Cubo Matemática Educacional
Vol. 5., N° 1, Enero 2003

Water Waves

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Abstract

This article describes some aspects of linearised theory of water waves in the context of current research interests in this area. After a brief introduction, the basic equations in the linearised theory of water waves alongwith their general solutions, concepts of scattering and radiation problems, breakwaters and finally trapped waves are presented without going through the details of mathematical analysis involved in these problems. This may motivate the readers to feel interested in water wave problems.

1. Introduction

The phenomena of wave propagation are encountered in almost all branches of mathematical physics such as continuum mechanics, quantum mechanics, acoustics, electromagnetic theory etc. It is somewhat difficult to give a precise definition of what constitute a wave, but to cover the whole range of phenomenon without being restrictive, it can be stated intuitively that wave is a recognizable signal that is transferred from one part of the medium to another part with a recognizable velocity of propagation. The signal may be any feature of disturbance which is clearly recognizable and its location at any time can be determined.

According to the famous theoretical physicist A. Somerfeld, 'ever since waves were studied, *water waves* have served the natural scientist as a model for wave theory in general' (cf. Segel(1977) p 299). Consequently the wave phenomena in water with a free surface under gravity have attracted the attention of a number of famous mathematicians and physicists such as A. L. Cauchy (1759-1857), S. D. Poisson (1781-1840), J. L . Lagrange (1736-1813), G. B. Airy (1801-1892), G. G. Stokes (1819-1903), Lord Kelvin (Sir William Thomson) (1824-1907), J. H. Michell (1836-1921) and many others. This has resulted in a systematic development of the theory of water waves from the latter half of the eighteenth century. This theory has provided a background for a somewhat rich development of some important mathematical concepts and techniques and consequently it has become an important branch of applied mathematics as well as mathematical physics.

The theory of water waves is most varied and is a fascinating subject, it includes a wide range of natural phenomena in oceans, rivers and lakes. The research activities in this area accelerated after the second world war due to explosive growth in ocean-related industries such as offshore drilling for oil production, construction of offshore structures, extraction of wave energy from ocean waves, design and manufacturing of large ships, oil tankers, breakwaters to protect ports, sea resorts from the impact of rough sea, etc. Interest in wave interaction with fixed and floating stuctures was somewhat stimulated after the unsuccessful attempts to utilize portable and floating breakwaters in the surprise amphibious landing at Normandy, France, during the second world war.

Floating docks, breakwaters, burges, ships, oil tankers, submersibles supporting oil drilling rigs, very large floating structures (VLFS) etc. are all stuctures which are useful in the ocean-related industries. Safety and performance of these structures depend on how they respond to ocean waves, particularly when the ocean is rough during a storm. In a calm ocean these structures are in static equilibrium. However, if the ocean is rough, then the waves are being diffracted by a structure present in the ocean and the structure is being subjected to considerable thrust produced by ocean waves and thus oscillates, and further radiates waves, and experineces additional thrust from the surrounding fluid. Thus investigation of problems on interaction of water waves and structures is very important. From the time of Lord Kelvin and J. H. Michell interaction of water waves with floating or submerged bodies (or structures) has been an active area of research in fluid mechanics. Investigation of problems on wave interaction with large structures needed for oil exploration in high seas has become very significant during the last few decades. These investigations are obviously concerned with development of analytical and computational techniques to predict hydrodynamic interaction of waves with floating or submerged structures. The *linearised* theory of water waves is generally utilized in the mathematical analysis for most cases although the free surface boundary condition produces severe complication. The problems are usually tackled by judicious combination of analytical and numerical techniques.

Below we present briefly the basic equations of the linearised theory of water waves and the form of general solutions, concepts of water wave scattering and radiation problems, breakwaters and trapped waves. The mathematical analysis is kept at a minimum level to create interest in the topic to a reader without having any background knowledge on water waves.

