n}{x_{n-1}}, \quad n = 0, 1, \ldots \tag{28}$$

$$x_{n+1} = \frac{x_n}{x_{n-1}}, \quad n = 0, 1, \ldots \tag{29}$$

are remarkable in the following sense.

Every positive solution of Eq.(26) is **periodic with period 2**.

Every positive solution of Eq.(27) is **periodic with period 4**.

Every positive solution of Eq.(28) is **periodic with period 5**.

Every positive solution of Eq.(29) is **periodic with period 6**.

The following result characterizes all possible special cases of equations of the form of Eq.(25) with the property that **every solution of the equation is periodic with the same period**.

Theorem 10.1 (See [19].) *Let $p \geq 2$ be a positive integer, and assume that every positive solution of Eq.(25) is periodic with period p. Then the following statements are true.*

1. *Suppose that $C > 0$. Then $A = B = \gamma = 0$.*
2. *Suppose that $C = 0$. Then $\gamma(\alpha + \beta) = 0$.*

Proof: Consider the solution $\{x_n\}_{n=-1}^{\infty}$ of Eq.(25) with

$$x_{-1} = 1 \quad \text{and} \quad x_0 \in (0, \infty).$$

Then clearly

$$x_{p-1} = x_{-1} = 1 \quad \text{and} \quad x_p = x_0$$

and so by Eq.(25)

$$x_0 = x_p = \frac{\alpha + \beta + \gamma x_{p-2}}{A + B + C x_{p-2}}.$$

Thus we see that

$$(A + B)x_0 + (Cx_0 - c)x_{p-2} = \alpha + \beta. \tag{30}$$

(a) Assume $C > 0$. Then we claim that

$$A = B = 0. \tag{31}$$

Otherwise, $A + B > 0$. So by choosing

$$x_0 > \max\left\{\frac{\alpha+\beta}{A+B}, \frac{\gamma}{C}\right\}$$

we see that (30) is impossible. Hence Eq.(31) is true. In addition to Eq.(31), we now also claim that

$$c = 0. \tag{32}$$

If not, then $\gamma > 0$. So by choosing

$$x_0 < \min\left\{\frac{\alpha+\beta}{A+B}, \frac{\gamma}{C}\right\}$$

we see again that (30) is impossible. Thus (32) also holds.

(b) Assume $C = 0$ and for the sake of contradiction, suppose that

$$\gamma(\alpha + \beta) > 0.$$

Then again by choosing x_0 sufficiently small, we see that (30) is impossible.

□

The following corollary of Theorem 10.1 states that Eqs.(26)-(29) are the only special cases of Eq.(25) with the property that every positive solution is periodic with the same period.

Corollary 10.2 (See [19].) *Let $p \in \{2,3,4,5,6\}$. Assume that $B+C>0$, and that every positive solution of Eq.(25) is periodic with period p. Then up to a change of variables of the form $x_n = \lambda y_n$ Eq.(25) reduces to one of the equations (26)-(29).*

For the more general difference equation

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1} + \delta x_{n-2}}{A + Bx_n + Cx_{n-1} + Dx_{n-2}} \quad , \quad n = 0, 1, \dots \tag{33}$$

with nonnegative initial conditions and nonnegative parameters, **what are all special equations with the property that every solution is periodic with the same period?** In addition to Eqs.(26)-(29) and Todd's equation (13), are there any other **surprises**?

11 Convergence of solutions of rational equations to period two solutions

Consider the difference equation

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1}}{A + Bx_n} \quad , \quad n = 0, 1, \dots \tag{34}$$

with

$$\alpha, \beta, \gamma, A, B \in [0, \infty)$$

and with nonnegative initial conditions. This is Eq.(25) with $C = 0$.

The following result is known. See [19].

Theorem 11.1 (See [19].) *Every positive solution of Eq.(34) converges to a period two solution if and only if* $\gamma = \beta + A$.

Open Problem 11.1 What is the "limiting" period two solution of Eq.(34) corresponding to a given initial condition $(x_{-1},\ x_0)$?

For the more general second order rational difference equation (25) with nonnegative coefficients and positive initial conditions, we offer the following conjecture.

