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The Wick calculus of pseudo-differential operators and some of its applications

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1. Introduction

In the present survey paper, we are willing to give an overview of the so-called wave packets methods in PDE. These methods were revived in western mathematics by the papers [CF],[U], although they go back actually to a long tradition initiated by Berezin [Be] in the Soviet Union. The same type of transformation was used in [La] to handle PDE in infinite dimension. We are using here the name Wick quantization as a reminiscence of the so-called Wick symbols as they are described in the book [Sh].

In section 2, we start from scratch and we give a self-contained introduction to Fourier analysis, using as an essential feature the wave packets transformation.

**Key words and phrases.* Poisson formula, Wave Packets, Fourier transform, Gabor wavelet, Wick symbol, FBI transform, Energy estimates

To illustrate the versatility of this tool, we provide a simple proof of the Poisson summation formula. Section 3 describes more systematically various positive quantizations which are widely used, sometimes implicitely, in the Physics literature under the generic name of coherent states. Some composition formulas for these quantizations are given along with their proofs, since we believe that they are original and quite simple and were useful to proving various a priori estimates for PDE. In section 4, we recall the remarkable fact that the regularity of a function can be read on its wave packets transform, for the Sobolev regularity as well as for the Gevrey regularity. The investigation of the analytic regularity via the Fourier-Bros-Iagolnitzer transformation [BI] has a now long history and is closely related to our approach. Section 5 opens the wide topic of energy estimates; we show on two typical theorems how the Wick quantization can be useful to proving a priori estimates.

2. Elementary Fourier analysis via wave packets

Let u be a function in the Schwartz class of rapidly decreasing functions $\mathcal{S}(\mathbb{R}^n)$: it means that u is a C^∞ function on $\mathbb{R}^n$ such that for all multi-indices† α, β

$$\sup_{x \in \mathbb{R}^n} |x^\alpha \partial_x^\beta u(x)| = C_{\alpha\beta} < \infty.$$

A simple example of such a function is $e^{-|x|^2}$, ($|x|$ is the Euclidean norm of x) and more generally if A is a symmetric positive definite $n \times n$ matrix the function

$$v_A(x) = e^{-\pi \langle Ax, x \rangle} \tag{2.1}$$

belongs to the Schwartz class. The Fourier transform of u is defined as

$$\hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-2i\pi x \cdot \xi} u(x) dx. \tag{2.2}$$

It is an easy matter to check that the Fourier transform sends $\mathcal{S}(\mathbb{R}^n)$ into itself ‡. Moreover, for A as above, we have

† $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n, x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$, $\beta \in \mathbb{N}^n, \partial_x^\beta = \partial_{x_1}^{\beta_1} \dots \partial_{x_n}^{\beta_n}$.

‡ Just notice that

$$\xi^\alpha \partial_\xi^\beta \hat{u}(\xi) = \int e^{-2i\pi x \xi} \partial_x^\alpha (x^\beta u)(x) dx (2i\pi)^{|\beta| - |\alpha|} (-1)^{|\beta|}.$$

(2.3) $$\hat{v}_A(\xi) = (detA)^{-1/2} e^{-\pi\langle A^{-1}\xi,\xi\rangle}.$$

In fact, diagonalizing the symmetric matrix A, it is enough to prove the one-dimensional version of (2.3), i.e. to check

$$\int e^{-2i\pi x\xi}\, e^{-\pi x^2}\, dx = \int e^{-\pi(x+i\xi)^2}\, dx e^{-\pi\xi^2} = e^{-\pi\xi^2},$$

where the second equality can be obtained by taking the ξ-derivative of $\int e^{-\pi(x+i\xi)^2}$ dx. Using (2.3) we calculate for $u \in \mathcal{S}(\mathbb{R}^n)$ and $\epsilon > 0$, dealing with absolutely converging integrals,

$$\begin{aligned} u_\epsilon(x) &= \int e^{2i\pi x\xi} \hat{u}(\xi) e^{-\pi\epsilon^2|\xi|^2} d\xi \\ &= \iint e^{2i\pi x\xi}\, e^{-\pi\epsilon^2|\xi|^2} u(y) e^{-2i\pi y\xi} dy d\xi \\ &= \int u(y) e^{-\pi\epsilon^{-2}|x-y|^2} \epsilon^{-n} dy \\ &= \int \underbrace{(u(x+\epsilon y) - u(x))}_{\text{with absolute value} \le \epsilon|y|\|u'\|_{L^\infty}} \quad e^{-\pi|y|^2}\, dy + u(x). \end{aligned}$$

Taking the limit when ϵ goes to zero, we get the Fourier inversion formula

(2.4) $$u(x) = \int e^{2i\pi x\xi} \hat{u}(\xi) d\xi.$$

So far we have just proved that the Fourier transform is an isomorphism of the Schwartz class and provided an explicit inversion formula. This was devised to refresh our memory on this topic and we want now to move forward with the definition of our wave packets. We define for $x \in \mathbb{R}^n, (y,\eta) \in \mathbb{R}^n \times \mathbb{R}^n$

(2.5) $$\varphi_{y,\eta}(x) = 2^{n/4} e^{-\pi(x-y)^2} e^{2i\pi(x-y)\cdot\eta} = 2^{n/4} e^{-\pi(x-y-i\eta)^2} e^{-\pi\eta^2}$$

where for $\zeta = (\zeta_1, \dots, \zeta_n) \in \mathbb{C}^n$, we set

(2.6) $$\zeta^2 = \sum_{1\le j\le n} \zeta_j^2 \,.$$

We note that the function $\varphi_{y,\eta}$ is in $\mathcal{S}(\mathbb{R}^n)$ and with L^2 norm 1. In fact, $\varphi_{y,\eta}$ appears as a *phase translation* of a normalized Gaussian. The following lemma introduces the wave packets transform as a Gabor wavelet.

Lemma 2.1. *Let u be a function in the Schwartz class $\mathcal{S}(\mathbb{R}^n)$. We define*

$$W u(y,\eta) = \langle u, \varphi_{y,\eta}\rangle_{L^2(\mathbb{R}^n)} = 2^{n/4}\int u(x)e^{-\pi(x-y)^2}e^{-2i\pi(x-y)\cdot\eta}dx$$

$$= 2^{n/4}\int u(x)e^{-\pi(y-i\eta-x)^2}dx e^{-\pi\eta^2}. \tag{2.7}$$

For $u\in L^2(\mathbb{R}^n)$, the function Tu defined by

$$(Tu)(y+i\eta) = e^{\pi\eta^2}Wu(y,-\eta) = 2^{n/4}\int u(x)e^{-\pi(y+i\eta-x)^2}dx$$

is an entire function. The mapping $u\mapsto Wu$ is continuous from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^{2n})$ and isometric from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^{2n})$. Moreover, we have the reconstruction formula

$$u(x) = \iint_{\mathbb{R}^n\times\mathbb{R}^n} Wu(y,\eta)\varphi_{y,\eta}(x)dyd\eta. \tag{2.8}$$

Proof. For u in $\mathcal{S}(\mathbb{R}^n)$, we have

$$Wu(y,\eta) = e^{2i\pi y\eta}\widehat{\Omega}^1(\eta,y)$$

where $\widehat{\Omega}^1$ is the Fourier transform with respect to the first variable of the $\mathcal{S}(\mathbb{R}^{2n})$ function $\Omega(x,y) = u(x)e^{-\pi(x-y)^2}2^{n/4}$. Thus the function Wu belongs to $\mathcal{S}(\mathbb{R}^{2n})$. It makes sense to compute

$$2^{-n/2}\langle Wu, Wu\rangle_{L^2(\mathbb{R}^{2n})} =$$

$$\lim_{\epsilon\to 0_+}\int u(x_1)\overline{u}(x_2)e^{-\pi[(x_1-y)^2+(x_2-y)^2+2i(x_1-x_2)\eta+\epsilon^2\eta^2]}dyd\eta dx_1dx_2.$$