2 Linrearised Theory of Water Waves

Basic Equations

The basic equations in the linearised theory of water waves are remarkably simple. These are derived under assumptions that water is an incompressible, inviscid and homogeneous fluid and the motion in it is under the action of gravity and is assumed to be irrotational and small. The assumption on smallness of motion means that the velocity components together with their partial derivatives are quantities of first order of smallness so that their products, squares and higher powers can be neglected. Also there is a free surface which is horizontal when the fluid is at rest. When a motion is set up, the free surface deviates smoothly from its horizontal position and this deviation is small compared to some physical length. The assumption of smallness of the velocity components and the free surface deviation (elevation or depression above or below the mean horizontal level of the free surface when at rest) together with their partial derivatives allow us to *linearise* the various equations of fluid mechanics and thus constitute the basis for the linearised theory of water waves.

We choose a rectangular cartesian co-ordinate system in which the y-axis is taken vertically downwards, xz- plane is the position of the undisturbed free surface so that water occupies the half space $y \geq 0$ if it is infinitely deep or the region $0 \leq y \leq h$ if it is of finite depth h. We will assume h to be constant in our discussion here. However, for water of variable depth, h is generally a function of x and z. If the irrotational motion in water is described by the velocity potential $\Phi(x, y, z, t)$, where t is the time, then the equation of continuity produces

$$\nabla_1^2 \Phi = 0 \quad \text{in the fluid region} \tag{1}$$

where ∇_1^2 is the three dimensional Laplacian operator, and the linearised form of the Bernoulli's equation is

$$\frac{\partial \Phi}{\partial t} = gy - \frac{p}{\rho} \tag{2}$$

where p is the presure and ρ is the density of water.

Let $y = \eta(x, z, t)$ denote the free surface depression below the plane $y = 0$, then η together with its partial dertivatives are smooth functions and quantities of first order of smallness, so that on the free surface $y = \eta$, the condition (2) gives rise to

$$\frac{\partial \Phi}{\partial t} = g\eta \text{ on } y = 0 \tag{3a}$$

since p is constant on the free surface (the atmospheric pressure) and can be taken to be zero by suitable choice of the pressure scale. The equation (3a) is in fact the *linearised dynamical condition at the free surface.* The *linearised kinematical condition at the free surface* is

$$\frac{\partial \Phi}{\partial t} = \frac{\partial \eta}{\partial t} \text{ on } y = 0 \tag{3b}$$

which is obtained from the fact that the velocity of the fluid particles at the free surface normal to it must be equal to the velocity of the free surface at that point.

Elimination of η between (3a) and (3b) produces the *linearised free surface condition*

$$\frac{\partial^2 \Phi}{\partial t^2} = g\frac{\partial \Phi}{\partial y} \text{ on } y = 0. \tag{4}$$

The condition of no motion at the bottom gives

$$\nabla_1 \Phi \to 0 \text{ on } y \to \infty \tag{5}$$

if the water extends infinitely downwards. However, if the water is of uniform finite depth h below the mean free surface, then

$$\frac{\partial \Phi}{\partial y} = 0 \text{ on } y = h. \tag{5$'$}$$

Thus the *basic equations* of the linearised theory of water waves are described by the partial differential equation (1), the free surface condition (4) and the bottom condition (5) or (5′). As mentioned earlier, these are quite simple in their appearances.

For simple harmonic motion with angular frequency σ, we may assume Φ (x, y, z, t) to be of the form $Re\{\phi(x, y, z)e^{-i\sigma t}\}$, then the complex valued function $\phi(x, y, z)$ satisfies

$$\nabla_1^2 \phi = 0 \text{ in the fluid region,} \tag{6}$$

the free surface condition

$$K\phi + \frac{\partial \phi}{\partial y} = 0 \text{ on } y = 0 \tag{7}$$

where $K = \sigma^2/g(> 0)$, and the bottom condition

$$\nabla_1\phi \to 0 \text{ on } y \to \infty \tag{8}$$

for infinitely deep water, or

$$\frac{\partial\phi}{\partial y} = 0 \text{ on } y = h \tag{8'}$$

for water of uniform finite depth h.

