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EUCLID'S *ELEMENTS* IN CULTURAL CONTEXT

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I

The historical importance of Euclid's *Elements* (c. 300 B.C.) is well known. This is the oldest extant book which presents an entire branch of learning in what is called "axiomatic-deductive" form. Definitions and axioms are given at the outset, and the theorems follow in a chain, each inferred from these beginnings and/or from results already proved. Since the Greeks conceived the axioms not as mere assumptions but as statements self-evidently true, and the propositions followed by strict logic, the resulting collection of theorems seemed in many eyes to give objective knowledge with total sureness. Euclid's great influence in succeeding centuries lay in people's hopes that they could attain equally certain insight in *their* branches of knowledge by following his methods.

Given such influence, it is natural to feel curious about how and why Euclid came to deal with mathematics in this axiomatic-deductive way. I shall summarize some possible paths to the *Elements* in mathematical and philosophical contexts; then I shall describe in more detail some features of the wider social and cultural life of contemporary Greece, and argue that these too may have played a role.

Unfortunately the surviving remains of pre-Euclidean mathematics are very sparse, and we lack the evidence that would give a definitive answer to our inquiry.

We may first ask whether at least some of the impulse might have come from within mathematics itself. Regrettably, Euclid's own book provides no answer – no comments of *any* kind interrupt the resolute march of his theorems; so we must look further back in time. We possess some earlier examples of deductive mathematics, notably the impressive "lune quadratures" of Hippocrates (c. 430 B.C.), but these are inferred from *theorems* assumed known, not from axioms. Indeed the meagre remains of 5th-century mathematics *nowhere* hint at axiomatization, or even at any *interest* in the idea.[1] On the other hand a famous passage in the philosopher Proclus (5th century A.D.) relates that a number of mathematicians before Euclid, beginning with the same Hippocrates, compiled books of "elements".[2] We have no sure idea what these contained, but it is tempting, and reasonable, to imagine them as stages or successive approximations toward Euclid's masterpiece. That picture is consistent with a precious glimpse offered by Plato, who could observe contemporary mathematicians at close range in his Academy. In the *Republic*, which probably dates from around 380 B.C., he makes Socrates say that

> students of geometry and reckoning and such subjects first postulate the odd and the even and the various figures and other things akin to these in each branch of science, regard them as known, and treating them as absolute assumptions, do not deign to render any further account of them to themselves or others, taking it for granted that they are obvious to everybody. They take their start from these, and pursuing the inquiry from this point on consistently, conclude with that for the investigation of which they set out.[3]

Except for the apparent choice of *things* rather than *propositions* as starting points, which seems odd to us, this does indeed sound like the axiomatic-deductive method evolving toward its definitive form and use.

But the question of *motivation* remains. *Why* did the mathematicians want to treat their subject so? One possibility is that Euclid or his predecessors sought in this way to respond to problems arising in the subject's development: rigorous inference from explicit premisses might make diagnosis and cure much easier. A favorite candidate for an "internal stress" that might have turned attention to organizational issues has been the discovery of "incommensurability", probably about 430 B.C. This was the discovery (by another early example of deductive reasoning) that given, say, two line segments, one cannot always find a unit segment that measures both of the given ones exactly. Undoubtedly, the impact of this revelation on Greek mathematics was very great. The realization that (given the Greeks' restriction of the number concept to positive integers) some line segments, for example the diagonal of a unit square, had no numerical length, gave their mathematics its characteristic bent toward geometrical as opposed to "algebraic" formulations. Perhaps the discovery of incommensurability also spurred attempts at axiomatization, though the dramatic idea, once commonly voiced, that it triggered a "crisis" in the foundations of mathematics seems now to be increasingly discounted.[4]