Now the last integral on $\mathbb{R}^{4n}$ converges absolutely and we can use without shame the Fubini theorem. Integrating with respect to η involves the Fourier transform of a Gaussian function and we get $\epsilon^{-n}e^{-\pi\epsilon^{-2}(x_1-x_2)^2}$. Since

$$2(x_1-y)^2+2(x_2-y)^2 = (x_1+x_2-2y)^2+(x_1-x_2)^2,$$

integrating with respect to y yields a factor $2^{-n/2}$. We are left with

$$\langle Wu, Wu\rangle_{L^2(\mathbb{R}^{2n})} = \lim_{\epsilon\to 0_+} \int u(x_1)\, \overline{u}(x_2) e^{-\pi(x_1-x_2)^2/2} \epsilon^{-n} e^{-\pi\epsilon^{-2}(x_1-x_2)^2} dx_1 dx_2.$$

Changing the variables, the integral is

$$\lim_{\epsilon\to 0_+} \int u(s+\epsilon t/2)\, \overline{u}(s-\epsilon t/2) e^{-\pi\epsilon^2 t^2/2} e^{-\pi t^2} dt ds = ||u||^2_{L^2(\mathbb{R}^n)}$$

by Lebesgue's dominated convergence theorem: the triangle inequality and the estimate $|u(x)| \le C(1+|x|)^{-n-1}$ imply, with $v = u/C$,

$$\begin{aligned} |v(s+\epsilon t/2)\, \overline{v}(s-\epsilon t/2)| &\le (1+|s+\epsilon t/2|)^{-n-1}(1+|s+\epsilon t/2|)^{-n-1} \\ &\le (1+|s+\epsilon t/2| + |s-\epsilon t/2|)^{-n-1} \\ &\le (1+2|s|)^{-n-1}. \end{aligned}$$

Eventually, this proves that

$$||Wu||^2_{L^2(\mathbb{R}^{2n})} = ||u||^2_{L^2(\mathbb{R}^n)} \tag{2.9}$$

i.e

$$W : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^{2n}) \qquad \text{with} \qquad W^*W = \mathrm{id}_{L^2(\mathbb{R}^n)}.$$

Noticing first that $\iint Wu(y,\eta)\varphi_{y,\eta} dy d\eta$ belongs to $L^2(\mathbb{R}^n)$ (with a norm smaller than $||Wu||_{L^1(\mathbb{R}^{2n})}$) and applying Fubini's theorem, we get from the polarization of (2.9) for $u, v \in \mathcal{S}(\mathbb{R}^n)$,

$$\begin{aligned} \langle u, v\rangle_{L^2(\mathbb{R}^n)} &= \langle Wu, Wv\rangle_{L^2(\mathbb{R}^{2n})} \\ &= \iint Wu(y,\eta)\langle \varphi_{y,\eta}, v\rangle_{L^2(\mathbb{R}^n)} dy d\eta \\ &= \left\langle \iint Wu(y,\eta)\varphi_{y,\eta} dy d\eta, v \right\rangle_{L^2(\mathbb{R}^n)} \end{aligned}$$

yielding the result of the lemma $u = \iint Wu(y,\eta)\varphi_{y,\eta} dy d\eta$.

The following lemma is in fact the Poisson summation formula for Gaussian functions in one dimension.

Lemma 2.2. *For all complex numbers z, the following series are absolutely converging and*

$$\sum_{m\in\mathbb{Z}} e^{-\pi(z+m)^2} = \sum_{m\in\mathbb{Z}} e^{-\pi m^2} e^{2i\pi m z}. \tag{2.10}$$

Proof. We set $\omega(z) = \sum_{m\in\mathbb{Z}} e^{-\pi(z+m)^2}$. The function ω is entire and 1-periodic since for all $m \in \mathbb{Z}$, $z \mapsto e^{-\pi(z+m)^2}$ is entire and for $R > 0$

$$\sup_{|z|\le R} |e^{-\pi(z+m)^2}| \le \sup_{|z|\le R} |e^{-\pi z^2}| e^{-\pi m^2} e^{2\pi|m|R} \in l^1(\mathbb{Z}).$$

Consequently, for $z \in \mathbb{R}$, we obtain, expanding ω in Fourier series [§]

$$\omega(z) = \sum_{k\in\mathbb{Z}} e^{2i\pi k z} \int_0^1 \omega(x) e^{-2i\pi k x} dx.$$

We also check, using Fubini's theorem on $L^1(0,1) \times l^1(\mathbb{Z})$

$$\begin{aligned}\int_0^1 \omega(x) e^{-2i\pi k x} dx &= \sum_{m\in\mathbb{Z}} \int_0^1 e^{-\pi(x+m)^2} e^{-2i\pi k x} dx \\ &= \sum_{m\in\mathbb{Z}} \int_m^{m+1} e^{-\pi t^2} e^{-2i\pi k t} dt \\ &= \int_{\mathbb{R}} e^{-\pi t^2} e^{-2i\pi k t} = e^{-\pi k^2}.\end{aligned}$$

So (2.10) is proved for real z and since both sides are entire functions, we conclude by analytic continuation. □

It is now straightforward to get the n-th dimensional version of lemma 2.2: for all $z \in \mathbb{C}^n$, using the notation (2.6), we have

$$\sum_{m\in\mathbb{Z}^n} e^{-\pi(z+m)^2} = \sum_{m\in\mathbb{Z}^n} e^{-\pi m^2} e^{2i\pi m\cdot z}. \tag{2.11}$$

Theorem 2.3. The Poisson summation formula *Let n be a positive integer and u be a function in $\mathcal{S}(\mathbb{R}^n)$. Then*

[§] Note that we use this expansion only for a C^∞ 1-periodic function. The proof is simple and requires only to compute $1 + 2\mathrm{Re}\sum_{1\le k\le N} e^{2i\pi k x} = \frac{\sin\pi(2N+1)x}{\sin\pi x}$. Then one has to show that for a smooth 1-periodic function ω such that $\omega(0) = 0$,

$$\lim_{\lambda\to+\infty} \int_0^1 \frac{\sin\lambda x}{\sin\pi x}\omega(x)dx = 0,$$

which is obvious since for a smooth ν (here we take $\nu(x) = \omega(x)/\sin\pi x$), $|\int_0^1 \nu(x)\sin\lambda x dx| = O(\lambda^{-1})$ by integration by parts.

$$\sum_{k\in\mathbb{Z}^n} u(k) = \sum_{k\in\mathbb{Z}^n} \hat{u}(k), \tag{2.12}$$

where $\hat{u}$ stands for the Fourier transform (2.2).

Proof. We write, according to (2.8) and to Fubini's theorem

$$\begin{aligned}\sum_{k\in\mathbb{Z}^n} u(k) &= \sum_{k\in\mathbb{Z}^n} \iint Wu(y,\eta)\varphi_{y,\eta}(k)dyd\eta \\ &= \iint Wu(y,\eta)\sum_{k\in\mathbb{Z}^n}\varphi_{y,\eta}(k)dyd\eta.\end{aligned} \tag{2.13}$$

Now, (2.11), (2.5) and (2.3) give

$$\sum_{k\in\mathbb{Z}^n}\varphi_{y,\eta}(k) = \sum_{k\in\mathbb{Z}^n}\widehat{\varphi}_{y,\eta}(k),$$

so that (2.13), (2.8) and Fubini's theorem imply (2.12). □

It is a simple matter to introduce at this point the dual space of the Fréchet $\mathcal{S}(\mathbb{R}^n)$, that is the space $\mathcal{S}'(\mathbb{R}^n)$ of tempered distributions (the continuous linear forms on $\mathcal{S}(\mathbb{R}^n)$). We can define the Fourier transform on $\mathcal{S}'(\mathbb{R}^n)$ by duality¶:

$$\langle \widehat{T},\varphi\rangle_{\mathcal{S}'(\mathbb{R}^n),\mathcal{S}(\mathbb{R}^n)} = \langle T,\widehat{\varphi}\rangle_{\mathcal{S}'(\mathbb{R}^n),\mathcal{S}(\mathbb{R}^n)}, \tag{2.14}$$

so that the inversion formula (2.4) still holds for $T\in\mathcal{S}'(\mathbb{R}^n)$ and reads

$$T = \check{\hat{\hat{T}}}, \qquad \text{with} \qquad \langle\check{T},\varphi\rangle = \langle T,\check{\varphi}\rangle, \qquad \check{\varphi}(x) = \varphi(-x).$$

Using duality, it is a matter of routine left to the reader to give a version of lemma 2.1 for tempered distributions. Now theorem 2.3 can be given a more compact version saying that the tempered distribution $D_0 = \sum_{k\in\mathbb{Z}^n}\delta_k$ is such that $\widehat{D_0} = D_0$.

