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Topological Methods for ODE's: Symplectic Differential Systems*

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Abstract

Taking inspiration from the geometrical ideas behind the classical Sturmian theory for ordinary differential equations in $\mathbb{R}$, in this paper we review some recent topological techniques to study some properties of systems of ODE's in higher dimension. More specifically, we will discuss the notion of *Maslov index* for *symplectic differential systems*, i.e., those systems of differential equations in $\mathbb{R}^n \oplus \mathbb{R}^{n*}$ whose flow preserves the canonical symplectic form. Such systems appear naturally in association with the Jacobi equation along a semi-Riemannian geodesic, or, more generally, with solutions of possibly time-dependent Hamiltonians on symplectic manifolds. In this paper we review some recent results in the theory of symplectic differential systems, with special emphasis on those systems arising from semi-Riemannian geometry.

1. Introduction

Geometry and Topology offer very powerful tools in the study of qualitative and also quantitative properties of differential equations. The main idea behind these theories is that some equations, or better, some classes of equations can be studied by means of their *symmetries*, where by symmetry we mean generically any algebraic or geometric structure which is preserved by their flow. Once such

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invariant structures are determined for a class of differential equations, many properties of the solutions of the class can be read off from the geometry of the curve obtained by the flow, taking values in the space (typically a Lie group) of all structure-preserving morphisms.

A simple, but instructive, example is given by the Sturmian theory for second order ordinary differential equations in $\mathbb{R}$ (see for instance [7, Chapter 8]). The classical Oscillation Theorem gives an equality between the number of oscillations (i.e., of zeroes) of a solution of a Sturm equation with the number of negative eigenvalues of the associated second order differential operator. In Section 2 we will show how to obtain a proof of the Sturm oscillation theorem by showing that the two quantities involved in the thesis can be obtained as the *winding number* of two homotopic closed curves in the real projective line.

The class of differential equations that we we will consider in this paper consists in the so called "symplectic differential systems"; these are linear systems in $\mathbb{R}^n \oplus \mathbb{R}^{n*}$ whose flow preserve the canonical *symplectic form*, given by $\omega\big((v,\alpha),(w,\beta)\big) = \beta(v) - \alpha(w)$. Recall that a symplectic form is a nondegenerate skew-symmetric bilinear form on a (necessarily even dimensional) vector space. These differential systems appear naturally in a great variety of fields of pure and applied mathematics, and many areas of mathematics and physics, like Calculus of Variations, Hamiltonian systems, (semi-)Riemannian Geometry, Symplectic Geometry, Mechanics and Optimal Control Theory produce examples of symplectic systems as basic objects of investigation. For instance, Morse–Sturm systems are special cases of symplectic systems; such systems are obtained from the Jacobi equation along any semi-Riemannian geodesic by means of a parallel trivialization of the tangent bundle of the semi-Riemannian manifold along the geodesic. More in general, symplectic systems are obtained by considering the linearized Hamilton equations along any solution of a (possibly time-dependent) Hamiltonian problem, using a symplectic trivialization along the solution of the tangent bundle of the underlying symplectic manifold. Another large class of examples where the theory leads naturally to the study of symplectic systems is provided by Lagrangian variational theories in manifolds, possibly time-dependent, even in the case of *constrained* variational problems. Indeed, under a suitable *invertibility* assumption called *hyper-regularity*, the solutions to such problems correspond, via the *Legendre transform*, to the solutions of an associated Hamiltonian problem in the cotangent bundle.

The fundamental matrix of a symplectic system is a curve in the *symplectic group*, denoted by $Sp(2n,\mathbb{R})$, which is a closed subgroup of the general linear group $GL(2n,\mathbb{R})$, hence it has a Lie group structure. This structure is extremely rich, due to the fact that symplectic forms on a vector space are intimately related to its complex structures, and such relation produces other invariant geometric and algebraic structures, such as inner products and Hermitian products.

Many interesting questions can be answered by studying solutions of symplectic systems whose initial data belong to a fixed *Lagrangian subspace* of $\mathbb{R}^n \oplus \mathbb{R}^{n*}$. Recall that a Lagrangian subspace of a symplectic space is a maximal subspace on which the symplectic form vanishes. Such initial conditions are obtained, for instance, in Riemannian or semi-Riemannian geometry when one considers Jacobi fields along a geodesic that are variations made of geodesics starting orthogonally at a given submanifold. Since symplectic maps preserve Lagrangian subspaces, the image of the initial Lagrangian by the flow of a symplectic system is a curve in the set Λ of all Lagrangian subspaces of $\mathbb{R}^n \oplus \mathbb{R}^{n*}$. The set Λ is a smooth (indeed, real-analytic) submanifold of the *Grassmannian* $G_n(\mathbb{R}^n \oplus \mathbb{R}^{n*})$ of all n-dimensional subspaces of $\mathbb{R}^n \oplus \mathbb{R}^{n*}$; Λ is called the *Lagrangian Grassmannian* of the symplectic space $\mathbb{R}^n \oplus \mathbb{R}^{n*}$.

The original interest of the authors was the study of conjugate points along geodesics in a semi-Riemannian manifold and their stability (see [16, 18]), with the aim of developing an infinite dimensional Morse Theory (see [11, 15, 17]) for semi-Riemannian geodesics. A few decades ago a new integer valued homological invariant, called the *Maslov index*, was introduced by the Russian school (see for instance [1] and the references therein) for closed curves in a Lagrangian submanifold M of the space $\mathbb{R}^{2n}$ endowed with its canonical symplectic structure. The notion of Maslov index has been immediately recognized as an important tool in the study of conjugate points, and it has has been thoroughly investigated and extended in several directions by mathematical-physicists, geometers and analysts. There is nowadays a very extensive literature about the subject, and it is almost impossible to acknowledge the work of all the many authors who have given significant contributions to the field.

Periodic or non periodic solutions of Hamiltonian systems, like for instance geodesics in a semi-Riemannian manifold, define a curve in the symplectic group, or in the Lagrangian Grassmannian, hence they define a Maslov index. Roughly speaking, the Maslov index gives a sort of *algebraic count* of the conjugate points along a solution; here are some of the main properties of this invariant:

- it is always finite (even when the number of conjugate points is infinite);
- it is *stable* by "small" perturbations of the data;
- it coincides with the *geometric index* in the case of a causal (timelike or lightlike) Lorentzian geodesic;
- it is related to the *analytic index* (or, more in general, to the relative index)

of the solution, which is the index of the second variation of an associated Lagrangian action functional;

- it is related to the spectral properties of the associated Hamiltonian second order differential operator.

Conjugate and focal points appear naturally in Optics, both classical and relativistic, and the Maslov index provides a new topological invariant.

In this article we review the notion of symplectic systems and their basic properties, and we will briefly discuss some recent applications of the theory of the Maslov index. Basic references for all the material exposed in this paper are [8, 11, 16, 19, 20, 24, 25].

2. A topological Proof of the Sturm Oscillation Theorem

The classical Sturmian theory for differential equations (see for instance [7, Chapter 8]) deals with equations of the form

$$-(px')' + rx = 0 \tag{2.1}$$

and the corresponding eigenvalue problem

$$-(px')' + rx = \lambda x, \tag{2.2}$$

where p and r are continuous functions on $[a, b]$, $p > 0$, and λ is a real parameter. We recall the following:

2.1 **Oscillation Theorem.** *The number of zeroes in $]a, b[$ of any non trivial solution of (2.1) satisfying $x(a) = 0$ equals the number of the negative eigenvalues of the corresponding differential operator in the space of C^2 functions vanishing at a and b, i.e., the number of negative λ's for which (2.2) admits a non zero solution $x : [a, b] \to \mathbb{R}$ satisfying $x(a) = x(b) = 0$.*

An alternative statement of the Oscillation Theorem can be given in terms of symmetric bilinear forms; in this new form the Oscillation Theorem can be generalized to systems of ODE's in $\mathbb{R}^n$.

Denoting by $C^1_o[\alpha, \beta]$ the space of C^1-functions on $[\alpha, \beta]$ vanishing at α and β, let us consider the *index form* of (2.1), which is the symmetric bilinear form

$$B(x, y) = \int_a^b [px'y' + rxy] \, dt \tag{2.3}$$

defined in the Banach space $C^1_o[a, b]$; its *index* [1] is precisely the number of negative eigenvalues of (2.2). On the other hand, an easy integration by parts shows that

[1]Recall that the index of a symmetric bilinear form B on a vector space V is the (possibly infinite) supremum of the dimensions of all subspaces W of V on which B is negative definite.

an instant $t \in]a, b]$ is a zero for one (hence for any) non trivial solution x of (2.1) satisfying $x(a) = 0$ if and only if the *restricted index form:*

$$B^t(x, y) = \int_a^t [px'y' + rxy]\,dt$$

defined in $C_0^1[a, t]$, has non trivial kernel. Thus, we have the following:

2.2 Oscillation Theorem (alternative statement). *The index of the index form B equals the sum over all $t \in]a, b[$ of the dimension of the kernel of the restricted index form B^t.*

As an instructive example of topological methods in the theory of ODE's, we will present below a proof of the Oscillation Theorem based on an argument in homotopy theory. More precisely, we will show that the two quantities that are claimed equal in the theorem, namely, the number of zeros of any non trivial solution of (2.1) and the number of negative eigenvalues of (2.2) are the winding numbers of two homotopic curves in the circle S^1.

For all $\lambda \in \mathbb{R}$ fixed, denote by $[a, b] \ni t \mapsto x(t, \lambda)$ the solution of (2.2) satisfying the initial condition

$$x(a, \lambda) = 0, \tag{2.4}$$

$$p(a)\tfrac{d}{dt}x(a, \lambda) = 1, \tag{2.5}$$

and denote by $[a, b] \times \mathbb{R} \ni (t, \lambda) \mapsto \theta(t, \lambda)$ the map taking values in the projective real line $\mathbb{R}P^1$ such that $\theta(t, \lambda)$ is the line through the point $\big(x(t, \lambda), p(t)\tfrac{d}{dt}x(t, \lambda)\big)$. Observe that the functions x, $\frac{dx}{dt}$ and $\frac{\partial x}{\partial \lambda}$ are continuous in (t, λ). The prime symbol $'$ will be used to denote derivatives with respect to t. Let us assume for simplicity that $\lambda = 0$ is not an eigenvalue of the Sturm equation (2.1), i.e., that there is no nonzero solution x of (2.1) satisfying $x(a) = x(b) = 0$.

Denote by θ_* the "vertical line" in $\mathbb{R}P^1$, i.e., the line in $\mathbb{R}^2$ through the origin and the point $(0, 1)$. We observe that the index of B is the number of negative λ_*'s for which the curve $\lambda \mapsto \ell_1(\lambda) = \theta(b, \lambda)$ passes through θ_*; on the other hand, the kernel of B^{t_*} is non trivial (hence, unidimensional) exactly at those instants t_* for which the curve $t \mapsto \theta(t, 0)$ passes through θ_*. We also observe that $\theta(t, \lambda) \neq \theta_*$ for all $t \in]a, b]$ and for all λ such that:

$$\lambda < -\|r\|_\infty = -\max_{t \in [a,b]} |r(t)|; \tag{2.6}$$

indeed, $\theta(t, \lambda) = \theta_*$ if and only if the index form:

$$B_\lambda^t(x, y) = \int_a^t [px'y' + (r - \lambda)xy]\,dt$$

has non trivial kernel in $C_o^1[a,t]$. Now, under the assumption (2.6), B_λ^t is positive definite:

$$B_\lambda^t(z,z) = \int_a^t [p(z')^2 + (r-\lambda)z^2]\,dt \geq \int_a^t p(z')^2\,dt > 0,$$

for all $z \in C_o^1[a,t]$, $z \neq 0$. In particular, the curve ℓ_3 given by $]a,b] \ni t \mapsto \theta(t, -\|r\|_\infty)$ does not pass through θ_*. By (2.5), it is easy to see that there exists $\varepsilon \in]0, b-a[$ such that $x(t,\lambda) \neq 0$ for all $\lambda \in [-\|r\|_\infty, 0]$ and for all $t \in]a, a+\varepsilon]$; moreover, since $\lambda = 0$ is not an eigenvalue of (2.1), one has $\theta(b,0) \neq \theta_*$.

Hence, a proof of the Oscillation Theorem is obtained by showing that the curves $[-\|r\|_\infty, 0] \ni \lambda \mapsto \ell_1(\lambda)$ and $[a+\varepsilon, b] \ni t \mapsto \ell_2(t) = \theta(t,0)$ have the same number of passages through θ_*. The concatenation of ℓ_3 and ℓ_1 is homotopic to ℓ_2, moreover, as we have seen, ℓ_3 does not pass through θ_*; we will establish the equality of the number of passages through θ_* of ℓ_1 and ℓ_2 by showing that:

$$\ell_2(t_*) = \theta_* \implies \frac{d}{dt}\tan(\ell_2(t_*)) > 0, \quad \text{and} \quad \ell_1(\lambda_*) = \theta_* \implies \frac{\partial}{\partial\lambda}\tan(\ell_1(\lambda_*)) > 0,$$

where $\tan(\ell)$ denotes the tangent of the oriented angle from ℓ to θ_*.

The inequality $\frac{d}{dt}\tan(\ell_2(t_*)) > 0$ is easy: if $x(t_*) = 0$ one has

$$\frac{d}{dt}\tan(\ell_2(t_*)) = \frac{d}{dt}\left(\frac{x}{px'}\right) = \frac{p(x')^2 - (px')'x}{(px')^2} = \frac{1}{p} > 0.$$

In order to compute $\frac{\partial x}{\partial\lambda}$, by differentiating (2.2), (2.4) and (2.5) with respect to λ we observe that, for all λ fixed, the map $z = \frac{\partial x}{\partial\lambda}(\cdot,\lambda)$ is the solution of the non homogeneous equation:

(2.7) $$-(pz')' + (r-\lambda)z = x,$$

satisfying the initial conditions:

(2.8) $$z(a) = 0, \quad z'(a) = 0.$$

Denote by $y = y(t,\lambda)$ the solution of (2.2) satisfying the initial conditions:

$$y(a,\lambda) = 1, \quad y'(a,\lambda) = 0;$$

a straightforward computations using (2.2) shows that:

(2.9) $$p(t)x'(t,\lambda)\,y(t,\lambda) - p(t)y'(t,\lambda)\,x(t,\lambda) = 1, \quad \forall t,\lambda,$$

and that the solution of (2.7) and (2.8) is given by:

(2.10) $\frac{\partial x}{\partial \lambda}(t,\lambda) = y(t,\lambda)\int_a^t x(s,\lambda)^2\,ds - x(t,\lambda)\int_a^t x(s,\lambda)y(s,\lambda)\,ds.$

By (2.9), we get that at those points where $x = 0$ one has:

(2.11) $y = \frac{1}{px'}.$

From (2.10) and (2.11) we get immediately:

$$\frac{\partial}{\partial\lambda}\tan(\ell_1(\lambda.)) =$$

$$\frac{\partial}{\partial\lambda}\left(\frac{x}{px'}\right) = \frac{\frac{\partial x}{\partial\lambda}px' - x\frac{\partial}{\partial\lambda}(px')}{(px')^2} = \frac{\frac{\partial x}{\partial\lambda}}{px'} = \frac{y}{px'}\int_a^t x^2 = \frac{1}{(px')^2}\int_a^t x^2 > 0,$$

and this concludes the proof of the Oscillation Theorem.

For the general case of symplectic systems, equality (2.9) will be replaced by the property that the flow of the system preserves a symplectic form, while the curve θ in the real projective line will be replaced by a curve in the Grassmannian of all Lagrangian subspaces of a fixed symplectic space.

3. The symplectic group and its Lie algebra

A *symplectic vector space* is a (finite dimensional) real vector space V endowed with a nondegenerate anti-symmetric bilinear form $\omega : V \times V \to \mathbb{R}$; a symplectic vector space is necessarily even dimensional and the set $Sp(V,\omega)$ of linear endomorphisms $T : V \to V$ that preserve ω (symplectomorphisms) is a closed and connected Lie subgroup of the general linear group $GL(V)$. The Lie algebra $Sp(V,\omega)$ of $Sp(V,\omega)$ consists of those linear endomorphisms X of V such that $\omega(X\cdot,\cdot)$ is a *symmetric* bilinear form on V, i.e., $\omega(Xv,w) = \omega(Xw,v)$ for all $v,w \in V$.