The equations (6), (7) and (8) or (8′), may also be regarded as the basic equations of the linearised theory of water waves.

It may be noted that one can introduce the effect of surface tension at the free surface. In this case the free surface condition (7) modifies to

$$K\phi + \frac{\partial\phi}{\partial y} + M\frac{\partial^3\phi}{\partial y^3} = 0 \text{ on } y = 0 \tag{9}$$

where $M = T/\rho g$, T being the coefficient of surface tension. It is interesting to note that although the governing partial derivatives upto second order only, the free surface condition involves the third order partial derivative $\frac{\partial^3\phi}{\partial y^3}$.

General solutions

For simplicity we confine our attention to two-dimensional motion so that ϕ is a function of x, y only. Then the basic equations satisfied by ϕ are described by the two-dimensional Lapalce equation

$$\nabla^2\phi = 0 \text{ in the fluid region}$$

where ∇^2 is the two-dimensional Laplace operator, the free surface condition

$$K\phi + \frac{\partial\phi}{\partial y} = 0 \text{ on } y = 0$$

and the bottom condition

$$\nabla\phi \to 0 \text{ as } y \to \infty$$

for deep water, or

$$\frac{\partial\phi}{\partial y} = 0 \text{ on } y = h$$

for water of uniform finite depth h.

The solutions for $\phi(x,y)$ can be obtained by the simple method of separation of variable.

For the case of infinitely deep water, using this method, we can easily find that $\phi(x,y)$ has the following two forms

$$\phi(x,y) = e^{-Ky \pm iKx} \tag{10a}$$

and

$$\phi(x,y) = (k\cos ky - K\sin ky)e^{-k|x|} \tag{10b}$$

where k is a non negative real parameter. Looking at the time-dependence factor $e^{-i\sigma t}$, we can interpret the term $e^{-Ky+iKx}$ to represent a progressive wave train propagating along the positive x-direction while $e^{-Ky-iKx}$ to represent a progressive wave train propagating along the negative x-direction. The solution represented by (10b) dies exponentially as $|x|$ become large so that this can be interpreted as a local solution having no effect as $|x| \to \infty$. In fact the set of functions

$$e^{-Ky+iKx}, e^{-Ky-iKx}, (k\cos ky - K\sin ky)e^{-k|x|} (k > 0)$$

forms a *complete* set of basis functions for the expansion of the complex valued function $\phi(x,y)$ in the half space $y \geq 0$. This expansion is given by

$$\begin{aligned} \phi(x,y) &= C_1 e^{-Ky+iKx} + C_2 e^{-Ky-iKx} \\ &+ \begin{cases} \displaystyle\int_0^\infty A(k)(k\cos ky - K\sin ky)e^{-kx}dk, & x > 0 \\ \displaystyle\int_0^\infty B(k)(k\cos ky - K\sin ky)e^{kx}dk, & x < 0 \end{cases} \end{aligned} \tag{11}$$

where C_1, C_2 are constants and $A(k)$, $B(k)$ are functions of k. The expansion (11) is generally referred to as the Havelock's expansion of water wave potential in deep water. The corresponding integral expansion formula for a function $f(y)$ defined for $y \geq 0$ and satisfying the Dirichlet's condition of Fourier integral expansion is given by

$$f(y) = A_0 e^{-Ky} + \int_0^\infty A(k)(k\cos ky - K\sin ky)dk \tag{12}$$

where

$$\begin{aligned} A_0 &= 2K\int_0^\infty f(y)e^{-ky}dy, \\ A(k) &= \frac{2}{\pi}\frac{1}{k^2+K^2}\int_0^\infty (k\cos ky - K\sin ky)f(y)dy. \end{aligned} \tag{13}$$