We shall need as well a parametric version of wave packets, and we state here a lemma analogous to lemma 2.1, whose proof is left to the reader. We define for $x\in\mathbb{R}^n$, $(\lambda,y,\eta)\in\mathbb{R}_+^*\times\mathbb{R}^n\times\mathbb{R}^n$,

¶In the formula below, we deal with real duality, so that, if T,φ are in $L^2(\mathbb{R}^n)$, $\langle T,\overline{\varphi}\rangle_{\mathcal{S}'(\mathbb{R}^n),\mathcal{S}(\mathbb{R}^n)} = \langle T,\varphi\rangle_{L^2(\mathbb{R}^n)}$.

(2.15) $\varphi_{y,\eta}^{\lambda}(x) = (2\lambda)^{n/4} e^{-\pi\lambda(x-y)^2} e^{2i\pi(x-y)\cdot\eta} = (2\lambda)^{n/4} e^{-\pi\lambda(x-y-i\lambda^{-1}\eta)^2} e^{-\pi\lambda^{-1}\eta^2}$.

We note that the function $\varphi_{y,\eta}^{\lambda}$ is in $\mathcal{S}(\mathbb{R}^n)$ and with L^2 norm 1.

Lemma 2.4. *Let u be a function in the Schwartz class $\mathcal{S}(\mathbb{R}^n)$. We define, for $(\lambda, y, \eta) \in \mathbb{R}_+^* \times \mathbb{R}^n \times \mathbb{R}^n$,*

$$W_\lambda u(y,\eta) = \langle u, \varphi_{y,\eta}^{\lambda}\rangle_{L^2(\mathbb{R}^n)} = (2\lambda)^{n/4} \int u(x) e^{-\lambda\pi(x-y)^2} e^{-2i\pi(x-y)\cdot\eta} dx$$

$$= (2\lambda)^{n/4} \int u(x) e^{-\pi\lambda(y-i\lambda^{-1}\eta-x)^2} dx e^{-\pi\lambda^{-1}\eta^2}. \tag{2.16}$$

For $u \in L^2(\mathbb{R}^n)$, the function $T_\lambda u$ defined by

$$(T_\lambda u)(y+i\eta) = \lambda^{-n/4} e^{\pi\lambda\eta^2} W_\lambda u(y, -\lambda\eta) = 2^{n/4} \int u(x) e^{-\pi\lambda(y+i\eta-x)^2} dx \tag{2.17}$$

is an entire function. The mapping $u \mapsto W_\lambda u$ is continuous from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^{2n})$ and isometric from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^{2n})$. Moreover, we have the reconstruction formula for each positive λ,

$$u(x) = \iint_{\mathbb{R}^n \times \mathbb{R}^n} W_\lambda u(y,\eta) \varphi_{y,\eta}^{\lambda}(x) dy d\eta. \tag{2.18}$$

In the next section, we shall clarify the role played by the Gaussian functions in these formulas.

3. A family of non-negative quantizations

A few facts on classical and Weyl quantizations. Let $a(x,\xi)$ be a classical Hamiltonian defined on $\mathbb{R}^n \times \mathbb{R}^n$. The Weyl quantization rule associates to this function the operator a^w defined on functions $u(x)$ as

$$(a^w u)(x) = \iint e^{2i\pi(x-y)\cdot\xi} a(\frac{x+y}{2}, \xi) u(y) dy d\xi. \tag{3.1}$$

For instance we have $(x\cdot\xi)^w = (x \cdot D_x + D_x \cdot x)/2$, with $D_x = \frac{1}{2i\pi}\frac{\partial}{\partial x}$ whereas the *classical* quantization rule would map the Hamiltonian $x\cdot\xi$ to the operator $x \cdot D_x$. A nice feature of the Weyl quantization rule, introduced in 1928 by Hermann Weyl in [Wy], is the fact that real Hamiltonians get quantized by (formally) self-adjoint operators. Let us recall that the classical quantization of the Hamiltonian $a(x,\xi)$ is given by the operator $Op(a)$ acting on functions $u(x)$ by

$$(3.2)\qquad (Op(a)u)(x) = \int e^{2i\pi x\cdot\xi}\, a(x,\xi)\, \widehat{u}(\xi) d\xi.$$

In fact, introducing the following one-parameter group $J^t = \exp 2i\pi t D_x \cdot D_\xi$, given by the integral formula

$$(J^t a)(x,\xi) = |t|^{-n} \iint e^{-2i\pi t^{-1} y\cdot\eta} a(x+y, \xi+\eta) dy d\eta,$$

we see that

$$(Op(J^t a)u)(x) = \iint e^{2i\pi(x-y)\cdot\xi}\, a((1-t)x + ty, \xi) u(y) dy d\xi.$$

In particular one gets $a^w = Op(J^{1/2}a)$. Moreover since $(Op(a))^* = Op(J\bar{a})$ we obtain

$$(a^w)^* = Op(J(\overline{J^{1/2}a})) = Op(J^{1/2}\bar{a}) = (\bar{a})^w,$$

yielding formal self-adjointness for real a. Formula (3.1) can be written as

$$(3.3)\qquad (a^w u, v) = \iint a(x,\xi)\mathcal{H}(u,v)(x,\xi) dx d\xi,$$

where the *Wigner function* $\mathcal{H}$ is defined as

$$(3.4)\qquad \mathcal{H}(u,v)(x,\xi) = \int u(x+\frac{y}{2})\bar{v}(x-\frac{y}{2}) e^{-2i\pi y\cdot\xi} dy.$$

The mapping $(u,v) \mapsto \mathcal{H}(u,v)$ is sesquilinear continuous from $\mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^{2n})$ so that a^w makes sense for $a \in \mathcal{S}'(\mathbb{R}^{2n})$ (here $u, v \in \mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}^*$ stands for the antidual):

$$\langle a^w u, v\rangle_{\mathcal{S}^*(\mathbb{R}^n), \mathcal{S}(\mathbb{R}^n)} = \langle a, \mathcal{H}(u,v)\rangle_{\mathcal{S}'(\mathbb{R}^{2n}), \mathcal{S}(\mathbb{R}^{2n})}.$$

The Wigner function also satisfies, since $\mathcal{H}(u,v)$ is the partial Fourier transform of the function $(x,y) \mapsto u(x+y/2)\bar{v}(x-y/2)$,

$$||\mathcal{H}(u,v)||_{L^2(\mathbb{R}^{2n})} = ||u||_{L^2(\mathbb{R}^n)} ||v||_{L^2(\mathbb{R}^n)},$$

$$(3.5)\qquad \mathcal{H}(u,v)(x,\xi) = 2^n \langle \sigma_{x,\xi} u, v\rangle_{L^2(\mathbb{R}^n)},$$

$$\text{with}\quad (\sigma_{x,\xi} u)(y) = u(2x-y)\exp -4i\pi(x-y)\cdot\xi,$$

and the phase symmetries σ_X are unitary and selfadjoint operators on $L^2(\mathbb{R}^n)$. We have also ([U], [Wy]),

$$a^w = \int_{\mathbb{R}^{2n}} a(X)2^n\sigma_X dX = \int_{\mathbb{R}^{2n}} \widehat{a}(\Xi)\exp(2i\pi\Xi\cdot M)d\Xi, \tag{3.6}$$

where $\Xi\cdot M = \hat{x}\cdot x + \hat{\xi}\cdot D_x$ (here $\Xi = (\hat{x},\hat{\xi})$). These formulas give in particular

$$||a^w||_{\mathcal{L}(L^2)} \le \min(2^n||a||_{L^1(\mathbb{R}^{2n})}, ||\widehat{a}||_{L^1(\mathbb{R}^{2n})}), \tag{3.7}$$

where $\mathcal{L}(L^2)$ stands for the space of bounded linear maps from $L^2(\mathbb{R}^n)$ into itself.

