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Abstract Homotopy Theory: The Interaction of Category Theory and Homotopy Theory

Timothy Porter
Mathematics Division
School of Informatics
University of Wales Bangor
Bangor, Gwynedd, LL57 1UT
Wales, U.K.

Abstract

This article is an expanded version of notes for a series of lectures given at the *Corso estivo Categorie e Topologia* organised by the *Gruppo Nazionale di Topologia del M.U.R.S.T.* in Bressanone, 2 - 6 September 1991. Those notes have been brought up to date by the addition of new references and a summary of what has happen in the area in the last ten years.

In 1991, the Italian *Gruppo Nazionale di Topologia* organised a summer course on Category Theory and Topology in the delightful setting of the town of Bressanone. There were several series of lectures planned and they asked me to give some on abstract homotopy theory and its interaction, both with category theory and with topology. The notes for the course were typed up and were made available as preprints from both Genova and Bangor. In the ten years since that meeting, I have been asked on several occasions if there was a published version of the course, or was there a version online somewhere. It seemed that others had found the notes to be of some use as a way into the area, giving, as I had hoped, some intution about what the problems were, what the potential for new applications is and so on. There has been no published version, nor web version available until now and my stock of hard copies was diminishing, and the contents was not as up to date as they might be, so ... .

When it was suggested that I write an 'overview' article on my own area of research for Cubo, I realised that there was here a good opportunity to prepare an updated version of the notes. It would fulfil the request from the editor and would make the notes more widely available, as a similar version could be put 'on-line'. In fact, in retyping the notes I found that not many changes did need making and this article has retained the form of notes for an informal postgraduate course (complete with suggestions for exercises etc). The main body of the material has been updated only in as much as the references have been changed to account for publication of material previously in preprint form, but in the last ten years several excellent texts and papers on the subject matter have been published so I have added another section with a discussion of the view of the various topics today and have given a supplementary bibliography.

I hope the notes are still found useful. They were aimed at postgraduate students one year into their studies, so are intended to be approachable.

T.P. Bangor, 2001.

Aims of the course.

- To give the background and some historical perspective on *Abstract Homotopy Theory.*

- To introduce some of the key ideas of abstract homotopy, its different 'schools' and how they interact and to indicate the sources in which they may be found.

- To try to help build up categorical and geometrical intuition on the subject which, by its very abstraction, can seem too unapproachable to be of use.

- To provide, where possible, a unifying 'overview', by concentrating on certain 'themes' that illustrate:

    1. what is abstract homotopy theory,
    2. why the problems that form the subject matter of abstract homotopy theory can be of interest outside the confines of that theory,

  but they will not attempt to prove deep results nor do more than skate over the surface of the subject.

Prerequisites

- A basic knowledge of homotopy theory with some intuition about the fundamental group (or fundamental groupoid), polyhedra, etc. A knowledge of the higher homotopy groups would be useful, but is not essential. In the last section, some covering space theory is used, but this can be approached via sheaf theory for the reader with a categorical or algebraic geometric background or can be found in books such as Massey [73]. All the material on groupoids and much more can be found in Brown [13].

- A first course in category theory covering categories, functors, natural transformations, limits (and colimits) and the idea of adjoint functors. (At the *Corso estivo* such a concentrated course was given by R. Betti.) This subject matter corresponds approximately to the first five chapters of Mac Lane [69], but we will not need the depth given there.
- For the last section of these notes, some ideas about sheaves and toposes are assumed. At the *Corso estivo*, such material was beautifully covered in lectures by F. Borceux and the notes are available (Borceux, [8]).

1 Introduction to Abstract Homotopy Theory

To understand the possible aims of abstract homotopy theory, it will help to list some general 'problems', some of which are of a philosophical or metatheoretic nature, others are very pragmatic and practical.

- Notions of deformation and of homotopy occur in many parts of mathematics. Such ideas are central in topological homotopy theory, but what is homotopy theory? What is 'a homotopy theory'? As one talks of homology and cohomology theories, what explicit structures, that exist in the topological case, are needed for homotopy theory? In other words: 'what is homotopy theory?' and 'what makes homotopy theory work?'

  In algebraic topology, one often looks for algebra that models the topology. How can one look for algebraic models of homotopy types and once one has found them, can one 'do homotopy theory' with them? This leads to a related problem:

- Early models for algebraic information on homotopy types, such as chain complexes, led to the study of chain complexes of modules, etc., which represented 'algebraic homotopy types'. Their study led to the growth of homological algebra. Finding newer, fuller algebraic models for homotopy types that take account of non-abelian behavior would seem to open the door to a process of adaptation which could have implications in non-abelian cohomology, homotopical algebra and hence, in particular, in algebraic geometry. This depends on being able to 'do' homotopy theory with such models of algebraic homotopy types, and not just at a trivial level.

- To manipulate algebra, one uses categorical tools such as limits and colimits. What are the good categorical tools for working with homotopy theory?
- In some of the applications of abstract homotopy theory, one is studying geometric objects such as spaces; one is using a geometric notion of homotopy, but the tools used, and even the language used, seems simpler when viewed from the abstract viewpoint (cf. Edwards and Hastings [36] for strong shape theory, Porter [79] and Hernańdez-Porter [46], [47] for proper homotopy theory using inverse systems and Baues [5], [6] for an approach to classical homotopy questions with remarkable new results, using in part insight gained by the abstract homotopy theory he develops. There is also work by Baues with Ayala, Marquez, and Quintero which is in preprint form and uses a direct abstract homotopy theory approach to proper homotopy theory).

2 First Theme: What is a Homotopy Theory?

To provide possible answers to this question, we start by listing some examples. Some of these will be examined in more detail later.

A brief list might include Spaces, Groupoids, Simplicial Sets, Simplicial Objects in other categories, Cubical Sets, Chain Complexes and Small Categories.

In more detail:

Spaces: There is perhaps no real need for explanation here, but there are various points worth making that can serve as an introduction to some of the ideas that come later. We consider a 'suitable' category, **Top**, of topological spaces and continuous maps. Homotopy between maps is defined by maps from a cylinder:

'$H : f_0 \simeq f_1 : X \to Y$' can be read as '$f_0$ is homotopic to f_1 by the homotopy H'; it interprets as '$f_0, f_1 : X \to Y$ in **Top**, $H : X \times I \to Y$ (where $I = [0, 1]$) and $H|X \times \{0\} = f_0$, $H|X \times \{1\} = f_1$.'

From the notion of homotopy one gets that of 'homotopy equivalence':

$f : X \to Y$ is a homotopy equivalence if there is a $g : Y \to X$ and homotopies $H : gf \simeq Id_X$, $K : fg \simeq Id_Y$. We also say X and Y have the same 'homotopy type' in this case.

Following the usual convention, we put $[X, Y] = \mathbf{Top}(X, Y)/\simeq$. The category $\mathbf{Top}/\simeq$ has spaces as objects but these $[X, Y]$ as sets of morphisms. Homotopy types correspond to isomorphism classes in $\mathbf{Top}/\simeq$.

There are special classes of maps called *cofibrations* and *fibrations*. These are defined by the *homotopy extension* and *homotopy lifting properties*, respectively.

For example if $f : X \to Y$ is a cofibration then given any homotopy $H : X \times I \to Z$ and any map $g : Y \to Z$ such that $H|X \times \{0\} = gf$, there is a homotopy $K : Y \times I \to Z$ extending H and starting with g, i.e. $K(f \times I) = H$ and $K|Y \times \{0\} = g$. This can be written more simply if we write $e_0(X) : X \to X \times I$ for the map $e_0(X)(x) = (x, 0)$, then it reads f is a cofibration if and only if, given $g : Y \to Z$ so that

$$\begin{tikzcd}
X \arrow[r, "e_0(X)"] \arrow[d, "f"'] & X \times I \arrow[d, "H"] \\
Y \arrow[r, "g"'] & Z
\end{tikzcd}$$

commutes, there is a $K : Y \times I \to Z$ such that $K(f \times I) = H$ and $f = He_0(X)$. We also say that f satisfies the *homotopy extension property*.

Note that fibrations are not a dual notion in the naïve sense!

Our final points on **Top** relate to pointed spaces and homotopy groups. A *pointed space* is a pair (X, x_0), where $x_0 \in X$. 'Pointed maps' of pointed spaces $f : (X, x_0) \to (Y, y_0)$ send x_0 to y_0 and give a category $\mathbf{Top}_0$. Homotopies between such 'pointed' maps are assumed to be constant on these base points. Writing S^n for the n-dimensional sphere, $n \geq 0$, special attention is given to the sets $[(S^n, 1), (X, x_0)]$. For $n = 0$, S^0 is a two point space $\{-1, 1\}$ and we write $\pi_0(X, x_0) = [(S^0, 1), (X, x_0)]$. It is the set of arcwise connected components of X, pointed at the component corresponding to x_0. This notation π_0 is also used in the unpointed case. In general we write $\pi_n(X, x_0) = [(S^n, 1), (X, x_0)]$. For $n \geq 1$, $\pi_n(X, x_0)$ has a natural group structure and for $n \geq 2$, this structure is abelian. A map $f : X \to Y$ of spaces is called a *weak equivalence* if $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ is a bijection and for all $n \geq 1$ and all possible basepoints $x_0 \in X$, $\pi_n(f) : \pi_n(X) \to \pi_n(Y)$ is an isomorphism. Any homotopy equivalence is a weak equivalence but the converse does not hold in general. For 'locally nice' spaces such as polyhedra and more generally for CW-complexes (which are built up inductively by attaching 'cells' to others of lower dimensions), the two concepts coincide and any weak equivalence between such spaces has a homotopy inverse.

Groupoids: These are small categories in which all morphisms are invertible. Groups correspond to the special case in which there is only one object. A special class of examples are the fundamental groupoids of spaces.

Let X be a space and let X_0 be a subspace of X (often one takes $X_0 = X$

or X_0 to be a convenient finite set of base points). The fundamental groupoid $\Pi_1 X X_0$ consists of all homotopy classes of paths $\alpha : [0,1] \to X$ where $\alpha(0)$, $\alpha(1) \in X_0$ and where if $\alpha, \beta : [0,1] \to X$ are two such paths, they are homotopic as paths if they are homotopic as maps with domain $[0,1]$ *and* the homotopy fixes the ends of the paths, i.e. $H : \alpha \simeq \beta$ satisfies $H(0,t) = \alpha(0)$, $H(1,t) = \alpha(1)$ for all $t \in [0,1]$ so we must have $\alpha(0) = \beta(0)$ and $\alpha(1) = \beta(1)$. The proof that $\Pi_1 X X_0$ is a groupoid is only slightly different from the proof that $\pi_1(X, x_0)$ is a group. If you have not seen it before, try to prove it or *worse*, look it up in some textbook.

The groupoid $\Pi_1 I\{0,1\}$ consists of two objects 0 and 1, their identities, and two morphisms $\iota : 0 \to 1$, $\iota^{-1} : 1 \to 0$. We will denote this groupoid by I as well. Homotopy of groupoids can be defined by a cylinder $G \times I$ or by a 'cocylinder' H^I, since, if G and H are groupoids, there is a natural isomorphism

$$Gpd(G \times I, H) \cong Gpd(G, H^I).$$

This 'cocylinder' is just the category of functors from I to H. It is easily seen to be a groupoid. (Checking this is left to you!) (Reference for groupoids: Brown, [13].)

Note in some categories of *spaces*, there is a cocylinder X^I which behaves in the same way as above, but as it does not exist in *all* the usual categories of spaces that you might need, we have only handled the cylinder based theory above.

Simplicial Sets: These form an extremely useful category in which to do homotopy theory. The basic theory can be found in the first half of the survey article by Curtis, [31], whilst Gabriel and Zisman, [39] look at the theory in a more categorical way.