For water of uniform finite depth h, the solutions for $\phi(x, y)$ are

$$\cosh k_0(h-y)e^{\pm ik_0x}, \cos k_n(h-y)e^{-k_n|x|}(n=1,2..) \tag{14}$$

where $\pm k_0, \pm ik_n(n=1,2..)$ are the roots of the transcendental equation

$$k \tanh kh = K.$$

As before, the first terms in (14) represent progressive wave solutions propagating along the positive or negative x-directions while the second terms represent local solutions which die away exponentially as $|x|$ becomes large. The set of functions given by (14) forms a *complete* set of basis functions for the expansion of the water wave potential $\phi(x, y)$ in the strip $0 \le y \le h$. This expansion is given by

$$\begin{aligned}\phi(x,y) &= (C_1e^{ik_0x} + C_2e^{-ik_0x})\cosh k_0(h-y) \\ &+ \begin{cases} \sum_1^\infty A_n \cos k_n(h-y)e^{-k_nx}dk, & x>0 \\ \sum_1^\infty B_n \cos k_n(h-y)e^{k_nx}dk, & x<0 \end{cases}\end{aligned} \tag{15}$$

where C_1, C_2 and $A_n, B_n(n=1,2,..)$ are constants. This expansion is also known as the Havelock's expansion of water wave potential in finite depth water. The corresponding expansion formula for a function $f(y)$ defined in the strip $0 \le y \le h$ and satisfying the Dirichlet's condition for the Fourier series expansion is given by

$$f(y) = A_0 \cosh k_0(h-y) + \sum_1^\infty A_n \cos k_n(h-y)$$

where

$$\begin{aligned} A_0 &= \frac{4k_0}{2k_0h + \sinh 2k_0h}\int_0^h f(y)\cosh k_0(h-y)dy, \\ A_n &= \frac{4k_n}{2k_nh + \sin 2k_nh}\int_0^h f(y)\cos k_n(h-y)dy. \end{aligned} \tag{16}$$

The formulae (11), (12), (13) as well as (14), (15), (16) play a very significant role in the mathematical analysis of various water wave problems in deep water as well as finite depth water.

For *three-dimensional* motion with harmonic variation along the z-direction, we can represent $\Phi(x, y, z, t)$ as

$$\Phi(x,y,z,t) = Re\{\phi(x,y)e^{i\nu z - i\sigma t}\} \tag{17}$$

where now $\phi(x,y)$ satisfies the modified Helmholtz's equation

$$(\nabla^2 - \nu^2)\phi = 0 \text{ in the fluid region}$$

along with the appropriate free surface and bottom conditions. For deep water, the solutions for $\phi(x,y)$ in this case are

$$e^{-Ky\pm i\mu x}, (k\cos Ky - K\sin ky)e^{-(k^2+\mu^2)^{1/2}|x|} \tag{19}$$

where $\mu = (K^2-\nu^2)^{1/2}(\nu < K)$ and k is real positive so that the general solution is

$$\begin{aligned} \phi(x,y) &= C_1e^{-Ky+i\mu x} + C_2e^{-Ky-i\mu x} \\ &+ \begin{cases} \int_0^\infty A(k)(k\cos ky - K\sin ky)e^{-(k^2+\nu^2)^{1/2}x}dk, & x > 0, \\ \int_0^\infty B(k)(k\cos ky - K\sin ky)e^{(k^2+\nu^2)^{1/2}x}dk, & x < 0. \end{cases} \end{aligned} \tag{20}$$

For water of uniform finite depth h, the general solution in this case is

$$\begin{aligned} \phi(x,y) &= (C_1e^{i(k_0^2-\nu^2)^{1/2}x} + C_2e^{-i(k_0^2-\nu^2)^{1/2}x})\cosh k_0(h-y) \\ &+ \begin{cases} \sum_1^\infty A_n\cos k_n(h-y)e^{(k_n^2+\nu^2)^{1/2}x}dk, & x > 0, \\ \sum_1^\infty B_n\cos k_n(h-y)e^{(k_n^2+\nu^2)^{1/2}x}, & x < 0 \end{cases} \end{aligned} \tag{21}$$

where it has been assumed that $\nu < k_0$.