As shown below, the symplectic invariance of the Weyl quantization is actually its most important property. Let us consider a finite dimensional real vector space E (the configuration space $\mathbb{R}^n_x$) and its dual space E^* (the momentum space $\mathbb{R}^n_\xi$). The phase space is defined as $\Phi = E\oplus E^*$; its running point will be denoted in general by a capital letter $(X = (x,\xi), Y = (y,\eta))$. The symplectic form on Φ is given by

$$[(x,\xi),(y,\eta)] = \langle\xi,y\rangle_{E^*,E} - \langle\eta,x\rangle_{E^*,E}, \tag{3.8}$$

where $\langle\cdot,\cdot\rangle_{E^*,E}$ stands for the bracket of duality. The symplectic group is the subgroup of the linear group of Φ preserving (3.8). With

$$\sigma = \begin{pmatrix} 0 & \mathrm{Id(E^*)} \\ -\mathrm{Id(E)} & 0 \end{pmatrix},$$

we have for $X,Y\in\Phi$, $[X,Y] = \langle\sigma X, Y\rangle_{\Phi^*,\Phi}$, so that the equation of the symplectic group is $A^*\sigma A = \sigma$. One can describe a set of generators for the symplectic group $Sp(n)$, identifying Φ with $\mathbb{R}^n_x\times\mathbb{R}^n_\xi$: the mappings

(i) $(x,\xi)\mapsto(Tx,{}^tT^{-1}\xi)$, where T is an automorphism of E,

(ii) $(x_k,\xi_k)\mapsto(\xi_k, -x_k)$, and the other coordinates fixed,

(iii) $(x,\xi)\mapsto(x,\xi+Sx)$, where S is symmetric from E to E^*.

We then describe the metaplectic group, introduced by André Weil [Wi]. The metaplectic group $Mp(n)$ is the subgroup of the group of unitary transformations of $L^2(\mathbb{R}^n)$ generated by

(j) $(M_T u)(x) = |\det T|^{-1/2}u(T^{-1}x)$, where T is an automorphism of E,

(jj) Partial Fourier transformation, with respect to x_k for $k = 1,\dots,n$, (jjj) Multiplication by $\exp(i\pi\langle Sx, x\rangle))$ where S is symmetric from E to E^*.

There exists a two-fold covering (the π_1 of both $Mp(n)$ and $Sp(n)$ is $\mathbb{Z}$)

$$\pi : Mp(n) \to Sp(n)$$

such that, if $\chi = \pi(M)$ and u, v are in $L^2(\mathbb{R}^n), \mathcal{H}(u, v)$ is their Wigner function,

$$\mathcal{H}(Mu, Mv) = \mathcal{H}(u, v) \circ \chi^{-1}.$$

This is Segal formula [S] which could be rephrased as follows. Let $a \in \mathcal{S}'(\mathbb{R}^{2n})$ and $\chi \in Sp(n)$. There exists M in the fiber of χ such that

$$(a \circ \chi)^w = M^* a^w M. \tag{3.9}$$

In particular, the images by π of the transformations (j), (jj), (jjj) are respectively (i), (ii), (iii). Moreover, if χ is the phase translation, $\chi(x, \xi) = (x + x_0, \xi + \xi_0)$, (3.9) is fulfilled with $M = \tau_{x_0,\xi_0}$, the phase translation given by

$$(\tau_{x_0,\xi_0} u)(y) = u(y - x_0)\ e^{2i\pi\langle y - \frac{x_0}{2}, \xi_0\rangle}.$$

If χ is the symmetry with respect to (x_0, ξ_0), M in (3.9) is, up to a unit factor, the phase symmetry σ_{x_0,ξ_0} defined above. This yields the following composition formula $a^w b^w = (a\sharp b)^w$ with

$$(a\sharp b)(X) = 2^{2n} \iint e^{-4i\pi[X-Y, X-Z]} a(Y) b(Z) dY dZ,$$

with an integral on $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$. We can compare this with the classical composition formula,

$$\mathrm{Op\,(a)Op\,(b)} = \mathrm{Op(a \circ b)}$$

(cf.(3.2)) with

$$(a \circ b)(x, \xi) = \iint e^{-2i\pi y \cdot \eta} a(x, \xi + \eta) b(y + x, \xi) dy d\eta,$$

with an integral on $\mathbb{R}^n \times \mathbb{R}^n$. It is convenient to give an asymptotic version of these compositions formulae, e.g. in the semi-classical[‖] case. Let m be a real number. A smooth function $a(x, \xi, \lambda)$ defined on $\mathbb{R}^n_x \times \mathbb{R}^n_\xi \times [1, +\infty)$ is in the

[‖] We have developed a taste for using a *large* parameter λ instead of a small Planck constant h. writing $\lambda = 1/h$ will give back the more familiar picture.

symbol class $S^m_{\rm scl}$ if

$$\sup_{(x,\xi)\in\mathbb{R}^{2n},\lambda\ge 1}|D_x^\alpha D_\xi^\beta a(x,\xi,\lambda)|\lambda^{-m+|\beta|}<\infty. \tag{3.10}$$

Then one has for $a\in S^{m_1}_{\rm scl}$ and $b\in S^{m_2}_{\rm scl}$, the expansion

$$(a\sharp b)(x,\xi)=\sum_{0\le k<N}2^{-k}\sum_{|\alpha|+|\beta|=k}\frac{(-1)^{|\beta|}}{\alpha!\beta!}D_\xi^\alpha\partial_x^\beta a\ D_\xi^\beta\partial_x^\alpha b+r_N(a,b), \tag{3.11}$$

with $r_N(a,b)\in S^{m_1+m_2-N}_{\rm scl}$. The beginning of this expansion is thus $ab+\frac{1}{2\iota}\{a,b\}$, where

$$\{a,b\}=\sum_{1\le j\le n}\partial_{\xi_j}a\ \partial_{x_j}b-\partial_{x_j}a\ \partial_{\xi_j}b$$

is the Poisson bracket and $\iota=2\pi i$. The sums inside (3.11) with k even are symmetric in a,b and skew-symmetric for k odd. This can be compared to the classical expansion formula

$$(a\circ b)(x,\xi)=\sum_{|\alpha|<N}\frac{1}{\alpha!}D_\xi^\alpha a\ \partial_x^\alpha b+t_N(a,b),$$

with $t_N(a,b)\in S^{m_1+m_2-N}_{\rm scl}$.

Definition and first properties of the Wick quantization. Let Γ be an Euclidean norm on $\mathbb{R}^{2n}$, identified with a $2n\times 2n$ symmetric matrix ; we define $\Gamma^\sigma=\sigma^*\Gamma^{-1}\sigma$, where $\sigma=\begin{pmatrix}0 & I_n\\ -I_n & 0\end{pmatrix}$. We shall say that Γ is a symplectic norm whenever $\Gamma=\Gamma^\sigma$. The basic examples of symplectic norms that we are going to use are

$$\Gamma_\lambda=\lambda|dx|^2+\frac{|d\xi|^2}{\lambda}=\begin{pmatrix}\lambda I_n & 0\\ 0 & \lambda^{-1}I_n\end{pmatrix}, \tag{3.12}$$

where λ is a positive parameter. Our construction of the Wick quantization could be carried out for any symplectic norm, however, for simplicity, we shall limit ourselves to the norms (3.12). The following definition contains also some classical properties.