Let $[n] = \{0 < 1 < \ldots < n\}$, considered as an ordered set or as a small category. Looked at for small values of n, it is clear why it is considered as a categorical simplex.

e.g. $[2] =$

$$\begin{tikzcd} & 2 \arrow[dr] & \\ 0 \arrow[ur] \arrow[rr] & & 2 \end{tikzcd}$$

Writing **Cat** for the category of small categories and $\mathbf{\Delta}$ for the full subcategory of **Cat** determined by the objects, $[n]$ for $n \geq 0$, a simplicial set is a functor $K : \mathbf{\Delta}^{op} \to \mathbf{Sets}$ and $\mathbf{S} = Simp(\mathbf{Sets})$ is just the notation for $\mathbf{Sets}^{\mathbf{\Delta}^{op}}$, the category of contravariant functors from $\mathbf{\Delta}$ to **Sets** and all natural transformations between them. A simplicial set K is often written diagrammatically as

$$\cdots\cdots \; K_2 \;\underset{s_0,\, s_1}{\overset{d_0,\, d_1,\, d_2}{\rightleftarrows}}\; K_1 \;\underset{s_0}{\overset{d_0,\, d_1}{\rightleftarrows}}\; K_0$$

The maps $d_i : K_n \to K_{n-1}$, $0 \leq i \leq n$, are called the *face maps*, the maps $s_i : K_n \to K_{n+1}$, $0 \leq i \leq n$ are called the *degeneracies*. (You should be asking yourself why "contravariant" and why only these maps are shown). The d_i and s_i satisfy certain "simplicial identities", which can be found in many books (e.g. Gabriel and Zisman [39]) or, better still can be worked out for yourself from the definition of d_i and s_i (see below). As a simple example of a simplicial set, we can take $\Delta[n] = \mathbf{\Delta}(-, [n])$, the n-simplex. Here are some other examples as exercises.

Exercises:

1. Consider the maps

$$\delta_i : [n-1] \to [n] \qquad 0 \leq i \leq n,$$
$$\sigma_i : [n+1] \to [n] \qquad 0 \leq i \leq n,$$

where δ_i is injective with image not containing the element i, for instance $\delta_0 : [1] \to [2]$, is given by $\delta_0(0) = 1$, $\delta_0(1) = 2$, so referring back to the 2-simplex diagram above δ_i gives the face which is opposite i; and where σ_i is surjective repeating i, (e.g. $\sigma_0 : [2] \to [1]$, is $\sigma_0(0) = 0$, $\sigma_0(1) = 0$, and $\sigma_0(2) = 1$).

Prove that these maps for all n and i generate all the maps in $\mathbf{\Delta}$, i.e. any $f : [m] \to [n]$ can be written as a composite of σs and δs, usually in many different ways. Find some relationships between pairwise composites such as $\delta_i\delta_j$ = some other composite of δ_ks. Then compare your answer with any list of the simplicial identities (e.g. in [31]), but BEWARE of the dual identities!

As any f in $\mathbf{\Delta}$ can be factorised in this way, $K \in \mathbf{S}$ can be specified by specifying the $d_i = K(\delta_i)$ and $s_i = K(\sigma_i)$.

2. Let Δ^n be the topological n-simplex, represented by

$$\{(t_0, \ldots, t_n) \in \mathbf{R}^{n+1} \mid \sum_{i=0}^{n} t_i = 1\}.$$

There are maps $\delta_i : \Delta^n \to \Delta^{n+1}$ given by

$$\delta_i(t_0, \ldots, t_n) = (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_n),$$

for $0 \leq i \leq n$, which insert Δ^n as the i^{th} face of Δ^{n+1}. Similarly let $\sigma_i : \Delta^n \to \Delta^{n-1}$ be the map

$$\sigma_i(t_0, \dots, t_n) = (t_0, \dots, t_{i-1} + t_i, \dots, t_n).$$

Let X be a space and set $Sing(X)_n = Top(\Delta^n, X)$. Using the compositions with the δ_i and σ_i, prove that $Sing(X)$ is a simplicial set. It is called the *singular complex* of X.

The singular complex construction gives a functor

$$Sing : \mathbf{Top} \to \mathbf{S}.$$

This has a left adjoint $|\ \ | : \mathbf{S} \to \mathbf{Top}$ called *geometric realisation*:

$$\mathbf{Top}(|K|, X) \simeq \mathbf{S}(K, Sing(X))$$

and the natural map

$$|Sing(X)| \to X$$

is a weak equivalence (that is, it induces isomorphisms for all π_ns). Check this for yourselves.

It is sometimes the case that within a category of spaces, **Top**, one can form mapping spaces Y^X, but in any case provided that for each n, Δ^n is in our chosen category, **Top**, and for all X in **Top**, $X \times \Delta^n$ is there as well, we can form something that plays the rôle of $Sing(Y^X)$ namely $\underline{\mathbf{Top}}(X, Y)$, where $\underline{\mathbf{Top}}(X, Y)_n = \mathbf{Top}(X \times \Delta^n, Y)$. (Checking that this is a simplicial set is again left to you.) A similar device works in the category **Cat**, of small categories, $\underline{\mathbf{Cat}}(C, D)_n = \mathbf{Cat}(C \times [n], D)$, and even inside **S** itself (i.e. $\underline{\mathbf{S}}(K, L)_n = \mathbf{S}(K \times \Delta[n], L)$).

2. Let C be a small category and put $Ner(C)_n = \mathbf{Cat}([n], C)$. Check that $Ner(C)$ is a simplicial set. It is called the *nerve of* C. (The space $BC = |Ner(C)|$ is called the *classifying space* of C and for special cases of C is often used in algebraic K-theory.)

Simplicial Objects in Other Categories.

If **A** is any category, we can form $Simp(\mathbf{A}) = \mathbf{A}^{\Delta^{op}}$. These categories often have a good notion of homotopy. Of particular use are:

(i) $Simp(\mathbf{Ab})$, the category of simplicial abelian groups. (This is equivalent to the category of chain complexes by the Dold-Kan theorem (cf. Curtis, [31])).

(ii) $Simp(\mathbf{Gps})$, the category of simplicial groups. (This 'models' all connected

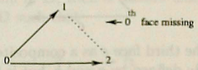
homotopy types, by Kan [62] (cf. Curtis, [31])). There are adjoint functors $G : \mathbf{S}_{conn} \to Simp(\mathbf{Gps})$, $\bar{W} : Simp(\mathbf{Gps}) \to \mathbf{S}_{conn}$, with the two natural maps $G\bar{W} \to Id$ and $Id \to \bar{W}G$ being weak equivalences. There are fairly recent new results on simplicial groups by Carrasco [22] that generalise the Dold-Kan theorem to the non-abelian case, (cf. Carrasco and Cegarra [23]).
(iii) $Simp(\mathbf{Gpds})$: in 1984, Dwyer and Kan [35] (and also Joyal and Tierney, and Duskin and van Osdol, cf. Nan Tie [74, 75]) noted how to generalise the $(G, \bar{W})$ adjoint pair to handle all simplicial sets, not just the connected ones. (Beware there are several important printing errors in their paper [35].) This category 'models' all homotopy types using a mix of algebra and combinatorial structure.

Kan complexes and Kan fibrations.

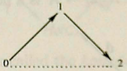
Within the category of simplicial sets, there is an important subcategory determined by those objects that satisfy the Kan condition, that is the *Kan complexes*.

As before we set $\Delta[n] = \Delta(-, [n]) \in \mathbf{S}$, then for each i, $0 \leq i \leq n$, we can form a subsimplicial set, $\Lambda^i[n]$ of $\Delta[n]$ by discarding the top dimensional n-simplex (given by the identity map on $[n]$) and its i^{th} face. We must also discard all the degeneracies of the simplices. This informal definition does not give a 'picture' of what we have, so we will list the various cases for $n = 2$.

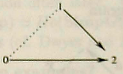
$\Lambda^0[2] =$



$\Lambda^1[2] =$



$\Lambda^2[2] =$

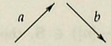


Any map $p : E \to B$ is a *Kan fibration* if given any n, i as above and any (n, i)-horn in E, i.e. any map $f_1 : \Lambda^i[n] \to E$, and n-simplex, $f_0 : \Delta[n] \to B$, such that

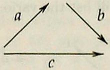
$$\begin{tikzcd} \Lambda^i[n] \arrow[r, "f_1"] \arrow[d, "inc"'] & E \arrow[d, "p"] \\ \Delta[n] \arrow[r, "f_0"'] & N \end{tikzcd}$$

commutes, then there is an $f : \Delta[n] \to E$ such that $pf = f_0$ and $f.inc = f_1$, i.e. f lifts f_0 and extends f_1.

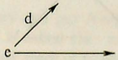
A simplicial set, K, is a *Kan complex* if the unique map $K \to \Delta[0]$ is a Kan fibration. This is equivalent to saying that every horn in K has a filler, i.e. any $f_1 : \Lambda^i[n] \to Y$ extends to an $f : \Delta[n] \to Y$. This condition looks to be purely of a geometric nature but in fact has a strong algebraic flavour; for instance if $f_1 : \Lambda^1[2] \to Y$ is a horn, it consists of a diagram



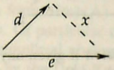
of 'composable' arrows in K. If f is a filler, it looks like



and one can think of the third face c as a composite of a and b. This 'composite' c is not usually uniquely defined by a and b, but is 'up to homotopy'. If we write $c = ab$ as a shorthand then if $g_1 : \Lambda^0[2] \to K$ is a horn, we think of g_1 as being



and to find a filler is to find a diagram



and thus to 'solve' the equation $dx = e$ for x in terms of d and e. This 'algebraic' aspect of Kan complexes is very important for understanding recent developments in the area (cf. Ashley [4], for instance, or the work of Carrasco mentioned earlier.)

Examples and Exercises.

1) $Sing(X)$ and $\underline{\mathbf{Top}}(X, Y)$ are always Kan complexes. Why?

2) $Ner(\mathbf{C})$ is Kan if and only if $\mathbf{C}$ is a groupoid. Prove it.

3) A simplicial set $\mathbf{K}$ is *weakly Kan* if for any n and $0 < k < n$, any (n, k)-horn in K has a filler. Prove that for any $\mathbf{C}$ any small category, $Ner(\mathbf{C})$ is weakly Kan.

Back to ' What is homotopy?'

In each of these settings, and in many more, one has a notion of equivalence (weak equivalence or homotopy equivalence depending on the context). Suppose $\mathbf{C}$ is a category with a collection, Σ, of maps called *weak equivalences* (no properties are thought of as being attached to the name, for the moment, it is just a name). We can form a new category $Ho(\mathbf{C}) = \mathbf{C}(\Sigma^{-1})$ by 'formally inverting' the morphisms in Σ. We do this by taking for each $f \in \Sigma$, say $f \in \mathbf{C}(X, Y)$, a new symbol f^{-1} and we add it into $\mathbf{C}(Y, X)$. We then form composite words in the old arrows together with all these new 'inverses' and if we ever see a pair, ff^{-1} or $f^{-1}f$, we cancel it out. (See Gabriel and Zisman, [39] for a proper description of this process). The resulting category comes with a functor $\gamma : \mathbf{C} \to Ho(\mathbf{C})$ with the nice universal property that if $\alpha : \mathbf{C} \to \mathbf{D}$ is any functor such that for all $f \in \Sigma$, $\alpha(f)$ is an isomorphism in $\mathbf{D}$, then α factors uniquely through γ, i.e. there is a unique $\bar{\alpha} : Ho(\mathbf{C}) \to \mathbf{D}$ such that

$$\begin{tikzcd} C \arrow[rr, "\alpha"] \arrow[dr, "\gamma"'] & & D \\ & Ho(C) \arrow[ur, "\bar{\alpha}"'] & \end{tikzcd}$$

commutes. This category will be called the ***homotopy category*** of $\mathbf{C}$ (and usually we will miss out mention of Σ).

We will need this construction often later. We note that using it one can prove, for instance, that $Ho(\mathbf{Top}) \simeq Ho(\mathbf{SimpGpds})$ for a suitable definition of weak equivalence in **SimpGpds** (cf. Dwyer and Kan [35]).

Common structure in the examples?

There are various interacting structures and therein lies the problem in deciding exactly what is an 'abstract homotopy theory'. We note various attempts to encode at least part of that structure.

a) **Quillen**: [82, 83, 84]

This is one of the most widespread of the structures by which I mean that it is elegant and quite powerful, so has often been considered as the basic abstract homotopy theory to use. It considers a category C with infinite limits and colimits and three classes of morphisms called ***weak equivalences***, ***fibrations*** and ***cofibrations***, whose behaviour, and in particular whose interaction, is governed by various axioms. (We do not give them here as they can be found in many sources in the list of references.) The origins of the work may be found in deformation theory and the need for a cohomology of commutative algebras (cf. Quillen, [84]). Once developed, Quillen used it highly successfully in [83] to produce new results on rational homotopy theory.