3 Scattering and radiation problems

Problems of water wave scattering and radiation by bodies (structures) of various geometrical configurations constitute two very important classes of problems in the linearised theory of water waves. In fact, investigations of the these classes of problems perhaps constitute the most of this theory. A general formulation of these classes of problems are given below. We consider only the two-dimensional case.

Scattering problem

If a train of surface water waves with angular frequency σ represented by $Re\{\phi^{inc}(x,y)e^{-i\sigma t}\}$ and propagating from the direction of negative infinity is incident on a floating or submerged body, then a part of it is reflected back by the body and the remaining part is transmitted forward below or over the body. If the motion is described by the velocity potential $Re\{\phi(x,y)e^{-i\sigma t}\}$ then ϕ satisfies the Laplace equation

$$\nabla^2\phi = 0 mbox in the fluid region,$$

the free surface condition

$$K\phi + \frac{\partial\phi}{\partial y} = 0 \quad \text{on } y = 0,$$

the condition on the wetted surface of the body

$$\frac{\partial\phi}{\partial n} = 0 \quad \text{on } W$$

where W is the wetted surface of the body and n denotes the outward drawn normal to it, the bottom condition

$$\nabla\phi \to 0 \quad \text{on} \quad y \to \infty$$

for infinitely deep water, or

$$\frac{\partial\phi}{\partial y} = 0 \quad \text{on} \quad y = h$$

for water of uniform finite depth h, and finally the infinity condition given by

$$\phi(x,y) \sim \begin{cases} \phi^{inc}(x,y) + R\phi^{inc}(-x,y) & \text{as } x \to -\infty, \\ T\phi^{inc}(x,y) & \text{as } \to \infty \end{cases}$$

where R and T denote the reflection and transmision coefficeints (unknown) respectively, and

$$\phi^{inc}(x,y) = \begin{cases} e^{-Ky+iKx} & \text{for deep water,} \\ \dfrac{\cosh k_0(h-y)}{\cosh k_0 h} e^{ik_0 x} & \text{for water of uniform finite depth.} \end{cases}$$

Determination of the potential function $\phi(x,y)$ explicitly for any configuration of the body is an almost impossible task. However, for some special configurations, such as thin barriers, it may be possible to determine $\phi(x,y)$ completely. Even if $\phi(x,y)$ cannot be determined is closed forms, R and T can be obtained by some approximate methods for some other configurations. For most scattering problems, we will be happy if numerical estimates for these coefficients can be obtained by some numerical procedure. For bodies having a submerged sharp edge, and additional condition, known as *edge condition*, has also to be satisfied. This will be discussed shortly.

Radiation problem

If a body present in water is forced to execute a small prescribed motion, then a wave motion is set up in the fluid, which radiates away from the body towards infinity (in the directions of positive and negative x-axis). If the motion in the fluid is described by $Re\{\phi(x,y)e^{-i\sigma t}\}$ then in this case $\phi(x,y)$ satisfies

$$\nabla^2\phi = 0 \text{ in the fluid region,}$$

$$K\phi + \frac{\partial\phi}{\partial y} = 0 \text{ on } y = 0,$$

$$\frac{\partial\phi}{\partial n} \text{ is prescribed on } W$$

where W is the wetted surface of the body and n denotes the outward drawn normal to it, the bottom condition

$$\nabla\phi \to 0 \text{ on } y \to \infty$$

for infinitely deep water, or

$$\frac{\partial\phi}{\partial y} = 0 \text{ on } y = h$$

for water of uniform finite depth h, and finally the radiation condition

$$\phi(x,y) \sim A_{\pm}e^{Ky+iK|x|} \; as \text{ as } x \to \pm\infty$$

for deep water, or

$$\phi(x,y) \sim A_{\pm}\frac{\cosh k_0(h-y)}{\cosh k_0 h}e^{ik_0|x|} \; as \text{ as } x \to \pm\infty$$

for water of uniform finite depth h, where $A_{\pm}$ is the amplitude of the wave motion set up at $x = \pm\infty$. These are unknown, and their determination, like the determination of R and T in scattering problems, is the most important task for a radiation problem.