All symplectic vector spaces of the same dimension are isomorphic; the standard example that we will consider in this paper is $V = \mathbb{R}^n \oplus \mathbb{R}^{n*}$ (here * denotes the dual space) endowed with the *canonical symplectic form* $\omega((v,\alpha),(w,\beta)) = \beta(v) - \alpha(w)$, for $v,w \in \mathbb{R}^n$ and $\alpha,\beta \in \mathbb{R}^{n*}$. In this case, the symplectic group is denoted by $Sp(2n,\mathbb{R})$ and its Lie algebra by $Sp(2n,\mathbb{R})$; in block matrix notation, a $2n \times 2n$ matrix X of the form

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

belongs to $Sp(2n,\mathbb{R})$ iff the following relations are satisfied:

$$D^*A - B^*C = \mathrm{I}, \quad A^*C \text{ and } B^*D \text{ are symmetric,}$$

where A, B, C, D are $n \times n$ matrices, I denotes the identity $n \times n$ matrix and * denotes the transpose. Similarly, elements of $Sp(2n, \mathbb{R})$ are identified with $2n \times 2n$ real matrices of the form:

$$\begin{pmatrix} A & B \\ C & -A^* \end{pmatrix},$$

where A, B, C are $n \times n$ matrices, with B and C symmetric.

Identifying $n \times n$ complex matrices with $2n \times 2n$ real matrices via the map

$$D + Ei \longmapsto \begin{pmatrix} D & -E \\ E & D \end{pmatrix},$$

where D, E are real $n \times n$ matrices, then the *unitary group* $U(n)$ is identified with a closed subgroup of $Sp(2n, \mathbb{R})$. Moreover, the inclusion $i : U(n) \to Sp(2n, \mathbb{R})$ is a homotopy equivalence, hence:

$$\pi_1(Sp(2n, \mathbb{R})) \cong \pi_1(U(n)) \cong \mathbb{Z};$$

recall that the latter isomorphism in the above formula is induced by the determinant map $\det : U(n) \to S^1$.

4. The Lagrangian Grassmannian

A subspace L of $\mathbb{R}^n \oplus \mathbb{R}^{n*}$ is said to be *isotropic* if $\omega|_{L \times L} = 0$; L is a *Lagrangian* subspace if L is *maximal isotropic*, i.e., if it is isotropic and it is not properly contained in any other isotropic subspace. It is easy to see that an isotropic subspace L is Lagrangian if and only if $\dim(L) = n$; the set

$$\Lambda = \left\{ L \subset \mathbb{R}^n \oplus \mathbb{R}^{n*} : L \text{ is Lagrangian} \right\}$$

is contained in the Grassmannian $G_n(\mathbb{R}^n \oplus \mathbb{R}^{n*})$ and it is called the *Lagrangian Grassmannian* of the symplectic space $(\mathbb{R}^n \oplus \mathbb{R}^{n*}, \omega)$. The Lagrangian Grassmannian Λ has the structure of a compact, connected, real-analytic embedded submanifold of $G_n(\mathbb{R}^n \oplus \mathbb{R}^{n*})$ having dimension $\frac{1}{2}n(n+1)$; for $L \in \Lambda$, there is a *natural* identification of the tangent space $T_L\Lambda$ with the space of symmetric bilinear forms on L.

There is a one-to-one correspondence between elements $\ell_0 \in \Lambda$ and pairs (P, S), where $P \subset \mathbb{R}^n$ is any subspace and $S : P \times P \to \mathbb{R}$ is a symmetric bilinear form on P; ℓ_0 and (P, S) are related by the following:

$$\ell_0 = \left\{ (v, \alpha) \in \mathbb{R}^n \oplus \mathbb{R}^{n*} : v \in P,\ \alpha|_P + S(v, \cdot) = 0 \right\}. \tag{4.1}$$

If $L_0 \in \Lambda$ is fixed, for each $k = 0, \dots, n$ we denote by $\Lambda_k(L_0)$ the subset of Λ defined by:

$$\Lambda_k(L_0) = \left\{L \in \Lambda : \dim(L \cap L_0) = k\right\};$$

for each k, $\Lambda_k(L_0)$ is a connected embedded real-analytic submanifold of Λ having codimension $\frac{1}{2}k(k+1)$ in Λ. The set $\Lambda_0(L_0)$ is a dense, open and contractible subset of Λ; its complementary set:

$$\Lambda_{\geq 1}(L_0) = \bigcup_{k=1}^{n} \Lambda_k(L_0)$$

is an *analytic subset* of Λ, whose *regular part* is $\Lambda_1(L_0)$, which is a dense open subset of $\Lambda_{\geq 1}(L_0)$. For the rest of the paper we will denote by L_0 the following Lagrangian space:

$$L_0 = \{0\} \oplus \mathbf{R}^{n*}. \tag{4.2}$$

There is a natural action of $Sp(2n, \mathbf{R})$ on Λ, given by

$$Sp(2n, \mathbf{R}) \times \Lambda \ni (T, L) \longmapsto T(L) \in \Lambda,$$

this action is real-analytic and transitive. The restriction of this action to the unitary group $\mathrm{U}(n) \subset Sp(2n, \mathbf{R})$ is also transitive, and the isotropy group of the Lagrangian $\mathbf{R}^n \oplus \{0\}$ can be identified with the orthogonal group $\mathrm{O}(n)$. Hence, Λ is a homogeneous space, and it is diffeomorphic to the quotient $\mathrm{U}(n)/\mathrm{O}(n)$. Consider the homomorphism

$$d = \det{}^2 : \mathrm{U}(n) \longrightarrow S^1.$$

Then, d induces by passage to the quotient a map:

$$\overline{d} : \mathrm{U}(n)/\mathrm{O}(n) \longrightarrow S^1; \tag{4.3}$$

using the homotopy exact sequence of the fibration (4.3) it is not hard to see that the map $\overline{d}$ induces an isomorphism between the fundamental groups, and so we have:

$$\pi_1(\Lambda) \cong \mathbf{Z}.$$

In particular, using the Hurewicz homomorphism, we get an isomorphism for the first homology group $H_1(\Lambda) \cong \mathbf{Z}$; finally, since $\Lambda_0(L_0)$ is contractible, using the long exact reduced homology sequence of the pair $(\Lambda, \Lambda_0(L_0))$ we get that the inclusion $\mathfrak{q} : (\Lambda, \emptyset) \to (\Lambda, \Lambda_0(L_0))$ induces an isomorphism:

$$\mathfrak{q}_* : H_1(\Lambda) \longrightarrow H_1(\Lambda, \Lambda_0(L_0)),$$

hence we have an isomorphism

$$\mu_{L_0} : H_1(\Lambda, \Lambda_0(L_0)) \longrightarrow \mathbb{Z}. \tag{4.4}$$

Even though Λ may fail to be orientable (it is orientable iff n is odd), $\Lambda_1(L_0)$ has a natural *transverse orientation* in Λ; the choice of a generator of $H_1(\Lambda)$ depends on the choice of such orientation, and the choice of a transversal orientation can be made canonically using the symplectic form ω. For a differentiable curve $l : [a,b] \to \Lambda$ with endpoints in $\Lambda_0(L_0)$ and such that each intersection of l with $\Lambda_{\geq 1}(L_0)$ occurs at a point of $\Lambda_1(L_0)$ and it is *transversal* to $\Lambda_1(L_0)$, the integer number corresponding to the homology class of l is equal to the number of *positive intersections* minus the number of *negative intersections* of l with $\Lambda_1(L_0)$.

The geometry of the symplectic group, its Lie algebra and the Lagrangian Grassmannian is very well known in the classical literature. A complete and elementary exposition of the results presented in Section 3 and in 4 can be found in References [16, 19].

5. Symplectic systems

A *symplectic differential system* in $\mathbb{R}^n \oplus \mathbb{R}^{n*}$ is a linear homogeneous system of differential equations of the form:

$$\begin{cases} v'(t) = A(t)\,v(t) + B(t)\,\alpha(t), \\ \alpha'(t) = C(t)\,v(t) - A^*(t)\,\alpha(t), \end{cases} \qquad v : [a,b] \to \mathbb{R}^n,\ \alpha : [a,b] \to \mathbb{R}^{n*} \tag{5.1}$$

where

$$X = \begin{pmatrix} A & B \\ C & -A^* \end{pmatrix} : [a,b] \longrightarrow Sp(2n, \mathbb{R}) \tag{5.2}$$

is a smooth curve in the Lie algebra $Sp(2n,\mathbb{R})$ such that $B(t)$ is *nonsingular* for all $t \in [a,b]$. Identifying $B(t)$ with a symmetric bilinear form on $\mathbb{R}^{n*}$, the assumption of nonsingularity means that $B(t)$ is *nondegenerate* for all t; in this case $B(t)^{-1}$ is identified with a symmetric bilinear form on $\mathbb{R}^n$.

Recall that the *index* (resp., the *co-index*) of a symmetric bilinear form $\mathfrak{b}$ on a vector space V, denoted by $n_-(\mathfrak{b})$ (resp., $n_+(\mathfrak{b})$) is the dimension of a maximal subspace $W \subset V$ such that $\mathfrak{b}|_{W\times W}$ is negative (resp., positive) definite; if either the index or the co-index of $\mathfrak{b}$ is finite, we define the *signature* of $\mathfrak{b}$ to be the difference $sgn(\mathfrak{b}) = n_+(\mathfrak{b}) - n_-(\mathfrak{b})$. Since $B(t)$ is nondegenerate for all $t \in [a,b]$, the function $n_-(B(t))$ is constant in $[a,b]$, and so there exists a non negative integer k_0, $0 \leq k_0 \leq n$, with:

$$n_-(B(t)) = n_-(B(t)^{-1}) = k_0, \quad \forall t \in [a,b];$$

we then say that (5.1) is a symplectic differential system *of index* k_0.
We denote by $\Phi : [a,b] \to \mathrm{GL}(\mathbb{R}^n \oplus \mathbb{R}^{n*})$ the *fundamental matrix* of the system (5.1), i.e., $\Phi(t)(v_0, \alpha_0) = (v(t), \alpha(t))$, where $(v(t), \alpha(t))$ is the unique solution of (5.1) satisfying $(v(a), \alpha(a)) = (v_0, \alpha_0)$. Then Φ satisfies:

$$\Phi' = X\Phi, \quad \Phi(a) = \mathrm{Id},$$

where Id is the identity map of $\mathbb{R}^n \oplus \mathbb{R}^{n*}$; since X takes values in the Lie algebra $Sp(2n, \mathbb{R})$, Φ takes values in the symplectic group $Sp(2n, \mathbb{R})$.

Let $\ell_0 \in \Lambda$ be a Lagrangian subspace of $\mathbb{R}^n \oplus \mathbb{R}^{n*}$; we will consider an initial condition for (5.1) of the form:

$$(v(a), \alpha(a)) \in \ell_0; \tag{5.3}$$

if (P, S) is the pair corresponding to ℓ_0 as in 4.1, then (5.3) is equivalent to:

$$v(a) \in P, \quad \alpha(a)|_P + S(v(a), \cdot) = 0. \tag{5.4}$$

We will say that a map $(v, \alpha) : [a, b] \to \mathbb{R}^n \oplus \mathbb{R}^{n*}$ is a *solution of* X if it satisfies the symplectic system (5.1), and that it is a *solution of* (X, ℓ_0) if it is a solution of X that satisfies the initial condition (5.3). We say that the initial condition (5.3) (or equivalently (5.4)) is *nondegenerate* if the bilinear form $B(a)^{-1}$ is nondegenerate on the space $P \subset \mathbb{R}^n$; this is equivalent to the bilinear form $B(a)$ being nondegenerate on the *annihilator* $P^o \subset \mathbb{R}^{n*}$ of P in $\mathbb{R}^{n*}$.

There is a natural notion of *equivalence* in the class of symplectic systems. We denote by $Sp(2n, \mathbb{R}; L_0)$ the closed subgroup of $Sp(2n, \mathbb{R})$ consisting of those symplectomorphisms ϕ_0 such that $\phi_0(L_0) = L_0$. If $X, \widetilde{X} : [a, b] \to Sp(2n, \mathbb{R})$ are coefficients maps for symplectic differential systems, we say that these systems are *isomorphic* if there exists a smooth map $\phi_0 : [a, b] \to Sp(2n, \mathbb{R}; L_0)$ such that:

$$\widetilde{X} = \phi_0' \phi_0^{-1} + \phi_0 X \phi_0^{-1}. \tag{5.5}$$

It is easy to see that if Φ and $\widetilde{\Phi}$ denote respectively the fundamental matrices of the symplectic systems with coefficient matrices X and $\widetilde{X}$ respectively related by (5.5), then we have:

$$\widetilde{\Phi}(t) = \phi_0(t)\Phi(t)\phi_0(a)^{-1}. \tag{5.6}$$

Given isomorphic symplectic systems with coefficient matrices X and $\widetilde{X}$ related by (5.5) and given Lagrangian subspaces ℓ_0 and $\widetilde{\ell}_0$ that determine initial conditions for the two systems, we say that also the initial conditions for the two systems are isomorphic if $\widetilde{\ell}_0 = \phi_0(a)(\ell_0)$.

The map ϕ_0 should be thought of as a "change of variables" map, i.e., it takes solutions $(v(t), \alpha(t))$ of X to solutions $(\tilde{v}(t), \tilde{\alpha}(t)) = \phi_0(t)(v(t), \alpha(t))$ of $\widetilde{X}$; the condition $\phi_0(L_0) = L_0$ is needed in order to ensure that X and $\widetilde{X}$ have the same conjugate instants with respect to isomorphic initial conditions.

5.1. **Example.** Let $(M, \mathfrak{g})$ be a semi-Riemannian manifold with $\dim(M) = n$ and with $\mathfrak{g}$ a metric tensor of index $k_0 \leq n$. Let $\gamma : [a, b] \to M$ be a geodesic in $(M, \mathfrak{g})$ and $\mathcal{P} \subset M$ be a submanifold with $\gamma(a) \in \mathcal{P}$ and $\dot\gamma(a) \in T_{\gamma(a)}\mathcal{P}^\perp$. Denote by ∇ the covariant derivative of the Levi-Civita connection of $\mathfrak{g}$ and by $\mathcal{R}$ the curvature tensor of $\mathfrak{g}$ chosen with the following sign convention: $\mathcal{R}(X, Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}$.

Let $\mathcal{S}_{\dot\gamma(a)}$ be the *second fundamental form* of $\mathcal{P}$ at $\gamma(a)$ in the normal direction $\dot\gamma(a)$. A $\mathcal{P}$-Jacobi field along γ is a smooth vector field V along γ that satisfies the *Jacobi equation*:

$$-V'' + \mathcal{R}(\dot\gamma, V)\dot\gamma = 0, \tag{5.7}$$

(here prime means covariant derivative along γ) and satisfying the initial conditions:

$$V(a) \in T_{\gamma(a)}\mathcal{P}, \quad \mathfrak{g}(V'(a), \cdot)|_{T_{\gamma(a)}\mathcal{P}} + \mathcal{S}_{\dot\gamma(a)}(V(a), \cdot) = 0. \tag{5.8}$$

Using a *parallel trivialization* of TM along γ we obtain an identification $T_{\gamma(t)}M \cong \mathbb{R}^n$ for all t and a bijection between vector fields V along γ and maps $v : [a, b] \to \mathbb{R}^n$ in such a way that the covariant derivative V' corresponds to standard derivative v'. The space $T_{\gamma(a)}\mathcal{P}$ is identified with a subspace $P \subset \mathbb{R}^n$, the second fundamental form $\mathcal{S}_{\dot\gamma(a)}$ with a symmetric bilinear form $S : P \times P \to \mathbb{R}$, the metric $\mathfrak{g}$ with a *constant* symmetric bilinear form $g : \mathbb{R}^n \to \mathbb{R}^{n*}$ in $\mathbb{R}^n$ of index k_0, and the curvature tensor $\mathcal{R}(\dot\gamma(t), \cdot)\,\dot\gamma(t)$ with a smooth curve $R(t)$ of g-symmetric endomorphisms of $\mathbb{R}^n$. The Jacobi equation (5.7) turns into the second order equation $v''(t) = R(t)\,v(t)$, which is equivalent to the system in $\mathbb{R}^n \oplus \mathbb{R}^{n*}$:

$$\begin{cases} v'(t) = g^{-1}\alpha(t), \\ \alpha'(t) = gR(t)\,v(t) \end{cases} \tag{5.9}$$

and the initial conditions (5.8) are given by (5.4). Clearly, (5.9) is a symplectic differential system with $A = 0$, $C(t) = gR(t)$ (which is symmetric because $R(t)$ is g-symmetric) and $B(t)$ is the constant map g^{-1}. The nondegeneracy of $B(t)$ corresponds to the nondegeneracy of the metric tensor $\mathfrak{g}$; the initial condition determined by (5.8) is nondegenerate if the metric $\mathfrak{g}$ is nondegenerate on the tangent space $T_{\gamma(a)}\mathcal{P}$. Symplectic differential systems of the form (5.9), i.e., for which $A = 0$, are called *Morse–Sturm systems*.

Different parallel trivializations of TM along γ produce isomorphic symplectic systems and initial conditions. If one chooses non parallel trivializations, then the Jacobi equation (5.7) produces a more general symplectic system, i.e., its coefficient matrix does not necessarily have $A = 0$ and B constant.