Definition 3.1. Let $Y=(y,\eta)$ be a point in $\mathbb{R}^{2n}$ and $\lambda>0$. We define first the operator

(3.13) $$\Sigma_Y^\lambda = [2^n e^{-2\pi\Gamma_\lambda(\cdot - Y)}]^w.$$

This is a rank-one orthogonal projection: using the notations (2.15-16), we have

(3.14) $$\Sigma_Y^\lambda u = (W_\lambda u)(Y)\varphi_Y^\lambda = \langle u, \varphi_Y^\lambda\rangle_{L^2(\mathbb{R}^n)}\varphi_Y^\lambda.$$

Let a be in $L^\infty(\mathbb{R}^{2n})$. The Wick($\lambda$) quantization of a is defined as

(3.15) $$a^{\mathrm{Wick}(\lambda)} = \int_{\mathbb{R}^{2n}} a(Y)\Sigma_Y^\lambda dY.$$

To check (3.13), starting from (3.14) is an easy exercise on the Weyl quantization left to the reader.

Proposition 3.2. *Let λ be a positive number and a be in $L^\infty(\mathbb{R}^{2n})$. Then*

(3.16) $$a^{\mathrm{Wick}(\lambda)} = W_\lambda^* a^\mu W_\lambda, \quad 1^{\mathrm{Wick}(\lambda)} = \mathrm{Id}_{L^2(\mathbb{R}^n)}$$

where W_λ is the isometric mapping from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^{2n})$ given in (2.16), and a^μ the operator of multiplication by a in $L^2(\mathbb{R}^{2n})$. The operator $\pi_{H_\lambda} = W_\lambda W_\lambda^*$ is the orthogonal projection on a closed proper subspace H_λ of $L^2(\mathbb{R}^{2n})$. Moreover, we have

(3.17) $$\|a^{\mathrm{Wick}(\lambda)}\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \le \|a\|_{L^\infty(\mathbb{R}^{2n})},$$

(3.18) $$a(X) \ge 0 \Longrightarrow a^{\mathrm{Wick}(\lambda)} \ge 0,$$

(3.19) $$\|\Sigma_Y^\lambda \Sigma_Z^\lambda\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \le 2^n e^{-\frac{\pi}{2}\Gamma_\lambda(Y-Z)}.$$

Proof. Here we assume that $\lambda = 1$ and omit the indexation by λ. The calculations are analogous for other positive values of λ. The first properties and (3.18) are immediate consequences of lemma 2.4. The operator π_H is an orthogonal projection on its range, which is the same as the range of W and the latter is closed since W is isometric. On the other hand, π_H is not onto, otherwise π_H would be the identity of $L^2(\mathbb{R}^{2n})$ and for all $u \in \mathcal{S}(\mathbb{R}^n)$, we would have

$$\|u\|^2_{L^2(\mathbb{R}^n)} = 2\mathrm{Re}\langle D_{x_1}u, ix_1 u\rangle_{L^2(\mathbb{R}^n)} = 2\mathrm{Re}\langle \xi_1^{\mathrm{Wick}}u, ix_1^{\mathrm{Wick}}u\rangle_{L^2(\mathbb{R}^n)}$$

$$= 2\mathrm{Re}\langle \xi_1 Wu, i\pi_H x_1 Wu\rangle_{L^2(\mathbb{R}^{2n})} = 2\mathrm{Re}\langle \xi_1 Wu, ix_1 Wu\rangle_{L^2(\mathbb{R}^{2n})} = 0.$$

Now, with $L^2(\mathbb{R}^n)$ dot-products, we have

$$\begin{aligned}|(a^{\mathrm{Wick}}u,v)| &= \left|\int_{\mathbb{R}^{2n}} a(Y)(\Sigma_Y u, v)dY\right| = \left|\int_{\mathbb{R}^{2n}} a(Y)(\Sigma_Y u, \Sigma_Y v)dY\right| \\ &\leq \|a\|_{L^\infty(\mathbb{R}^{2n})}\int_{\mathbb{R}^{2n}} \|\Sigma_Y u\|_{L^2(\mathbb{R}^n)}\|\Sigma_Y v\|_{L^2(\mathbb{R}^n)}dY \\ &\leq \|a\|_{L^\infty(\mathbb{R}^{2n})}\left(\int_{\mathbb{R}^{2n}} \|\Sigma_Y u\|_{L^2(\mathbb{R}^n)}dY\right)^{1/2}\left(\int_{\mathbb{R}^{2n}} \|\Sigma_Y v\|^2_{L^2(\mathbb{R}^n)}dY\right)^{1/2} \\ &= \|a\|_{L^\infty(\mathbb{R}^{2n})}\|u\|_{L^2(\mathbb{R}^n)}\|v\|_{L^2(\mathbb{R}^n)},\end{aligned}$$

yielding (3.17). For $Y, Z \in \mathbb{R}^{2n}$ a straightforward computation shows that the Weyl symbol of $\Sigma_Y\Sigma_Z$ is, as a function of the variable $X \in \mathbb{R}^{2n}$, setting $\Gamma_1(T) = |T|^2$

$$e^{-\frac{\pi}{2}|Y-Z|^2}e^{-2i\pi[X-Y,X-Z]}2^n e^{-2\pi|X-\frac{Y+Z}{2}|^2}.$$

Since for the Weyl quantization, one has $\|a^w\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \leq 2^n\|a\|_{L^1(\mathbb{R}^{2n})}$, we get the result (3.19). □

Remark: The positivity property (3.18) is not satisfied for the Weyl quantization since the Wigner function $\mathcal{H}(u,u)$ (see (3.4)) is not always non-negative, although it is actually positive if u is a Gaussian function. We leave to the reader the computation of $\mathcal{H}(u_1,u_1)(x,\xi)$ which is negative in a neighborhood V of the origin in $\mathbb{R}^{2n}$ for the choice $u_1(x) = x_1 e^{-\pi|x|^2}$. Now, choosing a non-negative $a(x,\xi) \in C_c^\infty(V)$ and using (3.3) we get $\langle a^w u_1, u_1\rangle < 0$.

Proposition 3.3. *Let m be a real number and $p(x,\xi,\lambda)$ be a symbol in S^m_{scl} (see (3.10)). Then*

$$p^{Wick(\lambda)} = p^w + r(p)^w, \tag{3.20}$$

with $r(p) \in S^{m-1}_{scl}$ so that the mapping $p \mapsto r(p)$ is continuous. Moreover, $r(p) = 0$ if p is a linear form or a constant.

Proof. From the definition 3.1, one has $P^{\mathrm{Wick}(\lambda)} = \tilde{p}^w$, with

$$\tilde{p}(X) = \int_{\mathbb{R}^{2n}} p(X+Y)\, e^{-2\pi\Gamma_\lambda(Y)}2^n dY \tag{3.21}$$

$$= p(X) + \underbrace{\int_0^1 \int_{\mathbb{R}^{2n}} (1-\theta) p''(X+\theta Y) Y^2 e^{-2\pi\Gamma_\lambda(Y)} 2^n dY d\theta}_{r(p)(X)}.$$

We note now that the estimates (3.10) of $S^m_{\rm scl}$ on p are equivalent to

$$|p^{(k)}(X)T^k| \le C_k \lambda^{m-\frac{k}{2}} \Gamma_\lambda(T)^{\frac{k}{2}} \quad \text{or} \quad |p^{(k)}(X)|_{\Gamma_\lambda} \le C_k \lambda^{m-\frac{k}{2}}.$$

Thus we get

$$|r(p)^{(k)}(X)|_{\Gamma_\lambda} \le C_{k+2} \lambda^{m-\frac{k+2}{2}} \int_{\mathbb{R}^{2n}} \Gamma_\lambda(Y) e^{-2\pi\Gamma_\lambda(Y)} 2^{n-1} dY,$$

and since $\det(\Gamma_\lambda) = 1$, the integral above is a constant and this implies that $r \in S^{m-1}_{\rm scl}$. The last point in the proposition follows from the formula (3.21) showing that $r(p)$ depends linearly on p''. □