If there is a submerged sharp edge in the body , then in the radiation problem also, an edge condition on ϕ has to be imposed. A simple derivation of the edge condition is given below.

Edge condition

Without any loss of generality, we choose the origin at the tip of the submerged edge and let $y = 0$ ($x > 0$) be one side of the edge and the other side of the edge makes an angle α with the x-axis. α is the angle of the wedge formed between the two sides of the edge. Using polar co-ordinate r, θ, the potential function $\phi(r, \theta)$ satisfies the differential equation

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 0 \quad 0 \leq \theta \leq \alpha$$

and the boundary conditions

$$\frac{1}{r}\frac{\partial \phi}{\partial \theta} = 0 \text{ on } \theta = 0, \alpha.$$

In a small beighbourhood of the origin, let

$$\phi(r, \theta) = r^\lambda f(\theta)$$

where $\lambda (> 0)$ is to be chosen suitably and $f(\theta)$ is a twice differentaiable function. Thus $f(\theta)$ satisfies the differential equation

$$f''(\theta) + \lambda^2 f(\theta) = 0, \quad 0 \leq \theta \leq \alpha$$

with the boundary conditions

$$f'(\theta) = 0 \text{ on } \theta = 0, \alpha.$$

Thus if we choose

$$f(\theta) = A \cos \lambda\theta + B \sin \lambda\theta$$

then we must have

$$B = 0, \quad \lambda = \frac{n\pi}{\alpha}$$

where n is a positive integer. In order to satisfy the requirement that energy must be bounded everywhere, we must choose n to be the smallest positive integer (cf. Karp and Karal (1962)). Thus we choose $n = 1$ so that

$$\lambda = \frac{\pi}{\alpha}$$

and hence, near the origin,

$$\phi \to A r^{\frac{\pi}{\alpha}} \cos \frac{\pi}{\alpha}\theta \text{ as } r \to 0.$$

Thus near the edge

$$\frac{\partial\phi}{\partial r} = 0\left(r^{\frac{\pi}{\alpha}-1}\right) \quad \text{as} \quad r \to 0.$$

where α is the angle of the wedge formed at the submerged edge.

For a thin barrier, near a submerged edge, the wedge angle is 2π, so that

$$\frac{\partial\phi}{\partial r} = 0\left(r^{-1/2}\right) \quad \text{as} \quad r \to 0.$$

For an obstacle in the form of a thick rectangular barrier, near a submerged edge, the wedge angle is $\frac{3\pi}{2}$, so that

$$\frac{\partial\phi}{\partial r} = 0\left(r^{-3/2}\right) \quad \text{as} \quad r \to 0.$$

These edge conditions play crucial role in the mathematical analysis of water wave scattering or radiation problems involving thin or thick barriers.

