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On Additive Groups of Rings

Shalom Feigelstock
Department of Mathematics
Bar-Ilan University
Ramat-Gan, Israel
feigel@macs.biu.ac.il

1. Introduction

Since the additive group R^+ of a ring R is abelian, a connection between abelian groups and rings is implicit in the definition of a ring. Two dual general problems arise from this connection.

Problem 1. Given a class of abelian groups, determine which rings R have additive group R^+ belonging to the class.

Problem 2. Given a class of rings, determine which abelian groups are additive groups of rings belonging to the class.

A solution to Problem 2 for a class of rings, can yield ring theoretic information about the class, as will be shown in the application at the end of this survey. A problem which is often easier to solve than Problem 2 is the following: Given a class C of rings, describe the abelian groups G for which **every** ring R with $R^+ = G$ belongs to C. A survey of some of the results on these questions will be given in this note. A more detailed treatment of this topic may be found in [10] and [11].

2. Abelian Groups

All groups in this note are abelian, with addition the group operation. An element a in a group G is a torsion element if a has finite order, and is a *torsion free* element if a has infinite order. If every element in a group G is a torsion element, then G is a *torsion group*. If every non-zero element in a group G is torsion free, then G is *torsion free*. A group G possessing both torsion, and torsion free non-zero elements is a *mixed group*. For G a group, $G_t = \{a \in G \mid a$

is a torsion element$\}$, is a subgroup of G, the *torsion subgroup* of G. For p a prime, $G_p = \{a \in G \mid |a| = \text{a power of } p\}$ is a subgroup of G, the *p-component* of G. It is well-known that $G_t = \oplus_{p \text{ a prime}} G_p$, [14], Theorem 8.4.

For n an integer, and G a group, $nG = \{na \mid a \in G\}$, and $G[n] = \{a \in G \mid na = 0\}$ are subgroups of G. If $nG = G$ for every positive integer n, then G is a *divisible group*. The additive group of integers, and the additive group of rational numbers will be denoted by $\mathbb{Z}$, and $\mathbb{Q}$, respectively; and $\mathbb{Z}(n)$ is a cyclic group of order n. For p a prime, the *Prufer group*, $\mathbb{Z}(p^\infty)$, is $(\mathbb{Q}/\mathbb{Z})_p$; it is isomorphic to the subgroup of the multiplicative group of complex numbers $\mathbb{C}$, consisting of all p-th power roots of unity. A group G is divisible if and only if $G \simeq \oplus_\alpha \mathbb{Z}(p^\infty) \oplus \oplus_\beta \mathbb{Q}$, with α and β cardinals, [14], Theorem 23.1. If D is a divisible subgroup of a group G, then D is a direct summand of G, [14], Theorem 21.2. A group which does not possess a non-zero divisible subgroup is a *reduced group*. Every group is a direct sum of a unique maximal divisible subgroup, and a reduced group, [14], Theorem 21.3.

Let G be a torsion free group. The cardinality of a maximal subset of G, linearly independent over $\mathbb{Z}$, is called the *rank* of G, and is denoted $r(G)$. The rank of G is the dimension of $\mathbb{Q}\otimes G$, viewed as a vector space over $\mathbb{Q}$. Let p be a prime, and let $a \in G$. If there exists an integer $n \geq 0$, such that $a \in p^n G \setminus p^{n+1}G$. then a is said to have *p-height*, $h_p(a) = n$. If $a \in p^n G$ for every positive integer n, then $h_p(a) = \infty$. The *height*, or *height sequence* of a is the sequence $h(a) = (h_{p_i}(a))_{i=1}^\infty$, with $(p_i)_{i=1}^\infty$, the sequence of primes ordered in increasing magnitude. Two height sequences $(x_i)_{i=1}^\infty$ and $(y_i)_{i=1}^\infty$ are *equivalent* if the set of indices i, such that $x_i \neq y_i$, is finite, and $x_i = y_i$ whenever $x_i = \infty$. This equivalence is indeed an equivalence relation. The equivalence class $t(a)$ of $h(a)$ is called the *type* of a, and is often identified with $h(a)$. A torsion free group G is *homogeneous* of type $t(G) = \tau$ if all its non-zero elements have the same type τ. Rank 1 torsion free groups are homogeneous. A type τ is *idempotent* if it possesses a height sequence with all its components either 0 or ∞. Let $x = (x_i)_{i=1}^\infty$ and $y = (y_i)_{i=1}^\infty$ be two height sequences. The relation $x > y$ means $x_i > y_i$ for all positive integers i. Two types τ, σ satisfy $\tau > \sigma$, if τ, σ, respectively, possess height sequences x, y satisfying $x > y$. The relation $\geq$ is similarly defined. Let x, y be height sequences as above. The sum $x + y$ is the height sequence $(x_i + y_i)_{i=1}^\infty$. If τ, σ are types possesing x, y, respectively, then $\tau + \sigma$ is the type possesing the height sequence $x + y$.