It is also interesting to observe that a similar construction of a Morse–Sturm system corresponding to the Jacobi equation along a semi-Riemannian geodesic γ can be obtained by considering a parallel trivialization of the *normal bundle* $\dot\gamma^\perp$ along γ, or of the *quotient bundle* $\dot\gamma^\perp/\mathbb{R}\dot\gamma$ if γ is a lightlike geodesic.

5.2 Example. Let $\langle\cdot,\cdot\rangle$ denote the canonical inner product in $\mathbb{R}^n \oplus \mathbb{R}^{n*}$ and $J : \mathbb{R}^n \oplus \mathbb{R}^{n*} \to \mathbb{R}^n \oplus \mathbb{R}^{n*}$ be the canonical complex structure for which the following relation holds:

$$\omega = \langle J\cdot,\cdot\rangle.$$

Let $H = H(t,q,p)$ be a *regular* smooth Hamiltonian function, possibly time dependent, defined on an open subset $A \subset \mathbb{R} \times (\mathbb{R}^n \times \mathbb{R}^{n*})$; the Hamiltonian vector field $\vec{H}_t$ is defined by:

$$\vec{H}_t = -J\,\nabla H_t,$$

where $H_t = H(t,\cdot,\cdot)$ and ∇ is the usual gradient operator corresponding to $\langle\cdot,\cdot\rangle$. Recall that regularity for a Hamiltonian function means that the second derivative $\frac{\partial^2 H}{\partial p^2}$ is always nondegenerate. Let $\mathcal{P} \subset \mathbb{R}^n$ be a smooth submanifold and let $\Gamma = (q,p) : [a,b] \to \mathbb{R}^n \oplus \mathbb{R}^{n*}$ be an integral curve of $\vec{H}_t$, i.e., a solution of the *Hamilton equations* $\dot q = \frac{\partial H}{\partial p}$, $\dot p = -\frac{\partial H}{\partial q}$; assume that $\Gamma(a)$ is in the annihilator $T\mathcal{P}^o$ of $T\mathcal{P}$ in $\mathbb{R}^n \oplus \mathbb{R}^{n*}$. The *linearized Hamilton equations* along Γ are given by:

$$\rho'(t) = -J\,\mathrm{Hess}H_t(\Gamma(t))(\rho(t)); \tag{5.10}$$

setting $\rho = (v,\alpha)$, (5.10) can be written in the form of the symplectic system (5.1), with:

$$A = \frac{\partial^2 H}{\partial q \partial p}, \quad B = \frac{\partial^2 H}{\partial p^2}, \quad C = -\frac{\partial^2 H}{\partial q^2}.$$

The nondegeneracy assumption for the coefficient B is given by the regularity of H; the Lagrangian subspace ℓ_0 that gives the initial condition for the symplectic system (5.10) is given by the tangent space $T_{\Gamma(a)}(T\mathcal{P}^o)$.

More generally, symplectic differential systems are obtained as linearization of the Hamilton equation along the solution Γ of a possibly time-dependent Hamiltonian H on a symplectic manifold $(\mathfrak{M},\omega)$ by a *symplectic trivialization* of $T\mathfrak{M}$ along Γ compatible with a given Lagrangian distribution. Again, different trivializations of the tangent bundle $T\mathfrak{M}$ along Γ produce isomorphic symplectic systems.

As a matter of fact, the class of examples arising from the Jacobi equation along a semi-Riemannian geodesic is as general as possible:

5.3. **Proposition.** *Every symplectic differential system is isomorphic to a Morse–Sturm system with the block B constant. Moreover, any Morse–Sturm system with B constant and with nondegenerate initial condition ℓ_0 can be obtained by a parallel trivialization of the normal bundle $\dot\gamma^\perp$ (see Example 5.1) from the Jacobi equation along a geodesic γ starting orthogonally to a nondegenerate submanifold $\mathcal{P}$ of a semi-Riemannian manifold $(M,\mathfrak{g})$.*

The notion of symplectic system was introduced in [19], where the reader will find all the details on the material of this section. A proof of Proposition 5.3 can be found in [13, Section 3] or [16, Proposition 2.3.1].

6. Conjugate points, the focal index and the Maslov index

For the rest of the paper we will consider a fixed symplectic system (5.1) of index k_0 with a nondegenerate initial condition (5.3) (or, equivalently, (5.4)); denoting by X the coefficient matrix of the system, we will refer to the pair system/initial conditions with the symbol (X,ℓ_0).

If $v:[a,b]\to\mathbb{R}^n$ is an absolutely continuous map, denote by $\alpha_v:[a,b]\to\mathbb{R}^{n*}$ the map

$$(6.1)\qquad \alpha_v(t)=B(t)^{-1}\big(v'(t)-A(t)v(t)\big);$$

clearly, if (v,α) is a solution of (5.1) then $\alpha=\alpha_v$. Consider the following space:

$$\mathbb{V}=\big\{v\in C^2([a,b],\mathbb{R}^n):(v,\alpha_v)\text{ is a solution of }(X,\ell_0)\big\},$$

and, for all $t\in[a,b]$, set

$$\mathbb{V}[t]=\{v(t):v\in\mathbb{V}\};$$

a simple dimension counting argument shows that, for all $t\in[a,b]$, the annihilator $\mathbb{V}[t]^o$ is given by:

$$\mathbb{V}[t]^o=\{\alpha_v(t):v\in\mathbb{V},\ v(t)=0\}.$$

6.1. **Definition.** An instant $t\in\,]a,b]$ is said to be *conjugate*, or *focal* (it is customary to use the term "conjugate" in the case that the initial condition is given by the Lagrangian $\ell_0=L_0=\{0\}\oplus\mathbb{R}^{n*}$; this corresponds to the case of geodesics with *fixed* initial point in Example 5.1), for (X,ℓ_0) if there exist

$v \neq 0$ in $\mathbf{V}$ such that $v(t) = 0$, i.e., if $\mathbf{V}[t] \neq \mathbf{R}^n$. The *multiplicity* $mul(t)$ of a focal instant t is the dimension of the space of those $v \in \mathbf{V}$ such that $v(t) = 0$ or, equivalently, the codimension of $\mathbf{V}[t]$ in $\mathbf{R}^n$. The *signature* $sgn(t)$ of a focal instant t is the signature of the restriction of the symmetric bilinear form $B(t)$ to the annihilator $\mathbf{V}[t]^o$, or, equivalently, the signature of the restriction of $B(t)^{-1}$ to the $B(t)^{-1}$-orthogonal complement $\mathbf{V}[t]^{\perp}$ of $\mathbf{V}[t]$ in $\mathbf{R}^n$. For notational purposes, it is convenient to define the multiplicity and the signature of any $t \in \,]a,b]$ by setting $mul(t) = sgn(t) = 0$ if t is not focal. A focal instant t is said to be *nondegenerate* if such restriction is nondegenerate. Observe that $\mathbf{V}[a] = P$, and the nondegeneracy assumption for the initial condition ℓ_0 means that the initial instant $t = a$ is a nondegenerate focal point of the system. If (X, ℓ_0) admits only a finite number of focal instants, then we define the *focal index* $\mathrm{i}_{\mathrm{foc}}(X, \ell_0)$ of (X, ℓ_0) to be the sum of the signatures of all the focal points:

$$\mathrm{i}_{\mathrm{foc}}(X, \ell_0) = \sum_{t \in]a,b]} sgn(t).$$

In Example 5.1, the set $\mathbf{V}$ corresponds to the space of $\mathcal{P}$-*Jacobi fields* along γ; an instant $t \in \,]a,b]$ is focal iff the corresponding point $\gamma(t)$ is a $\mathcal{P}$-*focal point* along the geodesic γ, in which case the multiplicity of t as a focal instant of (X, ℓ_0) coincides with multiplicity of $\gamma(t)$ as a $\mathcal{P}$-focal point. The signature of the focal instant t coincides with the signature of the restriction of the metric tensor g to the subspace of $T_{\gamma(t)}\mathcal{M}$ which is the g-orthogonal complement of the space given by the evaluation at t of all the $\mathcal{P}$-Jacobi fields along γ. Clearly, if g is *Riemannian* (i.e., positive definite), then every focal instant is nondegenerate, and its multiplicity equals its signature; hence, the focal index of a Riemannian geodesic coincides with the *geometric index* of the geodesic, which is defined as the sum of the multiplicities of all the $\mathcal{P}$-focal points along γ. It is not hard, although somewhat more involved, to prove that the same conclusions hold also in the case that the metric g is *Lorentzian* (i.e., $n_-(g) = 1$) and γ is a *causal geodesic* (i.e., timelike or lightlike).

There are many situations where the number of focal instants is finite; for instance, this is always the case if the coefficient matrix $X(t)$ is *real analytic* in t. Nondegenerate focal instants are *isolated*; similarly, the initial instant $t = a$ is isolated as a focal instant. In particular, if (X, ℓ_0) admits only nondegenerate focal points then their number is finite. This is the case of Riemannian or causal Lorentzian geodesics.

Let us now consider the fundamental matrix Φ of our symplectic system; since $\Phi(t) \in Sp(2n, \mathbf{R})$, it follows that $\Phi(t)(\ell_0)$ is a Lagrangian subspace of $\mathbf{R}^n \oplus \mathbf{R}^{n*}$ for all t. We therefore get a smooth curve $t \mapsto \ell(t) = \Phi(t)(\ell_0)$ in the Lagrangian

Grassmannian Λ.

It is easily seen that the focal instants of (X, ℓ_0) correspond to *intersections* of the curve ℓ with the subvariety $\Lambda_{\geq 1}(L_0)$. More precisely, an instant $t \in]a, b]$ is focal of multiplicity k iff $\ell(t) \in \Lambda_k(L_0)$; the signature of t is equal to the signature of the restriction of $\ell'(t)$ to $\ell(t) \cap L_0$ (here we are identifying tangent vectors in $T_{\ell(t)}\Lambda$ with symmetric bilinear forms on $\ell(t)$), and the nondegeneracy of the focal instant t is equivalent to the nondegeneracy of such restriction.

As we have observed, the instant $t = a$ is an isolated intersection of ℓ with $\Lambda_{\geq 1}(L_0)$, hence there exists $\varepsilon > 0$ such that $\ell|_{]a, a+\varepsilon]}$ has no intersection with $\Lambda_{\geq 1}(L_0)$. Moreover, if $t = b$ is not a focal instant, then $\ell(b) \notin \Lambda_{\geq 1}(L_0)$. We can therefore give the following definition:

6.2. Definition. Suppose that $t = b$ is not a focal instant for the pair (X, ℓ_0); then, the *Maslov index* $\mathrm{i}_{\mathrm{Maslov}}(X, \ell_0)$ is defined to be the integer number corresponding to the relative homology class of the curve $\ell|_{[a+\varepsilon, b]}$ in $H_1(\Lambda, \Lambda_{\geq 1}(L_0))$ by the isomorphism (4.4).

The definition of Maslov index can obviously be extended to the case that $t = b$ is a nondegenerate (hence isolated) focal instant. By standard arguments in Differential Topology, it is not hard to show that every continuous curve in Λ with endpoints in $\Lambda_0(L_0)$ is homotopic in $\Lambda_{\geq 1}(L_0)$ to a smooth curve whose intersections with $\Lambda_{\geq 1}(L_0)$ occur only at points of $\Lambda_1(L_0)$, and such that these intersections are always transversal to $\Lambda_1(L_0)$. For these curves, the homology class is computed easily as the difference between the number of positive and the number of negative intersections with $\Lambda_1(L_0)$. More generally, the Maslov index of pairs (X, ℓ_0) that have only nondegenerate focal instants can be computed as follows:

6.3. Theorem. *Suppose that $t = b$ is not a focal instant of (X, ℓ_0) and that all the focal instants of (X, ℓ_0) are nondegenerate. Then the Maslov index and the focal index of (X, ℓ_0) coincide.*

The proofs and many details concerning the material of this section are contained in references [16, 19, 24]. A counterexample to the equality $\mathrm{i}_{\mathrm{Maslov}}(X, \ell_0) = \mathrm{i}_{\mathrm{foc}}(X, \ell_0)$ for a system having exactly one degenerate focal instant is given in [16]. Finally, it is important to remark that the notions of focal instants, multiplicity, signature, focal index and Maslov index are preserved by isomorphisms of the pair (X, ℓ_0).

7. The Maslov index and the index form

Recall that we are denoting by A, B and C the $n \times n$ blocks of X as in (5.2), by k_0 the index of B and by (P, S) the pair corresponding to the Lagrangian ℓ_0 as in (4.1). We denote by $H^1([a,b],\mathbb{R}^n)$ the Sobolev space of all absolutely continuous $\mathbb{R}^n$-valued maps on $[a,b]$ having square-integrable derivative; let $\mathcal{H}$ be the Hilbert subspace of $H^1([a,b],\mathbb{R}^n)$ given by:

$$\mathcal{H} = \{v \in H^1([a,b],\mathbb{R}^n) : v(a) \in P,\ v(b) = 0\}. \tag{7.1}$$

7.1. **Definition** The *index form* I of (X, ℓ_0) is the following bounded symmetric bilinear form on $\mathcal{H}$:

$$I(v,w) = \int_a^b \Big[B(t)\big(\alpha_v(t), \alpha_w(t)\big) + C(t)\big(v(t), w(t)\big)\Big]\, \mathrm{d}t - S\big(v(a), w(a)\big), \tag{7.2}$$

where α_v and α_w are defined by (6.1).

For instance, the index form of a Morse–Sturm system arising from the Jacobi equation along a semi-Riemannian geodesic as in Example 5.1 is given explicitly by:

$$I(V,W) = \int_a^b \mathfrak{g}(V', W') + \mathfrak{g}(\mathcal{R}(\dot\gamma, V)\dot\gamma, W)\, \mathrm{d}t - S_{\gamma(a)}(V(a), W(a)),$$

which is the well known index form of a geodesic, i.e., the second variation of the semi-Riemannian action functional. More generally, if the symplectic system is obtained as in Example 5.2 from the linearized Hamilton equations along the solution of a *hyper-regular* Hamiltonian H on the cotangent bundle TM^* of some manifold M, then I corresponds to the second variation of the associated Lagrangian action functional.

The well known Morse index theorem states that for Riemannian or causal Lorentzian geodesics, the index of I on $\mathcal{H}$ is finite, and it is equal to the number of focal points counted with multiplicity, which is also equal to the Maslov index of the corresponding Morse–Sturm system. If $B(t)$ is neither positive nor negative definite, then the index and the co-index of I in $\mathcal{H}$ is infinite:

7.2 **Proposition.** The index (resp., the co-index) of I in $\mathcal{H}$ is finite iff $k_0 = 0$ (resp., iff $k_0 = n$).

7.3. **An Index Theorem.** A relation between the Maslov index $\mathrm{i}_{\mathrm{Maslov}}(X, \ell_0)$ and the index form I in the case $0 < k_0 < n$ is to be found by considering *restrictions* of I to suitable subspaces of $\mathcal{H}$. For this construction, we now choose a smooth family $\mathcal{D} = \{\mathcal{D}_t\}_{t \in [a,b]}$ of k_0-dimensional subspaces of $\mathbb{R}^n$ with the property that $B(t)^{-1}$ is negative definite on $\mathcal{D}_t \times \mathcal{D}_t$ for all $t \in [a,b]$. By smooth, we mean that there exist smooth maps $Y_1, \ldots, Y_{k_0} : [a,b] \to \mathbb{R}^n$ such that $Y_1(t), \ldots, Y_{k_0}(t)$ is a basis of $\mathcal{D}_t$ for all t; we will call such a family $Y_1, \ldots, Y_{k_0}$ a *frame* of $\mathcal{D}$.

We now define two closed subspaces of $\mathcal{H}$, denoted by $\mathcal{K}^{\mathcal{D}}$ and $\mathcal{S}^{\mathcal{D}}$, as follows. The space $\mathcal{S}^{\mathcal{D}}$ is given by those maps in $\mathcal{H}$ "taking values in $\mathcal{D}$":

$$\mathcal{S}^{\mathcal{D}} = \{v \in \mathcal{H} : v(a) = 0,\ v(t) \in \mathcal{D}_t \ \text{ for all } t \in [a,b]\}; \tag{7.3}$$

the space $\mathcal{K}^{\mathcal{D}}$ is given by the "solutions of (X, ℓ_0) in the directions of $\mathcal{D}$", and it is more conveniently defined with the help of a frame $Y_1, \ldots, Y_{k_0}$ of $\mathcal{D}$:

$$\begin{aligned} \mathcal{K}^{\mathcal{D}} = \{v \in \mathcal{H} : \alpha_v(Y_i) \in H^1([a,b],\mathbb{R}) \text{ and } \\ \alpha_v(Y_i)' = B(\alpha_v, \alpha_{Y_i}) + C(v, Y_i), \quad \forall i = 1, \ldots, k_0\}. \end{aligned} \tag{7.4}$$

Indeed, it is not hard to prove that the above definition of the space $\mathcal{K}^{\mathcal{D}}$ does *not* depend on the choice of the frame $Y_1, \ldots, Y_{k_0}$. Observe that if $v \in \mathcal{H}$ is of Sobolev class H^2, then $v \in \mathcal{K}^{\mathcal{D}}$ iff it satisfies:

$$\alpha_v' - Cv + A^*\alpha_v \perp \mathcal{D},$$

which is the reason why elements in $\mathcal{K}^{\mathcal{D}}$ are considered "solutions in the directions of $\mathcal{D}$".