4 Breakwaters

Breakwaters are being constructed to protect a sheltered area from the impact of the rough sea. There are numerous models of breakwaters. The simplest models are perhaps thin vertical barriers. Water wave interaction with such models have been investigated in the water wave literature extensively for more than last five decades because of their simplicity in the engineering design and most importantly due to the ability to solve the associated scattering problems explicitly for normally incident surface water waves in infinitely deep water. The velocity potential of the resulting fluid motion can be obtained in closed form and the quantities of physical interest, namely the reflection and transmission coefficients, can also be obtained in terms of known functions or some definite integrals. A variety of mathematical techniques have been utilized to obtain the explicit solutions. These have enriched enoromously the classical applied mathematics in general and the linearised theory of water waves in particular. The reason for the existence of explicit solutions for the aforesaid class of problems lies in the fact that each of these problems is equivalent to solving the two-dimensional Laplace equation in a half plane with the condition of zero normal derivative of the function being sought for on the barrier and the mixed condition on the free surface. By the use of the complex variable theory, each problem can be reduced to finding a complex function satisfying certain conditions, which is somewhat straight forward in principle. For obliquely incident waves, the governing partial differential equation is no longer the Laplace equation, and as such the complex variable theory is no longer applicable. Thus it is perhaps no longer possible to obtain explicit solutions. The same conclusion also applies if the water is not infinitely deep even when the waves are normally incident on a barrier. In these cases, the associated problems are generally tackled by some approximate methods to obtain numerical estimates for the reflection and transmission coefficients.

For obliquely incident waves in deep water the scattering problems involving thin vertical barriers, complementary bounds for the reflection coefficient can be obtained by utilizing Galerkin approximations for solving two integral equations of first kind arising in each problem in complementary intervals, one in terms of the difference of potential function across the barrier and the other in terms of the horizontal component of velocity across the gap. For the case of a partially immersed or completely submerged thin vertical plate, single term approximation involving the corresponding explicit solution for normally incident wave training deep water provides very close bounds for the reflection coefficient and as such their average produces very accurate numerical estimates for this. However, for other barrier configurations, such single term approximations do not provide close bounds and as such multi-term approximations in terms of some basis functions are utilized. Although in principle any set of independent functions would serve as the basis functions, in practice, the basis functions are chosen in some judicious manner. For thin barriers in finite depth water these are chosen in terms of Chebyshev polynomials. For thick barriers with rectangular cross section in finite depth water, the same Galerkin approximation technique can be utilized. However, in this case, the basis functions are chosen in terms of functions involving ultraspherical Gegenbauer polynomials of order one sixth to account for the cube root of singularity of the velocity near a submerged edge which was mentioned earlier.

In the monograph by Mandal and Chakrabarti (2000) and the handbook by Linton and McIver(2001) almost exhaustive list of references of research work in this area are available.

5 Trapped waves

Study of *trapped waves* or *edge waves* is an area of intense and active research in recent years. Trapped waves are confined within a finite region of the fluid. Nearly one and half century back Stokes discovered the existence of edge waves over a sloping beach. These waves travel unchanged in the direction of the shore line but decays exponentially in the seaward direction. Thus the waves are *trapped* by a sloping boundary inspite of the unbounded fluid region. For almost a century after this discovery of Stokes, it was felt that such waves are not important and no study on these were made until exactly fifty years back when Ursell (1951) proved the existence of trapped waves above a submerged horizontal circular cylinder of small radius. He showed that these trapped waves travel along the top generator of the cylinder without changes while they decay to zero in a horizontal direction perpendicular to the generator.

There is a close relationship between the problem of *uniqueness* of solution to the linearised water wave problem for time harmonic motions involving a body or bodies and the problem of existence of trapped waves in the presence of the body or bodies. Obviously if the water wave problem possesses non-unique solution or no solution then there will perhaps exist trapped waves. Considerable effort has been made by many researchers to obtain a general uniqueness theorem valid for any general body, but success could not be achieved. Finally the question of uniqueness of solution was put to rest by M. McIver (1996) who provided an explicit example of non-uniqueness by considering two identical line sources at the free surface placed at such a distance that their effect at either infinity is cancelled. She showed that there are certain streamlines of the resulting fluid motion which enter the sources from above the free surface. These can be replaced by pairs of rigid bodies enclosing the sources and having an open free surface. After the publication of this landmark work, there is a tremendous amount of interest amongst the researchers on water waves to discover different situations in which trapped wave can exist within the framework of linearised theory of water waves.

Acknowledgement

This work is supported by the National Board for Higher Mathematics, Bombay.

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