3. Ring Multiplications on Groups

A ring multiplication on a group G is bilinear, (the distributive laws), and therefore factors through the tensor product $G\otimes G$. Conversely, if $f : G\otimes G \to G$

is a homomorphism, then the products $ab = f(a \otimes b)$ for all $a, b \in G$ induce a non-associative ring structure with additive group G. Let $Mult(G)$ denote the set of (non-associative) ring multiplications on G, and let $*_1$, $*_2, \in Mult(G)$. Define $a(*_1, +*_2)b = (a *_1 b) + (a *_2 b)$ for all $a, b \in G$. It is readily seen that $(*_1 + *_2) \in Mult(G)$, that $Mult(G)$ is a group under this operation, and that $Mult(G) \simeq Hom(G \otimes G, G) \simeq Hom(G, E(G))$, where $E(G)$ is the group of endomorphisms of G.

Determining the rings R with R^+ belonging to the class of cyclic groups is trivial. Let $G = (a)$ be a cyclic group generated by a. If $*$ is a ring multiplication on G, then $a * a = ka$ for some integer k. Conversely, for any integer k, the product $a *_k a = ka$ induces the ring multiplication $(na) *_k (ma) = nmka$ for all $na, ma \in G$. If $|a| = \infty$, then the map $\mathbb{Z} \to Mult(G)$ via $k \to *_k$ is an isomorphism, i.e., $Mult(G) \simeq \mathbb{Z}$. If $|a| = n$, then the above map has kernel $n\mathbb{Z}$, and $Mult(G) \simeq \mathbb{Z}/n\mathbb{Z}$. All rings R with R^+ cyclic are associative.

For G a direct sum of cyclic groups, $G = \oplus_{i \in I}(a_i)$, it is not difficult to describe the non-associative rings R with $R^+ = G$. For each pair of indices $i, j \in I$, let $b_{i,j} \in G$, with $|b_{i,j}| \mid |a_i \otimes a_j|$. Then the products $a_i a_j = b_{i,j}$ induce a ring structure on G. Conversely, all rings R with $R^+ = G$ are obtained in this manner. Describing the associative rings R with $R^+ = G$ is more difficult, see [2], or [25].

Another class of groups for which the rings with additive group belonging to the class can be completely described, is the class of rank 1 torsion free groups. Let G be a rank 1 torsion free group, and let $t(G) = \tau$. It is easily seen that $G \otimes G$ is also a rank 1 torsion free group, with $t(G \otimes G) = 2\tau$. Therefore if τ is not idempotent, then $t(G \otimes G) > t(G)$. Let R be a ring with $R^+ = G$. There exists $\varphi \in Hom(G \otimes G, G)$ such that $ab = \varphi(a \otimes b)$ for all $a, b \in R$. Since $t[\varphi(a \otimes b)] \geq t(a \otimes b) = 2\tau$, and $t(x) = \tau < 2\tau$ for all $x \in G$. $x \neq 0$, is follows that $\varphi(a \otimes b) = 0$. Therefore R is the *zero-ring*, i.e.. $ab = 0$ for all $a, b \in R$. If $t(G)$ is idempotent, and P is the set of primes p for which non-zero elements of G have infinite p-height, then, up to isomorphism the only ring R with additive group G, other than the zero-ring, is a ring isomorphic to the subring of the field $\mathbb{Q}$ generated by $\{1/p | p \in P\}$. These result were obtained by Redei and Szele, [20], and independently by Beaumont and Zuckerman, [8]. The rings with rank 2 torsion free additive group were studied by Beaumont, Pierce, and Wisner, [6], [7].

The above discussion of rank 1 torsion free groups G suggests the definition of two classes of groups. A group G is a nil group if the only ring with additive group G is the zero-ring. If was seen above that if G is a rank 1 torsion free group, and $t(G)$ is not idempotent then G is a nil group. A group G is *quasi-nil* if there are only finitely many non-isomorphic rings with additive group G. The rank 1 torsion free groups are quasi-nil.