One may criticise it from various viewpoints. For instance, the weak equivalences, etc. are given right from the start and no guidance is given how these classes might arise. Thus weak equivalence are often given by 'external' information (e.g. $f : K \to L$ in $\mathbf{S}$ is a weak equivalence if $|f|$ is a weak equivalence in **Top**). This does not help one when interpreting the notions geometrically. Linked to this is the fact that several important ideas such as that of fundamental groupoid, homotopy limits and even homotopy equivalence are 'out of place' in the theory. If one wants the homotopy theory primarily to study some notion of homotopy, Quillen's theory is not designed with your application in mind. It can do a lot, but it is not the universal solution. (Of course I do not claim to know if a universal solution exists, let alone what it is!)

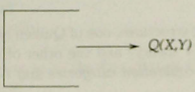
b) **Kan**: [61].

Kamps: numerous articles from 1968 onwards, see [50, 51, 52, 53, 54, 55, 56, 57, 58, 59] and, of course [60].

Here the 'primitive' idea is that of abstracting the structure of the functor 'X goes to $X \times [0, 1]$' used as the basis for topological homotopy theory. Dually one can use 'X goes to X^I' when that exists.

Let $\mathbf{C}$ be a category. A cylinder functor on $\mathbf{C}$ is a functor $\mathbf{I} : \mathbf{C} \to \mathbf{C}$ together with natural transformations $e_0, e_1 : Id_C \to I$ and $p : I \to Id_C$ with $pe_0 = pe_1 = Id$. This defines a notion of homotopy in an obvious way, and hence a relation on the sets $\mathbf{C}(X, Y)$, etc. This relation need <u>not</u> be an equivalence relation, but one can still form $\mathbf{C}(\Sigma^{-1})$ for Σ the class of homotopy equivalences.

To control extra structure, form $Q(X, Y)_n = \mathbf{C}(I^n X, Y)$. These sets together with the maps $Q(X, Y)_n \to Q(X, Y)_{n+1}$ induced by the $e_i(X)$, for $i = 0, 1$ and $Q(X, Y)_n \to Q(X, Y)_{n-1}$ induced by $p(X)$, give $Q(X, Y)$ the structure of a *cubical* set. Cubical sets are somewhat similar in definition to simplicial sets, but one replaces simplices, $[n]$, by cubes $\{0 < 1\}^n$, to get the basic category. One can define (n, i)-boxes and Kan conditions somewhat as before and the existence of fillers for certain boxes gives properties of homotopy relations. For instance, if any box of the form



fills to give an element in $Q(X,Y)_2$ then the homotopy relation on $\mathbf{C}(X,Y)$ is symmetric. We sketch a proof.

Suppose $f, g : X \to Y$, $H : f \simeq g$, then $H : IX \to Y$ and satisfies $f = He_0(X)$, $g = He_1(X)$. Form the box

$$\begin{tikzcd} f \arrow[r, dash, "fp"] \arrow[d, dash, "fp"'] & f \\ f \arrow[r, dash, "H"'] & g \end{tikzcd}$$

As we are supposing this has a filler, there is some $K \in Q(X,Y)_2$ with form

$$\begin{tikzcd} f \arrow[r, dash, "fp"] \arrow[d, dash, "fp"'] \arrow[dr, phantom, "K"] & f \arrow[d, dash, "H'"] \\ f \arrow[r, dash, "H"'] & g \end{tikzcd}$$

Restricting to the last face (via an e_1 induced face map) gives

$$H' ; g \cong f.$$

Exercise: What fillers, if any, are needed to prove $\simeq$ (i) reflexive, (ii) transitive?

If you look back at Kan's paper or the papers of Kamps, you will see that they refer to fillers as solutions to the 'equations', which are the 'boxes'. This idea is very useful as was mentioned before. Kamps and myself [60] include a discussion of how various fillers tegether with preservation of pushouts and the initial object give classes of homotopy equivalences and cofibrations that satisfy a weakened form of Quillen's axioms that are due to K. Brown [12]. Baues has a similar result.

c) **Baues** [5]

Baues uses interacting structures, one of Quillen type (or rather of K. Brown's version of half of Quillen's theory) and the other of cylinder functor type. The two structures are called *cofibration categories* and I-categories.

Cofibration category: $(C,\ cof,\ w.e.)$

i.e. a category C with two classes of morphisms, cof of *cofibrations* and $w.e.$ of *weak equivalences*. These are to satisfy:

C 1) Isomorphisms are both cofibrations and weak equivalences. If

$$A \xrightarrow{f} B \xrightarrow{g} C$$

are composable morphisms in C, then if two of f, g gf are in $w.e.$, so is the third; if $f, g \in cof$ then $gf \in cof$.

C2) Pushout axiom

Given $B \to A$ in cof and any $f : B \to Y$, the pushout

$$\begin{tikzcd} B \arrow[r] \arrow[d, "i"'] & Y \arrow[d, "\bar{i}"] \\ A \arrow[r, "\bar{f}"'] & A \sqcup_B Y \end{tikzcd}$$

exists and $\bar{i}$ is in cof; if f is a w.e., so is $\bar{f}$; if i is also a w.e., so is $\bar{i}$.

C3) Factorisation axiom

Given $B \xrightarrow{f} Y$ there is a factorisation

$$\begin{tikzcd} B \arrow[rr, "f"] \arrow[dr, "i"'] & & Y \\ & A \arrow[ur, "g"'] & \end{tikzcd}$$

with $i \in cof$, $g \in w.e.$

C4) Axiom on fibrant models

Using the terminology: "f is a *trivial cofibration*" to mean "$f \in cof \cap w.e.$", and "RX is *fibrant*" to mean "Given any trivial cofibration $i : RX \xrightarrow{\simeq} Q$, there is a retraction $r : Q \to RX$, $ri = Id_{RX}$", the axiom states:

Given $X \in C$, there is trivial cofibration $X \to RX$ with RX fibrant.

I-**category** (**C**, cof, **I**, $\emptyset$).

Here **C** is a category, 'cof' is a class of 'cofibrations', $\emptyset$ is the initial object of **C** and I is a cylinder functor.

These are required to satisfy:

I 1) I is a cylinder functor;

I 2) Pushout axiom (almost as in the first part of C2 above, but I is also to preserve pushouts, and $I\emptyset = \emptyset$;

I 3) Cofibration axiom:

$Iso \subset cof$; $\emptyset \to X$ is always in cof; a composition of cofibrations is a cofibration and all morphisms in cof satisfy the homotopy extension property (cf. the discussion of cofibration in **Top** earlier in these notes.)

I 4) Relative cylinder axiom.

If $i : B \to A$ is a cofibration and one forms the pushout

$$\begin{tikzcd} B \cup B \arrow[r] \arrow[d, "e_0(B)+e_1(B)"'] & A \cup A \arrow[d] \\ B \times I \arrow[r] & A \cup_B (B \times I) \cup_B A \end{tikzcd}$$

then the natural map

$$A \cup_B (B \times I) \cup_B A \to A \times I$$

is a cofibration;

I 5) The interchange axiom.

For all objects X, there is a map

$$T : I^2 X \to I^2 X$$

interchanging the two copies of I, i.e.

$$Te_i(IX) = Ie_i(X) \qquad T(Ie_i(X)) = e_i(IX)$$

for $i = 0, 1$. (This corresponds to exchanging the first and second I-coordinates of $X \times I \times I$, that is,

$$(x, s, t) \to (x, t, s).$$

These axioms are generally intuitive and are easy to use. (The relative cylinder axiom is the only one that may require thought if you are from an algebraic or categorical rather than a topological background.)

Theorem 1 *(cf. Baues [5])*
If ($\mathbf{C}$, *cof*, $\mathbf{I}$, $\emptyset$) *is an I-category and we let w.e. be the class of homotopy equivalences with repect to I, then* ($\mathbf{C}$, *cof*, *w.e.*) *is a cofibration category.*

Baues develops a large segment of homotopy theory in this setting and gives a large number of examples. He then goes on to get deep new results on the homotopy theory of spaces via this abstract homotopy approach. (See also Baues, [6]).

3 Second Theme: Algebraic Models for Homotopy Types

Algebraic Homotopy

> The ultimate aim of algebraic homotopy is to construct a purely algebraic theory, which is equivalent to homotopy theory in the same sort of way that 'analytic' is equivalent to 'pure' projective geometry.
>
> J.H.C.Whitehead, [91], (quoted in Baues, [5])

A statement of the aims of 'algebraic homotopy' might thus include the following homotopy classification problem (from the same source, J.H.C.Whitehead, [91]):

> Classify the homotopy types of polyhedra, X, Y, ..., by algebraic data.
> Compute the set of homotopy classes of maps, $[X, Y]$, in terms of the classifying data for X, Y.

These aims, with the possible enlargement of the class of objects of study to include many other types of spaces, are still valid. One may summarise them by saying that one searches for a nice "algebraic" category **A** together with a functor or functors

$$\mathbf{F} : \mathbf{Spaces} \to \mathbf{A}$$

and an algebraically defined notion of homotopy in **A** such that
a) if $X \simeq Y$ in **Spaces**, then $F(X) \simeq F(Y)$ in **A**;
b) if $f \simeq g$ in **Spaces**, then $F(f) \simeq F(g)$ in **A**,
and F induces an equivalence of homotopy categories

$$Ho(\mathbf{Spaces}) \simeq Ho(\mathbf{A}).$$

[Here **Spaces** is a category, perhaps of topological spaces such as polyhedra or CW-complexes, but it may be larger than this and may contain the sort of 'generalised space' used in other contexts such as algebraic geometry.]

Ideal Scenarios

i) If we know how X is constructed from simpler objects (e.g. from 'cells' or 'simplices') and if we know F on these simple objects, then we can 'calculate' $F(X)$ completely, (e.g. not just up to extensions of groups or its analogue in **A**). For this to be the case, we would need a result of the form of the *van Kampen Theorem*:

Recall (cf. Brown, [13] or Massey, [73]), the van Kampen theorem says that if $X = A \cup B$, A, B are open, then

$$\begin{tikzcd} \Pi_1(A \cap B) \arrow[r] \arrow[d] & \Pi_1(A) \arrow[d] \\ \Pi_1(B) \arrow[r] & \Pi_1(X) \end{tikzcd}$$

is a pushout of groups (or groupoids).

(For the group version, one needs $A \cap B$ arcwise connected; for the groupoid version $A \cap B$ is a union of arcwise connected components and one bases the groupoids on at least one point in each component, cf. Brown, [13]). It is worth noting that the usual proofs of the van Kampen theorem use ideas of 'subdivision' and 'filling' similar to those we have looked at earlier. (It is a useful exercise to examine the proofs from this viewpoint. There are other proofs using gluing of covering spaces that do not use these ideas explicitly - find the connection!)

ii) It is to be hoped that the 'algebra' in **A** reflects the 'geometry' in the spaces. This is evident in the fundamental groupoid where the algebraic composition is defined via a geometric construction.

iii) Ideally the detailed 'homotopy structure' of **Top**, **Poly**, or your favourite category of 'spaces' will be reflected in **A**, ..., but what does 'detailed homotopy structure' mean? We are back with a version of our original question, now perhaps with richer intuition of ways of approaching it.

These 'ideal scenarios' are, at present, unrealistic. There are however various ideas that may reduce the problem to more manageable proportions.

We could:

a) restrict the spaces being considered using geometric properties, e.g. having dimension $\leq n$ (cf. Whitehead, [90] or more recently Baues [5] and [6]); or

b) find a model which models fully only certain homotopy types (typically those having some condition such as $\pi_i(X) = 0$ if $i > n$); or

c) find a model that classifies <u>all</u> spaces and maps, but up to a weaker relation than homotopy, (e.g. up to n-equivalence, cf. Whitehead [89], but beware the definition of n-equivalence will be slightly different in more recent work).

The specific examples of these strategies are, of course, not the only ones possible, but they have the merit of being linked in the idea of n-type (Fox, [37], Whitehead, [89], Loday, [67], etc.) The idea is that n-equivalence measures information detectable with maps coming from polyhedra of dimension $\leq n$, so the $\pi_i(X)$ for $i > n$ do not have as much significance for this notion of equivalence. Each n-equivalence class of spaces ($= n$-type) has a representative X with $\pi_i(X) = 0$ for $i > n$, so here the three ideas are strongly linked.

Examples:

(For simplicity, assume that X is a connected CW-complex or polyhedron.)

$n = 1$: the fundamental group $\pi_1(X)$ or groupoid $\Pi_1(X)$, completely models the 1-type of X, classifies maps from 1-dimensional complexes into X and also classifies covering spaces of X.