It is easy to see that the spaces $\mathcal{K}^{\mathcal{D}}$ and $\mathcal{S}^{\mathcal{D}}$ span a finite codimensional closed subspace of $\mathcal{H}$; however, their sum may fail to be direct because the intersection $\mathcal{K}^{\mathcal{D}} \cap \mathcal{S}^{\mathcal{D}}$ may be non zero. In fact, using a frame $Y_1, \ldots, Y_{k_0}$ the maps in such intersection can be described as the solutions of a symplectic system in $\mathbb{R}^{k_0}$ vanishing at the endpoints of the interval $[a,b]$. Such system is called the ***reduced symplectic system*** determined by the frame $Y_1, \ldots, Y_{k_0}$ and it has index 0 (i.e., its upper right block is positive definite); different frames produce isomorphic reduced symplectic systems, and one has $\mathcal{H} = \mathcal{K}^{\mathcal{D}} \oplus \mathcal{S}^{\mathcal{D}}$ precisely when $t = b$ is not a conjugate instant of the reduced symplectic system.

7.3. **Definition.** Given a nondegenerate smooth family of subspaces $\mathcal{D}$ for a symplectic differential system X and given a frame $(Y_i)_{i=1}^r$ for $\mathcal{D}$ we define the ***reduced symplectic differential system*** corresponding to X, $\mathcal{D}$ and $(Y_i)_{i=1}^r$ to be the following symplectic differential system in $\mathbb{R}^r$:

$$\begin{cases} f' = -(\mathfrak{B}^{-1} \circ \mathcal{A})f + \mathfrak{B}^{-1}\varphi, \\ \varphi' = (\mathcal{C} - \mathcal{A}^* \circ \mathfrak{B}^{-1} \circ \mathcal{A})f + (\mathcal{A}^* \circ \mathfrak{B}^{-1})\varphi, \end{cases} \tag{7.5}$$

where $\mathcal{A}(t), \mathfrak{B}(t), \mathcal{C}(t) \in \mathrm{Lin}(\mathbb{R}^r, \mathbb{R}^{r*})$ are the linear operators represented by the following matrices:

$$\mathfrak{B}_{ij}\ (t) = B(t)^{-1}(Y_i(t), Y_j(t)), \qquad \mathcal{A}_{ij}(t) = \alpha_{Y_j}(t)Y_i(t) \tag{7.6}$$

$$C_{ij}(t) = B(t)(\alpha_{Y_i}(t), \alpha_{Y_j}(t)) + C(t)(Y_i(t), Y_j(t)).$$

The fact that $\mathcal{D}$ is nondegenerate for X implies that $\mathfrak{B}(t)$ is indeed nondegenerate, so that $\mathfrak{B}(t)^{-1}$ in (7.5) makes sense. Moreover, the index of the reduced system (7.5) equals the index of $\mathcal{D}$ with respect to X. We will denote by X_{red} the coefficient matrix of (7.5) and by A_{red}, B_{red}, C_{red} the coefficients of X_{red}.

It is interesting to observe that the index form of the reduced system can be identified with the restriction of $-I$ to $S^{\mathcal{D}} \times S^{\mathcal{D}}$. Since the conjugate instants of a symplectic system of index 0 are isolated, we can say that $\mathcal{H}$ is *direct* sum of $\mathcal{K}^{\mathcal{D}}$ and $S^{\mathcal{D}}$ under *generic* circumstances.

7.4. *Remark*. In some situations it is useful to consider the symplectic differential system $\tilde{X}_{red}$ which is isomorphic to X_{red} and whose coefficients $\tilde{A}_{red}$, $\tilde{B}_{red}$, $\tilde{C}_{red}$ are given by:

$$\tilde{A}_{red}(t) = -\mathfrak{B}(t)^{-1} \circ \mathcal{A}_{ant}(t), \quad \tilde{B}_{red} = \mathfrak{B}(t)^{-1}, \tag{7.7}$$

$$\tilde{C}_{red}(t) = \mathcal{C}(t) - \mathcal{A}'_{sym}(t) + \mathcal{A}_{ant}(t) \circ \mathfrak{B}(t)^{-1} \circ \mathcal{A}_{ant}(t),$$

for all $t \in [a, b]$, where $\mathcal{A}_{sym}$, $\mathcal{A}_{ant}$ denote respectively the symmetric and anti-symmetric components of $\mathcal{A}$:

$$\mathcal{A}_{sym}(t) = \tfrac{\mathcal{A}(t)+\mathcal{A}(t)^*}{2}, \quad \mathcal{A}_{ant}(t) = \tfrac{\mathcal{A}(t)-\mathcal{A}(t)^*}{2}. \tag{7.8}$$

7.5. *Remark*. If a nondegenerate smooth family of subspaces $\mathcal{D}$ for a symplectic differential system X admits a frame $(Y_i)_{i=1}^r$ consisting of *solutions* of X satisfying the symmetry condition

$$\alpha_{Y_i}(Y_j) = \alpha_{Y_j}(Y_i), \quad i, j = 1, \ldots, r,$$

then the coefficients of the reduced symplectic system $\tilde{X}_{red}$ defined in Remark 7.4 are $\tilde{A}_{red} = 0$, $\tilde{B}_{red} = \mathfrak{B}^{-1}$, $\tilde{C}_{red} = 0$. The system $\tilde{X}_{red}$ becomes the differential equation

$$\mathfrak{B} f' \equiv \text{constant}. \tag{7.9}$$

An instant $t \in [a, b]$ is conjugate for $\tilde{X}_{red}$ iff the integral:

$$\mathbf{B}^f(t) = \textstyle\int_a^t \mathfrak{B}(s)^{-1}\, ds \tag{7.10}$$

is a degenerate (symmetric) bilinear form in $\mathbb{R}^{r*}$, in which case the multiplicity of t equals the degeneracy of $\mathbf{B}^f(t)$. If $t = b$ is not conjugate for $\tilde{X}_{red}$ then the Maslov index of $\tilde{X}_{red}$ is given by:

$$\mathrm{iMaslov}(\tilde{\mathrm{X}}_{red}) = \textstyle\sum_{t \in]a,b[} \mathrm{sgn}\Big(\mathfrak{B}(t)|_{\mathrm{Im}(\mathbf{B}^f(t))^\perp}\Big), \tag{7.11}$$

provided that $\mathfrak{B}(t)$ is nondegenerate on the image of $\mathbf{B}(t)$ for those $t \in]a,b]$ such that $\mathbf{B}^f(t)$ is degenerate. In (7.11) we have denoted by $\perp$ the orthogonal complement with respect to $\mathfrak{B}(t)$.

7.6 Theorem. *For any choice of the family $\mathcal{D}$, the index form I has finite index in $\mathcal{K}^{\mathcal{D}}$ and finite co-index in $\mathcal{S}^{\mathcal{D}}$. Moreover, if $t = b$ is not focal the following equality holds:*

$$\text{(7.12)}\qquad \mathrm{i}_{\mathrm{Maslov}}(X,\ell_0) = n_-\big(I|_{\mathcal{K}^{\mathcal{D}}\times\mathcal{K}^{\mathcal{D}}}\big) - n_+\big(I|_{\mathcal{S}^{\mathcal{D}}\times\mathcal{S}^{\mathcal{D}}}\big) - n_-\big(B(a)^{-1}|_{P\times P}\big).$$

Observe that, even though the right side of the equality (7.12) does not depend on the choice of $\mathcal{D}$, the two terms $n_-\big(I|_{\mathcal{K}^{\mathcal{D}}\times\mathcal{K}^{\mathcal{D}}}\big)$ and $n_+\big(I|_{\mathcal{S}^{\mathcal{D}}\times\mathcal{S}^{\mathcal{D}}}\big)$ may indeed depend on $\mathcal{D}$.

The result of Theorem 7.6 is announced in [20]; a proof that holds in the case that the reduced symplectic system does not have focal instants can be found in [19]. The proof of the general case can be found in [24]. A different index theorem, whose statement is somewhat similar to that of Theorem 7.6, is stated in [13, Theorem 7.1]. The result of Theorem 7.6 can also be extended to the case of a solution with a Lagrangian boundary condition at both endpoints (see for instance [18, 19]).

Let us now make a few comments on the meaning of the result of Theorem 7.6.

If the symplectic system has index 0, then clearly there is no choice of $\mathcal{D}$ to be done, $\mathcal{S}^{\mathcal{D}} = \{0\}$, $\mathcal{K}^{\mathcal{D}} = \mathcal{H}$ and $n_-\big(B(a)^{-1}|_{P\times P}\big) = 0$. Moreover, as we have observed the Maslov index coincides with the number of focal instants counted with multiplicity, and the equality (7.12) gives us back the classical Morse index theorem for Riemannian geodesics and for solutions of convex Hamiltonian systems.

Let us consider now the case of a Morse–Sturm system of index 1 arising from the Jacobi equation along a timelike geodesic γ in a Lorentzian manifold $(\mathcal{M},\mathfrak{g})$. Let us consider the family $\mathcal{D}$ in such a way that $\mathcal{D}_t$ is the unidimensional subspace of $\mathbb{R}^n$ corresponding to the space $\mathbb{R}\cdot\dot\gamma(t)$ through the parallel trivialization of $T\mathcal{M}$ along γ. In this case, it is easy to see that $\mathcal{S}^{\mathcal{D}}$ corresponds to the space of vector fields along γ that are everywhere tangent to γ, and $\mathcal{K}^{\mathcal{D}}$ corresponds to the space of vector fields along γ that are everywhere orthogonal to γ. Moreover, since the initial submanifold $\mathcal{P}$ is orthogonal to γ, hence $\mathcal{P}$ is a spacelike submanifold, it is $n_-\big(B(a)^{-1}|_{P\times P}\big) = 0$; also, one easily computes $n_+\big(I|_{\mathcal{S}^{\mathcal{D}}\times\mathcal{S}^{\mathcal{D}}}\big) = 0$. Again, the Maslov index of the Morse–Sturm system is given by the number of focal instants counted with multiplicity, and the equality (7.12) gives us the Timelike Morse Index Theorem of Beem and Ehrlich [3, Theorem 10.27]. A similar construction allows to obtain from Theorem 7.6 also the *lightlike* Morse index theorem for

Lorentzian manifolds.

Let us consider now the case of a geodesic γ of any causal character in a Lorentzian manifold $(\mathcal{M}, \mathfrak{g})$ such that there exists a timelike Jacobi field Y along γ. If one takes $\mathcal{D}_t$ to be the one dimensional space generated by $Y(t)$, then it is not hard to prove that the index form I is negative semidefinite on $\mathcal{S}^{\mathcal{D}}$, and so $n_+(I|_{\mathcal{S}^{\mathcal{D}}\times\mathcal{S}^{\mathcal{D}}}) = 0$. If one consider the case of fixed endpoints, i.e., $P = \{0\}$, then from (7.12) we obtain that the Maslov index is equal to the index of the restriction of I to $\mathcal{K}^{\mathcal{D}}$. This fact has an interesting geometrical interpretation in the case that Y is the restriction of a (locally defined) timelike Killing vector field. Namely, in this case $\mathcal{K}^{\mathcal{D}}$ is the space of variational vector fields along γ that correspond to variations $\{\gamma_s\}_{s\in]-\epsilon,\epsilon[}$ of γ made of curves γ_s that are "geodesics in the direction of Y", i.e., satisfying the *conservation law* $\frac{d}{dt}\mathfrak{g}(\dot\gamma_s, Y) = \mathfrak{g}(\nabla_{\dot\gamma_s}\dot\gamma_s, Y) = 0$. This idea has been exploited in [11], where the authors develop an infinite dimensional Morse theory for geodesics in a stationary Lorentzian manifold.

7.2. **A Generalized Index Theorem.** We will now consider the following setup:

- (X, ℓ_0) is a symplectic differential system with initial data on $\mathbb{R}^n$ over the interval $[a, b]$;
- $\mathcal{D}$ and Δ are nondegenerate smooth families of subspaces for X with $\Delta_t \subset \mathcal{D}_t$ for all t;
- $(Y_i)_{i=1}^r$ is a frame for $\mathcal{D}$ such that $(Y_i)_{i=1}^k$ is a frame for Δ;
- X_{red} is the reduced symplectic system corresponding to $\mathcal{D}$ and $(Y_i)_{i=1}^r$;
- $I : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ is the index form of (X, ℓ_0) and $\mathcal{K}^{\mathcal{D}}$, $\mathcal{S}^{\mathcal{D}}$ (resp., $\mathcal{K}^{\Delta}$, $\mathcal{S}^{\Delta}$) are the subspaces of $\mathcal{H}$ defined in analogy with (7.4) and (7.3) for (X, ℓ_0) and $\mathcal{D}$ (resp., Δ);
- $I_{red} : \mathcal{H}_{red} \times \mathcal{H}_{red} \to \mathbb{R}$ is the index form of X_{red};
- $\lambda : \mathcal{H}_{red} \to \mathcal{S}^{\mathcal{D}}$ is the continuous isomorphism defined by:

(7.13) $$\lambda : \mathcal{H}_{red} \ni f = (f_i)_{i=1}^r \longmapsto \sum_{i=1}^r f_i Y_i \in \mathcal{S}^{\mathcal{D}}$$

- Δ_{red} is the (constant) smooth family of subspaces $\Delta_{red} \equiv \mathbb{R}^k \oplus \{0\} \subset \mathbb{R}^r$ in $\mathbb{R}^r$ over the interval $[a, b]$;
- $\mathcal{K}^{\Delta_{red}}$ and $\mathcal{S}^{\Delta_{red}}$ are the subspaces of $\mathcal{H}_{red}$ defined in analogy with (7.4) and (7.3) for the symplectic differential system with initial data $(X_{red}, \{0\}\oplus\mathbb{R}^{r*})$ relatively to the smooth family of subspaces Δ_{red};

The following facts are immediate:

(1) $\mathcal{K}^{\mathcal{D}} \subset \mathcal{K}^{\Delta}$ and $\mathcal{S}^{\Delta} \subset \mathcal{S}^{\mathcal{D}}$;
(2) $\lambda(\mathcal{S}^{\Delta_{red}}) = \mathcal{S}^{\Delta}$;
(3) Δ_{red} is a nondegenerate family of subspaces for X_{red}.

We prove the following preparatory lemma:

7.7. Lemma. *An absolutely continuous map $f : [a, b] \to \mathbb{R}^r$ is a solution of X_{red} along Δ_{red} iff $v = \sum_{i=1}^{r} f_i Y_i$ is a solution of X along Δ. In particular, $\lambda(\mathcal{K}^{\Delta_{red}}) = \mathcal{K}^{\Delta} \cap \mathcal{S}^{\mathcal{D}}$.*

Proof. The map f is a solution of X_{red} along Δ_{red} iff $(\mathfrak{B} f')_i$ is absolutely continuous for $i = 1, \dots, k$ and

$$[(\mathfrak{B} f' + \mathcal{A} f)_i]' = (\mathcal{C} f + \mathcal{A}^* f')_i, \quad i = 1, \dots, k.$$

It is easy to check that this is also the condition for v to be a solution of X along Δ.

We have the following generalized index theorem:

7.8. Theorem (generalized index theorem). *Consider a triple $(X, \ell_0, \mathcal{D})$ where (X, ℓ_0) is a symplectic differential system with initial data in $\mathbb{R}^n$ over the interval $[a, b]$ such that ℓ_0 defines a nondegenerate initial condition for X and $\mathcal{D}$ is a smooth family of subspaces in $\mathbb{R}^n$ over $[a, b]$ whose index with respect to X equals the index of X. Denote by X_{red} the reduced symplectic system corresponding to X and $\mathcal{D}$. If $t = b$ is neither a focal instant for (X, ℓ_0) nor a conjugate instant for X_{red} then:*

$$n_-(I|_{\mathcal{K}^{\mathcal{D}}}) = i_{\text{Maslov}}(X, \ell_0) - i_{\text{Maslov}}(X_{red}) + n_-(B(a)^{-1}|_P). \tag{7.14}$$

Proof. See [25, Theorem 3.2].

8. Geodesics in semi-Riemannian Manifolds

We will now go back to the setup of Example 5.1, and we consider the Morse–Sturm system arising from the Jacobi equation along a geodesic γ in a semi-Riemannian manifold $(M, \mathfrak{g})$.