4. Nil groups

Let G be a divisible torsion group, and let R be a ring with $R^+ = G$. For $a, b \in R$, $b \neq 0$, there exists $c \in R$ such that $a = |b|c$. Thus $ab = (|b|c)b = c(|b|b) = 0$. Therefore every divisible torsion group is nil. If a torsion group G is not divisible, then $G = (a) \otimes H$, with (a) a cyclic group of finite order n, [14], Corollary 27.3. The direct sum of a ring isomorphic to $\mathbb{Z}/n\mathbb{Z}$, with additive group (a), and the zero-ring with additive group H is a ring R with $R^+ = G$. Since R is not a zero-ring, it follows that a torsion group is nil if and only if it is divisible. If G is a mixed group then G is not nil. It is known, [14]. Corollary 27.3 that a mixed group G has a direct summand $\mathbb{Z}(p^k)$ with k a positive integer, or $k = \infty$. If k is a positive integer then there is a ring R with $R^+ = G$, and $\mathbb{Z}/p^k\mathbb{Z}$ is a ring direct summand of R. If $k = \infty$, then $\mathbb{Z}(p^\infty)$ is a direct summand of G. For $a \neq 0$ a torsion free element in G, the subgroup $(a \otimes a)$ of $G \oplus G$ is infinite cyclit, so there exists a non-trivial homomorphism $\varphi : (a \otimes a) \to \mathbb{Z}(p^\infty)$. Since $\mathbb{Z}(p^\infty)$ is inyective in the category of abelian groups,[14], Theorem 21.1, there is an extension of φ to a homomorphism $\varphi : G \oplus G \to \mathbb{Z}(p^\infty)$. The ring R with $R^+ = G$, and multiplication defined by $xy = \varphi(x \oplus y)$ for all x, $y \in R$, is not a zero-ring. The classification of torsion free nil groups remains an open problem. An argument similar to that used above in the rank 1 case, shows that every homogeneous torsion free group with non-idempotent type is nil. A group G is *associative nil* if the only associative ring R with $R^+ = G$, is the zero-ring. A torsion group is nil if and only if it is associative nil. There are no mixed, associative nil group. It is an open question whether or not there exists an associative nil torsion free group which is not nil. The investigation of nil groups was initiated by Szele, [22].

5. Quasi-nil groups

The quasi-nil groups were introduced by Fuchs, [13]. He showed that a torsion group is quasi-nil if and only if it is the direct sum of a finite group and a divisible torsion group. He also proved that a ring with torsion, quasi-nil additive group is associative. Investigations of Fuchs, and Borho, [13], [9], reduce the problem of classifying quasi-nil groups, to the problem of classifying torsion free nil groups. A torsion free group G is the additive group of precisely two non-isomorphic rings if and only if $G \simeq \mathbb{Q}$. Both rings are of course associative. A torsion free group G the additive group of precisely three non-isomorphic associative rings if and only if $G = \mathbb{Q} \otimes H$ with H a rank 1 torsion free group whose type is not idempotent. Every other torsion free group is either nil, or is additive group of infinitely many non-isomorphic associative rings. A mixed group G is quasi-nil if and only if either $G \simeq B \otimes H$ with B a finite group, and H a torsion free quasi-nil group, or $G \simeq B \oplus D \oplus H$ with B a finite group, D a divisible torsion group satisfying $D_p = \{0\}$ for all but finitely many primes p, and either $H \simeq \mathbb{Q}$, or H is a nil rank one torsion free group, satisfying $pH = H$ for all primes p such that $D_p \neq \{0\}$.