Can we locate some of the origins of axiomatic and deductive methods *outside* mathematics? Some rough and informal use of them must be almost as old as people's attempts to debate and persuade. In everyday conversation one often tries to argue from positions accepted by the other side, and to pass to a desired conclusion by convincing inference. Some thinkers in early Greece sought to formalize such practices by making axioms a recognized part of proper procedure. Thus Diogenes of Apollonia, in the second half of the fifth century B.C., wrote that "in starting any thesis, it seems to me, one should put forward as one's point of departure something incontrovertible"; and a medical writer of the same period declared, rather more vaguely, that any inquiry needs a "starting point" to be truly scientific.[5] And if axiomatization had early beginnings outside mathematics, so too did the conscious use of deductive argument. A wide scholarly consensus credits the first sustained example to the philosopher Parmenides (c. 515 B.C.). This is a poem, of legendary obscurity, which reaches and accepts the unsettling conclusion that the familiar phenomena of motion, change and plurality are all delusions of the senses, that in fact all reality is one and immutable.[6] The familiar paradoxes with which Zeno (c. 450 B.C.) sought to defend his teacher Parmenides were in the same vein. The Hungarian philologist Árpád Szabó argued at length that this "Eleatic" tradition in philosophy – it is named for the Italian "home town" of Parmenides and Zeno – was the example and inspiration for the mathematicians' adoption of axiomatic-deductive methods. Szabó's thesis was persuasively countered by the distinguished American historian Wilbur Knorr, but no final verdict is possible.[7]

It may be relevant to note that acceptance of the arguments of Parmenides and Zeno, and/or of the proof of incommensurability, proclaims a very strong respect for deductive inference. The former reasoning ends of course in drastic contradictions of everyday experience, while for many people the latter result runs directly counter to intuitive expectations. Perhaps then for many Greek thinkers an almost magical aura clung to a method of revelation so powerful that it apparently allowed the mind to overturn the evidence of the senses (as in Parmenides and Zeno) or to discover truth where the senses were simply helpless (as in the discovery of incommensurability). Every line in Parmenides, wrote the great classical scholar Werner Jaeger, "pulsates with his ardent faith in the newly discovered powers of pure reason".[8] Perhaps something of this nearly religious feeling helped to motivate Euclid and his precursors.

Still, we may ask: how close are we here to the *Elements*, even so? It is one thing to base a single deductive argument on premisses adopted for the purpose, as in Parmenides' poem and the proofs of incommensurability, and quite another *kind* of achievement to build a *whole body of knowledge* from a few appropriate axioms. Let me widen the scope of the inquiry.

II

I begin with an observation whose relevance may not at first be obvious. The ancient Greeks were highly *competitive* people. The joy of the *agon* or organized contest, the zestful matching of one's skill or strength against another's, are conspicuous already in Homer, and of course were given full scope in the Olympic Games (776 B.C. ff.). This combativeness naturally extended to intellectual life. Much of the age's philosophical writing, for example, attempts to establish positions by explicit rebuttal of theorists earlier in the field. More relevant to our present theme is a strong tendency for the practitioners of various professions or "arts" to try to place their own pursuits higher, in one sense or another, than all the others. Which activities had the credentials of an "art" (Greek *techne*)? Which could make the (even better) claim to be a "science" (*episteme*)? Which boasted the best methodology? Which attained true knowledge? Which conferred the greatest blessings on mankind? Claims and counter-claims filled the air. Rivalries sprang up not only *between* professions but *within* them, subgroups arguing variations in outlook or technique. Sometimes such confrontations became public spectacles, as competing schools of medicine or rival sects of philosophy aired their differences before audiences, taking the law courts as models for procedure. Geoffrey Lloyd of Cambridge University has written fascinating pages linking this adversarial spirit, this constant urge to compete, with the development of Greek philosophy and science. He points out that in the absence (for the most part) of state patronage or private philanthropy, a public demonstration of professional competence, a public victory over one's opponents, might be the best way for an individual or a group to build a reputation – or even a necessary condition of sheer survival. Convincing arguments for the superior value of one's ideas or activities might be the surest way to attract students or followers, to win influence and prestige.[9]