Remark. For further understanding of our results, it would be better to use symbol classes defined by a metric in the phase space, as introduced in chapter 18 of [H1]. As we have seen above,

$$S^m_{\rm scl} = S(\lambda^m, \lambda^{-1}\Gamma_\lambda),$$

that is symbols such that

$$|a^{(k)}(X)T^k| \le \gamma_k(a) \lambda^{m-\frac{k}{2}} \Gamma_\lambda(T)^{\frac{k}{2}},$$

or more accurately, for all $k \in \mathbb{N}$,

$$\gamma_k(a) = \sup_{\substack{X\in\mathbb{R}^{2n}, \lambda\ge 1, \\ T\in\mathbb{R}^{2n}, \Gamma_\lambda(T)=1}} |a^{(k)}(X)T^k| \lambda^{-m+\frac{k}{2}} < +\infty.$$

Proposition 3.4. *Let $a \in L^\infty(\mathbb{R}^{2n}), b \in S^1_{scl}$, be real-valued functions. Then*

$$\text{(3.22)} \qquad Re\left(a^{Wick(\lambda)} b^{Wick(\lambda)}\right) = \left[ab - \frac{1}{4\pi} a'(Y) \cdot b'(Y)\right]^{Wick(\lambda)} + S,$$

where $\|S\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \le d_n \|a\|_{L^\infty} \gamma_2(b)$. Here $\gamma_2(b)$ is a semi-norm of b in S^1_{scl} and d_n depends only on the dimension.

Proof. We omit in the proof below the superscripts λ. We have

$$\begin{aligned} a^{\mathrm{Wick}} b^{\mathrm{Wick}} &= \iint a(Y)b(Z)\Sigma_Y\Sigma_Z dY dZ \\ &= \iint a(Y)\left[b(Y) + b'(Y)\cdot(Z-Y)\right]\Sigma_Y\Sigma_Z dY dZ \\ &\quad + \iint a(Y)\left[\int_0^1 (1-\theta)b''(Y+\theta(Z-Y))\,d\theta (Z-Y)^2\right]\Sigma_Y\Sigma_Z dY dZ \\ &= \int a(Y)b(Y)\Sigma_Y dY + \iint a(Y)b'(Y)\cdot(Z-Y)\Sigma_Y\Sigma_Z dY dZ + R, \end{aligned}$$

with

$$R = \iint \alpha(Y,Z)(Z-Y)^2\,\Sigma_Y\Sigma_Z\, dY dZ,$$

where the norm of the quadratic form $\alpha(Y,Z)$ is less than $||a||_{L^\infty}\gamma_2(b)$; here $\gamma_2(b)$ is a semi-norm of the symbol b. We need now to use the celebrated Cotlar's lemma in a version given in the paper [BL](lemme 4.2.3′) (see also [H1],[U]).

Lemma 3.5 (Cotlar's lemma). *Let $(\Omega, \mathcal{M}, \mu)$ be a σ-finite measured space where μ is a positive σ-finite measure and let* H *be a Hilbert space. Let $\omega \mapsto A_\omega$ be a weakly measurable mapping from Ω into $\mathcal{L}(\mathrm{H})$. We assume that*

$$M = max\left(sup_{\omega\in\Omega}\int_\Omega ||A_\omega^* A_{\omega'}||^{1/2} d\mu(\omega'),\quad sup_{\omega\in\Omega}\int_\Omega ||A_\omega A_{\omega'}^*||^{1/2} d\mu(\omega')\right) < +\infty.$$

Then the operator $A = \int_\Omega A_\omega d\mu(\omega)$ is bounded on H *with norm less than M.*

From (3.19) and lemma 3.5, using

$$\Sigma_Y\Sigma_Z\Sigma_{Y'}\Sigma_{Z'} = (\Sigma_Y\Sigma_Z)(\Sigma_Z\Sigma_{Y'})(\Sigma_{Y'}\Sigma_{Z'}),$$

one gets that

$$||R||_{\mathcal{L}(L^2(\mathbb{R}^n))} \le C(n)||a||_{L^\infty}\gamma_2(b), \tag{3.23}$$

where $C(n)$ depends only on the dimension. We check now the second term in the expression of $a^{\mathrm{Wick}} b^{\mathrm{Wick}}$, using definition 3.1,

$$\int b'(Y)\cdot(Z-Y)\,\Sigma_Z\, dZ = b'(Y)\cdot\left[\int(\overbrace{Z-X}^{\text{will give }0} + X - Y)2^n e^{-2\pi\Gamma(X-Z)}\, dZ\right]^w = b'(Y)\cdot L_Y^w, \tag{3.24}$$

where L_Y is the (vector-valued) linear form $X - Y$. Note that, from proposition 3.3, $L^w = L^{\mathrm{Wick}}$. We get then

$$\text{(3.25)}\qquad \operatorname{Re}\left(a^{\text{Wick}}b^{\text{Wick}}\right) = (ab)^{\text{Wick}} + \int a(Y)b'(Y)\cdot \operatorname{Re}(L_Y^w\Sigma_Y)dY + \operatorname{Re}R.$$

Now, since L_Y is a real linear form, we have

$$\text{(3.26)}\qquad \operatorname{Re}(L_Y^w\Sigma_Y) = \left[(X-Y)2^n e^{-2\pi\Gamma(X-Y)}\right]^w = \frac{1}{4\pi}\nabla_Y^\Gamma(\Sigma_Y).$$

An integration by parts, in the distribution sense, gives what we expect in proposition 3.4, except possibly for

$$\text{(3.27)}\qquad -\frac{1}{4\pi}\int a(Y)\operatorname{Trace}(b''(Y))\Sigma_Y dY + \operatorname{Re}R.$$

The estimate of R in (3.23), $a \in L^\infty, b \in S(\lambda, \lambda^{-1}\Gamma)$ and the estimate of $||a^{\text{Wick}}||$ in (3.17) applied to the integral in (3.27) prove the statement on S in proposition 3.4, whose proof is now complete.

Remark. Under the assumptions $a \in L^\infty(\mathbb{R}^{2n}), b \in S^1_{\text{scl}}$, we have actually proved that

$$\text{(3.28)}\qquad a^{\text{Wick}(\lambda)}b^{\text{Wick}(\lambda)} = \left[ab - \tfrac{1}{4\pi}\, a'(Y)\cdot b'(Y) + \frac{1}{4i\pi}\{a,b\}\right]^{\text{Wick}(\lambda)} + S,$$

where $||S||_{\mathcal{L}(L^2(\mathbb{R}^n))} \le d_n||a||_{L^\infty}\gamma_2(b)$.

The Wick and Berezin-Wick quantization. A typical problem in the analysis of PDE is to deal with a Hamiltonian $a(x,\xi)$ which is homogeneous with respect to ξ. This is the reason which led to the study of the most classical class of pseudo-differential operators: a smooth function a defined on $\mathbb{R}^{2n}$ belongs to the class S^m (m is a given real number) whenever

$$\text{(3.29)}\qquad |(\partial_x^\alpha\partial_\xi^\beta a)(x,\xi)| \le C_{\alpha\beta}(1+|\xi|)^{m-|\beta|}.$$

A very nice tool to study the properties of this class of operators is the so-called Littlewood-Paley decomposition, which reduces the problem to a semi-classical one. We give here a short integral version of this tool. Let $\theta \in C_c^\infty((1,2),\mathbb{R}_+)$ such that $\int_0^{+\infty}\theta(1/\lambda)d\lambda/\lambda = 1$. For $\chi_0 \in C_c^\infty(\{|\xi| < 2\})$, identically equal to 1 in the unit ball, $\omega = 1-\chi_0, a \in S^m$ we have

$$a(x,\xi) = a(x,\xi)\chi_0(\xi) + \int_0^{+\infty}\theta\left(\frac{|\xi|}{\lambda}\right)\omega(\xi)a(x,\xi)\frac{d\lambda}{\lambda}.$$