Before we go to higher values of n, we need some more notation and terminology.

We write X^n for the union of the i-cells for $i \leq n$. 'Recall' that if (X, A) is a pair of spaces, with $A \subset X$, $x_0 \in A$, then $\pi_n(X, A, x_0)$ is the n^{th} relative homotopy group of (X, A). It consists of homotopy classes of maps from an n-cube I^n into X that map all but one face of I^n to x_0 and the remaining face into A. The detailed description will not be needed here, but can be found in most books on homotopy theory. Restricting the maps to the last face gives a homomorphism

$$\partial : \pi_n(X, A, x_0) \to \pi_{n-1}(A, x_0).$$

We can now handle the case $n = 2$:

MacLane and Whitehead [70] showed that the algebraic structure of

$$\partial : \pi_n(X, X^1, x_0) \to \pi_{n-1}(X^1, x_0)$$

models the 2-type of X (Their 3-type is our 2-type - the terminology has changed in the years since their work was published.)

The structure referred to is that of a crossed module (see below). We note that

(i) Ker $\partial \cong \pi_2(X)$;

(ii) Im $\partial \lhd \pi_1(X^1)$

and

(iii) Coker $\partial \cong \pi_1(X)$,

so the usual invariants π_1 and π_2 can be found from this data.

Crossed Modules

A crossed module, (C, G, θ), consists of groups C, G, an action of G on C (written $(g, c) \to {}^g c$) and a homomorphism $\theta : C \to G$ such that

CM1 $\theta({}^{g}c) = g.\theta c.g^{-1}$ for all $g \in G$, $c \in C$;
CM2 ${}^{\theta c}c' = c.c'.c^{-1}$ for all $c, c' \in C$.

Morphisms are pairs of maps preserving structure. These give a category, **CMod**, of crossed modules.

Examples and Exercises.

1. If G is a group, and $N \lhd G$ with $i : N \to G$ the inclusion, then (N, G, i) is a crossed module, where G acts by conjugation, ${}^{g}n = gng^{-1} \in N$.

Exercise: If (C, G, θ) is a crossed module, prove that $Im\ \theta \lhd G$.

2. If M is a left G-module (so M is an abelian group with an action of G on it), let $\theta : M \to G$ be the morphism with, for all $m \in M$, $\theta(m) = 1_G$, the identity of G, then (M, G, θ) is a crossed module.

Exercise: If (C, G, θ) is a crossed module, prove that $Ker\ \theta$ is a G-module on which $Im\ \theta$ acts trivially.

3. Let G be any group. There is a natural homomorphism

$$\alpha : G \to Aut(G),$$

where $Aut(G)$ is the automorphism group of G. The homomorphism α is given by : if $g \in G$, $\alpha(g)$ is the inner automorphism given by conjugation by g, i.e.

$$\alpha(g)(x) = gxg^{-1} \quad \text{for } x \in G.$$

Exercise: Prove that $(G, Aut(G), \alpha)$ is a crossed module.

Digression on Internal Categories.

Let **C** be a category with finite limits. An internal category in **C** is a diagram

$$C_1 \underset{t}{\overset{s}{\rightrightarrows}} C_0 \xrightarrow{i} C_1$$

where

$$\begin{aligned} s &= \text{‘source of - ’} \\ t &= \text{‘target of - ’} \\ i &= \text{‘identity on - ’} \end{aligned}$$

and $si = ti = Id_{C_0}$, together with a composition map (in **C**)

$$C_1 \,{}_s\!\times_t C_1 \to C_1,$$

whose domain is given by the pullback

$$\begin{array}{ccc} C_1\,{}_s\!\times_t C_1 & \longrightarrow & C_1\,, \\ \downarrow & & \downarrow \\ C_1 & \longrightarrow & C_0 \end{array}$$

satisfying the usual associativity and identity rules. We say C_1 is the object of arrows and C_0 the object of objects, then $C_1\ {}_s\!\times_t C_1$ is the object of composable pairs of arrows.

[Thinks : $(b,a) \in C_1\ {}_s\!\times_t C_1$ if and only if $ta = sb$, i.e.

$$\left(\begin{array}{ll} s(a) \xrightarrow{a} t(a) & \\ \qquad = s(b) \xrightarrow{b} t(b) \end{array} \right) = s(a) \xrightarrow{ba} t(b)$$

In general, objects of **C** may not have elements - in the case we need this however, they do as we need **C** = **Grps**, the category of groups.]

We write **Cat**(**C**) for the category of internal categories in **C**.

Proposition 1 (cf. Brown-Spencer [21] and see remarks in that paper.) *The categories* **CMod** *and* **Cat**(**Grps**) *are equivalent.*

Sketch Proof.

With the same notation as before, set $A = Ker\ s$ and let $\theta : A \to C_0$ be given by $\theta(a) = t(a)$, i.e. θ is the restriction of t to A. The action is

$${}^g a = i(g)ai(g)^{-1}.$$

The verification of CM1 and CM2 is left to you.

Now suppose (A, G, θ) is a crossed module. Take $C_0 = G$, $C_1 = A \rtimes G$, the semidirect product of A and G. [Look this up in a good Group Theory book for fuller details, but here you will need to know the multiplication in this group:

$$(a,g).(a',g') = (a.{}^g a', gg').]$$

Now define $s(a,g) = g$, $t(a,g) = \theta(a)g$, $i(g) = (1_A, g)$ and the composition by the composite of

$$g \overset{(a,g)}{\Longrightarrow} \theta(a)g \overset{(a',\theta(a)g)}{\Longrightarrow} \theta(a'a)g$$

is to be $(a'a, g) : g \to \theta(a'a)g$. Again the job of checking that s, t and i and the composition are all group homomorphisms and that they make up an internal category in **Grps** is left to you. (In fact it is an internal groupoid in **Grps** - prove it).

We saw that $\Pi_2(X, A) \to \Pi_1(A)$ is a crossed module and so there will be a *Cat*(*Group*) as well. It is fairly easy to see how to generalise these ideas from groups to groupoids, so we get a crossed module of groupoids corresponds to a special type of double groupoids, that is a 'groupoid with two independant compositions'. This can be interpreted as homotopy classes of maps of squares. We thus have that : 1-types correspond to groupoids; 2-types correspond to double groupoids (but with a bit more extra structure in fact). For more details see Brown-Higgins, [15], which describes a van Kampen theorem for crossed mouldes.

To try to approach our 'ideal scenario', our crossed modules should carry an abstract homotopy theory that at least partially reflects that of spaces. In fact to get a richer structure than we have here, it is best to go to 'crossed complexes'. These were introduced by Whitehead in [89], where they are called 'homotopy systems'. Approximately they are obtained by attaching to a crossed module

$$\partial : C \to G$$

a chain complex of modules (over a groupoid)

$$\ldots \xrightarrow{\partial} C_n \xrightarrow{\partial} C_{n-1} \ldots \xrightarrow{\partial} C_3$$

to get a 'chain complex'

$$\ldots \xrightarrow{\partial} C_n \xrightarrow{\partial} C_{n-1} \ldots \xrightarrow{\partial} C_3 \xrightarrow{\partial} C \xrightarrow{\partial} G$$

(so $\partial\partial = 0$) of groupoids (but C and G may be non-abelian). Writing **CRS** for the category of crossed complexes, we get a functor

$$\Pi : \mathbf{CW} \to \mathbf{CRS}$$

$$\Pi(X)_n = \{\pi_n(X^n, X^{n-1}, p) | p \in X^0\}$$

$$\Pi(X)_1 = \Pi_1 X^1 X^0 \rightrightarrows X^0$$

the ***homotopy*** or ***fundamental crossed complex*** functor (cf. for instance Brown and Higgins, [19]).

Note the following:

a) Π satisfies a van Kampen theorem (Brown-Higgins, [16]) and **CRS** supports, a Quillen model category structure (Brown and Golasinski, [14]).

b) **CRS** is equivalent to several algebraic-geometric categories, e.g. $\omega-$ **gpds**, $\infty-$ **gpds**, (which are cubical sets with extra algebraic structure), simplicial T-complexes (which are Kan complexes whose fillers are nicely behaved, (cf. Ashley [4])) and their cubical analogues. These T-complexes remind one of the structures that Kamps imposes in his abstract homotopy theory to get nice homotopy properties, but the structure used here is literally 'infinitely richer'.
c) **CRS** has a tensor product and an internal hom-structure

$$\mathbf{CRS}(X \otimes Y, Z) \cong \mathbf{CRS}(X, \underline{\mathbf{CRS}}(Y, Z))$$

(cf. Brown-Higgins, [17]).
d) ΠX gives *complete* information on the 2-type of X *and* on the chain complex of modules over $\Pi_1(X)$ given by the chains on the universal cover of X, (cf. Brown-Higgins, [18]).
e) There is a simply defined functor from simplicial groups (or simplical groupoids) to crossed complexes which gives an algebraic version of Π, (cf. Carrasco and Cegarra[23] and Carrasco, [22]).

Crossed complexes thus model more information than chain complexes and have the additional nice feature that they can easily be adapted to other algebraic settings (e.g. crossed complexes of commutative algebras were used by Lichtenbaum and Schlessinger, [65], in work on the cotangent complex construction). They may thus be useful in non-abelian homological algebra and 'homotopical algebra'.

Other extensions of crossed modules have also been used.

The categories **CMod** and **Cat(Grps)** have finite limits so we can form

$$\mathbf{Cat^2(Grps)} = \mathbf{Cat(Cat(Grps))}$$

and so on to get $\mathbf{Cat^n(Grps)}$ (Loday, [67])

These model all $(n+1)$-types extending the Mac Lane - Whitehead result and they satisfy a form of van Kampen theorem, (Brown-Loday, [20]). Cat^2-groups correspond to crossed squares:

$$\begin{tikzcd} L \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & M \arrow[d, "\mu"] \\ N \arrow[r, "\nu"] & P \end{tikzcd} \quad \text{with} \quad \mu\lambda = \nu\lambda'$$

in which λ, λ', μ, ν, and $\mu\lambda$ are crossed modules and there is an 'h-map', $h : M \times N \to L$ that behaves like a commutator. An almost typical example is when

μ and ν are inclusions of normal subgroups and $L = M \cap N$ with λ, λ', inclusions and $h(m, n) = [m, n] \in L$.

A useful observation is that if G is a simplicial group and $M \lhd G$ then

$$\pi_0(M) \to \pi_0(G)$$

is a crossed module (try to prove this yourself - it's easy!) and up to isomorphism all crossed modules arise in this way (cf. Loday, [67] or Porter, [81]). Now if $N \lhd G$ as well, form the square

$$\begin{tikzcd} \pi_0(M \cap N) \arrow[r] \arrow[d] & \pi_0(M) \arrow[d] \\ \pi_0(N) \arrow[r] & \pi_0(G) \end{tikzcd}$$

This is a crossed square and up to isomorphism all crossed squares arise in this way - and so on.

It seems likely that to get a nice homotopy theory of crossed squares one would need to add a chain complex as a 'tail' in some way. Current research is investigating categorical and homotopical properties of such objects.

4 Third Theme: Categorical Structures of Homotopy Theory

(We use **Ho(Top)** as an example; the same sort of things happen in other homotopy categories.)

a) *Homotopy categories have few limits and colimits in general.*

(To prove this requires more algebraic topology than I am assuming and in fact specific proofs of this are quite few in the literature. Heller ([45], p.32.) gives the example

$$\begin{tikzcd} S^2 \arrow[r, "f"] \arrow[d] & S^2 \\ * & \end{tikzcd}$$

where the degree of f is 2. This diagram cannot have a pushout in **Ho(Top)** for if P was a pushout then the cohomology of P with coefficients $\mathbb{Z}$ and $\mathbb{Z}/2$ would violate the Universal Coefficient Theorem.)