Conjecture 11.1 *Assume that*

$$C > 0.$$

Then every solution of Eq.(25) converges to the positive equilibrium $\bar{x}$ *of Eq.(25), or else there exists a 2-cycle*

$$(\varphi,\ \psi)$$

of Eq.(25), and every solution of Eq.(25) converges either to $\bar{x}$ *or to* $(\varphi,\ \psi)$.

12 Convergence of solutions of systems to period two solutions

Consider the system of difference equations

$$\left\{\begin{array}{rcl} x_{n+1} & = & \dfrac{a}{x_n} + \dfrac{b}{y_n} \\ \\ y_{n+1} & = & \dfrac{c}{x_n} + \dfrac{d}{y_n} \end{array}\right. , \quad n = 0, 1, \ldots \tag{35}$$

where

$$a, b, c, d \in (0, \infty).$$

The following result was established in [15]. See also [33] and [34].

Theorem 12.1 (See [15].) *Every positive solution of Eq.(35) converges to a periodic solution with period 2.*

Open Problem 12.1 What is the **"limiting" period two solution** of Eq.(35) corresponding to a given initial condition $(x_{-1},\ x_0)$?

Conjecture 12.1 *Assume that*

$$a_{ij} \in (0, \infty) \quad \textit{for} \quad i, j \in \{1, 2, 3\}.$$

Show that every positive solution of the system

$$\begin{cases} x_{n+1} & = & \dfrac{a_{11}}{x_n} + \dfrac{a_{12}}{y_n} + \dfrac{a_{13}}{z_n} \\[2ex] y_{n+1} & = & \dfrac{a_{21}}{x_n} + \dfrac{a_{22}}{y_n} + \dfrac{a_{23}}{z_n} \\[2ex] z_{n+1} & = & \dfrac{a_{31}}{x_n} + \dfrac{a_{32}}{y_n} + \dfrac{a_{33}}{z_n} \end{cases} , \quad n = 0, 1, \ldots \tag{36}$$

converges to a period 2 solution. Extend and generalize this result to systems with real parameters and nonzero initial conditions.

13 On the difference equation

$$x_{n+1} = \max\left\{\frac{A_0}{x_n}, \frac{A_1}{x_{n-1}}, \cdots, \frac{A_k}{x_{n-k}}\right\}$$

Consider the difference equation

$$x_{n+1} = \max\left\{\frac{1}{x_n}, \frac{A}{x_{n-1}}\right\} \quad , \quad n = 0, 1, \ldots \tag{37}$$

where the parameter A and the initial conditions x_{-1} and x_0 are nonzero real numbers.

It was shown in [2] that every solution of Eq.(37) is eventually periodic with period 2, 3, or 4. (*Throughout this section, we are using the convention that a* *solution which is eventually constant is eventually periodic with any period.)* More precisely, the following statements are true.

Theorem 13.1 (See [2].)

(i) Assume that $A < 0$. Then every solution of Eq.(37) is eventually periodic with period 2 and is of the form $\left(p, \frac{1}{p}\right)$ for some positive number p which depends upon A and the initial conditions x_{-1} and x_0.

(ii) Assume that $A > 0$. Then every solution of Eq.(37) is eventually periodic with the following period.

(a) 2 if $A \in (0,1)$ and the initial conditions are not both negative, or if $A \in (1,\infty)$ and the initial conditions are both negative.

(b) 3 if $A = 1$

(c) 4 if $A \in (1,\infty)$ and the initial conditions are not both negative, or if $A \in (0,1)$ and the initial conditions are both negative.

Motivated by the above results, we pose some **open problems and conjectures** about the behavior of the solutions of the difference equation

$$x_{n+1} = \max\left\{\frac{A_0}{x_n}, \frac{A_1}{x_{n-1}}, \cdots, \frac{A_k}{x_{n-k}}\right\} \quad , \quad n = 0, 1, \ldots \tag{38}$$

where the parameters $A_0, A_1, \ldots, A_k$ are real numbers and where the initial conditions are nonzero real numbers. See [23].