Another generalization of nil groups due to Szele, [22], is the following: Let n be a positive integer. A group G has *nilstufe* $n(G) = n$, if there exists a ring R with $R^+ = G$, such that $R^n \neq \{0\}$, but every ring R with $R^+ = G$ satisfies $R^{n+1} = 0$. If no such positive integer exists, then $n(G) = \infty$. For every positive integer n, Szele constructed torsion free, and mixed groups G. satisfying $n(G) = n$. If a group G has finite nilstufe, then clearly every ring R with $R^+ = G$ is nilpotent. If G is a torsion free group with finite rank $r(G) = n$, and R is a ring with $R^+ = G$, then for positive integers $k < l$, if R^k properly contains R^l, then $r(R^{k+}) > r(R^{l+})$. Therefore every associative nilpotent ring R with $R^+ = G$ satisfies $R^n = \{0\}$, and every, not necessarily associative ring R with $R^+ = G$, satisfies $R^{2^n-1} = \{0\}$. This yields that a finite rank torsion free group has finite nilstufe, if and only if every ring R with $R^+ = G$ is nilpotent. As was shown above, if a torsion group G is not divisible, then there exists a ring R with $R^+ = G$, having a ring direct summand isomorphic to $\mathbb{Z}/n\mathbb{Z}$ for some positive integer n. Clearly R is not nilpotent. Therefore for G a torsion group the following conditions are equivalent: i) G is divisible, ii) G has finite nilstufe, iii) every ring R with $R^+ = G$ is nilpotent. Wickless, [26], reduced the problem of determining the groups G satisfying: every ring R with $R^+ = G$ is nilpotent, to the torsion free case. He showed that every ring R with $R^+ = G$ is nilpotent if and only if $G = D \oplus H$ with D a divisible torsion group, and H a reduced torsion free group such that every ring with additive group H is nilpotent.

6. Other Classes of Rings

For n a positive integer, the subgroups $G[n]$, and nG of a group G, are ideals in every ring R with $R^+ = G$. Therefore the following conditions are equivalent: i) G is the additive group of a field, ii) G is the additive group of a division ring, iii) G is the additive group of a simple ring, iv) $G \simeq \oplus_\alpha \mathbb{Q}$, or $G \simeq \oplus_\alpha \mathbb{Z}(p)$, with p a prime, and α a cardinal.

A ring R is semiprimitive if its Jacobson radical, $J(R) = \{0\}$. A torsion group G is the additive group of a semiprimitive ring if and only if there exists a set of primes P such that $G \simeq \oplus_{p\in P} \oplus_{\alpha_p} \mathbb{Z}(p)$ with α_p a cardinal for each $p \in P$. A rank 1 torsion free group is the additive group of a semiprimitive ring if and only if either $G \simeq \mathbb{Q}$, or $t(G)$ is idempotent, with infinitely many components equal 0. The finite rank torsion free additive groups of semiprimitive rings have been explored using two different methods in [3], and in [19].

A ring R is a radical ring if $J(R) = R$. For any group G the zero-ring with additive group G is a radical ring. The groups G which are additive groups of radical rings R satisfying $R^2 \neq \{0\}$ have been studied by Haimo, [17].

Investigations of Beaumont and Pierce, [4], [5], [18], have yielded much information about rings with finite rank torsion free groups. Considerable background material on subrings of algebraic number fields, and on finite rank torsion free abelian groups is necessary in order to describe their results. Therefore this important topic is not being presented in this short survey. This theory of finite rank torsion free rings has been developed in a different way in [1].

Additive groups of principal ideal rings have been studied in [12]. Let G be a torsion group. G is the additive group of a (an associative) principal ideal ring if and only if G is bounded, i.e., $nG = \{0\}$ for some positive integer n. Every (associative) ring with torsion additive group G is a principal ideal ring if and only if either G is cyclic, or $G \simeq \mathbb{Z}(p) \oplus \mathbb{Z}(p)$, with p a prime. If a mixed group G is the additive group of a principal ideal ring, then G_t is bounded, and G/G_t is the additive group of a principal ideal ring. Conversely, if G_t is bounded, and G/G_t is the additive group of a principal ideal ring with unity, then G is the additive group of a principal ideal ring.

A ring R is *Artinian* if every non-empty set of left ideals of R has a minimal element. Fuchs and Szele, [24], have shown that a group G is the additive group of an Artinian ring if and only if $G \simeq \oplus_\alpha \mathbb{Q} \oplus \oplus_{\text{finite}} \mathbb{Z}(p_i^\infty) \oplus \oplus_\beta \mathbb{Z}(p_j^{k_j})$, with α, β cardinals, p_i, p_j primes, $p_j^{k_j} | m$, and m is a fixed positive integer.

7. An Application

The structure of the additive groups of Artinian rings was given in the previous paragraph. The following two purely ring theoretical questions were solved by Fuchs, Halperin, and Szele. The answer to both questions depends entirely on the additive groups of Artinian rings.

Question 1. When can an Artinian ring be embedded in an Artinian ring with unity?

A ring R is Noetherian if every non-empty set of left ideals of R has a maximal element.

Question 2. When is an Artinian ring Noetherian?