What is much easier to prove is:

b) *Pullbacks (and pushouts) in* **Top** *are not preserved on passage to* **Ho(Top)**

e.g. Suppose X is a space and $A \subseteq X$ with $i : A \to X$ the inclusion. Let $\{*\}$ be a one point space, $x \in X \setminus A$, $a \in A$, and $\alpha : x \simeq a$, a path.

$$\begin{tikzcd} \{a\} \arrow[r] \arrow[d] & A \arrow[d, "i"] \\ \{*\} \arrow[r, "a"'] & X \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \emptyset \arrow[r] \arrow[d] & A \arrow[d, "i"] \\ \{*\} \arrow[r, "x"'] & X \end{tikzcd}$$

are both pulbacks in **Top**, where we have used a and x also to denote the functions picking out those elements, and having domain $\{*\}$. These two functions are homotopic, in fact the path α <u>is</u> a homotopy between them, so the two diagrams

$$\begin{tikzcd} & A \arrow[d, "i"] \\ \{*\} \arrow[r, "a"'] & X \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & A \arrow[d, "i"] \\ \{*\} \arrow[r, "x"'] & X \end{tikzcd}$$

considered as diagrams in **Ho(Top)** are isomorphic diagrams, hence their pullbacks (in **Ho(Top)**) if they exist, are isomorphic. Clearly such a pullback cannot be isomorphic (i.e. homotopy equivalent) both to a one-point set and the empty set, since there are no maps with the empty set as codomain except the unique identity function having the empty set as both domain and codomain.

c) *Basic constructions go wrong in homotopy categories*

e.g. Suppose G is a group and X is a left G-space (i.e. X has an action

$$G \times X \to X$$

written

$$(g, x) \mapsto {}^g x$$

which is continuous - we will give G the discrete topology). This can also be thought of as a functor

$$\mathbf{X} : \mathbf{G} \to \mathbf{Top}.$$

(**Exercise:** Check that by considering a group as a groupoid and hence as a category, a G-space *is* a functor as stated).

If we replace X by a homotopy equivalent Y, i.e. an isomorphic object in **Ho(Top)**, we cannot claim that Y will be a G-space (see later as well) so the notion of G-space is not homotopy invariant.

Some simple exercises on limits and colimits - for later use.

Let $\mathbf{X} : \mathbf{G} \to \mathbf{Sets}$ be a G-set (or more generally a G-space).

1) The colimit of X satisfies:

$$\mathbf{Sets}(Colim X, Y) \cong G - \mathbf{Sets}(X, k(Y))$$

where, if Y is a set, $k(Y)$ is Y with trivial G-action.

Prove that $Colim X$ is isomorphic to X/G, the set of G-orbits of X (i.e. the set of equivalence classes for the relation $x \sim {}^g x$).

2) The limit of X satisfies

$$G - \mathbf{Sets}(k(Y), X) \simeq \mathbf{Sets}(Y, Lim X).$$

By taking Y to be a singleton set, $\{*\}$, or otherwise, prove that $Lim X$ is $Fix_G X$, the fixed point set of X:

$$Fix_G X = \{x \in X : x = {}^g x \text{ for all } g \in G\}.$$

The corresponding constructions for G-spaces, of course, work in the same way.

Many problems in algebraic topology are linked to the problem of calculating the invariants of the homotopy type of X/G or $Fix_G X$ given similar information on X. An important variation on this is to consider homotopic group actions and to find invariants of X/G or $Fix_G X$ which *are* invariants for all related actions.

Exercise: Find an example of a G-space X such that within $\mathbf{Ho(Top)}$, the limit and colimit of $\gamma X : \mathbf{G} \to \mathbf{Ho(Top)}$ do not exist. (Recall: $\gamma : \mathbf{Top} \to \mathbf{Ho(Top)}$ is the natural quotient functor, see earlier).

The next subject we need to look at in this section also needs a bit more category theory, namely, the theory of Kan extensions. [Historical thought: D. M. Kan whilst working in abstract homotopy theory made a great contribution to 'pure' category theory. He was one of the first to identify the fundamental properties of adjoint functors and also looked at the extension problem that is now central to 'Kan extensions', (cf. Mac Lane [69]).]

Problem:

Given

$$\begin{tikzcd} A \arrow[r, "K"] \arrow[d, "F"'] & B, \\ C & \end{tikzcd}$$

find the 'best' $\bar{F} : \mathbf{B} \to \mathbf{C}$ completing the diagam, i.e. to 'extend' F along K.

In general an exact extension, i.e. $\check{F}K = F$, is not possible and 'best' is interpreted, as with adjoints, etc., as a universal property.

Two solutions:

(i)

$$\begin{tikzcd} A \arrow[r, "K"] \arrow[d, "F"'] & B \arrow[dl, "Ran_K F"] \\ C & \end{tikzcd}$$

$Ran_K F$, the right Kan extension of F along K, (when it exists) comes with an isomorphism

$$Nat(SK, F) \cong Nat(S, Ran_K F)$$

natural in $S : \mathbf{B} \to \mathbf{C}$. [Thinks: it is a *right* Kan extension because it occurs in the righthand part of the $Nat(\ ,\ )$ expression].

In particular, there is a natural transformation $(Ran_K F)K \to F$ that corresponds to the identity on $Ran_K F$.

(ii) The left Kan extension, $Lan_K F$, comes with a natural isomorphism

$$Nat(F, SK) \cong Nat(Lan_K F, S).$$

[Thinks: Yes, a left Kan extension is in the left hand position of the Nat].

Note: if $\mathbf{A}$, $\mathbf{B}$ are small, then $\mathbf{C}^{\mathbf{A}}$ and $\mathbf{C}^{\mathbf{B}}$, that is the functor categories, exist and $K : \mathbf{A} \to \mathbf{B}$ induces a functor

$$\mathbf{C}^{\mathbf{K}} : \mathbf{C}^{\mathbf{B}} \to \mathbf{C}^{\mathbf{A}}$$

by composition with K. If right and left Kan extensions along K exist for all F in $\mathbf{C}^{\mathbf{A}}$, this is the same as saying that $\mathbf{C}^{\mathbf{K}}$ has right and left adjoints.

Exercise:

Suppose G, H are groups (considered as categories), $K : \mathbf{G} \to \mathbf{H}$ a functor, (i.e. a group homomorphism), $X : \mathbf{G} \to \mathbf{Sets}$, a G-set. What are $Ran_K X$ and $Lan_K X$? (These correspond to quite well known constructions in group representation theory.) It may help to notice that when $H = \{1\}$, the trivial group, these are $Lim X$ and $Colim X$. The way to try to calculate $Ran_K X$ and $Lan_K X$ is to imitate the arguments used with $Lim X$ and $Colim X$ earlier.

Back to homotopy theory: **Homotopy Limits and Colimits.** (Bousfield-Kan [11], Illusie, [48], Boardman-Vogt, [7], Vogt [88], Edwards and Hastings [36], Bourn-Cordier, [10], etc).

Idea:
$Lim : \mathbf{C}^J \to \mathbf{C}$ is right adjoint to $k : \mathbf{C} \to \mathbf{C}^J$, the 'constant diagram' functor.

If $\mathbf{C}$ has a homotopy structure (e.g. notion of weak equivalence) so that $\mathbf{Ho}(\mathbf{C})$ can be formed, then k induces

$$\mathbf{Ho}(k) : \mathbf{Ho}(\mathbf{C}) \to \mathbf{Ho}(\mathbf{C}^J)$$

for a suitable notion of weak equivalence in $\mathbf{C}^J$. [In fact one uses the level weak equivalences in $\mathbf{C}^J$) which are those

$$f : X \to Y \qquad\qquad X, Y \in \mathbf{C}^J$$

for which each $f(j) : X(j) \to Y(j)$ is a weak equivalence]. In many cases $\mathbf{Ho}(k)$ has a right adjoint, *Holim*

$$\mathbf{Ho}(\mathbf{C}^J)(k(X), Y) \cong \mathbf{Ho}(\mathbf{C})(X, Holim Y).$$

Problem: understand the construction of $Holim Y$ in sufficient generality.

Of course, as we can interpret homotopy structure in several ways, so we can interpret *Holim*. We shall look at $\mathbf{Ho}(\mathbf{Top})$ and homotopy fibres as we have already looked at 'fibres'.

Example:

If $f : X \to Y$ is a fibration and Y is arcwise connected, then the fibre $F(f)$ is determined up to homotopy

$$\begin{tikzcd} F(f)_y \arrow[r] \arrow[d] & X \arrow[d] \\ \{*\} \arrow[r, "y"'] & Y \end{tikzcd} \qquad \textit{(pullback)}$$

If F is not a fibration, then we can make it into one using factorisation (compare Baues' Axiom C3 and dualise) and then take fibres, *or* ... we can do it directly and get a ***homotopy fibre*** using a ***homotopy pullback***.

Let $F_h(f)_y = \{(\lambda, x) | \lambda : I \to Y, \lambda(0) = y, \lambda(1) = f(x)\}$, then we have a diagram

$$\begin{tikzcd} F_h(f)_y \arrow[r, "p"] \arrow[d] & X \arrow[d, "f"] \\ \{*\} \arrow[r, "y"'] & Y \end{tikzcd} \qquad p(l,x) = x$$

which is *not* commutative, but there is a homotopy $H : F_h(f)_y \to Y^I$ given by $H(\lambda, x) = \lambda$, between the two composites of the sides, $H : y \simeq pf$. Note that if we change y along a path (i.e. change the map y within its homotopy class, then the homotopy type of $F_h(f)_y$ does not change. Thus $F_h(f)_y$ is more robust than the fibre itself. We tend therefore to omit the y from the notation where possible.

For a more concrete example, take $f : A \to X$ to be the inclusion, then

$$\pi_2(X, A) \cong \pi_1(F_h(f))$$

as can easily be checked directly. [Hint: pick $a_0 \in A$ as basepoint so $F_h(f)$ is $\{(\lambda, a) | \lambda : I \to Y, \lambda(0) = a_0, \lambda(1) = f(a)\}$. Base $F_h(f)$ at the constant path at a_0, i.e. (λ, a_0) with $\lambda(t) = a_0$ for all $t \in I$. Now describe a loop in $F_h(f)$ at this basepoint].

Exercise: (Returning to a general $f : X \to Y$.)

Suppose we try to fill in the corner with another square plus homotopy

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "K \nearrow"] & X \arrow[d, "f"] \\ \{*\} \arrow[r, "y"'] & Y \end{tikzcd}$$

$K : A \times I \to Y$ or, if you prefer, $K : A \to Y^I$.

Is there some universal property for the original square that leads, say, to a map $A \to F_h(f)$, together with factorisations of the homotopy K via the natural homotopy H? e.g. something like

$$\begin{tikzcd} A \arrow[dr] \arrow[drr, bend left, "g"] \arrow[ddr, bend right] & & \\ & F_h(f) \arrow[r] \arrow[d] \arrow[dr, phantom, "H \nearrow"] & X \arrow[d, "f"] \\ & \{*\} \arrow[r] & Y \end{tikzcd}$$

This suggests that homotopy limits should 'store' more homotopy information than ordinary limits allow - but in **Ho**(**C**), this information has been destroyed. This further suggests that homotopy limits correspond to structure in **C** rather than in **Ho**(**C**) as such, and this would seem to be the case. (Some authors use *Holim* for the functor at the homotopy category level and *holim* for a functorial construction in **C** that gives representative objects for *Holim*.)

All these ideas can be dualised to give homotopy colimits

$$\mathbf{Ho}(\mathbf{C}^J)(X, k(Y)) \cong \mathbf{Ho}(\mathbf{C})(Hocolim X, Y)$$

and more generally homotopy Kan extensions:

if **A B** are small, $k : \mathbf{A} \to \mathbf{B}$ induces $\mathbf{Ho}(\mathbf{C}^k) : \mathbf{Ho}(\mathbf{C}^\mathbf{B}) \to \mathbf{Ho}(\mathbf{C}^\mathbf{A})$, homotopy Kan extensions are left and right adjoints to this functor, if they exist.

Grothendieck [44] started a discussion of the existence of homotopy Kan extensions as 'integrators' in derived category theory (cf. Verdier, [87], Deligne, [32], Illusie, [48]; the theory of derived categories is a rich homotopy structure on categories of chain complexes in various settings of, for instance, algebraic geometry). His idea relates to his '*Pursuit of stacks*'. Stacks (cf. Giraud [40] or Deligne - Mumford [33]) are 'fibred categories' of a certain type, but Grothendieck thinks of them as a varying family of models of homotopy types much as a sheaf can be thought of as a varying family of sets; the gluing of sections is also slightly different. To handle the homotopy theory of stacks easily, he needs to be able to form extensions along maps between the 'bases' (i.e. whatever objects are playing the role of space, or site, in the theory of stacks). His manuscript [44] contains an enormous amount of abstract homotopy theory.