Vivid pictures of debate over the various arts' claims to distinction survive in Plato's dialogues. In the *Philebus* the criterion for excellence is mathematical: any pursuit ranks high to the extent that it makes systematic use of "number, measure and weight".[10] Plato of course ranked mathematics itself near (not quite *at*) the very pinnacle of studies, as for example in the curriculum that would train his "philosopher-kings"; and probably his high estimate was widely shared by educated contemporaries. A line in Thucydides' magnificent *History of the Peloponnesian War* hints that already by the late 5th century mathematics was for many (as it remains for us) the model of an exact science.[11] On the other hand its lofty position was by no means uncontested. For one thing it seems that no *practitioner* of *any* of what we would now call the sciences enjoyed on that account any special social elevation.[12] More to the present point, some thinkers felt that they could contest the supremacy of mathematics in purely intellectual terms. The deepest challenge of that sort came from a direction which in modern eyes might seem very surprising: from the orators, the men who earned their bread by public speaking.

Tradition placed the rise of "rhetoric" as an organized discipline in Syracuse in the early 5th century; in the succeeding decades it witnessed an explosive development, in theory and in practice alike. That is no mystery: two pervasive Greek institutions, flourishing in this age, offered great rewards to oratorical skill. The "democratic" assembly and the law courts valued verbal persuasiveness to an extent that we now find hard to imagine. Modes and techniques of argument multiplied, and virtuoso talkers like Demosthenes (4th century B.C.) won vast acclaim.

In this climate it is not surprising that some ranked rhetoric as the highest of all the arts, above even mathematics. Amid many obvious differences these two pursuits have one strong similarity, which may have invited weighing of their respective merits: a rhetorical argument formally resembles a mathematical proof, in passing (ideally) by a chain of valid inference from agreed assumptions to a targeted conclusion. But in 5th-century Greece the possibility of comparison, and the temptation to claim superiority, were increased by another aspect of the "mindset" of the age. In those long-ago days the world had not been as fully "sorted out", so to say, as it seems to us: now-familiar differences and boundaries between things were not yet fully drawn. One case of this blurring of distinctions is especially germane here. It seems that people then had a (to us) incomplete understanding of the idea of a *proof*. We see a clear difference between the *logical* use of the techniques of proof, as in a theorem of geometry, and the *psychological* use of persuasion, as in a court of law; we distinguish a necessary conclusion from one that is merely plausible or probable. By Plato's time, or at any rate in Plato's own powerful mind, these contrasts became increasingly clear, and (as we shall see in a moment) he urged them strongly; but Plato could still entertain, early in his career, the possibility that an orator's technique could rise toward "science", provided only that he acquire a sufficiently deep knowledge of the human soul that was the target of his arguments.[13] Conversely,there are hints that some geometers may have sought or accepted merely *probabilistic* arguments for their theorems.[14] Technical terms suited only (*we* would say) for strict demonstration slipped without apparent resistance into looser contexts.[15] Against this background the famous rhetorician Gorgias (late 5th century) could declare his calling supreme among all the arts;[16] and Plato's contemporary Isocrates established, as a rival of the mathematically oriented Academy, a school that gave priority to rhetoric as the best training possible for youthful minds.

But this rivalry of professions had a darker side, important for our story. To understand this we need a bit more background on the cultural life of 5th-century Greece.

This was an age of great intellectual ferment. Old certainties were challenged, deep problems of the human condition troubled thoughtful minds. In particular the perennial questions about the possibility and the nature of our knowledge of the world were widely aired. What, if anything, can we know? Are there objective truths, independent of our minds, or are even our most cherished convictions mere decrees of human convention? If, in our pursuit of knowledge, our reason and our senses come into conflict, which should we trust? These and other dilemmas, alive still in our own age, were under vigorous and urgent debate.