Let $\mathcal{P} \subset M$ be a smooth submanifold with $\gamma(a) \in \mathcal{P}$ and $\gamma'(a) \in T_{\gamma(a)}\mathcal{P}^{\perp}$; a *$\mathcal{P}$-Jacobi field* along γ is a Jacobi field $\mathfrak{v}$ satisfying the initial conditions:

$$\mathfrak{v}(a) \in T_{\gamma(a)}\mathcal{P}, \quad \mathfrak{g}(\mathfrak{v}'(a), \cdot)|_{T_{\gamma(a)}\mathcal{P}} + \mathrm{II}_{\gamma'(a)}(\mathfrak{v}(a), \cdot) = 0 \in T_{\gamma(a)}\mathcal{P}^*, \tag{8.1}$$

where $\mathrm{II}_{\gamma'(a)} \in \mathit{Bil}_{sym}(T_{\gamma(a)}\mathcal{P})$ denotes the second fundamental form of $\mathcal{P}$ in the normal direction $\gamma'(a)$. If $P \subset \mathbb{R}^n$, $S \in \mathit{Bil}_{sym}(P)$ correspond to $T_{\gamma(a)}\mathcal{P}$ and $\mathrm{II}_{\gamma'(a)}$

by means of the chosen parallel trivialization of TM along γ then $\mathcal{P}$-Jacobi fields correspond to (X, ℓ_0)-solutions, where X is the Morse–Sturm system (5.9) and $\ell_0 \subset \mathbf{R}^n \oplus \mathbf{R}^{n*}$ is the Lagrangian (4.1). Also, the index form of the pair (X, ℓ_0) corresponds to the second variation of the *geodesic action functional*

$$E(z) = \tfrac{1}{2} \int_a^b \mathfrak{g}(z', z')\, dt \tag{8.2}$$

at the critical point γ. The domain of E is the Hilbert manifold $\Omega_{\mathcal{P}q}(M)$ consisting of H^1 curves $z : [a, b] \to M$ with $z(a) \in \mathcal{P}$, $z(b) = q$, where $q = \gamma(b)$. Recall that the critical points of E in $\Omega_{\mathcal{P}q}(M)$ are the geodesics starting orthogonally at $\mathcal{P}$ and ending at q.

The Lagrangian ℓ_0 defines a nondegenerate initial condition for X iff the submanifold $\mathcal{P}$ is nondegenerate at $\gamma(a)$, i.e., if $\mathfrak{g}$ is nondegenerate on $T_{\gamma(a)}\mathcal{P}$. Focal instants for (X, ℓ_0) correspond to $\mathcal{P}$-focal points along γ. The case where the initial submanifold $\mathcal{P}$ is a single point corresponds to the case where $\ell_0 = L_0 = \{0\} \oplus \mathbf{R}^{n*}$; in this case, the initial condition defined by ℓ_0 is always nondegenerate.

When $\mathcal{P}$ is nondegenerate at $\gamma(a)$ and $\gamma(b)$ is not $\mathcal{P}$-focal along γ then we can define the *Maslov index* $i_{\rm Maslov}(\gamma, \mathcal{P})$ of the geodesic γ with respect to the initial submanifold $\mathcal{P}$ to be the Maslov index of the pair (X, ℓ_0); the Maslov index of γ with respect to $\mathcal{P}$ does not depend on the parallel trivialization used to produce the pair (X, ℓ_0). When $\mathcal{P}$ is a single point we call $i_{\rm Maslov}(\gamma, \mathcal{P})$ the *Maslov index of* γ and we write simply $i_{\rm Maslov}(\gamma)$.

If $(\mathcal{Y}_i)_{i=1}^r$ are smooth vector fields along γ such that $(\mathcal{Y}_i(t))_{i=1}^r$ is the basis of a nondegenerate subspace of $T_{\gamma(t)}M$ for all $t \in [a, b]$ then the parallel trivialization along γ produce maps $Y_i : [a, b] \to \mathbf{R}^n$ which form a frame for a nondegenerate smooth family of subspaces for X. The operators $\mathcal{A}, \mathfrak{B}, \mathcal{C} \in Lin(\mathbf{R}^r, \mathbf{R}^{r*})$ which appear in the corresponding reduced symplectic system (7.5) are given by:

$$\mathfrak{B}_{ij} = \mathfrak{g}(\mathcal{Y}_i, \mathcal{Y}_j), \quad \mathcal{A}_{ij} = \mathfrak{g}(\mathcal{Y}_j', \mathcal{Y}_i), \tag{8.3}$$
$$\mathcal{C}_{ij} = \mathfrak{g}(\mathcal{Y}_i', \mathcal{Y}_j') + \mathfrak{g}(\mathcal{R}(\gamma', \mathcal{Y}_i)\gamma', \mathcal{Y}_j).$$

8.1. **Definition.** Let $\gamma : [a, b] \to M$ be a geodesic and $(\mathcal{Y}_i)_{i=1}^r$ smooth vector fields along γ such that $(\mathcal{Y}_i(t))_{i=1}^r$ is the basis of a nondegenerate subspace of $T_{\gamma(t)}M$ for all $t \in [a, b]$. Consider the symplectic differential system X_{red} defined in (7.5) with $\mathcal{A}$, $\mathfrak{B}$, $\mathcal{C}$ defined in (8.3). If $t = b$ is not conjugate for X_{red} then the *reduced Maslov index* of the geodesic γ (with respect to the fields $\mathcal{Y}_i$) is defined by:

$$i_{\rm Maslov}^{\rm red}(\gamma) = i_{\rm Maslov}(X_{\rm red}).$$

In this geometrical context, the Index Theorem 7.6 gives a generalized Morse index theorem for semi-Riemannian geodesics. Observe that the term $n_-(B(a)^{-1}|_P)$

appearing in equality (7.12) is the index of the metric $\mathfrak{g}$ in the tangent space $T_{\gamma(a)}\mathcal{P}$ of the initial submanifold.

8.1. **A variational principle for semi-Riemannian geodesics.** We now consider fixed an n-dimensional semi-Riemannian manifold $(M, \mathfrak{g})$ with metric tensor $\mathfrak{g}$ of index k, a smooth submanifold $\mathcal{P} \subset M$, a point $q \in M$ and smooth vector fields $(\mathcal{Y}_i)_{i=1}^r$ on M such that $\big(\mathcal{Y}_i(m)\big)_{i=1}^r$ is a basis for a nondegenerate subspace of T_mM for all $m \in M$. We say that an absolutely continuous curve $\gamma : [a, b] \to M$ is a *geodesic along the fields* $\mathcal{Y}_i$ if $\mathfrak{g}(\gamma', \mathcal{Y}_i)$ is absolutely continuous on $[a, b]$ and

$$\mathfrak{g}(\gamma', \mathcal{Y}_i)' = \mathfrak{g}(\gamma', \mathcal{Y}_i'),$$

for $i = 1, \ldots, r$. If γ is of class C^2 then γ is a geodesic along the fields $\mathcal{Y}_i$ iff γ'' is orthogonal to the distribution spanned by the $\mathcal{Y}_i$; in particular, if γ is a geodesic then γ is a geodesic along the fields $\mathcal{Y}_i$. Moreover, if the vector fields $\mathcal{Y}_i$ are Killing (recall that $\mathcal{Y}$ is a Killing vector field iff the bilinear form $\mathfrak{g}(\nabla\mathcal{Y}, \cdot)$ is skew-symmetric) then γ is a geodesic along the fields $\mathcal{Y}_i$ iff $\mathfrak{g}(\gamma', \mathcal{Y}_i)$ is constant for all $i = 1, \ldots, r$.

Consider the following subset of the Hilbert manifold $\Omega_{\mathcal{P}q}(M)$:

$$\mathcal{N}_{\mathcal{P}q}(M) = \big\{\gamma \in \Omega_{\mathcal{P}q}(M) : \gamma \text{ is a geodesic along the fields } \mathcal{Y}_i\big\}.$$

We are interested in determining conditions that imply that $\mathcal{N}_{\mathcal{P}q}(M)$ is a Hilbert submanifold of $\Omega_{\mathcal{P}q}(M)$ and that the critical points of the restriction of the geodesic action functional E to $\mathcal{N}_{\mathcal{P}q}(M)$ are the geodesics $\gamma : [a, b] \to M$ starting orthogonally to $\mathcal{P}$ and ending at q. These conditions are given in the following:

8.2. **Theorem.** *Let $\gamma \in \mathcal{N}_{\mathcal{P}q}(M)$ be fixed; consider the following homogeneous system of linear ODE's in $\mathbb{R}^r \oplus \mathbb{R}^{r*}$:*

$$\text{(8.4)} \qquad \begin{cases} f' = -(\mathfrak{B}^{-1} \circ \mathcal{A}) f + \mathfrak{B}^{-1}\varphi, \\ (\varphi + \mathcal{E}f)' = (\mathcal{C} + \overline{\mathcal{E}} - (\mathcal{A}^* + \mathcal{E}) \circ \mathfrak{B}^{-1} \circ \mathcal{A}) f + ((\mathcal{A}^* + \mathcal{E}) \circ \mathfrak{B}^{-1})\varphi, \end{cases}$$

where $\mathcal{A}, \mathfrak{B}, \mathcal{C} \in \mathrm{Lin}(\mathbb{R}^r, \mathbb{R}^{r*})$ are defined in (8.3) and $\mathcal{E}, \overline{\mathcal{E}} \in \mathrm{Lin}(\mathbb{R}^r, \mathbb{R}^{r*})$ are defined by:

$$\mathcal{E}_{ij} = \mathfrak{g}(\nabla_{\mathcal{Y}_j}\mathcal{Y}_i, \gamma'), \quad \overline{\mathcal{E}}_{ij} = \mathfrak{g}((\nabla_{\mathcal{Y}_j}\mathcal{Y}_i)', \gamma').$$

Assume that the system (8.4) does not admit a non zero solution (f, φ) with $f(a) = f(b) = 0$. Then:

(1) γ has a neighborhood $\mathfrak{V}$ in $\mathcal{N}_{\mathcal{P}_q}(M)$ which is a Hilbert submanifold of $\Omega_{\mathcal{P}_q}(M)$;
(2) γ is a critical point of $E|_{\mathfrak{V}}$ iff γ is a geodesic starting orthogonally at $\mathcal{P}$;
(3) if γ is a critical point of $E|_{\mathfrak{V}}$ then the degeneracy of the second variation of $E|_{\mathfrak{V}}$ at γ is equal to the multiplicity of $\gamma(b)$ as a $\mathcal{P}$-focal point along γ.

Assume that γ is a critical point of $E|_{\mathfrak{V}}$ such that $\gamma(b)$ is not $\mathcal{P}$-focal along γ and $\mathcal{P}$ is nondegenerate at $\gamma(a)$. If the index of $\mathfrak{g}$ restricted to the distribution spanned by $(\mathcal{Y}_i)_{i=1}^r$ equals the index of $\mathfrak{g}$ then the Morse index of $E|_{\mathfrak{V}}$ at γ is given by:

$$n_-(\mathrm{d}^2(\mathrm{E}|_{\mathfrak{V}})(\gamma)) = \mathrm{i}_{\mathrm{Maslov}}(\gamma) - \mathrm{i}^{\mathrm{red}}_{\mathrm{Maslov}}(\gamma) + \mathrm{n}_-\big(\mathfrak{g}|_{\mathrm{T}_{\gamma(a)}\mathcal{P}}\big). \tag{8.5}$$

Before getting into the proof of Theorem 8.2, which will take up almost entirely the rest of the subsection, we will make a few remarks about its statement. First we observe that if γ is a geodesic then $\overline{\mathcal{E}} = \mathcal{E}'$, so that (8.4) becomes the reduced symplectic system (7.5); in particular, the hypothesis of the theorem is satisfied precisely when $t = b$ is not a conjugate instant for the reduced symplectic system.

Let us look now at the particular case where each $\mathcal{Y}_i$ is a Killing vector field; this obviously implies that $\mathcal{E} = -\mathcal{A}^*$. Another remarkable equality that holds in this case is $\mathcal{C} = -\overline{\mathcal{E}}$; to prove it, recall that the *Hessian* of a vector field $\mathcal{Y}$ is the $(2,1)$-tensor field defined by $\mathrm{Hess}(\mathcal{Y}) = \nabla\nabla\mathcal{Y}$, i.e., $\mathrm{Hess}(\mathcal{Y})(\mathrm{V},\mathrm{W}) = \nabla_{\mathrm{V}}\nabla_{\mathrm{W}}\mathcal{Y} - \nabla_{\nabla_{\mathrm{V}}\mathrm{W}}\mathcal{Y}$. Observe that $\mathrm{Hess}(\mathcal{Y})(\mathrm{V},\mathrm{W}) - \mathrm{Hess}(\mathcal{Y})(\mathrm{W},\mathrm{V}) = \mathcal{R}(\mathrm{V},\mathrm{W})\mathcal{Y}$; moreover, if $\mathcal{Y}$ is Killing then $\mathfrak{g}(\mathrm{Hess}(\mathcal{Y})(\mathrm{V},\mathrm{W}),\mathrm{Z})$ is skew-symmetric in the variables W and Z, because $\mathfrak{g}(\nabla\mathcal{Y},\cdot)$ is skew-symmetric. Using all these formulas we compute:

$$\begin{aligned}\overline{\mathcal{E}}_{ij} &= \mathfrak{g}((\nabla_{\mathcal{Y}_j}\mathcal{Y}_i)',\gamma') = \mathfrak{g}(\mathrm{Hess}(\mathcal{Y}_i)(\gamma',\mathcal{Y}_j),\gamma') + \mathfrak{g}(\nabla_{\mathcal{Y}_j'}\mathcal{Y}_i,\gamma')\\ &= \mathfrak{g}(\mathcal{R}(\gamma',\mathcal{Y}_j)\mathcal{Y}_i,\gamma') + \mathfrak{g}(\nabla_{\mathcal{Y}_j'}\mathcal{Y}_i,\gamma') = -\mathcal{C}_{ij}.\end{aligned}$$

We have proven that if the fields $\mathcal{Y}_i$ are Killing then the system (8.4) becomes:

$$\mathfrak{B}f' + 2\mathcal{A}_{\mathrm{ant}}f \equiv \text{constant}.$$

Moreover, if the fields $\mathcal{Y}_i$ *commute*, i.e., $[\mathcal{Y}_i,\mathcal{Y}_j] = \nabla_{\mathcal{Y}_i}\mathcal{Y}_j - \nabla_{\mathcal{Y}_j}\mathcal{Y}_i = 0$ for all $i,j = 1,\dots,r$ then $\mathcal{A}$ is symmetric and (8.4) becomes (7.9).

Proof of Theorem 8.2. We start by considering the smooth map

$$\mathcal{F} : \Omega_{\mathcal{P}_q}(M) \longrightarrow L^2([a,b],\mathbb{R}^{r*})/\mathrm{Const} \tag{8.6}$$

defined by:

$$\mathcal{F}(\gamma)(t)_i = \mathfrak{g}(\gamma'(t), \mathcal{Y}_i(\gamma(t))) - \int_a^t \mathfrak{g}(\gamma', \mathcal{Y}_i') \, \mathrm{d}s \bmod \mathrm{Const},$$

for all $\gamma \in \Omega_{\mathcal{P}_q}(M)$, $t \in [a,b]$ and $i = 1, \ldots, r$. In (8.6) we have denoted by *Const* the subspace of $L^2([a,b], \mathbb{R}^{r*})$ consisting of constant maps. Obviously:

$$\mathcal{N}_{\mathcal{P}_q}(M) = \mathcal{F}^{-1}(0).) \tag{8.7}$$

The differential of $\mathcal{F}$ is computed as:

$$\begin{aligned} d\mathcal{F}_\gamma(\mathfrak{v})(t)_i &= \mathfrak{g}(\mathfrak{v}'(t), \mathcal{Y}_i(\gamma(t))) + \mathfrak{g}(\gamma'(t), \nabla_{\mathfrak{v}(t)}\mathcal{Y}_i) \\ &- \textstyle\int_a^t \mathfrak{g}(\mathfrak{v}', \mathcal{Y}_i') + \mathfrak{g}(\mathcal{R}(\gamma', \mathfrak{v})\gamma', \mathcal{Y}_i) + \mathfrak{g}((\nabla_{\mathfrak{v}}\mathcal{Y}_i)', \gamma') \, \mathrm{d}s \bmod \mathrm{Const}, \end{aligned} \tag{8.8}$$

for all $t \in [a,b]$, $i = 1, \ldots, r$ and all $\mathfrak{v} \in T_\gamma\Omega_{\mathcal{P}_q}(M)$. Consider the subspace $\mathcal{S}_\gamma$ of $T_\gamma\Omega_{\mathcal{P}_q}(M)$ consisting of vector fields that vanish at the endpoints and that take values in the span of the fields $\mathcal{Y}_i$, i.e.:

$$\mathcal{S}_\gamma = \Big\{ \sum_{i=1}^r f_i\mathcal{Y}_i : f_i : [a,b] \xrightarrow{H^1} \mathbb{R},\ f_i(a) = f_i(b) = 0 \Big\} \subset T_\gamma\Omega_{\mathcal{P}_q}(M).$$

The central point of the proof is showing that the restriction of $\mathrm{d}\mathcal{F}_\gamma$ to $\mathcal{S}_\gamma$ is an isomorphism; for $\mathfrak{v} = \sum_{i=1}^r f_i\mathcal{Y}_i \in \mathcal{S}_\gamma$, (8.8) can be rewritten as:

(8.9)

$$\mathrm{d}\mathcal{F}_\gamma(\mathfrak{v})(t) = \mathfrak{B}(t)f'(t) + (\mathcal{A}(t) + \mathcal{E}(t))f(t) - \textstyle\int_a^t (\mathcal{A}^* + \mathcal{E})f' + (\mathcal{C} + \overline{\mathcal{E}})f \, \mathrm{d}s \bmod \mathit{Const},$$

for all $t \in [a,b]$, where $f = (f_i)_{i=1}^r : [a,b] \to \mathbb{R}^r$. The righthand side of (8.9) defines an $L^2([a,b], \mathbb{R}^{r*})/\mathit{Const}$-valued Fredholm operator of index zero in the Hilbert space $H_0^1([a,b], \mathbb{R}^r)$ of H^1 maps $f : [a,b] \to \mathbb{R}^r$ with $f(a) = f(b) = 0$. This is proven using the compactness of the inclusion $W^{1,1} \hookrightarrow L^2$. Setting $\varphi = \mathcal{A}f + \mathfrak{B}f'$ then the righthand side of (8.9) vanishes iff f is a solution of (8.4) with $f(a) = f(b) = 0$; it follows that $\mathrm{d}\mathcal{F}_\gamma|_{\mathcal{S}_\gamma}$ is injective and therefore an isomorphism onto $L^2([a,b], \mathbb{R}^{r*})/\mathit{Const}$.