Although the motivation was different, the use of homotopy Kan extensions was also considered essential by Anderson [1] and, taking his ideas further, Heller, [45] has placed them at the centre of his ideas on abstract homotopy theory. He describes a homotopy theory as being the functorial assignment to each small category, **J** of a category **T**(**J**) (thought of, for us here, as $\mathbf{Ho}(\mathbf{C}^\mathbf{J})$), and adjoints, both right and left, to the induced functors

$$\mathbf{Ho}(\mathbf{C}^\mathbf{F}) : \mathbf{Ho}(\mathbf{C}^{\mathbf{J}'}) \to \mathbf{Ho}(\mathbf{C}^\mathbf{J})$$

if $F : \mathbf{J} \to \mathbf{J}'$, with these data satisfying certain axioms. The standard example is $\mathbf{Ho}(\mathbf{S}^\mathbf{J})$ and he shows that it acts on arbitrary homotopy theories in a natural and useful way. This theory is quite hard going, but it looks as if it is very rich, and a lot of further research needs to be done in it to exploit its potential for the applications that Heller just touches on.

5 Fourth Theme: Enriched Categories and Homotopy Theory

We saw how the Kan-Kamps approach to homotopy theory in a category leads to a cubical set $Q(X,Y)$ with $Q(X,Y)_0$ being the ordinary hom-set, in other words it enriches the 'hom' with more structure. The theory of enriched categories can be approached for the algebraic topologist via the simplicially enriched categories used in Quillen [82] or for the categorist via the book by Kelly [64]. We will not go into the theory in detail, but will give some examples:

- For each X, Y in **Top**, we can form a simplicial set $\underline{\mathbf{Top}}(X,Y)$ with

$$\underline{\mathbf{Top}}(X,Y)_n = \mathbf{Top}(X \times \Delta^n, Y).$$

This behaves as if it were $Sing(Y^X)$ even if Y^X does not always exist within the category of spaces being considered.

- For each X, $Y \in \mathbf{S}$, we get $\underline{\mathbf{S}}(X,Y)$ with

$$\underline{\mathbf{S}}(X,Y)_n = \mathbf{S}(X \times \Delta[n], Y).$$

If Y is Kan, then so is $\underline{\mathbf{S}}(X,Y)$, no matter what X is.

- For each X, $Y \in \mathbf{Cat}$, the category of small categories, then we similarly get $\underline{\mathbf{Cat}}(X,Y)$,

$$\underline{\mathbf{Cat}}(X,Y)_n = \mathbf{Cat}(X \times [n], Y).$$

These are all examples of S-enriched categories (or S-categories for short), i.e. categories $\mathbf{C}$, where $\mathbf{C}(X,Y) = \underline{\mathbf{C}}(X,Y)_0$ for some simplicial set $\underline{\mathbf{C}}(X,Y)$, where the composition map in $\mathbf{C}$ extends to one between these $\underline{\mathbf{C}}(X,Y)$s

$$\text{i.e. } \underline{\mathbf{C}}(X,Y) \times \underline{\mathbf{C}}(Y,Z) \to \underline{\mathbf{C}}(X,Z)$$

satisfying associativity and identity rules in the obvious way. Similarly, there is a notion of S-functor $\mathbf{F} : \mathbf{A} \to \mathbf{B}$ between general S-categories; this consists of an assignment, $\mathbf{F}$ on objects and for pairs A_0, A_1 of objects in $\mathbf{A}$, a simplicial map

$$\mathbf{F}_{A_0,A_1} : \underline{\mathbf{A}}(A_0, A_1) \to \underline{\mathbf{B}}(\mathbf{F}A_0, \mathbf{F}A_1)$$

satisfying axioms expressing compatibility with composition and identities. (For full detail, see Kelly [64]).

If $\mathbf{C}$ is an S-category, we can form a category $\pi_0\mathbf{C}$ with the same objects and having

$$(\pi_0\mathbf{C})(X,Y) = \pi_0(\underline{\mathbf{C}}(X,Y)).$$

For instance, if $\mathbf{C} = \mathbf{CW}$, then $\pi_0\mathbf{CW} = \mathrm{Ho}(\mathbf{CW})$, the homotopy category of CW-complexes. Similarly we could obtain a groupoid enriched category using the fundamental groupoid (cf. Gabriel and Zisman, [39]).

One can 'do' some elementary homotopy theory in any **S**-category, **C**, by saying that two maps $f_0, f_1 : X \to Y$ in **C** are homotopic if there is an $H \in \underline{\mathbf{C}}(X,Y)_1$ with $d_0H = f_1$, $d_1H = f_0$.

This theory will not be very rich unless at least some low dimensional Kan conditions are satisfied. The **S**-category, **C**, is called *locally Kan* if each $\underline{\mathbf{C}}(X,Y)$ is a Kan complex, *locally weakly Kan* if ..., etc.

The theory is 'geometrically' nicer to work with if **C** is *tensored* or *cotensored*: If for all $K \in \mathbf{S}$, $X, Y, \in \mathbf{C}$, there is an object $K \bar{\otimes} X$ in **C** such that

$$\underline{\mathbf{C}}(K \bar{\otimes} X, Y) \cong \mathbf{S}(K, \underline{\mathbf{C}}(X,Y))$$

naturally in K, X and Y, then **C** is said to be *tensored* over **S**.

Dually, if we require objects $\mathrm{C}(K,Y)$ such that

$$\underline{\mathbf{C}}(X, \mathrm{C}(K,Y)) \cong \mathbf{S}(K, \underline{\mathbf{C}}(X,Y))$$

then we say **C** is *cotensored* over **S**.

Proposition 2(*cf. Kamps and Porter, [60]*)
If **C** *is a locally Kan* **S**-*category tensored over* **S** *then taking* $IX = \Delta[1]\bar{\otimes}X$, *we get a good cylinder functor such that for the cofibrations relative to I and weak equivalences taken to be homotopy equivalences, the category* **C** *has a cofibration category structure.*

There are variants of this, due to Kamps alone, where **C** is both tensored and cotensored over **S** and the conclusion is that **C** has a Quillen model category structure. The examples of locally Kan **S**-categories include **Top**, **Kan**, **Grpd** and **CRS**, but not **Cat** or **S** itself.

6 Fifth Theme: Homotopy Coherence

(We will often use h.c. as an abbreviation for "homotopy coherent".)

We have already seen several problems in which homotopy coherence plays a part (e.g. changing a G-space X by a homotopy equivalence, or describing the universal property of a homotopy pullback). What is homotopy coherence?

Examples of h.c. diagrams in a category with cylinder.

1) A diagram indexed by the small category, [2].

$$\begin{tikzcd} & X(1) \arrow[dr, "X(12)"] & \\ X(0) \arrow[ur, "X(01)"] \arrow[rr, "X(02)"'] \arrow[rr, phantom, bend left=20, "X(012)"] & & X(2) \end{tikzcd}$$

is h.c. if there is specified a homotopy

$$X(012) : IX(0) \to X(2),$$

$$X(012) : X(02) \simeq X(12)X(01).$$

2) For a diagram indexed by $[3]$: Draw a 3-simplex, marking the vertices $X(0)$, $\ldots$, $X(3)$, the edges $X(ij)$, etc., the faces $X(ijk)$, etc. The homotopies $X(ijk)$ fit together to make the sides of a square

$$\begin{tikzcd} X(13)\,X(01) \arrow[r, "X(123)\,X(01)"] & X(23)\,X(12)\,X(01) \\ X(03) \arrow[u, "X(013)"] \arrow[r, "X(023)"'] & X(23)\,X(02) \arrow[u, "X(23)\,X(012)"'] \end{tikzcd}$$

and the diagram is made h.c. by specifying a second level homotopy

$$X(0123) : I^2X(0) \to X(3)$$

filling this square.

These can be continued for larger $[n]$, and the results glued together to make larger h.c. diagrams. Of course, this is not how it is done, but it provides some understanding of the basic idea. The basic theory was developed by Vogt, [88] following methods introduced with Boardman, [7] (see also the references in that source for other earlier papers on the area). Cordier [25] provides a simple **S**-category theory way of working with h. c. diagrams and hence released the 'arsenal' of categorical tools for working with h. c. diagrams (cf. Bourn-Cordier [10] and Cordier [26] for an h. c. description of homotopy limits of h. c. diagrams using **S**-categorical language, again generalising earlier work of Vogt).

Results

(i) If $X : \mathbf{A} \to \mathbf{Top}$ is a commutative diagram and we replace some of the $X(a)$ by homotopy equivalent $Y(a)$ with specified homotopy equivalence data:

$$f(a) : X(a) \to Y(a), \quad g(a) : Y(a) \to X(a)$$

$$H(a) : g(a)f(a) \simeq Id, \quad K(a) : f(a)g(a) \simeq Id,$$

then we can combine these data into the construction of a h. c. diagram Y based on the objects $Y(a)$ and homotopy coherent maps

$$f : X \to Y, \quad g : Y \to X, \text{ etc.},$$

making X and Y homotopy equivalent as h.c. diagrams.

Exercise: Investigate this in simple cases, e.g. when $\mathbf{A} = [1]$ or $[2]$.

(This applied to G-space X shows that if we replace X by a homotopy equivalent Y, then Y is a h. c. version of a G-space, i.e. a h. c. diagram of type $\mathbf{G}$).

(ii) **Vogt**[88] If $\mathbf{A}$ is a small category, there is a category $\mathbf{Coh}(\mathbf{A}, \mathbf{Top})$ of h.c. diagrams and homotopy classes of h. c. maps between them. Moreover there is an equivalence of categories

$$\mathbf{Coh}(\mathbf{A}, \mathbf{Top}) \xrightarrow{\simeq} \mathbf{Ho}(\mathbf{Top}^{\mathbf{A}})$$

(This 'explains' in part why homotopy limits can be interpreted in terms of h. c. cones, cf. Bourn-Cordier, [10] and Bousfield-Kan [11]

(iii) **Cordier (1980)** [25].

Given $\mathbf{A}$, a small category, then there is an $\mathbf{S}$-category $\mathbf{S}(\mathbf{A})$ (with $\pi_0\mathbf{S}(\mathbf{A}) \cong \mathbf{A}$) such that a h. c. diagram of type $\mathbf{A}$ in $\mathbf{Top}$ is given precisely by an $\mathbf{S}$-functor

$$F : \mathbf{S}(\mathbf{A}) \to \mathbf{Top}$$

This suggests the extension of h. c. diagrams to other contexts such as a general locally Kan $\mathbf{S}$-category, $\mathbf{B}$:

(iv) **Cordier-Porter**, [27]

Vogt's theorem generalises to a locally Kan, $\mathbf{B}$, replacing $\mathbf{Top}$.

(v) **Cordier-Porter**, [28]

If $\mathbf{B}$ is locally Kan, $f : F \to G$ an h. c. map between h. c. diagrams of type $\mathbf{A}$ in $\mathbf{B}$ and $H(a) : g(a) \simeq f(a)$, then there is a h. c. map $g : F \to G$ constructed from f and H having $g(a)$ at index a.

More recently, results seem to be revealing a rich theory of $\mathbf{S}$-categories, $\mathbf{S}$-functors and coherent maps between them, that runs parallel to ordinary category theory. There seem to be coherent versions of representability theorems, adjointness, monads etc. Evidence for these in special cases has been found many times before and has proved crucial in the development of algebraic topology. What is now appearing would seem to be a much more global view providing a lot more machinery and an underlying theory for handling examples.

7 Sixth Theme: A wider perspective and some surprising links.

Why bother developing such theories? Do they interact well with other parts of Mathematics or are they, as some people would suggest, the equivalent of 'contemplating one's navel', merely introspective Mathematics. Such questions are useful, even when difficult to answer. Interactions with different parts of maths seem to be a sign of health, although absence of such interactions often only means a lack of 'ripeness', of 'maturity' in some sense, of the subject. Following ideas of Grothendieck and others, I will attempt to show reasons for optimism in the future of abstract homotopy theory.

1. **Grothendieck**(SGA1, [42]): **The fundamental group of a topos.**

Let $\mathbb{E}$ be a Grothendieck topos (think of $\mathbb{E}$ as the category, $Sh(X)$, of set valued sheaves on a space X). Within $\mathbb{E}$, we can pick out a subcategory, $\mathbb{C}$, of locally finite, locally constant objects in $\mathbb{E}$. (If X is a space with $\mathbb{E} = Sh(X)$, $\mathbb{C}$ corresponds to those sheaves whose '*espace étalé*' (cf. Borceux [8]) is a finite covering space of X). Picking a base point in X generalises to picking a 'fibre functor' $F : \mathbb{C} \to \mathbf{Sets_{fin}}$, a functor satisfying various conditions implying that it is 'pro-representable'. (If $x_0 \in X$ is a base point $\{x_0\} \to X$ induces a 'fibre functor' $Sh(X) \to Sh\{x_0\} \cong \mathbf{Sets}$, by pullback).