Prominent in this lively atmosphere, indeed so typical of it that their arguments have been said to give the age its characteristic voice,[17] were the wandering teachers and scholars known as the "sophists". The Greek word behind their name has a complex set of meanings, but for our purposes "wisdom" is close enough: the sophists were "wise men". They taught everything, but they were associated especially with the theory and practice of rhetoric. Now in the debate over the possibility of objective and certain knowledge some at least of the sophists were deeply skeptical. Protagoras urged in a famous phrase that "man is the measure of all things" – a statement taken by modern scholars as declaring an extreme relativism in which each person's private judgment is his or her sufficient criterion of truth. Gorgias, mentioned earlier as affirming supreme status for rhetoric, gave an argument which claimed to prove that there is no absolute truth, that if there *were* such we could not know it, and if we *could* know it we could not communicate it.[18]

To such pessimistic conclusions the most famous of Greek philosophers – Socrates, and his student Plato, and his student Aristotle – reacted with revulsion, both intellectual and emotional. All three were deeply convinced that there *is* absolute and objective truth, that we can know it, and indeed that we *must* know it to live well. They came therefore to despise the sophists, both for undermining these convictions with their relativism and for the rhetorical trickery that made their pronouncements plausible; and they fought back.

The results were historic, for they included the birth of one of the enduringly important doctrines of all western philosophy. The Theory of "Forms" (or "Ideas") asserts that the world's true reality consists of eternal entities existing independently of us and accessible only to our minds, not to our senses, the things in the physical world are in some way imperfect, perishable shadows of this higher level of being. Probably there is no better *example* of such "Forms" than the objects of mathematics: the perfect triangle (say) present to the geometer's mind, in stark contrast to the crude approximation she draws in the sand. Hence lovers of mathematics may be tempted to suppose that contemplation of their science suggested the Theory of Forms to the philosophers; and perhaps in some measure it was so. But the *primary* impulse came from elsewhere. When Socrates (no keen student of mathematics!) took the first steps he was responding directly to the challenge posed by the sophists and other champions of relativism. He was seeking, in the moral realm, absolute standards, binding and guiding human beings for all time – an absolute Justice, an absolute Goodness, an absolute Virtue. The Theory of Forms was the ultimate outcome.

That theory met the longing of many for an objective order of reality; but it did more. In his quest for moral absolutes, Socrates struggled to isolate the common elements in "real-world" instances – to identify, for example, the defining "justness" in particular acts of justice – and then to capture such cores in words; in this way (as Aristotle generously declared)[19] the old gadfly began the serious study of definition and of induction. These in turn were early steps toward one of Aristotle's own most momentous achievements, the working out of a full *theory* of "scientific" demonstration. Meanwhile the reaction against the rhetoricians contributed to the same end in another way. Probably their excesses sharpened in Socrates and Plato – in the Greek mind in general – a growing sense of the distinctions which, as we saw, earlier views of demonstration had blurred: the differences between necessary and probable inference, between proof and mere persuasion. Several of Plato's dialogues urge just these discriminations, in passionate polemic against the orators.[20]

So developed some of the background of the astonishing feat of genius by which Aristotle for the first time made of logic an organized science, and in particular set out, mostly in his *Posterior Analytics*, the theory of "scientific" demonstration mentioned above. In his view this was the most rigorous and powerful of all modes of argument: it inferred necessary, eternal truth, by iron rules of logic, from premisses unquestionably true. His account of such premisses amounts to the first-ever *theory* of axiomatization, including for example an attempt to *classify* first principles into diverse types. He made another breakthrough by insisting on the fact (not obvious to Plato) that axioms must remain unproved, and he labored to show how we can *see* their certainty *without* proof, by acts of more or less immediate apprehension.[21]

(Again some of the most suggestive examples may have come from mathematics. Aristotle seems to have taken his cue, in part, from the way we grasp a fact such as the equality of the opposite angles formed by intersecting straight lines: we "see" it (either literally, with our eyes, or else with our minds), first in one or more particular cases and then in the appropriate generalization.) Meanwhile Aristotle created also what he took to be the appropriate logical apparatus for scientific *deduction* from axioms, namely his theory of inference by "syllogism"(e.g. "some A are B; all B are C; therefore some A are C"). Significantly, he recognized other *kinds* of demonstration than this "scientific" mode, including the "rhetorical", but all such alternatives he ranked as inferior. In rhetorical demonstration, for example, the premisses (he said) are not all explicit, and are merely probable rather than certain.[22] Here was struck a lasting blow against the old confusions in the art of reasoning.