We can now prove all the assertions made in the statement of the theorem. Assertion (1) follows from (8.7) and from the fact that γ is a regular point for $\mathcal{F}$. Moreover:

$$T_\gamma\mathfrak{V} = \mathrm{Ker}(\mathrm{d}\mathcal{F}_\gamma). \tag{8.10}$$

Since $\mathrm{d}\mathcal{F}_\gamma|_{\mathcal{S}_\gamma}$ is an isomorphism, we have:

$$T_\gamma\Omega_{\mathcal{P}_q}(M) = T_\gamma\mathfrak{V} \oplus \mathcal{S}_\gamma.$$

Assertion (2) will follow once we establish that $\mathrm{d}E_\gamma$ vanishes on $\mathcal{S}_\gamma$. To see this, recalling that γ is a geodesic along the fields $\mathcal{Y}_i$, we compute as follows for $\mathfrak{v} = \sum_{i=1}^r f_i\mathcal{Y}_i \in \mathcal{S}_\gamma$:

$$\mathrm{d}E_\gamma(\mathfrak{v}) = \int_a^b \mathfrak{g}(\gamma', \mathfrak{v}')\,\mathrm{d}t = \sum_{i=1}^r \int_a^b [f_i \mathfrak{g}(\gamma', \mathcal{Y}_i)]'\,\mathrm{d}t = 0.$$

Assume now that γ is a geodesic starting orthogonally at $\mathcal{P}$. As in the beginning of the section, we choose a parallel trivialization of TM along γ and consider the Morse–Sturm system with initial data (X, ℓ_0) corresponding to (8.1) and to the Jacobi equation along γ. As it was observed, the index form $I \in Bil_{\mathrm{sym}}(\mathcal{H})$ of (X, ℓ_0) corresponds to the second variation $\mathrm{d}^2E_\gamma \in Bil_{\mathrm{sym}}(T_\gamma\Omega_{\mathcal{P}q}(M))$; moreover, $\mathcal{S}_\gamma$ corresponds to the space $\mathcal{S}^{\mathcal{D}}$ in (7.3). Since γ is a geodesic, integration by parts in (8.8) shows that $\mathfrak{v} \in T_\gamma\Omega_{\mathcal{P}q}(M)$ is in the kernel of $\mathrm{d}\mathcal{F}_\gamma$ iff $\mathfrak{g}(\mathfrak{v}', \mathcal{Y}_i)$ is absolutely continuous and

$$\mathfrak{g}(\mathfrak{v}', \mathcal{Y}_i)' = \mathfrak{g}(\mathfrak{v}', \mathcal{Y}_i') + \mathfrak{g}(\mathcal{R}(\gamma', \mathfrak{v})\gamma', \mathcal{Y}_i),$$

for all $i = 1, \ldots, r$. From (8.19) we conclude that the tangent space $T_\gamma\mathfrak{V}$ corresponds by the chosen parallel trivialization of TM along γ to the space $\mathcal{K}^{\mathcal{D}}$ in (7.4) (here $\mathcal{D}$ is the nondegenerate family of subspaces for X which has as a frame the maps $Y_i : [a, b] \to \mathbb{R}^n$ corresponding to the fields $\mathcal{Y}_i$). Since γ is a geodesic, the system (8.4) coincides with the reduced symplectic system X_{red}, so that $t = b$ is not conjugate for X_{red}. The remaining assertions in the statement of the theorem now follow immediately from the generalized Index Theorem 7.8.

8.2. **Geodesics in stationary semi-Riemannian manifolds.** In this subsection we apply our theory to obtain Morse relations for geodesics in stationary semi-Riemannian manifolds. For simplicity, we consider the case of geodesics between two fixed points. A Ljusternik–Schnirelman theory for this situation was developed in [12].

Let $(M, \mathfrak{g})$ be an n-dimensional semi-Riemannian manifold with $\mathfrak{g}$ a metric tensor of index r. We will call $(M, \mathfrak{g})$ *stationary* if it admits Killing vector fields $(\mathcal{Y}_i)_{i=1}^r$ such that $[\mathcal{Y}_i, \mathcal{Y}_j] = 0$ for all $i, j = 1, \ldots, r$ and such that $(\mathcal{Y}_i(m))_{i=1}^r$ is the basis of a subspace $\mathcal{D}_m$ of T_mM on which $\mathfrak{g}$ is negative definite for all $m \in M$.

Let $p, q \in M$ be fixed and define $\mathcal{N}_{pq}(M)$ and $\Omega_{pq}(M)$ as in Subsection 8.1 with $\mathcal{P} = \{p\}$. Since $\mathfrak{g}$ is negative definite on $\mathcal{D}$, the bilinear form $\mathfrak{B}(t) \in Bil_{\mathrm{sym}}(\mathbb{R}^r)$ defined in (8.3) is always negative definite and therefore also $\mathrm{B}(t)$ is negative definite for all $t \in [a, b]$. It follows that the hypothesis of Theorem 8.2 is satisfied for every curve $\gamma \in \mathcal{N}_{pq}(M)$ and that the reduced Maslov index of any geodesic is zero. Theorem 8.2 implies the following facts:

- $\mathcal{N}_{pq}(M)$ is a Hilbert submanifold of $\Omega_{pq}(M)$;

- the critical points of $E|_{\mathcal{N}_{pq}(M)}$ (see (8.2)) are precisely the geodesics on M from p to q;
- if q is not conjugate to p then all the critical points of $E|_{\mathcal{N}_{pq}(M)}$ are nondegenerate;
- if γ is a nondegenerate critical point of $E|_{\mathcal{N}_{pq}(M)}$ then its Morse index is given by:

$$n_-\big(\mathrm{d}^2(E|_{\mathcal{N}_{pq}(M)})(\gamma)\big) = i_{\mathrm{Maslov}}(\gamma).$$

8.3. Definition. We say that E is *pseudo-coercive* on $\mathcal{N}_{pq}(M)$ if given a sequence $(\gamma_n)_{n\geq 1}$ in $\mathcal{N}_{pq}(M)$ with $\sup_{n\geq 1} E(\gamma_n) < +\infty$ then $(\gamma_n)_{n\geq 1}$ admits a uniformly convergent subsequence.

Examples and sufficient conditions for E to be pseudo-coercive on $\mathcal{N}_{pq}(M)$ are given in [12, Appendix B]GPS-JMAA.

Let $\mathcal{D}^\perp$ denote the orthogonal complement of $\mathcal{D}$ with respect to $\mathfrak{g}$ and let $\mathfrak{g}_+$ be the Riemannian metric in M such that $\mathcal{D}$ and $\mathcal{D}^\perp$ are $\mathfrak{g}_+$-orthogonal, $\mathfrak{g}_+$ equals $\mathfrak{g}$ on $\mathcal{D}^\perp$ and $\mathfrak{g}_+$ equals $-\mathfrak{g}$ on $\mathcal{D}$. We define a Riemannian metric on the Hilbert manifold $\mathcal{N}_{pq}(M)$ by:

$$\langle \mathfrak{v}, \mathfrak{w}\rangle_{H^1} = \int_a^b \mathfrak{g}_+(\mathfrak{v}', \mathfrak{w}')\,\mathrm{d}t, \quad \mathfrak{v}, \mathfrak{w} \in T_\gamma\mathcal{N}_{pq}(M),\ \gamma \in \mathcal{N}_{pq}(M),$$

where the prime denotes the covariant derivative along γ with respect to the Levi-Civita connection of $\mathfrak{g}_+$.

8.4. Proposition. *If E is pseudo-coercive on $\mathcal{N}_{pq}(M)$ then $E|_{\mathcal{N}_{pq}(M)}$ has complete sublevels, it is bounded from below and it satisfies the Palais–Smale condition. Moreover, if the fields $\mathcal{Y}_i$ are complete then $\mathcal{N}_{pq}(M)$ has the same homotopy type of the loop space of M.*

Proof. See [12, Proposition 3.3, Theorem 4.1, Proposition 4.3, Proposition 5.2]GPS-JMAA. □

8.5. Theorem *(Morse relations for geodesics in stationary semi-Riemannian manifolds). Let $(M, \mathfrak{g})$ be a stationary semi-Riemannian manifold and let p and q in M be two non conjugate points. For $i \in \mathbb{N}$, set:*

$$n_i(p,q) = \text{number of geodesics } \gamma \text{ in } M \text{ from } p \text{ to } q \text{ with } i_{\mathrm{Maslov}}(\gamma) = i.$$

Then, under all the assumptions of Proposition 8.4, we have the following equality of formal power series in the variable λ:

$$\sum_{i=0}^{+\infty} n_i(p,q)\lambda^i = \mathfrak{P}_\lambda(\Omega^{(0)}(M);\mathbb{K}) + (1+\lambda)Q(\lambda),$$

where $\mathbb{K}$ *is an arbitrary field,* $\Omega^{(0)}(M)$ *is the loop space of* M, $\mathfrak{P}_\lambda(\Omega^{(0)}(M);\mathbb{K})$ *is its* Poincaré polynomial *with coefficients in* $\mathbb{K}$ *and* $Q(\lambda)$ *is a formal power series in* λ *with coefficients in* $\mathbb{N}\cup\{+\infty\}$.

Proof. It follows from Proposition 8.4 using standard Morse theory on Hilbert manifolds (see for instance [5]). □

Geodesics in Gödel Type Spacetimes. Let $(M_0, \mathfrak{g}^0)$ be a Riemannian manifold and let $\rho : M_0 \to Bil_{\mathrm{sym}}(\mathbb{R}^r)$ be a smooth map such that $\rho(x)$ is a nondegenerate symmetric bilinear form of index k in $\mathbb{R}^r$ for all $x \in M_0$. Consider the product $M = M_0 \times \mathbb{R}^r$ endowed with the semi-Riemannian metric $\mathfrak{g}$ defined by:

$$\mathfrak{g}_{(x,u)}((\xi_1,\eta_1),(\xi_2,\eta_2)) = \mathfrak{g}^0_x(\xi_1,\xi_2) + \rho(x)(\eta_1,\eta_2),$$

for all $x \in M_0$, $u \in \mathbb{R}^r$, $\xi_1, \xi_2 \in T_xM_0$ and $\eta_1, \eta_2 \in \mathbb{R}^r$. In analogy with [6, Definition 1.1] we will call $(M, \mathfrak{g})$ a *semi-Riemannian manifold of Gödel type*. In [6] it is considered the case where $r = 2$ and $k = 1$.

Consider the commuting Killing vector fields $\mathcal{Y}_i = (0, \frac{\partial}{\partial u_i})$, $i = 1, \ldots, r$ in M. An absolutely continuous curve $\gamma = (\gamma_0, u) : [a,b] \to M$ is a geodesic along the fields $\mathcal{Y}_i$ iff

$$\rho(\gamma_0(t))u'(t) \equiv \text{constant} \in \mathbb{R}^{r*}, \tag{8.11}$$

for $t \in [a,b]$. Let $p = (p_0, u_0)$, $q = (q_0, u_1) \in M$ be fixed and define $\mathcal{N}_{pq}(M)$ and $\Omega_{pq}(M)$ as in Subsection 8.1 with $\mathcal{P} = \{p\}$. For $\gamma = (\gamma_0, u) \in \mathcal{N}_{pq}(M)$, the bilinear form $\mathfrak{B}(t) \in Bil_{\mathrm{sym}}(\mathbb{R}^r)$ corresponding to γ defined in (8.3) is given by $\mathfrak{B}(t) = \rho(\gamma_0(t))$; the bilinear form $\mathbf{B}(t) \in Bil_{\mathrm{sym}}(\mathbb{R}^{r*})$ defined in (7.10) is given by:

$$\mathbf{B}^{\mathcal{I}}_{\gamma_0}(t) = \int_a^t \rho(\gamma_0(s))^{-1}\,\mathrm{d}s \in Bil_{\mathrm{sym}}(\mathbb{R}^{r*}), \tag{8.12}$$

where we write $\mathbf{B}^{\mathcal{I}}_{\gamma_0}$ rather than $\mathbf{B}^{\mathcal{I}}_{\gamma_0}$ to keep the dependence on γ_0 explicit. The hypothesis of Theorem 8.2 is satisfied for γ iff $\mathbf{B}^{\mathcal{I}}_{\gamma_0}(b)$ is nondegenerate[1].

Assume now that for every $\gamma_0 \in \Omega_{p_0q_0}(M_0)$ the bilinear form $\mathbf{B}^{\mathcal{I}}_{\gamma_0}(b)$ is nondegenerate. Then a curve $\gamma = (\gamma_0, u) \in \mathcal{N}_{pq}(M)$ is uniquely determined by γ_0; namely, from (8.11), we get:

---

[1] In the notation of [6] (where it is considered the case $r = 2$, $k = 1$), the nondegeneracy of (8.2) is the condition "$|\mathcal{L}(x)| > 0$".

$$u(t) = u_0 + \mathbf{B}^{f}_{\gamma_0}(t)\big(\mathbf{B}^{f}_{\gamma_0}(b)\big)^{-1}(u_1 - u_0). \tag{8.13}$$

By Theorem 8.2, $\mathcal{N}_{pq}(M)$ is a Hilbert submanifold of $\Omega_{pq}(M)$; moreover, we obtain a diffeomorphism $\phi : \Omega_{p_0q_0}(M_0) \to \mathcal{N}_{pq}(M)$ given by $\phi(\gamma_0) = (\gamma_0, u)$ with u defined in (8.13). If E denotes the geodesic action functional of M (see (8.2)) then the composite map $E_0 = E \circ \phi : \Omega_{p_0q_0}(M_0) \to \mathbb{R}$ is given by:

$$E_0(\gamma_0) = \frac{1}{2}\int_a^b \mathfrak{g}_0(\gamma_0', \gamma_0')\,\mathrm{d}t + \frac{1}{2}\mathbf{B}^{f}_{\gamma_0}(b)^{-1}(u_1 - u_0, u_1 - u_0),$$

for all $\gamma_0 \in \Omega_{p_0q_0}(M_0)$. Theorem 8.2 implies that the critical points of E_0 are precisely the curves $\gamma_0 \in \Omega_{p_0q_0}(M_0)$ for which $\gamma = \phi(\gamma_0)$ is a geodesic; moreover, γ_0 is a nondegenerate critical point of E_0 iff q is not conjugate to p along γ. The index of the second variation of E_0 at a nondegenerate critical point γ_0 is given by:

$$n_-\big(\mathrm{d}^2E_0(\gamma_0)\big) = i_{\mathrm{Maslov}}(\gamma) - i^{\mathrm{red}}_{\mathrm{Maslov}}(\gamma).$$

The Palais-Smale condition and the boundedness from below for the functional E_0 are satisfied under certain technical hypothesis on $\mathfrak{g}$. In the result below we will assume that the Hilbert manifold $\Omega_{p_0q_0}(M_0)$ is endowed with the Riemannian metric:

$$\langle \xi_1, \xi_2\rangle_{H^1} = \int_a^b \mathfrak{g}_0(\xi_1', \xi_2')\,\mathrm{d}t, \quad \xi_1, \xi_2 \in T_{\gamma_0}\Omega_{p_0q_0}(M_0),\ \gamma_0 \in \Omega_{p_0q_0}(M_0),$$

where the prime denotes covariant derivative along γ_0 in the Levi-Civita connection of $(M_0, \mathfrak{g}_0)$. Recall that if $\mathfrak{g}_0$ is complete then the metric $\langle\cdot,\cdot\rangle_{H^1}$ is also complete (see [14]).