If F is 'pro-representable' by P, then $\pi_1(\mathbb{E}, F)$ is defined to be $Aut(P)$, which is a profinite group. (Usually I will simply write $\pi_1(\mathbb{E})$, for this). Grothendieck proves there is an equivalence of categories $\mathbb{C} \simeq \pi_1(\mathbb{E}) - \mathbf{Sets_{fin}}$, the category of finite $\pi_1(\mathbb{E})$-sets. If X is a locally nicely behaved space such as a CW-complex and $\mathbb{E} = Sh(X)$, then $\pi_1(\mathbb{E})$ is the profinite completion of $\pi_1(X)$. This profinite completion occurs only because Grothendieck considers locally finite objects. Without this restriction, a covering space Y of X would correspond to a $\pi_1(X)$-set, Y', but if Y is a finite covering of X then the homomorphism from $\pi_1(X)$ to the finite group of transformations of Y factors through the profinite completion of $\pi_1(X)$. This is defined by : if G is a group, $\hat{G} = lim(G/H : H \lhd G, H$ of finite index) is its profinite completion. This idea of using covering spaces or their analogue in $\mathbb{E}$ raises several important points:

a) these are homotopy theoretic results, but no paths are used. The argument involving sheaf theory, the theory of (pro)representable functors, etc., is of a purely categorical nature. This means it is applicable to spaces where the use of paths, and other homotopies is impossible because of bad (or unknown) local properties. Such spaces have been studied within Shape Theory and Strong Shape Theory, although not by using Grothendieck's fundamental group, nor using sheaf theory. (See 2 below for more on this connection and such sources as Lisica and

Mardesić [66], Edwards and Hastings, [36], Cordier and Porter, [29], Mardesić and Segal [71] for more information on Shape and Strong Shape).

b) As no paths are used, these methods can also be applied to 'non-spaces', e.g. locales (cf. Borceux [8]) and possibly to their non-commutative analogues, quantales. For instance, classically one could consider a field k and an algebraic closure K of k and then choose $\mathbb{C}$ to be a category of étale algebras over k, in such a way that $\pi_1(\mathbb{E}) \cong Gal(K/k)$, the Galois group of k. A beautiful treatment of this can be found in Douady and Douady, [34], and the link with locales (which is very strong) is explored in Joyal and Tierney, [49]. It, in fact, leads to a classification theorem for Grothendieck toposes. From this viewpoint, low dimensional homotopy theory is seen as being part of Galois theory, or *vice versa*.

c) This underlines the fact that $\pi_1(X)$ classifies covering spaces – but for $i > 1$, $\pi_i(X)$ does not seem to classify anything other than maps from S^i into X!

This is abstract homotopy theory *par excellence*, but where are the links with earlier work?

Artin- Mazur (1965-66), [3].

Their aim was to 'study the analogues of homotopy invariants which can be obtained from algebraic varieties by using the étale topology of Grothendieck, thus their starting point was a nice Grothendieck topos (namely sheaves for the étale Grothendieck topology on a nice scheme).

[Thinks: do I need to know a lot about schemes, varieties and the étale topology to understand their work? No, certain of their constructions *do* need an algebraic geometric or categorical viewpoint to be interpreted easily, but the general picture *can* be obtained without that. Specific results, of course, do refer to algebraic geometric properties, but the essence of these can be extracted without detailed knowledge].

Digression on Čech methods

Given a (compact) space X and a finite open cover, α, of X, we can form a simplicial set $C(X,\alpha)$, whose n-simplices are $(n+1)$-strings of open sets from α, i.e. $(U_0,\ldots,U_n)$, each $U_i \in \alpha$, satisfying $\cap U_i \neq \emptyset$.

If β is another cover such that for each $V \in \beta$, there is a $U \in \alpha$ with $V \subseteq U$, then the assignment $V \to U$ in this case defines a map

$$C(X,\alpha) \to C(X,\beta)$$

dependent on the choice of U for each V, but independent 'up to homotopy'. Applying one's favourite homotopy functor $F : \mathbf{S} \to \mathbf{A}$, to each $C(X,\alpha)$ and to these homotopy classes of induced transition maps, yields an inverse system of objects in $\mathbf{A}$, what is called, in Artin-Mazur, [3], a *pro-object* in $\mathbf{A}$. [Thinks: Profinite groups occurred in Grothendieck's work? Is there a connection?] Čech in the 1930s, applied homology and cohomology in this situation to extend simplicially-based homology to a much wider class of spaces. Lefshetz, without the language of category theory, studied, again in the 1930s, the formal properties of inverse systems of polyhedra and maps between them and his student Christie [24] looked at the homotopy groups in this setting. Borsuk [9] developed Shape Theory, which although initially very geometric in flavour turned out also to be described in terms of Christie's theory of the "Čech extensions" of homotopy theory, (see Cordier and Porter, [29] for a categorical approach to Shape Theory).

The use of modified Čech methods in algebraic geometry was well established when Grothendieck and his collaborators in Paris started adapting it to work with a Grothendieck topology. Verdier in SGA4 [2] introduced hypercoverings and we return to Artin and Mazur to find their use in homotopy theory. (The work of Lubkin, [68], should also be mentioned here as it contains much that is parallel to the development by Verdier, Artin and Mazur and is sometimes much easier to decipher for the non-specialist algebraic geometer).

Given that a Grothendieck topology is essentially about abstracting a notion of 'covering' (cf. Borceux, [8]), it is not surprising that modified Čech methods can be applied. Artin and Mazur [3] used Verdier's idea of a hypercovering to get, for each Grothendieck topos, $\mathbb{E}$, a pro-object in $Ho(\mathrm{S})$ (i.e. an inverse system of simplicial sets), which they call the étale homotopy type of the topos $\mathbb{E}$ (which for them is 'sheaves for the étale topology on a variety'). Applying homotopy group functors gives pro-groups $\pi_i(\mathbb{E})$ such that $\pi_1(\mathbb{E})$ is essentially the same as Grothendieck's $\pi_1(\mathbb{E})$. (Here you need to know that the category of profinite groups, i.e. inverse *limits* of finite groups each with the discrete topology, is equivalent to the category of pro-objects in the category of finite groups. The proof is not difficult and should be clear if you first check up on the definition of profinite groups. WARNING: if you remove 'finite' the result does not hold, *but* you can recover it in part by working with "pro-discrete localic groups" instead of topological groups, i.e. take limits of finite groups within the category of 'localic' rather than 'topological' groups, remembering that 'locales' are almost 'spaces without points', again see Borceux, [8]).

Grothendieck's nice π_1 has thus an interpretation as a limit of a Čech type, or shape theoretic, system of π_1s of 'hypercoverings'. Can shape theory (or its more powerfully structured 'strong' or 'coherent' version, cf. Lisica and Mardešić, [66], or the book [72], Edwards and Hastings, [36], Porter [76, 78]) be useful for studying étale homotopy type? Not without extra work, since the Artin-Mazur-Verdier approach leads one to look at inverse systems in *proHo*(**S**), i.e. inverse systems (diagrams) in a homotopy category not a homotopy category of inverse systems as in Strong Shape Theory (in the same references as above and for comparison Porter, [80]). Attempts to 'rigidify' the hypercovering approach, so as to get into $Hopro(\mathbf{S})$ have been made (e.g. by Lubkin, [68]) or using simplicial schemes in Friedlander, [38]) but none has yet come to light that can be said to be *the* definitive method.

One of the difficulties with this hypercovering approach is that 'hypercovering' is a difficult concept and to the 'non-expert' seem non-geometric and lacking in intuition. Thankfully for us, there is an alternative approach put forward by Ken Brown [12]. (Thinks: I've already seen that paper referred to earlier in these notes, haven't I?)

As the Grothendieck topos $\mathbb{E}$ 'pretends to be' the category of *Sets*, but with a strange logic, we can 'do' simplicial set theory in $Simp(\mathbb{E})$ as long as we take care of the arguments we use. To see a bit of this in action we can note that the object $[0]$ in $Simp(\mathbb{E})$ will be the constant simplicial sheaf with value the ordinary $[0]$, "constant" here taking on two meanings at the same time, (a) constant sheaf, i.e. not varying 'over X' if $\mathbb{E}$ is thought of as $Sh(X)$, and (b) constant simplicial object, i.e. each K_n is the same and all face and degeneracy maps are identities. Thus $[0]$ interpreted as an étale space is the identity map $X \to X$ as a space over X. Of course not all simplicial objects are constant and so $Simp(\mathbb{E})$ can store a lot of information about the space (or site) X. One can look at the homotopy structure of $Simp(\mathbb{E})$. Ken Brown [12] showed it had a fibration category structure (i.e. more or less dual to the axioms that Baues gives (see above) and giving half of Quillen's structure) and if we look at those fibrant objects K in which the natural map

$$p : K \to [0]$$

is a weak equivalence, we find that these K are exactly the hypercoverings. Global sections of p give a simplicial set, $\Gamma(K)$ and varying K amongst the hypercoverings gives a pro-simplicial set (still in $proHo(\mathbf{S})$ not in $Hopro(\mathbf{S})$ unfortunately) which determines the Artin-Mazur pro-homotopy type of $\mathbb{E}$

This makes the link between shape theoretic methods and derived category theory more explicit. In the first, the 'space' is resolved using 'coverings' and these, in a sheaf theoretic setting, lead to simplicial objects in $Sh(X)$ that are weakly equivalent to $[0]$; in the second, to evaluate the derived functor of some functor $F : \mathbf{C} \to \mathbf{A}$, say, on an object C, one takes the 'average' of the values of F on objects weakly equivalent to C, i.e. one works with the functor

$$F' : \mathbf{W}(C) \to \mathbf{A}$$

(where $\mathbf{W}(C)$ has objects, $\alpha : C \to C'$, α a weak equivalence, and maps, the commuting 'triangles', and this has a 'domain' functor $\delta : \mathbf{W}(C) \to \mathbf{C}$, $\delta(\alpha) = C'$ and F' is the composite $F\delta$). This is in many cases a pro-object in $\mathbf{A}$ - unfortunately standard derived functor theory interprets 'commuting triangles' in too weak a sense and thus corresponds to shape rather than strong shape theory - one thus, in some sense, arrives in $proHo(\mathbf{A})$ instead of in $Ho(pro\mathbf{A})$, (cf. [32]).

Grothendieck (letter to L. Breen, [43] mid 1970s; Pursuing Stacks [44], 600 page manuscript.)

Already within his non-abelian cohomology, Giraud had introduced **stacks**. These are fibred categories with a special sheaf-like property. A fibred category is a functor $f : \mathbb{E} \to \mathbb{B}$ with lifting properties that are rather like those of a Kan fibration in low dimensions. As an example, let $F : \mathbf{B} \to \mathbf{Cat}$ be a functor, then Grothendieck had introduced a semidirect product construction, as had Ehresmann (for which see his collected works).

Exercise:

Given $F : \mathbf{B} \to \mathbf{Cat}$, form up a category $F \rtimes \mathbf{B}$, that has a functor $p_F : F \rtimes \mathbf{B} \to \mathbf{B}$ with $p_F^{-1} = F(B)$, and there is an induced map so that if $\beta : B \to B'$ in $\mathbf{B}$, there is a functor $p_F^{-1}(\beta) : p_F^{-1}(B) \to p_F^{-1}(B')$ corresponding to $F(\beta)$. Investigate this structure more fully.

Thomason [85], trying to look for the structure of homotopy limits and colimits in $\mathbf{Cat}$ for potential applications in algebraic K-theory, proved that $hocolim(F) \simeq F \rtimes \mathbf{B}$, for a suitable and natural homotopy structure on $\mathbf{Cat}$.

We will not need to know the detailed definition of a stack in Giraud's sense, but will think of it as a varying family of homotopy types with a gluing rule that is a homotopy coherent version of the gluing rule for local sections in a sheaf. Stacks occur quite naturally in various situations, (cf. Deligne and Mumford, [33]).