The answer to both questions is the same. Let R be an Artinian ring. The following are equivalent: i) R can be embedded in an Artinian ring with unity, ii) R is Noetherian, iii) $R^+ \simeq \oplus_\alpha \mathbb{Q} \oplus \oplus_\beta \mathbb{Z}(p_j^{k_j})$, with α, β cardinals, p_i, p_j primes, $p_j^{k_j} | m$, and m is a fixed positive integer, i.e., R_t^+ is reduced.

References

[1] **Arnold, D.M.**, *"Finite Rank Torsion Free Abelian Groups and Rings"*, Lecture Notes in Mathematics 931, Springer Verlag, (1982).

[2] **Beaumont, R.A.**, *"Rings with additive group which is the direct sum of cyclic groups"*, Duke Math. J. 15, 367-369, (1948).

[3] **Beaumont, R.A. and Lawver, D.A.**, *"Strongly semisimple abelian groups"*, Pac. J. Math. 53, 327-336, (1974).

[4] **Beaumont, R.A. and Pierce, R.S.**, *"Torsion free rings"*, Ill. J. Math. 5, 61-98, (1961).

[5] **Beaumont, R.A. and Pierce, R.S.**, *"Subrings of algebraic number fields"*, Acta Sci. Math. Szeged 22, 202-216, (1961).

[6] **Beaumont, R.A. and Pierce, R.S.** *"Torsion Free Groups of Rank Two"*, Mem. Amer. Math. Soc. 38, Providence, (1961).

[7] **Beaumont, R.A. and Wisner, R.J.** *"Rings with additive group which is a torsion free group of rank two"*, Acta Sci. Math. Szeged 20, 105-116, (1959).

[8] **Beaumont, R.A. and Zuckerman, H.S.**, *"A characterization of the subgroups of the additive rationals"*, Pac. J. Math. 1, 169-177, (1951).

[9] **Borho, W.**, *"Uber die abelschen Gruppen auf denen sich nur endliche viele wesentlich verschiedene Ringe definieren lassen"*, Abh. Math. Sem. Univ. Hamburg 37, 197-214, (1972).

[10] **Feigelstock, S.**, *"Additive Groups of Rings"*, Pitman Research Notes in Math. 83,, Pitman, Boston-London, (1983).

[11] **Feigelstock, S.**, *"Additive Groups of Rings"*, vol. 2, Pitman Research Notes in Math. 169,, Longman, Harlow, (1988).

[12] **Feigelstock, S. and Schlussel, Z.** *"Principal ideal and Noetherian groups"*, Pac. J. Math. 75, 87-92, (1978).

[13] **Fuchs, L.**, *"On quasi-nil groups"*, Acta Sci. Math. Szeged 18, 33-43, (1957).

[14] **Fuchs, L.**, *"Infinite Abelian Groups"*, vol. 1, Academic Press, NY, (1971).

[15] **Fuchs, L.**, *"Infinite Abelian Groups"*, vol. 2, Academic Press, NY, (1973).

[16] **Fuchs, L. and Halperin, I.**, *"On the embedding of a regular ring in a regular ring with identity"*, Fund. Math. 54, 285-290, (1964).

[17] **Haimo, F.** *"Radical and antiradical groups"*, Rocky Mt. J. Math. 3, 91-106, (1973).

[18] **Pierce, R.S.**, *"Subrings of simple algebras"*, Mich. Math. J. 7, 241-243, (1960).

[19] **Reid, J.D.** *"On rings on groups"*, Pac. J. Math. 53, 229-237, (1974).

[20] **Redei, L.; Szele, T. and Ringe, D.**, *"ersten Ranges"*, Acta Sci. Math. Szeged 12, 18-29, (1950).

[21] **Ree, R. and Wisner, R.J.** *"A note on torsion free nil groups"*, Proc. Amer. Math. Soc. 7, 6-8, (1956).

[22] **Szele, T.**, *"Zur Theorie der Zeroringe"*, Math. Ann. 121, 242-246, (1949).

[23] **Szele, T.**, *"Gruppentheoretische Beziehungen bei gewissen Ringkonstruktionen"*, Math. Z. 54, 168-180, (1951).

[24] **Szele, T. and Fuchs, L.**, *"On Artinian rings"*, Acta Sci. Math Szeged 17, 30-40, (1956).

[25] **Toskey, B.R.**, *"Rings on direct sum of cyclic groups"*, Publ. Math. Debrecen 10, 93-95, (1963).

[26] **Wickless, W.J.**, *"Abelian groups which admit only nilpotent multiplications"*, Pac. J. Math. 40, 251-259, (1972).