Now, setting $a_\lambda(x,\xi) = a(x,\xi)\omega(\xi)\theta(\frac{|\xi|}{\lambda})$, we see that

$$a \in S^m_{\text{scl}},\quad \operatorname{supp} a_\lambda \subset \{\lambda \le |\xi| \le 2\lambda\}\quad \text{and}$$

$$\text{(3.30)}\qquad a(x,\xi) = \int_1^{+\infty} a_\lambda(x,\xi)\frac{d\lambda}{\lambda} + \chi(\xi)a(x,\xi)\quad \text{where } \chi \in C_c^\infty(\{|\xi| \le 2\}).$$

Now, we can set-up, with $\theta_1 \in C_c^\infty((2^{-1}, 2^2), \mathbb{R}_+), \theta_1 \equiv 1$ on $(1,2)$,

$$\phi_\lambda(x,\xi) = \theta(\frac{|\xi|}{\lambda})\omega(\xi), \qquad \psi_\lambda(x,\xi) = \theta_1(\frac{|\xi|}{\lambda})\omega(\xi), \tag{3.31}$$

so that we have

$$a = \int_1^{+\infty} a_\lambda \frac{d\lambda}{\lambda} + \chi a = \int_1^{+\infty} \psi_\lambda \sharp a_\lambda \sharp \psi_\lambda \frac{d\lambda}{\lambda} + S^{-\infty}, \tag{3.32}$$

with $S^{-\infty} = \cap_{m\in\mathbb{R}} S^m$.

As a matter of fact, since $\psi_\lambda = 1$ on the support of a_λ, we have, with $r_j \in S_{\rm scl}^{-\infty}$,

$$\psi_\lambda \sharp a_\lambda \sharp \psi_\lambda = \psi_\lambda \sharp (a_\lambda \psi_\lambda + r_1) = \psi_\lambda^2 a_\lambda + r_2 = a_\lambda + r_2.$$

Since r_2 satisfies $|\partial_x^\alpha \partial_\xi^\beta r_2| \le C_{\alpha\beta N} \min(|\xi|^{-N-|\beta|}, \lambda^{-N-|\beta|})$ we get that $\int_1^{+\infty} r_2 d\lambda/\lambda \in S^{-\infty}$. We can now modify (3.32) to get the Berezin-Wick quantization of a symbol $a \in S^m$:

$$a^{BW} = \int_1^{+\infty} \psi_\lambda^w a_\lambda^{\mathrm{Wick}(\lambda)} \psi_\lambda^w \frac{d\lambda}{\lambda}. \tag{3.33}$$

From (3.32) and proposition 3.3, we get that

$$\psi_\lambda^w a_\lambda^{\mathrm{Wick}(\lambda)} \psi_\lambda^w = \psi_\lambda^w a_\lambda^w \psi_\lambda^w + \psi_\lambda^w b_{m-1,\lambda}^w \psi_\lambda^w$$

with $b_{m-1,\lambda} \in S_{\rm scl}^{m-1}$; it is an easy exercise to prove that $\int_1^{+\infty} \psi_\lambda \sharp b_{m-1,\lambda} \sharp \psi_\lambda d\lambda/\lambda$ belongs to S^{m-1}. This implies that

$$a^{BW} = a^w + \mathrm{Op}(S^{m-1}) = a^w + b^w, \text{ with } b \in S^{m-1}. \tag{3.34}$$

and since the Wick quantization is positive, we get at once the standard Gårding inequality. Let $a \in S^1$ be a non-negative symbol; then, with an operator $R \in \mathrm{Op}(S^0)$, we get

$$\langle a^w u, u\rangle_{L^2(\mathbb{R}^n)} = \int_1^{+\infty} \underbrace{\langle a_\lambda^{\mathrm{Wick}\lambda} \psi_\lambda^w u, \psi_\lambda^w u\rangle}_{\ge 0} \frac{d\lambda}{\lambda} + \langle Ru, u\rangle \ge -C\|u\|^2_{L^2(\mathbb{R}^n)}. \tag{3.35}$$

Of course formula (3.34) can be iterated so that for any $a \in S^m$, there exists $\tilde a \in S^m$ such that $a - \tilde a \in S^{m-1}$ and

$$a^w = \tilde a^{BW} + r^w, \quad \text{with} \quad r \in S^{-\infty}. \tag{3.36}$$

Remarks. It is interesting to notice that the Wick and the Berezin-Wick quantization could be defined in a much more general framework, involving classes of pseudo-differential operators defined by metrics as in chaper 18 of [H1]. In fact, following the presentation in [BL], one can write for a symbol in the class $S(m, g)$ that

$$a = \int_{\mathbb{R}^{2n}} a\varphi_Y dY.$$

Using the metric g_Y, one can define a symplectic metric Γ_Y such that

$$g_Y \le \Gamma_Y = \Gamma_Y^\sigma \le g_Y^\sigma.$$

Then changing slightly our notations, we can define the Wick quantization for the symplectic metric Γ_Y, denoted by $\mathrm{Wick}(\Gamma_Y)$. Then the formula

$$a^{BW} = \int_{\mathbb{R}^{2n}} \psi_Y^w (a\varphi_Y)^{\mathrm{Wick}(\Gamma_Y)} \psi_Y^w dY$$

provides a positive quantization of the symbol a, assuming that the symbols ψ_Y are uniformly in $S(1, g)$ and supported in an neighborhood of the support of φ_Y. On the other hand, the difference $a^w - a^{BW}$ has a Weyl symbol in $S(mh, g)$.

A different and more elementary question is to understand to which extent Gaussian functions play an important role in the definition of the Wick quantization. As observed in the proof of Theorem 18.1.14 in [H1], the regularization of the symbol by the Wigner function of a Gaussian of the configuration space (which is also a Gaussian function of the phase space) can be replaced by regularization by $\Phi\sharp\Phi$ where Φ is even with $L^2(\mathbb{R}^{2n})$ norm 1. Definition 3.1 gives that the Weyl symbol of $a^{\mathrm{Wick}(\lambda)}$ is the convolution

$$a * 2^n \exp - 2\pi(\lambda|x|^2 + \lambda^{-1}|\xi|^2) = a * \mathcal{H}(\varphi, \varphi).$$

In fact the sole virtue of this mollifier is to correspond to a non-negative operator, here a rank-one projection whose Weyl symbol is the Wigner function $\mathcal{H}(\varphi, \varphi)$, where φ is a Gaussian function. Nonetheless φ could be replaced by any even (or odd) function, but in fact $\Phi\sharp\Phi$ would be an appropriate substitute for $\mathcal{H}(\varphi, \varphi)$. It seems that the fact that the computations with Gaussian functions can be made rather explicit is the only explanation to the popularity of Gaussian mollifiers for positive quantization.

As a final remark in this section, one could also say that the more refined lowerbounds given by the Fefferman-Phong inequality are out of reach of the tool introduced here. The microlocalization involved in this inequality use a Calderón-Zygmund decomposition, which depends heavily on the symbol under scope. However, the strength of the wave packets method relies on the fact that it provides a non-negative quantization.

4. Regularity of Functions

Going back to lemma 2.4, we recall that with $W_\lambda u$ defined in (2.16) and $\varphi^\lambda_{y,\eta}$ given in (2.15), we have for any positive λ, and $u \in \mathcal{S}'(\mathbb{R}^n)$,

$$(4.1) \quad u = \int_{\mathbb{R}^{2n}} W_\lambda u(y,\eta)\varphi^\lambda_{y,\eta} dy d\eta, \qquad (T_\lambda u)(y+i\eta) = \lambda^{-n/4} e^{\pi\lambda\eta^2} W_\lambda u(y, -\lambda\eta)$$

in such a way that T_λ is an entire function. Now the function $W_\lambda u$ is defined on the phase space $\mathbb{R}^{2n}$ and appears as a good representative of the function u at the frequency λ. In fact, using the Berezin-Wick quantization of section 3, we have

$$u = \int_1^{+\infty} \psi^w_\lambda \phi_\lambda^{\mathrm{Wick}(\lambda)} \psi^w_\lambda u \frac{d\lambda}{\lambda} + C^\infty.$$

The following proposition is proved in [G] (see also [D]), using the operators T_λ.