Grothendieck's suggestion in his letter to Breen was that there was a higher order covering space concept and a corresponding Galois theory. Covering spaces correspond to locally constant sheaves and the fibres of a covering space projection are sets, therefore representing "0-types" in the homotopy sense. A stack of groupoids (as in Deligne and Mumford, [33]) is a varying family of 1-types, so we could consider a suitable notion of "locally constant" stack of groupoids as corresponding to a 1st order generalisation of a covering space. Covering spaces are classified by G-sets, where $G = \pi_1(X)$, or more generally $\pi_1(\mathbb{E})$ or a groupoid version of it. Locally constant stacks of groupoids are, perhaps, classified by G-groupoids, where G is ...what? At this stage in Grothendieck's thought experiment, we can see a need for a concept.

We saw that if G was a group,

$$G \to Aut(G)$$

was a crossed module. If we form up the corresponding cat^1-group, $G \rtimes Aut(G)$ and call it $\mathbf{Aut}(G)$, then we have the possibility that a cat^1-group $\mathbf{C}$ could act on G via a map $\mathbf{C} \to \mathbf{Aut}(G)$ of cat^1-groups. This idea is however depending too much on a rather special consruction, but does suggest that automorphism objects of n-types might be $(n+1)$-types - a sort of higher order symmetry groupoid construction.

Exercises

1) Let $\mathbf{G}$ be a groupoid. Form the functor category $\mathbf{G}^{\mathbf{G}}$ and select those functors that are automorphisms of $\mathbf{G}$. Let $\mathbf{Aut}(\mathbf{G})$ be the corresponding full subcategory. Show that $\mathbf{Aut}(\mathbf{G})$ is a cat^1-group with group structure coming from composition of functors. Describe in detail the corresponding crossed module of groupoids.

2) Specialise the above construction to the case where $\mathbf{G}$ has only one object, i.e. is a group. Seen this somewhere before?

3) Generalise the construction of 1) in the following way. Let $\mathbf{Equiv}(\mathbf{G})$ be the full subcategory of $\mathbf{G}^{\mathbf{G}}$ determined by the self equivalences of $\mathbf{G}$, i.e. those $f : \mathbf{G} \to \mathbf{G}$ that are equivalences of categories. Find an adequate description of the structure that $\mathbf{Equiv}(\mathbf{G})$ has - it is not a group as such, since such an f need not have an inverse. Compare the resulting 'thing' with $\mathbf{Aut}(\mathbf{G})$.

The idea, then, that perhaps 'locally constant' stacks of groupoids might be equivalent in some way to that of G-groupoids, for some crossed module or cat^1-group, G does not seem so silly as all that. We have proved nothing, but the analogies suggest what approach might work. Before, however, we can do that various problems have to be resolved, for instance do we really know what 'locally constant' is to mean. 'Constant' might be imagined to mean each 'stalk' isomorphic to all the others, but 'stalks' are defined ony up to equivalence so perhaps 'equivalent' to the others might be more reasonable. Another problem is that we do not know the level of 'equivalence' to expect between locally constant stacks of groupoids and G-groupoids. One expects an equivalence of homotopy categories, but what is that to mean for these two situations and is it all that can be achieved.

These condsiderations are in any case only a simple case of the general ones. Grothendieck's problem was to prove a general version of this. Stacks are thought of as varying families of homotopy types. An n-stack will be a varying family of n-types (so the above discussion was about 1-stacks), whilst if G is a model for an n-type, one hopes to find an automorphism $(n+1)$-type so that $(n+1)$-type can act on an N-type G via a homomorphism to its automorphism $(n+1)$-type. Hopefully you have guess Grothendieck's idea by now. The generalisation of Galois theory to higher order should be:

Locally Constant n-stacks on $\mathbb{E} \leftrightarrow G(\mathbb{E})$-($n$-types)

where $G(\mathbb{E})$ is an $(n+1)$-type. We can list some of the problems and possible methods of attack. We have already seen some ideas:

0-types	1-types	2-types	...	n-types
Sets	Groupoids	Cat^1-groupoids	...	Cat^{n-1}-groupoids
		= crossed modules	...	crossed $(n-1)$-cubes
		= special double	...	= special n-fold
		groupoids	...	groupoids

but Grothendieck, although himself in favour of an n-fold groupoid model warns that there may be as many choices of model as there are mathematicians working with them and his n-fold groupoid is not constructed as Loday's $\mathrm{cat}(n-1)$-groups are, so even if the idea is the same, the model may be somewhat different. Grothendieck is thus suggesting an 'equivalence' between stacks of $(n-1)$-fold groupoids and $G(\mathbb{E})$-(n-1)-fold groupoid, where he hopes $G(\mathbb{E})$ can be constructed fairly directly from $\mathbb{E}$, will be a 'homotopy invariant' etc. and will be an n-fold groupoid.

Assuming that we picked models for n-types, what is to be an n-stack. That they exist is not in doubt. A stack, as was said earlier, is a special type of fibred category, small categories model all homotopy types (Thomason [86]), but in a somewhat non-geometric way, so we could possibly look for categories that correspond to n-types, but then where do n-fold groupoids with quite a bit of mixed algebraic and geometric structure in them, fit in? If we keep categories as our models, we need to describe n-types in those terms, if we abandon categories for n-fold groupoids, we do not know any longer what is a stack!

We still have the problems that we noted earlier with 1-stacks: what does 'locally constant' mean? and what does 'equivalence' mean? We must also point out that the automorphism object of an n-fold groupoid has yet to be shown to be a $(n+1)$-fold groupoid, although intuitively it seems 'obvious'!

There is some evidence for Grothendieck's idea since if X is nice, i.e. a CW-complex, the étale homotopy type of $\mathbb{E} = Sh(X)$ is given by the homotopy type of X, so the crossed module of the Mac Lane-Whitehead theorem yields a possible 2-fold groupoid associated to $\mathbb{E}$ in a natural way:

$$\Pi_2(X, X^1) \to \Pi_1(X^1),$$

but this is dependent on the choice of 1-skeleton, X^1, so is only really determined 'up to equivalence'.

There is a lovely elegant proof of the classical or groupoid van Kampen theorem which uses covering spaces. Brown and Loday [20] have proved a generalised van Kampen theorem for cat^n-groups. This may also be seen as evidence for some sort of Galois theory involving a 'fundamental' n-fold groupoid of a space and in that way giving a geometric interpretation of the Brown-Loday van Kampen theorem.

Grothendieck also, in [41], looked at cofibred additive categories and his results again suggest that there is, here again, the same sort of phenomenon occurring. The pay-off of a proof of a higher order Galois theorem would be considerable. The technical tools needed for producing such a proof may exist already, but they need adapting.

8 Conclusion (to the Bressanone notes)

I have tried to give you a taste of abstract homotopy theory, of some of its different facets and to show how a categorical viewpoint is essential for producing a 'coherent' view of homotopy theory. Abstract homotopy theory can produce new results of a calculatory nature in the hands of someone like Baues, [5], but is also very important in providing general categorical machinery for handling homotopy types. (I have not tried to be exhaustive in my coverage of the area). I have ended with an attempt to explain the core of Grothendieck's programme 'Pursuing Stacks'. Those notes mention quite a lot of abstract homotopy theory, of a very elegant type, but their motivation is best found in Grothendieck's higher order Galois theory, an idea that, if it could be pushed through, would have an enormous potential for applications in algebraic geometry, algebraic number theory, etc., as well as in more topological areas. To complete Grothendieck's programme, many of the ideas mentioned in these notes will be needed. The questions that that programme raise still need more formulation before it can be even classed as a conjecture, but it is a useful focus for developing further this area of abstract homotopy theory.

9 Conclusion : Ten years on.

Since the notes were first written, there has been considerable activity in the area of abstract homotopy theory and related areas of homotopical algebra. I cannot hope to do justice to all of it so will attempt merely to indicate some new sources, texts and results closely related to the themes of the various sections.

The material on homotopy theories, both generically and for specific examples has been significantly increased with, in particular, the book by Goerss and Jardine, [G&J], on simplicial homotopy theory, and Mark Hovey's [MH] on Model Categories. At an introductory level, there is also my own text with Kamps, [60], already cited, which was still in preparation when the original lectures were given. Other material can be found in James' Handbook, [IMJ], especially the article by Dwyer and Spalinski, [D&S], and my own one on proper homotopy theory.

For work on algebraic models for homotopy types, the Handbook article by Baues, [HJB] summarises a lot and is reasonably accessible with the background assumed here. Work extending that mentioned in the notes includes the beginnings of a clear theory relating the different models for 3-types. This extends early but unpublished work by Joyal and Tierney that discussed 'lax 3-groupoids' as such a model. (A 2-groupoid is equivalent to a cat^1-groupoid or a crossed module; a lax 3-groupoid similarly is related to the quadratic modules mentioned by Baues in [HJB]).

My own work with Mutlu, [M&P1, M&P2, M&P3, M&P4], has continued the analysis initiated by Carrasco, [22] and Cegarra, [23], and has tried to study the way in which the categorical/ algebraic structures inherent in the models of the different n-types, is already visible in the simplicial groupoid model.

The 2-groupoid/crossed module models mentioned in the notes are derived from filtered spaces, e.g. from the skeletal filtration of a CW-complex. The problem of finding 'absolute' models, independent of any additional structure and extending the construction of the fundamental groupoid, seemed to be a hard one. An elegant construction has now been given by Hardie, Kamps and Kieboom, [HK&K], and again the techniques should be reasonably accessible with the background assumed here.

Good source material on derived category theory and triangulated categories is now available. These areas are treated in Mark Hovey's book, but also in the book by Neeman, [AM]. This latter handles in detail the question of Brown representability, that is, approximately, representability of functors 'up to homotopy' in those settings.

When the original notes were being written, I was working with Cordier on an attempt to produce homotopy coherent analogues of much of elementary category theory, with a view to applying it to aspects of the Pursuit of Stacks. The amount of enriched category theory needed to read [C&P] is slightly more than has been assumed here but that paper does try to bridge the gap between the abstract categorical machinery and the interpretation in terms of homotopy coherence. The 'Pursuit' continues (see also later)!

The work of Grothendieck and its relationship with classical Galois theory etc.

is now treated in detail in a beautifully accessible text by Borceux and Janelidze, [B&J]. Some insight on the homotopy coherence of the nerve construction and the rigidification of hypercoverings has been given in a recent thesis (M. Alinor, University of Wales, Bangor, 2000), but this is not yet available in preprint form.

Gradually Grothendieck's 'Pursuit' has attracted more and more interest. Results on algebraic stacks are now much more readily available, [G&M-B], whilst simplicial versions of some of Grothendieck's ideas have been developed by Simpson, [CS], Toen, [BT] and others. (Check the math preprint archive at : http://arXiv.org/ and then look on the nearest mirror site to you to find these and many related papers).

'Globular' models for n-types were suggested by Grothendieck in [44]. Globular descriptions of (strict) n-categories and groupoids have been know for some time, but it has become clear that they do not model all homotopy types. For example, 3-groupoids do not model all 3-types, merely those with vanishing Whitehead product. On the other hand 'lax 3-groupoids' also called 'Gray-groupoids' do, as was mentioned earlier. Lax versions of ∞-categories were sought for avidly and now several different successful approaches have emerged. Papers by Batanin, [MB], Baez and Dolan, [B&D], and Tamsamani, [ZT] are amongst the earliest, whilst Hermida, Makkai and Power, [HM&P], and Leinster, [TL] have looked at comparisons between these models. (Again look up these authors on the Archive, and on their homepages, to see the current state of play on this). It is of note that the motivation for these papers is often very different. Some of the authors are motivated by considerations mentioned earlier in this article, but Baez and his co-workers are coming at the problem from the direction of, and with insights from, mathematical physics. Others approach it with applications in Logic and Computer Science in mind. [The Newsletter maintained by John Baez (http://math.ucr.edu/home/baez/TWF.html) is an excellent place to gain information on what is happening in the area).

As you can see, the area has exploded. (I have not done justice to much of the work that has been done, even in the limited area on which I have concentrated). There remains however, a lot to be done. The abstract homotopy theory of weak ∞-categories is little understood as yet. The evidence for Grothendieck's idea gets stronger and stronger, but explicit (and understandable) direct constructions of say, 'universal 2-stacks' or similar objects still seem some way off.

As I said in the introduction, I hope the notes and these brief additional comments will be useful and will persuade more researchers that abstract homotopy and its many facets, has something to offer them.

T.P., Bangor, 26 July 2001.

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