Conjecture 13.1 *Assume that $A_0, A_1, \ldots, A_{k-1} \in [0,\infty)$ and $A_k \in (0,\infty)$. Show that every positive solution of Eq.(38) is eventually periodic with period*

$$p \in \{2,\ 3, \ldots,\ 2(k+1)\}.$$

It is fascinating to observe how the period in the above conjecture is determined by the *"dominant"* parameter among $A_0, A_1, \ldots, A_k$. For example if for some j_0

$\in \{0, 1, \ldots, k\}$, $A_{j_0} > \max\{A_j : j \neq j_0\}$, then every positive solution of Eq.(38) is eventually periodic with period $2(j_0 + 1)$. Also if some consecutive string of the parameters $A_0, A_1, \ldots, A_k$ are equal and dominate the remaining ones, then every positive solution of Eq.(38) is eventually periodic with period equal to the average of the periods of the *"dominating difference equations."* In particular, it can be easily shown that every positive solution of the difference equation

$$x_{n+1} = \max\left\{\frac{1}{x_n}, \frac{1}{x_{n-1}}, \cdots, \frac{1}{x_{n-k}}\right\} \quad , \quad n = 0, 1, \ldots \tag{39}$$

is eventually periodic with period $(k+2)$. Note that $(k+2)$ is the **average eventual period** of the $k+1$ difference equations

$$y_{n+1} = \frac{1}{y_{n-j}} \quad , \quad n = 0, 1, \ldots$$

for $j = 0, 1, \ldots, k$.

Conjecture 13.2 *Assume that $A_0, A_1, \ldots, A_k \in \mathbf{R}$. Show that every bounded solution of Eq.(38) is eventually periodic.*

Open Problem 13.1 Assume that $A_0, A_1, \ldots, A_k \in \mathbf{R}$. Obtain necessary and sufficient conditions for every solution of Eq.(38) to be unbounded.

Open Problem 13.2 Assume that $A_0, A_1, \ldots, A_k \in \mathbf{R}$. Investigate the periodic character of the solutions of Eq.(38).

Conjecture 13.3 *Consider the difference equation*

$$x_{n+1} = \max\left\{\frac{A}{x_n}, \frac{B}{x_{n-1}}, \frac{C}{x_{n-2}}\right\} \quad , \quad n = 0, 1, \ldots \tag{40}$$

where

$$A, B, C, x_{-2}, x_{-1}, x_0 \in (0, \infty).$$

Show that every solution of Eq.(40) is eventually periodic with period $p \in \{2, 3, 4, 5, 6\}$.

More precisely, show that every solution is eventually periodic with period

(*i*) 2 *if* $A > \max\{B, C\}$;
(*ii*) 3 *if* $A = B > C$;
(*iii*) 4 *if* *either* $B > \max\{A, C\}$ *or* $A = C > B$;
(*iv*) 5 *if* $B = C > A$;
(*v*) 6 *if* $C > \max\{A, B\}$.

14 Max equations with periodic coefficients

Recall from Section 13 that every positive solution of the difference equation

$$x_{n+1} = \max\left\{\frac{1}{x_n}, \frac{A}{x_{n-1}}\right\} \quad , \quad n = 0, 1, \ldots$$

where $A \in (0, \infty)$, is eventually periodic with period 2 if $A < 1$, 3 if $A = 1$, 4 if $A > 1$.

The above result was extended in [6] to the difference equation with period 2 coefficient

$$x_{n+1} = \max\left\{\frac{1}{x_n}, \frac{A_n}{x_{n-1}}\right\} \quad , \quad n = 0, 1, \ldots \tag{41}$$

where

$$A_n = \begin{cases} A_0 & \text{if} \quad n \text{ is even} \\ A_0 & \text{if} \quad n \text{ is even} \end{cases} \quad \text{with} \quad A_0, A_1 \in (0, \infty).$$

More precisely, it was shown in [6] that every positive solution of Eq.(41) is eventually periodic with period 2 if $A_0A_1 < 1$, 6 if $A_0A_1 = 1$ 4 if $A_0A_1 > 1$.

In this section, we pose some **open problems and conjectures** about the solutions of the nonautonomous Eq.(41) when $\{A_n\}_{n=0}^{\infty}$ is a periodic sequence of positive real numbers. See [7].