8.6. **Proposition.** *Assume that $(M_0, \mathfrak{g}_0)$ is a complete Riemannian manifold, that $\mathbf{B}^{f}_{\gamma_0}(b)$ is nondegenerate for all $\gamma_0 \in \Omega_{p_0q_0}(M_0)$ and that:*

$$\sup_{x\in M_0}\|\rho(x)^{-1}\| < +\infty, \qquad \sup_{\gamma_0\in\Omega_{p_0q_0}(M_0)}\|\mathbf{B}^{f}_{\gamma_0}(b)^{-1}\| < +\infty.$$

Then the functional $E_0 : \Omega_{p_0q_0}(M_0) \to \mathbb{R}$ is bounded from below and it satisfies the Palais-Smale condition.

Proof. This is proved in [6, Lemmas 3.5 and 3.7] in the case $r = 2$, $k = 1$. The proof of the general case is analogous. □

The technical hypotheses in the statement of Proposition 8.6 are satisfied under suitable boundedness assumptions on ρ (see [6, Remark 1.4] for examples).

8.7. Theorem (Morse relations for geodesics in Gödel-type manifolds). *Let (M, g) be a semi-Riemannian manifold of Gödel-type. Let $p = (p_0, u_0)$ and $q = (q_0, u_1)$ in M be two non conjugate points; for $i \in \mathbb{N}$, set:*

$$n_i(p, q) = \text{number of geodesics } \gamma \text{ in } M \text{ from } p \text{ to } q \text{ with } i_{\text{Maslov}}(\gamma) - i^{\text{red}}_{\text{Maslov}}(\gamma) = i.$$

Then, under the assumptions of Proposition 8.6, we have the following equality of formal power series in the variable λ:

$$\sum_{i=0}^{+\infty} n_i(p, q)\lambda^i = \mathfrak{P}_\lambda(\Omega^{(0)}(M); \mathbb{K}) + (1 + \lambda)Q(\lambda),$$

where $\mathbb{K}$ is an arbitrary field, $\Omega^{(0)}(M)$ is the loop space of M, $\mathfrak{P}_\lambda(\Omega^{(0)}(M); \mathbb{K})$ is its *Poincaré polynomial* with coefficients in $\mathbb{K}$ and $Q(\lambda)$ is a formal power series in λ with coefficients in $\mathbb{N} \cup \{+\infty\}$.

Proof. It follows from Proposition 8.6 by using standard Morse theory on Hilbert manifolds (see for instance [5]) and observing that the loop space of M has the same homotopy type of $\Omega_{p_0q_0}(M_0)$.

9. The spectral index

We've seen in Theorem 6.3 that, under a nondegeneracy assumption, the Maslov index of a symplectic system gives an *algebraic count* of its conjugate instants. In this section we will consider a Morse–Sturm system (recall Proposition 5.3) of the type and we will give definition of another integer valued invariant such a system called the spectral index. The spectral index of a Morse–Sturm system is given by an algebraic count of the negative eigenvalues of the associated Jacobi differential operator $\mathcal{J}$.

As it is easy to see, the operator $\mathcal{J}$ is not in general self-adjoint. For the study of the spectral theory of a non self-adjoint operator, the use of complex Hilbert spaces is required; let us introduce some notation.

Given (real or complex) vector spaces V, W we denote by $Lin(V, W)$ the space of (real or complex) linear operators from V to W; we also set $Lin(V) = Lin(V, V)$. If V is a real vector space we denote by $\mathrm{Bil}(V)$ the space of bilinear forms $B : V \times V \to \mathbb{R}$ and by $Bil_{sym}(V)$ the subspace of $\mathrm{Bil}(V)$ consisting of symmetric bilinear forms. If the context indicates that V and W are Banach spaces then $Lin(V, W)$ will denote the space of *bounded* linear operators; similarly, $\mathrm{Bil}(V)$ and $Bil_{sym}(V)$ will denote respectively the space of bounded bilinear forms and symmetric bounded bilinear forms on V. Given a symmetric bilinear form $B \in Bil_{sym}(V)$ we define the *index* of B by:

$$n_-(B) = \sup\{\dim(W) : W \text{ is a subspace of } V \text{ and } B|_W \text{ is negative definite}\};$$

the *coindex* of B is defined by $n_+(B) = n_-(-B)$ and the *signature* $sgn(B)$ is defined as the difference $n_+(B) - n_-(B)$, provided that either $n_-(B)$ or $n_+(B)$ is finite.

Let g be a nondegenerate symmetric bilinear form on $\mathbb{R}^n$; denote by $Lin_g(\mathbb{R}^n)$ the space of *g-symmetric* endomorphisms of $\mathbb{R}^n$:

$$Lin_g(\mathbb{R}^n) = \{T \in Lin(\mathbb{R}^n) : g(T\cdot,\cdot) \text{ is symmetric}\}.$$

Let $R : [a,b] \to Lin_g(\mathbb{R}^n)$ be a continuous curve. We will consider Morse–Sturm systems in $\mathbb{R}^n$ of the form:

$$v'' = Rv, \tag{9.1}$$

with $v : [a,b] \to \mathbb{R}^n$. We will be mainly concerned with the case where g is *not* positive definite. We consider the differential operator

$$\mathcal{J}(v) = -v'' + Rv \tag{9.2}$$

corresponding to the equation (9.1) with Dirichlet boundary conditions:

$$v(a) = v(b) = 0. \tag{9.3}$$

The operator $\mathcal{J}$ is thought of as a densely defined unbounded operator on the Hilbert space

$$\mathcal{H} = L^2([a,b],\mathbb{R}^n)$$

of square-integrable maps $v : [a,b] \to \mathbb{R}^n$; the domain $\mathcal{D}$ of $\mathcal{J}$ is the space:

$$\mathcal{D} = \{v \in \mathcal{H} : v \text{ is of class } H^2 \text{ and } v(a) = v(b) = 0\}. \tag{9.4}$$

9.1. **Definition.** The operator $\mathcal{J}$ given in (9.2) with boundary conditions (9.3) will be called the *Jacobi differential operator* associated to the Morse–Sturm system (9.1).

Consider the following bounded nondegenerate bilinear form on $\mathcal{H}$:

$$\mathfrak{g}(v,w) = \int_a^b g(v(t),w(t))\,dt; \tag{9.5}$$

obviously $\mathcal{J}$ is symmetric with respect to $\mathfrak{g}$, i.e., $\mathfrak{g}(\mathcal{J}v,w) = \mathfrak{g}(v,\mathcal{J}w)$ for all $v,w \in \mathcal{D}$. If g is positive definite then $\mathfrak{g}$ is a Hilbert space inner product in $\mathcal{H}$ and $\mathcal{J}$ is indeed self-adjoint with respect to $\mathfrak{g}$. If g is not positive (or negative) definite then in general $\mathcal{J}$ may not even be normal with respect to *any* Hilbert space inner product in $\mathcal{H}$.

Consider the complex Hilbert space $\mathcal{H}^{\mathbb{C}} = L^2([a,b],\mathbb{C}^n)$ which is the complexification of $\mathcal{H}$ and denote by $\mathcal{J}^{\mathbb{C}}$ the unique complex linear extension of $\mathcal{J}$ to $\mathcal{H}^{\mathbb{C}}$. The domain of $\mathcal{J}^{\mathbb{C}}$ is the complex linear span of $\mathcal{D}$ in $\mathcal{H}^{\mathbb{C}}$, denoted by $\mathcal{D}^{\mathbb{C}} = \mathcal{D} \oplus i\mathcal{D}$. If we denote by $\hat{g}^{\mathbb{C}}$ the unique *sesqui-linear* extension of $\hat{g}$ to $\mathcal{H}^{\mathbb{C}}$ then $\hat{g}^{\mathbb{C}}$ is a nondegenerate Hermitian form on $\mathcal{H}^{\mathbb{C}}$ and $\mathcal{J}^{\mathbb{C}}$ is Hermitian with respect to $\hat{\mathbb{C}}$. The Hermitian form $\hat{\mathbb{C}}$ is given by the righthand side of (9.5) if we replace g by $\mathbb{C}$, the unique sesquilinear extension of g to $\mathbb{C}^n$. The operator $\mathcal{J}^{\mathbb{C}}$ is given by the righthand side of (9.2) if we replace R by $R^{\mathbb{C}}$, the unique complex linear extension of R to $\mathbb{C}^n$; $R^{\mathbb{C}}$ is clearly Hermitian with respect to $g^{\mathbb{C}}$.

In Proposition 9.2 below we will summarize the main spectral properties of the operator $\mathcal{J}^{\mathbb{C}}$. Recall that a densely defined unbounded operator T on a complex Hilbert space X is called *discrete* if there exists $\lambda \in \mathbb{C}$ such that the resolvent $(\lambda - T)^{-1}$ is compact. If T is discrete, the spectrum $\sigma(T)$ of T is a discrete subset of $\mathbb{C}$, and it consists only of eigenvalues; the resolvent $(\lambda - T)^{-1}$ will be compact for every $\lambda \in \mathbb{C} \setminus \sigma(T)$ and the *generalized eigenspace*

$$\mathcal{G}_\lambda(T) = \bigcup_{k=1}^{+\infty} \operatorname{Ker}(\lambda - T)^k$$

is finite dimensional for every $\lambda \in \sigma(T)$. The space $\mathcal{G}_\lambda(T)$ is the image of the projection (i.e., idempotent) operator $E_\lambda(T) \in \mathrm{Lin}(X)$ defined by the line integral

$$E_\lambda(T) = \frac{1}{2\pi i}\oint_{|z-\lambda|=\epsilon}(z - T)^{-1}\,\mathrm{d}z, \tag{9.6}$$

where $\epsilon > 0$ is small enough so that λ is the unique element of $\sigma(T)$ in the disc $|z - \lambda| \le \epsilon$. For $\lambda, \mu \in \sigma(T)$, $\lambda \ne \mu$ we have $E_\lambda(T)E_\mu(T) = 0$ and hence the sum $\sum_{\lambda\in\sigma(T)} \mathcal{G}_\lambda(T)$ is direct. A proof of the properties of discrete operators mentioned above can be found in [9, Chapter XIX].

9.2. Proposition. *The operator $\mathcal{J}^{\mathbb{C}}$ is discrete. Its spectrum $\sigma(\mathcal{J}^{\mathbb{C}})$ is contained in the strip:*

$$\{z \in \mathbb{C} : \Re(z) \ge -\|R\|_\infty,\ |\Im(z)| \le \|R\|_\infty\},$$

where $\|R\|_\infty$ denotes the supremum norm of R.

Proof. The bounds for the eigenvalues of $\mathcal{J}^{\mathbb{C}}$ are an easy consequence of the fact that $\mathcal{J}^{\mathbb{C}}$ is a perturbation of the positive operator $v \mapsto -v''$ by the bounded operator $v \mapsto R^{\mathbb{C}}v$. If $\lambda \in \mathbb{C}$ is not an eigenvalue of $\mathcal{J}^{\mathbb{C}}$ then the resolvent $\rho(\mathcal{J}^{\mathbb{C}};\lambda) = (\lambda - \mathcal{J}^{\mathbb{C}})^{-1}$ can be computed explicitly using the method of variation of constants as follows. For $u \in \mathcal{H}^{\mathbb{C}}$ we consider the non homogeneous equation

$$-v'' + (R^{\mathbb{C}} - \lambda)v = u \tag{9.7}$$

and we denote by $t \mapsto \Phi(t,\lambda)$ the flow of the corresponding homogeneous equation, i.e.:

$$\Phi(t,\lambda)\big(v(a), g^{\mathbb{C}}v'(a)\big) = \big(v(t), g^{\mathbb{C}}v'(t)\big), \tag{9.8}$$

for every $v : [a,b] \to \mathbb{C}^n$ such that the lefthand side of (9.7) vanishes. In (9.8) we denote by $g^{\mathbb{C}} : \mathbb{C}^n \to \mathbb{C}^{n*}$ the unique complex linear extension of $g : \mathbb{R}^n \to \mathbb{R}^{n*}$. Now the resolvent $\rho(\mathcal{J}^{\mathbb{C}};\lambda)$ can be written as:

$$\rho(\mathcal{J}^{\mathbb{C}};\lambda)\cdot u = \pi_1\left(\Phi(t,\lambda)\begin{pmatrix}0\\ \alpha_0\end{pmatrix} + \int_a^t \Phi(s,\lambda)^{-1}\begin{pmatrix}0\\ -gu(s)\end{pmatrix}\,\mathrm{d}s\right), \tag{9.9}$$

where $\pi_1 : \mathbb{C}^n \oplus \mathbb{C}^{n*} \to \mathbb{C}^n$ denotes projection onto the first coordinate and $\alpha_0 = \alpha_0(\lambda;u) \in \mathbb{C}^{n*}$ is the unique covector such that the righthand side of (9.9) vanishes at $t = b$. It follows that the spectrum of $\mathcal{J}^{\mathbb{C}}$ consists only of eigenvalues, and if $\lambda \notin \sigma(\mathcal{J}^{\mathbb{C}})$ it follows easily from (9.9) and the compact inclusion of the Sobolev space H^1 in L^2 that $\rho(\mathcal{J}^{\mathbb{C}};\lambda)$ is compact. This shows that $\mathcal{J}^{\mathbb{C}}$ is discrete and completes the proof.

For every $\lambda \in \sigma(\mathcal{J}^{\mathbb{C}})$ the generalized eigenspace $\mathcal{G}_\lambda(\mathcal{J}^{\mathbb{C}})$ is conjugate to the space $\mathcal{G}_{\bar\lambda}(\mathcal{J}^{\mathbb{C}})$; for $\lambda \in \mathbb{R}$ this implies that setting

$$\mathcal{H}_\lambda = \mathcal{G}_\lambda(\mathcal{J}^{\mathbb{C}}) \cap \mathcal{H}$$

then $\mathcal{G}_\lambda(\mathcal{J}^{\mathbb{C}})$ is the complex linear span of $\mathcal{H}_\lambda$ in $\mathcal{H}^{\mathbb{C}}$, i.e., $\mathcal{G}_\lambda(\mathcal{J}^{\mathbb{C}}) = \mathcal{H}_\lambda \oplus i\mathcal{H}_\lambda$. For $\lambda \in \mathbb{C}\setminus\mathbb{R}$ we denote by $\mathcal{H}_\lambda$ the real subspace of $\mathcal{H}$ whose complex linear span in $\mathcal{H}^{\mathbb{C}}$ is $\mathcal{G}_\lambda(\mathcal{J}^{\mathbb{C}}) \oplus \mathcal{G}_{\bar\lambda}(\mathcal{J}^{\mathbb{C}})$, i.e.:

$$\mathcal{H}_\lambda = \big(\mathcal{G}_\lambda(\mathcal{J}^{\mathbb{C}}) \oplus \mathcal{G}_{\bar\lambda}(\mathcal{J}^{\mathbb{C}})\big) \cap \mathcal{H}.$$

Observe that with the convention above we always have $\mathcal{H}_\lambda = \mathcal{H}_{\bar\lambda}$.