Proposition 4.1. **H^s regularity.** *Let s be a real number, Ω be an open set of $\mathbb{R}^n$, $u \in \mathcal{D}'(\Omega)$ and $(x_0, \xi_0) \in \Omega \times (\mathbb{R}^n \backslash \{0\}) = \dot{T}^*(\Omega)$. Then $u \in H^s_{x_0,\xi_0}$ iff there exists a neighborhood V_0 of $(x_0, \xi_0/|\xi_0|)$ such that for all $\chi \in C^\infty_c(\pi_1(V_0))$*

$$(4.2) \qquad \int_1^{+\infty} \iint_{\substack{\lambda \le |\xi| \le 2\lambda, \\ (x, \frac{\xi}{|\xi|}) \in V_0}} |W_\lambda(\chi u)(x,\xi)|^2 dx d\xi \lambda^{2s} \frac{d\lambda}{\lambda} < +\infty.$$

N.B. In fact, the left-hand-side of (4.2) appears as the H^s norm of $a^w \chi u$, where a is a symbol in S^0 whose essential support is included in V_0.

It is pretty remarkable that this tool could be used as well to describe Gevrey regularity (the analytic case is $\sigma = 1$).

Proposition 4.2. **Gevrey G^σ regularity.** *Let $\sigma > 0$, Ω be an open set of $\mathbb{R}^n$, $u \in \mathcal{D}'(\Omega)$ and $(x_0, \xi_0) \in \Omega \times (\mathbb{R}^n \backslash \{0\}) = \dot{T}^*(\Omega)$. Then $u \in G^\sigma_{x_0,\xi_0}$ iff there exists a neighborhood V_0 of $(x_0, \xi_0/|\xi_0|)$ such that, for all $\chi \in C^\infty_c(\pi_1(V_0))$ identically 1 near x_0, there exists $\rho > 0$ such that*

$$(4.3) \qquad \sup_{\substack{\lambda \ge 1,\ \lambda \le |\xi| \le 2\lambda, \\ (x, \frac{\xi}{|\xi|}) \in V_0}} |W_\lambda(\chi u)(x,\xi)| \exp(\rho \lambda^{1/\sigma}) < +\infty.$$

5. Energy estimates

Propagation estimates for micro-hyperbolic operators. Let us consider a principal type properly supported pseudo-differential operator P of order m

on an open set Ω of $\mathbb{R}^n$ and assume first that the principal symbol $p(x,\xi)$ is real-valued. Let $\Gamma = (\gamma(t))_{T_0 \le t \le T_1}$ be a piece of bicharacteristic curve, that is an integral curve of the Hamiltonian vector field of p,

$$\frac{d}{dt}\gamma(t) = H_p(\gamma(t)), \qquad H_p = \sum_{1\le j\le n} \frac{\partial p}{\partial \xi_j}\frac{\partial}{\partial x_j} - \frac{\partial p}{\partial x_j}\frac{\partial}{\partial \xi_j}.$$

The classical propagation-of-singularities result (see theorem 26.1.4 in [H1]) states that if $u \in \mathcal{D}'(\Omega)$ is such that

$$Pu \in H^s_\Gamma, \qquad u \in H^{s+m-1}_{\gamma(T_1)},$$

then

$$u \in H^{s+m-1}_{\gamma(T_0)}.$$

The same statement remains true if the imaginary part of p is non-negative. If the imaginary part of p remains non-positive, the rôle of T_0 and T_1 should be exchanged in the statement above. There are essentially two families of proofs of these results. The first one deals with Fourier integral operators reducing the problem to proving an easy estimate for operators of type

$$D_{x_1} + r_0(x_1, x', D')$$

where r_0 is a pseudo-differential operator of order 0. The second type of proof is more direct, but in fact equivalent, and consists of finding a zeroth-order "multiplier" M_0 to check

$$2Re\langle Pv, iM_0 v\rangle$$

for a compactly supported v. Whenever the imaginary part of p is non-negative, we choose a non-negative m_0 such that

$$H_{\mathrm{Re}p}(m_0) = \psi_0 - \psi_1$$

where ψ_j are non-negative symbols in S^0 supported respectively near $\gamma(T_j), j = 1, 2$. Using (3.28), it is possible to give a very simple approach to this kind of proof. The natural multiplier M_0 is the Wick quantization of a non-negative function $m_0(x,\xi)$ which is the characteristic function of an open set ω such that

$$H_{\mathrm{Re}p}(m_0) = \Delta_0 - \Delta_1$$

where Δ_j is a positive measure supported on an hypersurface $\Sigma_j \ni \gamma(T_j)$ transversal to $H_{\mathrm{Re}p}$. The fact that m_0 is only L^∞ is not an obstacle to using (3.28).

Solvability estimates for a class of operators with complex symbols. Although micro-hyperbolic operators are well-understood for quite a long time, the situation for pseudo-differential operators with complex symbols is far more complicated. We refer the reader to the surveys [H2],[L2] for an overview on these questions. However, we want to quote and comment in more details the result of [L1]. Let us consider the operator

$$L = D_t + iq(t, x, D_x) \tag{5.1}$$

where $t \in \mathbb{R}$ and $q(t, x, \xi)$ is a first order symbol such that

$$q(t, x, \xi) > 0 \text{ and } s \geq t \Longrightarrow q(s, x, \xi) \geq 0. \tag{5.2}$$

Condition (5.2) appears as the natural condition to get solvability for L^*, the adjoint operator of L: it is the so-called condition (ψ) on the symbol $\tau - iq(t, x, \xi)$. It is not known if this condition is sufficient for solvability of L^*, although the necessity is proved in [H1](Corollary 26.4.8). In the paper [L1], we were able to prove the following solvability result, supplementing (5.2) by an extra condition.

Theorem 5.1. *Let $q(t, x, \xi) \in C^0([-1, 1], C^\infty(\mathbb{R}^n_x \times \mathbb{R}^n_\xi))$ be real-valued satisfying (5.2) such that*

$$sup_{|t|\leq 1, (x,\xi)\in\mathbb{R}^{2n}} |\partial_x^\alpha \partial_\xi^\beta q(t, x, \xi)|(1 + |\xi|)^{-1+|\beta|} < +\infty. \tag{5.3}$$

We assume that there exists a constant C_0 such that, for $|\xi| \geq 1$,

$$|\xi|^{-1} \left|\frac{\partial q}{\partial x}(t, x, \xi)\right|^2 + |\xi| \left|\frac{\partial q}{\partial \xi}(t, x, \xi)\right|^2 \leq C_0 \frac{\partial q}{\partial t}(t, x, \xi) \quad at \quad q(t, x, \xi) = 0. \tag{5.4}$$

Then, there exist two positive constants C_1, ρ_0 such that, for all $u(t, x) \in C_c^\infty$ vanishing when $|t| \geq \rho_0$,

$$C_1 ||Lu||_{L^2(\mathbb{R}^{n+1})} \geq ||u||_{L^2(\mathbb{R}^{n+1})}. \tag{5.5}$$

The proof is too long and too technical to be reproduced here. However the idea is pretty simple and quite naïve, as a matter of fact. Condition (5.2) implies that there is a function $s(t, x, \xi)$ valued in $\{-1, 1\}$ such that

$$q(t, x, \xi)s(t, x, \xi) = |q(t, x, \xi)| \quad \text{and} \quad \frac{\partial s}{\partial t} \text{ is a non-negative measure.}$$

The function s is somehow a good choice for the sign of q. Then we choose as a multiplier the operator

$$s(t, x, \xi)^{\text{Wick}}$$

and we use proposition 3.4 to handle some of the computations. The fact that we deal with a non-negative quantization plays an important rôle, beyond the lowerbounds properties. Unfortunately, proposition 3.4 is not enough to get (5.5) assuming (5.4), except if C_0 is small enough. The argument gets more complicated to tackle large constants C_0 in (5.4) and we have to resort to the more refined Beals-Fefferman partitions of unity.

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