Open Problem 14.1 Let $\{A_n\}_{n=0}^{\infty}$ be a periodic sequence of positive real numbers with period $k \geq 3$. Find necessary and sufficient conditions for each of the following statements to be true.

(a) Every positive solution of Eq.(41) is bounded.

(b) Every positive solution of Eq.(41) is eventually periodic. In this case, determine all possible such periods.

Conjecture 14.1 *Let $\{A_n\}_{n=0}^{\infty}$ be a periodic sequence of positive real numbers with period $k \geq 2$. Assume that $A_n \in (0,1)$ for all $n \geq 0$. Then every positive solution of Eq.(41) is eventually periodic with period 2.*

For the cases $k = 2$ and 3, the above conjecture was shown to be true in [6] and [5], respectively.

Conjecture 14.2 *Let $\{A_n\}_{n=0}^{\infty}$ be a periodic sequence of positive real numbers with prime period $k \geq 2$. Assume that $A_n \in (1,\infty)$ for all $n \geq 0$. Then every positive solution of Eq.(41) is eventually periodic with period*

$$2k \quad \textit{if} \quad k \textit{ is even}$$

$$4k \quad \textit{if} \quad k \textit{ is odd.}$$

For the cases $k = 2$ and 3, the above conjecture was shown to be true in [6] and [14], respectively.

Conjecture 14.3 *Let $\{A_n\}_{n=0}^{\infty}$ be a periodic sequence of positive real numbers with prime period $k \geq 3$.*

(a) Assume k is not a multiple of 3. Then every positive solution of Eq.(41) is eventually constant or eventually periodic with prime period $p \in \{2,\ k,\ 2k,\ 4k\}$, and p is independent of the initial conditions.

(b) Assume k is a multiple of 3. Then one of the following statements is true.

(i) Every positive solution of Eq.(41) is eventually constant or eventually periodic with prime period $p \in \{2,\ k,\ 2k,\ 4k\}$, and p is independent of the initial conditions.

(ii) Every positive solution of Eq.(41) is eventually constant or unbounded.

In the case $k = 3$, the above conjectures were established in [14].

15 Periodicity in $x_{n+1} = \dfrac{A}{x_{n-k}} + \dfrac{B}{x_{n-l}}$

Consider the difference equation

$$x_{n+1} = \frac{A}{x_{n-k}} + \frac{B}{x_{n-l}}, \quad n = 0, 1, \ldots \tag{42}$$

where

$$A, B \in (0, \infty)$$

and

$$k, l \in \{0, 1, \ldots\} \quad \text{with} \quad k < l.$$

Note that Eq.(42) has the unique equilibrium point

$$\bar{x} = \sqrt{A + B}.$$

We have the following conjecture.

Conjecture 15.1

(a) Every positive solution of Eq.(42) converges to the equilibrium $\sqrt{A+B}$ if and only if all roots of the polynomial equation

$$\lambda^{l+1} + \frac{A}{A+B}\lambda^{l-k} + \frac{B}{A+B} = 0 \tag{43}$$

lie inside the unit disk

$$|\lambda| < 1.$$

(b) Every positive solution of Eq.(42) converges to a periodic solution of prime period $p > 1$ if and only if Eq.(43) has a root λ with $|\lambda| = 1$.

How is the number p in the above conjecture related to the roots of Eq.(43) which lie on the unit circle $|\lambda| = 1$?

16 Sharkovsky's Theorem

In this paper we have dealt almost exclusively with difference equations of order greater than or equal to 2. For a first order difference equation of the form

$$x_{n+1} = F(x_n) \quad , \quad n = 0, 1, \ldots \tag{44}$$

where

$$F : I \to I$$

is a continuous function mapping some interval of real numbers $I \subseteq \mathbf{R}$ into itself, the most glorified result known about periodic solutions of Eq.(44) is known as **Sharkovsky's Theorem**. (See [37]). For a good reference for this and other related theorems, read the historical remarks by M. Misiurewicz in [32].