Aristotle died in 322 B.C., so perhaps two generations separate his theory of scientific demonstration from Euclid's *Elements*, the definitive axiomatic-deductive presentation of mathematics. Much scholarly speculation has pondered the relation between the two achievements. The puzzle fascinates partly because Euclid's practice is closely similar to Aristotle's theory at some points, strikingly different at others. Aristotle's classification of first principles corresponds in some degree, but not exactly, to Euclid's familiar trio of definitions, "common notions" and postulates;[23] Aristotle's deductive steps (as we saw) are syllogistic, Euclid's certainly are not. Did Euclid draw on the Posterior Analytics? Conversely, was Aristotle guided by observation of the mathematics of his own time, in particular by some pre-Euclidean *Elements*? Could both possibilities be true? Direct evidence is lacking, and the question is complex. I cannot pursue it here except to report that a detailed study by Richard McKirahan concludes that indeed both kinds of influence *were* probably at work – not mere copying but selective borrowing and creative transformation.[24]

In any case some irony attaches to the respective fates of these two great books, the *Posterior Analytics* and the *Elements*. Aristotle, following Plato, was strongly spurred by the wish to separate true philosophy and science from rhetoric, proof from persuasion; and he came to believe that in any context his prescription for scientific argument was both necessary and sufficient for full understanding.[23] The irony is that despite this supposed power and universality of Aristotle's procedure it was the quite different methodology of the *Elements* that went on to enjoy the longer and deeper historical influence. Now I must stress again that of Euclid's motivations – unlike Aristotle's – we know nothing. Probably, of course, they were many and diverse. Perhaps, for example, he wished to secure the foundations of mathematics against internal stress; perhaps he tried to put his subject in the form best suited for teaching; perhaps he took delight in the austere beauty of his book's structure. But perhaps also he felt some impulse from the social and cultural background that I have here tried to sketch: the rivalries among professions for social standing and prestige, the fervent quest of some thinkers for objective and eternal truth. Perhaps Euclid hoped that he was confirming the claim of mathematics to be supreme among the arts of mankind, and a bastion of knowledge impervious to assault by potential critics. We cannot know – he gives no clue.

But we *can* plausibly picture such motives in at least some of his ancient emulators. In the last centuries of antiquity certain cultural trends posed ongoing challenges to defenders of the value and prestige of mathematics. In the old contest between rhetoric on the one hand and philosophy allied with mathematics on the other – the pedagogical opposition between Isocrates and Plato – rhetoric had generally the upper hand, as attested for example by its dominance in school curricula.[26] Meanwhile a persistent undercurrent of skepticism attacked the foundations, the very possibility, of objective knowledge.[27] In attempts to counter both these trends, the results and the methodology of mathematics in general, and of the *Elements* in particular, were obvious cards to play. Two distinguished voices offer especially striking statements. The great astronomer Ptolemy (2nd century A.D.), declared that "only mathematics can provide sure and unshakeable knowledge to its devotees ... for its kind of proof proceeds by indisputable methods".[28] Three centuries later, Proclus praised Euclid for the "irrefutable" character of his demonstrations,[29] and wrote an *Elements of Theology* in hopeful imitation. "Unshakeable", "indisputable", "irrefutable" – these words suggest stimuli going beyond the disinterested pursuit of knowledge for its own sake, vital though that urge must always have been. They hint also at the long cultural history here superficially described, and so at the extra emotional dimension to be gained by routing one's rivals for status and reputation, by displaying the supremacy of one's discipline, or by confounding the relativists and the skeptics with sure and necessary truth.