Proposition 9.2 implies that there are only a finite number of eigenvalues of $\mathcal{J}^{\mathbb{C}}$ with non positive real part; we can then give the following:

9.3. **Definition.** The *spectral index* of the Morse–Sturm system (9.1) is defined by:

$$\mathrm{i}_{\mathrm{spectral}}(g,R) = \sum_{\substack{\lambda\in\sigma(\mathcal{J}^{\mathbb{C}})\\ \lambda\in]-\infty,0]}} \mathrm{sgn}\big(\hat{g}|_{\mathcal{H}_\lambda}\big).$$

The notions of index, coindex and signature for symmetric bilinear forms in a real vector space can be extended to the case of sesqui-linear forms in a complex vector space. Obviously, for $\lambda \in \mathbb{R}$, the signature of $\hat{g}$ in $\mathcal{H}_\lambda$ coincides with the signature of $\hat{\mathbb{C}}$ in $\mathcal{G}_\lambda(\mathcal{J}^{\mathbb{C}}) = \mathcal{H}_\lambda \oplus i\mathcal{H}_\lambda$. By exploiting the symmetry of $\mathcal{J}$ with respect to $\hat{g}$ we obtain the following:

9.4. Proposition. *For $\lambda, \mu \in \sigma(\mathcal{J}^{\mathbb{C}})$, $\lambda \neq \bar{\mu}$, the spaces $\mathcal{G}_\lambda(\mathcal{J}^{\mathbb{C}})$ and $\mathcal{G}_\mu(\mathcal{J}^{\mathbb{C}})$ are $\hat{g}^{\mathbb{C}}$-orthogonal. In particular, the direct sum $\bigoplus_\lambda \mathcal{H}_\lambda$ is $\hat{g}$-orthogonal, where $\lambda \in \sigma(\mathcal{J}^{\mathbb{C}})$ runs over the eigenvalues of $\mathcal{J}^{\mathbb{C}}$ with non negative imaginary part.*

Proof. We show by induction in k that $\mathrm{Ker}(\lambda - \mathcal{J}^{\mathbb{C}})^k$ and $\mathcal{G}_\mu(\mathcal{J}^{\mathbb{C}})$ are $\hat{g}^{\mathbb{C}}$-orthogonal; the case $k = 0$ is trivial. Let $v \in \mathrm{Ker}(\lambda - \mathcal{J}^{\mathbb{C}})^k$ and $w \in \mathcal{G}_\mu(\mathcal{J}^{\mathbb{C}})$ be chosen; the induction hypothesis gives $\hat{g}^{\mathbb{C}}((\lambda - \mathcal{J}^{\mathbb{C}})^i v, w) = 0$ for $i \geq 1$. Choose l large enough so that $(\mu - \mathcal{J}^{\mathbb{C}})^l w = 0$ and compute as follows:

$$0 = \hat{g}^{\mathbb{C}}(v, (\mu - \mathcal{J}^{\mathbb{C}})^l w) = \hat{g}^{\mathbb{C}}((\bar{\mu} - \mathcal{J}^{\mathbb{C}})^l v, w) = \hat{g}^{\mathbb{C}}\Big([(\bar{\mu} - \lambda) + (\lambda - \mathcal{J}^{\mathbb{C}})]^l v, w\Big)$$

$$= \sum_{i=0}^{l} \binom{l}{i} (\bar{\mu} - \lambda)^{l-i} \hat{g}^{\mathbb{C}}((\lambda - \mathcal{J}^{\mathbb{C}})^i v, w) = (\bar{\mu} - \lambda)^l \hat{g}^{\mathbb{C}}(v, w).$$

This concludes the proof.

9.5. Lemma. *Let V be a real finite dimensional vector space, B be a nondegenerate symmetric bilinear form in V and let $T : V \to V$ be a B-symmetric linear endomorphism of V, i.e., $B(T\cdot, \cdot)$ is symmetric. If T has no real eigenvalues then the signature of B is zero.*

Proof. Let $V^{\mathbb{C}}$ be the complexification of V, $T^{\mathbb{C}} : V^{\mathbb{C}} \to V^{\mathbb{C}}$ be the unique complex linear extension of T and let $B^{\mathbb{C}} : V^{\mathbb{C}} \times V^{\mathbb{C}} \to \mathbb{C}$ be the unique sesqui-linear extension of B. As in the proof of Proposition 9.4, it follows that for $\lambda, \mu \in \sigma(T^{\mathbb{C}})$ with $\lambda \neq \bar{\mu}$ the generalized eigenspaces $\mathcal{G}_\lambda(T^{\mathbb{C}})$ and $\mathcal{G}_\mu(T^{\mathbb{C}})$ are $B^{\mathbb{C}}$-orthogonal. Since the signature of a Hermitian form is additive by orthogonal direct sum decompositions, it follows that we can assume without loss of generality that $\sigma(T^{\mathbb{C}}) = \{\lambda, \bar{\lambda}\}$, i.e., $V^{\mathbb{C}} = \mathcal{G}_\lambda(T^{\mathbb{C}}) \oplus \mathcal{G}_{\bar{\lambda}}(T^{\mathbb{C}})$. Since B (and hence $B^{\mathbb{C}}$) is nondegenerate we can write $V^{\mathbb{C}} = Z_+ \oplus Z_-$ as a direct sum of complex subspaces Z_+, Z_- such that $B^{\mathbb{C}}$ is respectively positive and negative definite in Z_+ and in Z_-. Since $B^{\mathbb{C}}$ vanishes on $\mathcal{G}_\lambda(T^{\mathbb{C}})$ we have $\mathcal{G}_\lambda(T^{\mathbb{C}}) \cap Z_+ = \mathcal{G}_\lambda(T^{\mathbb{C}}) \cap Z_- = \{0\}$ and

$$\frac{1}{2}\dim_{\mathbb{R}}(V) = \dim_{\mathbb{C}}(\mathcal{G}_\lambda(T^{\mathbb{C}})) \leq \min\{\dim_{\mathbb{C}}(Z_+), \dim_{\mathbb{C}}(Z_-)\};$$

this implies $\dim_{\mathbb{C}}(Z_+) = \dim_{\mathbb{C}}(Z_-)$ and hence $sgn(B^{\mathbb{C}}) = sgn(B) = 0$. □

9.6. Corollary. *If $\lambda \in \sigma(\mathcal{J}^{\mathbb{C}})$ is not real then the signature of $\hat{g}$ in $\mathcal{H}_\lambda$ is zero. In particular, the spectral index $\mathrm{i}_{\mathrm{spectral}}(g, R)$ equals the signature of the restriction of $\hat{g}$ to the subspace $\bigoplus_\lambda \mathcal{H}_\lambda \subset \mathcal{H}$, where λ runs over any subset of $\sigma(\mathcal{J}^{\mathbb{C}})$ consisting of all the real negative eigenvalues of $\mathcal{J}^{\mathbb{C}}$ and any finite set of non real eigenvalues of $\mathcal{J}^{\mathbb{C}}$.*

Proof. The first part of the statement is a direct consequence of Lemma 9.5 applied to the space $V = \mathcal{H}_\lambda$, to the operator T given by the restriction of $\mathcal{J}$ and to the bilinear form B given by the restriction of $\bar{g}$; we'll see later that $\bar{g}$ is nondegenerate in $\mathcal{H}_\lambda$. As to the second part, observe that the spaces $\mathcal{H}_\lambda$ are $\bar{g}$-orthogonal (see Proposition 9.4) and that the signature is additive by orthogonal sums.

10. The Equality between the Maslov and the Spectral Index

The Maslov and the spectral index of a Morse–Sturm system are equal. A detailed proof of this equality can be found in reference [8]; we recall here the main ideas. Three steps are needed:

Step 1. a direct proof of the equality $i_{\mathrm{Maslov}}(g, R) = i_{\mathrm{spectral}}(g, R)$ in the case where the negative eigenvalues of the Jacobi operator are simple;

Step 2. a proof of the stability of the indexes with respect to uniformly small perturbations of the coefficient R of the Morse–Sturm system;

Step 3. a proof of the *genericity* of the condition of simplicity for the negative eigenvalues of $\mathcal{J}$.

The idea of the proof of Step 1 is to show that the numbers $i_{\mathrm{Maslov}}(g, R)$ and $i_{\mathrm{spectral}}(g, R)$ are Maslov indexes of curves in the Lagrangian Grassmannian Λ that are homologous in $H_1(\Lambda, \Lambda_0(L_0))$. This is in analogy with the topological proof of the classical Oscillation Theorem given in Section 2.

We start by observing that $\lambda \in \mathbf{R}$ is an eigenvalue for the Jacobi operator $\mathcal{J}$ iff $t = b$ is a conjugate instant for the Morse–Sturm system

$$v'' = (R - \lambda)v. \tag{10.1}$$

We denote by $\Phi(t, \lambda)$ the flow of (10.1); $\Phi(t, \lambda)$ is the isomorphism of $\mathbf{R}^n \oplus \mathbf{R}^{n*}$ defined as in (9.8), replacing $g^{\mathbf{C}}$ with g. Observe that Φ is the solution of the matrix differential equation

$$\frac{\partial \Phi}{\partial t}(t, \lambda) = \begin{pmatrix} 0 & g^{-1} \\ g(R(t) - \lambda) & 0 \end{pmatrix} \Phi(t, \lambda), \quad t \in [a, b], \tag{10.2}$$

satisfying the initial condition $\Phi(a, \lambda) = \mathrm{Id}$. We define

$$\ell(t, \lambda) = \beta(\Phi(t, \lambda)) = \Phi(t, \lambda)(L_0).$$

where $L_0 = \{0\} \oplus \mathbb{R}^{n*}$. By definition, for $\varepsilon > 0$ small enough, the Maslov index of the curve $[a+\varepsilon, b] \ni t \mapsto \ell(t,0) \in \Lambda$ is the Maslov index of the Morse–Sturm system 9.1; moreover, the curve $\lambda \mapsto \ell(b,\lambda)$ intersects $\Lambda_{\geq 1}(L_0)$ precisely when λ is an eigenvalue of $\mathcal{J}$, so that one should expect that its Maslov index is somehow related to the spectral properties of $\mathcal{J}$. Our next goal is to determine precisely this relation. We have the following:

10.1. **Lemma.** *Let $\lambda \in \mathbb{R}$ be an eigenvalue of $\mathcal{J}$. The map $v \mapsto (0, gv'(b))$ is an isomorphism between* $\mathrm{Ker}(\lambda - \mathcal{J})$ *and the intersection $\ell(b,\lambda) \cap L_0$. Moreover, this isomorphism carries the restriction of $\mathfrak{g}$ to the restriction of $\frac{\partial \ell}{\partial \lambda}(b,\lambda)$, i.e.:*

$$\frac{\partial \ell}{\partial \lambda}(b,\lambda)\big((0,gv'(b)),(0,gw'(b))\big) = \mathfrak{g}(v,w),$$

for all $v, w \in \mathrm{Ker}(\lambda - \mathcal{J})$.

Proof. The first part of the statement is immediate.

Differentiating (10.2) with respect to λ we obtain that $t \mapsto \frac{\partial \Phi}{\partial \lambda}(t,\lambda)$ is a solution of a non homogeneous linear differential equation and it is easily computed by the method of variation of constants as follows:

$$\frac{\partial \Phi}{\partial \lambda}(b,\lambda) = \Phi(b,\lambda) \int_a^b \Phi(t,\lambda)^{-1} \begin{pmatrix} 0 & 0 \\ -g & 0 \end{pmatrix} \Phi(t,\lambda)\, \mathrm{d}t.$$

Since $\ell = \beta \circ \Phi$, one computes:

(10.3)

$$\frac{\partial \ell}{\partial \lambda}(b,\lambda) = \omega\left(\Phi(b,\lambda)\left[\int_a^b \Phi(t,\lambda)^{-1}\begin{pmatrix} 0 & 0 \\ -g & 0\end{pmatrix}\Phi(t,\lambda)\,\mathrm{d}t\right]\Phi(b,\lambda)^{-1}\cdot,\cdot\right)\Bigg|_{\ell(b,\lambda)}.$$

To conclude the proof, choose $v, w \in \mathrm{Ker}(\lambda - \mathcal{J})$ and apply both sides of (10.3) to $(0, gv'(b))$ and $(0, gw'(b))$, keeping in mind that Φ is a symplectomorphism.

10.2 **Corollary.** *Assume that $t = b$ is not a conjugate instant for the Morse–Sturm system (9.1), so that zero is not an eigenvalue of $\mathcal{J}$. If all the negative eigenvalues of $\mathcal{J}$ are simple, i.e.,* $\dim(\mathcal{H}_\lambda) \leq 1$ *for all $\lambda < 0$, then the spectral index of (9.1) equals the Maslov index of the curve $[-M, 0] \ni \lambda \mapsto \ell(b,\lambda) \in \Lambda$ if $M > 0$ is bigger than the supremum norm of R.*

Proof. For every negative eigenvalue λ of $\mathcal{J}$ the condition $\dim(\mathcal{H}_\lambda) = 1$ implies that the generalized eigenspace $\mathcal{H}_\lambda$ equals the standard eigenspace $\mathrm{Ker}(\lambda - \mathcal{J})$. Lemma 10.1 implies that the restriction of $\mathfrak{g}$ to $\mathcal{H}_\lambda$ corresponds by the isomorphism $v \mapsto (0, gv'(b))$ to the restriction of $\frac{\partial \ell}{\partial \lambda}(b,\lambda)$ to $\ell(b,\lambda) \cap L_0$; moreover, the restriction of $\bar{g}$ to $\mathcal{H}_\lambda$ is nondegenerate. The conclusion now follows using Theorem 6.3, keeping in mind that $\mathcal{J}$ has no negative eigenvalue with absolute value bigger than the supremum norm of R (see Proposition 9.2).

As to the stability of the Maslov index of a Morse–Sturm system, we have the following:

10.3. Proposition. (Stability of the Maslov index). *Assume that $t = b$ is not a conjugate instant for the Morse–Sturm system (9.1). Let $(R_k)_{k\geq 1}$ be a sequence of continuous curves $R_k : [a,b] \to \mathrm{Lin}_g(\mathbb{R}^n)$ of g-symmetric endomorphisms of $\mathbb{R}^n$ such that R_k converges uniformly to R. Then, for k sufficiently large, $t = b$ is not a conjugate instant for the Morse–Sturm system $v'' = R_k v$ and $\mathrm{i}_{\mathrm{Maslov}}(g, R_k) = \mathrm{i}_{\mathrm{Maslov}}(g, R)$.*

Proof. By standard results on the continuous dependence of the solution with respect to the data of an ODE, we have that the flow Φ_k of the Morse–Sturm system $v'' = R_k v$ converges uniformly (actually, in the C^1 topology) to the flow Φ of $v'' = Rv$; hence $\ell_k = \beta \circ \Phi_k$ converges uniformly to $\ell = \beta \circ \Phi$. The condition that $t = b$ is not conjugate for $v'' = Rv$ means that $\ell(b)$ belongs to the open subset $\Lambda_0(L_0) \subset \Lambda$, which proves that $t = b$ is not conjugate for $v'' = R_k v$ if k is sufficiently large. Let $\varepsilon > 0$ be such that $v'' = Rv$ has no conjugate instants in $]a, a+\varepsilon]$; an easy uniformity argument shows that $\varepsilon > 0$ can be chosen so that, for every $k \geq 1$, there are no conjugate instants in $]a, a+\varepsilon]$ for the system $v'' = R_k v$. To conclude the proof, observe that the uniform convergence of $\ell_k|_{[a+\varepsilon,b]}$ to $\ell|_{[a+\varepsilon,b]}$ implies that $\ell_k|_{[a+\varepsilon,b]}$ and $\ell|_{[a+\varepsilon,b]}$ determine the same homology class in $H_1(\Lambda, \Lambda_0(L_0))$ for k sufficiently large.

We now state the main theorem of the section.

10.4. Theorem. (Spectral index theorem). *Assume that $t = b$ is not a conjugate instant for the Morse–Sturm system (9.1). Then the Maslov index and the spectral index of (9.1) are equal:*

$$\mathrm{i}_{\mathrm{Maslov}}(g, R) = \mathrm{i}_{\mathrm{spectral}}(g, R). \tag{10.4}$$

Proof. For $\varepsilon > 0$ small enough, the Maslov index of the curve $[a+\varepsilon, b] \ni t \mapsto \ell(t, 0) \in \Lambda$ equals the Maslov index of the Morse–Sturm system (9.1). Assume that all the negative eigenvalues of $\mathcal{J}$ are simple; if $M > 0$ is bigger than the supremum norm of R then by Corollary 10.2, the Maslov index of the curve $[-M, 0] \ni \lambda \mapsto \ell(b, \lambda) \in \Lambda$ equals the spectral index of (9.1). An easy uniformity argument shows that if $\varepsilon > 0$ is small enough then the Morse–Sturm system (10.1) has no conjugate instants in $]a, a+\varepsilon]$ for all λ in the compact interval $[-M, 0]$ (see [16] for details). Now consider the rectangle $[a+\varepsilon, b] \times [-M, 0]$ in the domain of ℓ; the sides $[a+\varepsilon, b] \times \{-M\}$ and $\{a+\varepsilon\} \times [-M, 0]$ are mapped by ℓ into $\Lambda_0(L_0)$, which implies that the images by ℓ of the two remaining sides are homotopic in Λ with endpoints in $\Lambda_0(L_0)$. This proves the equality (10.4) in the case where all the negative eigenvalues of $\mathcal{J}$ are simple. For the general case, let $(R_k)_{k\geq 1}$ be a sequence of continuous curves $R_k : [a,b] \to Lin_g(\mathbb{R}^n)$ such that:

- R_k converges uniformly to R on $[a,b]$;
- all the negative eigenvalues of the Jacobi differential operator corresponding to the Morse–Sturm system $v'' = R_k v$ are simple.

The existence of such a sequence is proven in [8, Theorem 6.4]. Observe that, by Proposition 10.3, for k sufficiently large $t = b$ is not a conjugate instant for the Morse–Sturm system $v'' = R_k v$ and hence, by the first part of the proof, we have:

$$i_{\text{Maslov}}(g, R_k) = i_{\text{spectral}}(g, R_k), \tag{10.5}$$

for k sufficiently large. The conclusion follows by taking the limit $k \to +\infty$ in equality (10.5), keeping in mind that both the Maslov and the spectral index of a Morse–Sturm system are stable.

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