Before stating the theorem, we introduce the *Sharkovsky ordering* of the set of positive integers $\mathbf{N} = \{1, 2, \ldots\}$.

$$\begin{gathered} 3 \prec 5 \prec 7 \prec \cdots \prec 3\cdot 2 \prec 5\cdot 2 \prec 7\cdot 2 \prec \cdots \prec 3\cdot 2^2 \prec 5\cdot 2^2 \prec 7\cdot 2^2 \prec \cdots \\ \cdots \prec 2^4 \prec 2^3 \prec 2^2 \prec 2 \prec 1. \end{gathered}$$

Theorem 16.1 *(Sharkovsky's theorem) Let I be an interval of real numbers, and let $F : I \to I$ be a continuous function. Assume that Eq.(44) has a periodic solution of prime period k. Then Eq.(44) has solutions of prime period l for all positive integers l with $k \prec l$ in the Sharkovsky ordering.*

A special case of Sharkovsky's theorem is the celebrated theorem of Li and Yorke, **Period Three Implies Chaos** (see [27]), in which it was shown that if I is an interval of real numbers and $F \in C[I, I]$, and if Eq.(44) possesses a periodic solution of prime period 3, then Eq.(44) possesses solutions of prime period p for every positive integer p.

17 The Riccati difference equation

The *Riccati Difference Equation* is the difference equation

$$x_{n+1} = \frac{a + bx_n}{c + dx_n} \quad , \quad n = 0, 1, \ldots \tag{45}$$

where the parameters a, b, c, d are given real numbers and the initial condition x_0 is an arbitrary real number.

Note that when $d = 0$, Eq.(45) is a linear equation, while if

$$d \neq 0 \quad \text{and} \quad bc - ad = 0$$

Eq.(45) reduces to the trivial difference equation

$$x_{n+1} = \frac{\frac{bc}{d} + bx_n}{c + dx_n} = \frac{b(c + dx_n)}{d(c + dx_n)} = \frac{b}{d} \quad , \quad n = 0, 1, \ldots .$$

Also note that when

$$b + c = 0 \quad \text{and} \quad x_0 \neq -\frac{c}{d} \tag{46}$$

the solution $\{x_n\}_{n=0}^{\infty}$ of Eq.(45) is periodic with **period two**.

In the remainder of this section, we shall assume that

$$d \neq 0, \quad bc - ad \neq 0, \quad \text{and} \quad b + c \neq 0. \tag{47}$$

The change of variables

$$x_n = \frac{b + c}{d} w_n - \frac{c}{d}$$

transforms Eq.(45) into the difference equation with one parameter

$$w_{n+1} = 1 - \frac{\mathcal{R}}{w_n} \quad , \quad n = 0, 1, \ldots \tag{48}$$

where the parameter $\mathcal{R}$, which we call the *Riccati number* of Eq.(45), is the nonzero real number

$$\mathcal{R} = \frac{bc - ad}{(b+c)^2}.$$

When

$$\mathcal{R} \leq \frac{1}{4}$$

one can see that Eq.(45) has no periodic solutions of any prime period $p \geq 2$.

When

$$\mathcal{R} > \frac{1}{4}$$

let $\theta \in \left(0, \frac{\pi}{2}\right)$ be such that

$$\cos\theta = \frac{1}{2\sqrt{\mathcal{R}}} \quad \text{and} \quad \sin\theta = \frac{\sqrt{4\mathcal{R}-1}}{2\sqrt{\mathcal{R}}}$$

and define the set

$$\mathcal{F} = \left\{ \frac{b-c}{2d} - \frac{(b+c)\sqrt{4\mathcal{R}-1}}{2d}\cot(n\theta) : n \geq 1 \text{ and } \sin(n\theta) \neq 0 \right\}.$$

Then (see [15]) every solution $\{x_n\}_{n=0}^{\infty}$ of Eq.(45) with $x_0 \notin \mathcal{F}$ is periodic if and only if θ is a *rational multiple of* π, and no solution of Eq.(45) is periodic otherwise. Furthermore if

$$\theta = \frac{q}{p}\pi$$

where p and q are positive integers which are relatively prime, then every solution $\{x_n\}_{n=0}^{\infty}$ of Eq.(45) with $x_0 \notin \mathcal{F}$ is periodic with prime period p.

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