NOTES

1. This was the conclusion of an exhaustive survey by **Wilbur Knorr**, "On the early history of axiomatics: the interaction of mathematics and philosophy in Greek antiquity", in J. Hintikka, D. Gruender and E. Agazzi, eds., *Theory Change, Ancient Axiomatics and Galileo's Methodology* (Dordrecht, 1981), Vol. 1, pp. 145-86, at pp. 150-57.

2. **Proclus**, *A Commentary on the First Book of Euclid's Elements*, tr. Glenn R. Morrow (Princeton, 1992), sec. 68.

3. **Plato**, *Republic* 510c-d.

4. **E.g. Walter Burkert**, *Lore and Science in Ancient Pythagoreanism*, tr. Edwin Minar (Harvard U.P., 1972), p. 465ff, **G.E.R. Lloyd**, *Magic, Reason and Experience* (Cambridge, 1979), p. 113 n. 289.

5. **Diogenes of Apollonia**, frag. 1, in **Kathleen Freeman**, *Ancilla to the Pre-Socratic Philosophers*, Harvard U.P., 1983, p. 87; Anon., *Tradition in Medicine*, also called *On Ancient Medicine*, in **G.E.R. Lloyd**, ed., *Hippocratic Writings* (London, 1978), p. 71.

6. **Parmenides**, in **Freeman** (n. 5), pp. 45ff. His starting point is frag. 2, his conclusions are in frag. 8.

7. **Árpád. Szabó**, *The Beginnings of Greek Mathematics*, tr. A.M. Ungar (Budapest, 1978); Knorr (n. 1); cf. Lloyd (n. 4), p. 103.

8. **Werner Jaeger**, *Paideia: The Ideals of Greek Culture*, tr. Gilbert Highet (New York, 1974), p. 177

9. **G.E.R. Lloyd**, *Adversaries and authorities: Investigations into ancient Greek and Chinese science* (Cambridge, 1996), pp. 35ff, 130ff.

10. **Plato**, *Philebus* 55d-e; cf. *Republic* 602d.

11. **Thucydides**, Book V, "Melian Dialogue", 90.

12. **G.E.R. Lloyd**, "Science and Mathematics", in M.I. Finley, ed., *The Legacy of Greece* (Oxford U.P., 1984), pp. 261-62.

13. **Plato**, *Phaedrus* 271, 277.

14. **Plato**, *Phaedo* 92d, *Theaetelus* 162e.

15. **Lloyd** (n. 9), pp. 56, 58.

16. **Plato**, *Gorgias* 452e, 454e ff, 459a-c, *Philebus* 58a-b.

17. **G.R. Kerferd**, *The Sophistic Movement* (Cambridge, 198 1), p. 1.

18. **Protagoras**, frag. 1 Gorgias, frag. 3, both in Freeman (n. 4), pp. 125, 128-29.

19. **Aristotle**, *Metaphysics* 987b1-4, 1078b17-30.

20. **E.g. Gorgias** 452e ff, 458e ff, *Phaedrus* 259e ff, 272d ff, *Philebus* 58a ff.

21. **Aristotle**, *Posterior Analytics*, II, 19.

22. **Aristotle**, *Rhetoric* 1403b-1404a.

23. See, e.g., **Richard D. McKirahan**, *Principles and proofs: Aristotle's Theory of Demonstrative Science* (Princeton, 1992), pp. 133ff, **Thomas L. Heath**, *The Thirteen Books of euclid's Elements* (New York, 1956), Vol. 1, pp. 117ff.

24. **McKirahan** (n. 23), especially p. 134.

25. **Lloyd** (n. 9), p. 215; Aristotle, *Posterior Analytics* 71b9 ff.

26. **H.-I. Marrou**, *A History of Education in Antiquity*, tr. George Lamb (New York, 1964), pp 252-53.

27. See, e.g., **Julia Annas and Jonathan Barnes**, *The Modes of Scepticism* (Cambridge, 1985).

28. **Ptolemy**, *Almagest*, tr. G.J. Toomer (London, 1984), I.1 (H6).

29. **Proclus** (n. 2), sec. 70.