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Embeddings of Small Graphs on the Torus

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ABSTRACT

Embeddings of graphs on the torus are studied. All 2-cell embeddings of the vertex-transitive graphs on 12 vertices or less are constructed. Their automorphism groups and dual maps are also constructed. A table of embeddings is presented.

1 Toroidal Graphs

Let G be a 2-connected graph. The vertex and edge sets of G are $V(G)$ and $E(G)$, respectively. $E(G)$ is a multiset consisting of unordered pairs $\{u,v\}$, where $u,v \in V(G)$, and possibly ordered pairs (v,v), as the graphs G will sometimes have multiple edges and/or loops. We write the pair $\{u,v\}$ as uv, and the ordered pair (v,v) as vv, which represents a loop on vertex v. If $u,v \in V(G)$ then $u \to v$ means that u is adjacent to v (and so also $v \to u$). The reader is referred to Bondy and Murty [2], West [11], or Gross and Tucker [3] for other graph-theoretic terminology. An embedding of a graph on a surface is represented combinatorially by a *rotation system* [3]. This consists

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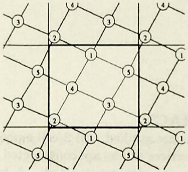
of a cyclic ordering of the incident edges, for each vertex v. Let v be a vertex of G, incident on edges $e_1, e_2, \ldots, e_k$. We write $v \rightarrow (e_1, e_2, \ldots, e_k)$ to indicate the cyclic ordering for v in a rotation system. If some e_i is a loop vv, then this loop must appear twice in the cyclic adjacency list $(e_1, e_2, \ldots, e_k)$, because walking around the vertex v along a small circle in the torus will require that a loop vv be crossed twice. Thus, we assume that if e_i is a loop vv, there is another e_i' in the list corresponding to the same loop vv. Since every loop must appear twice in the rotation system, a loop contributes two to the degree of a vertex. If e_i with endpoints uv is not a loop, then it will appear in the cyclic adjacency list of both vertices u and v. Given e_i in the list for u, the corresponding e_j in the list for v is given by the rotation sytem. Figure 1 shows an embedding of the complete graph K_5 on the torus, together with its rotation system. Embeddings on the torus are also called torus *maps*.



$$
\begin{array}{lcl}
1 & \longrightarrow & (13,15,14,12) \\
2 & \longrightarrow & (24,21,25,23) \\
3 & \longrightarrow & (35,32,31,34) \\
4 & \longrightarrow & (41,43,42,45) \\
5 & \longrightarrow & (52,54,53,51)
\end{array}
$$

Figure 1: An embedding of K_5 and its rotation system

The torus is represented here by a rectangle surrounded by a larger, shaded rectangle. See [6] for more details on this representation of the torus. The outer shaded rectangle contains copies of the vertices and edges of the embedding, thereby allowing us to easily visualize the faces of the embedding.

An embedding is called a *2-cell* embedding if every face is equivalent to a disc (ie, 2-cell). We require that all embeddings be 2-cell embeddings, and that all graphs be 2-connected. A 2-cell embedding on the torus satisfies

1.1 Euler's formula: $n + f - \varepsilon = 0$

where n is the number of vertices of G, f is the number of faces in the embedding, and ε is the number of edges. A cycle in the torus is said to be *essential* if cutting the torus along that cycle results in a cylinder (not a disk or a torus with a hole). A cycle which is not essential is called a *flat* cycle (equivalently: null-homotopic). We require that all facial cycles of embeddings contain at least 3 edges (ie, no digons or loops as facial cycles). Loops are allowed if they embed as essential cycles. Multiple edges are also allowed, so long as any cycle composed of two multiple edges is an essential cycle. This limits the number of loops allowed on a single vertex to 3, and the number of multiple edges connecting a pair of vertices to 4. We must allow loops and multiple edges, because the duals of the graphs we are interested in often have loops or multiple edges. See [6] for more information on loops and multiple edges in torus maps.

1.2 Dual Maps

Corresponding to every embedding of a graph G on the torus is a dual map, which we will denote by G^*, or dual(G) (see West [11]). The vertices of G^* are the faces of G. G^* is constructed by placing a new vertex in each face of G, and joining two new face-vertices by an edge whenever the corresponding faces of G share a common edge. Dual maps often have multipe edges or loops. Some embeddings are *self-dual*, that is, they are isomorphic to their duals. The embedding of K_5 shown in Figure 1 has this property.

A rotation system is sufficient to determine the faces and the dual map of an embedding [3, 6]. Given an edge $e_i = uv$ appearing in the rotation list for u, we can find the face to the right of e_i by executing the following loop:

```
given edge e_i incident on u
e := e_i
repeat
        v := other endpoint of e
        e' := edge corresponding to uv in the rotation list for v
        e := edge previous to e' in the rotation list for v
        u := v
until e = e_i
```

This loop walks around the boundary of the face determined by $e_i = uv$. By walking around all the faces in the embedding, we find the dual graph.

The recent book [4] by Jackson and Visentin contains a catalogue of small graphs on various surfaces. In this paper we compute all 2-cell embeddings of some small graphs on the torus. The results are summarized in a table, and a number of diagrams are also provided. In particular, we focus on the vertex-transistive graphs up to 12 vertices. (A graph G is vertex-transitive if its automoprhism group is transitive on $V(G)$). The tables also give information on the automorphism groups of the embeddings, the orientability of the embeddings, and on their dual maps in the torus. A number of other miscellaneous graphs are also included.

In order to calculate the automorphism group of an embedding and to distinguish different isomorphism types, we have converted each torus map to a digraph, and then used the graph isomorphism program of Kocay [5] to calculate the automorphism group, and to distinguish non-isomorphic embeddings.

2 Medial Digraphs, Automorphism Groups

Let G be a graph. In general, there are many possible rotation systems for G. Each rotation system will correspond to an embedding on some surface. Given a rotation system t, we write G^t to indicate G with the rotation system t. We will refer to G^t as an *embedding* of the graph G. Given two rotation systems t_1 and t_2, we need to distinguish whether the embeddings G^{t_1} and G^{t_2} are equivalent. To do this we need to consider mappings $\theta : V(G) \to V(G)$. Consider first a simple graph G (ie, no loops or multiple edges). If θ is a mapping of $V(G)$ and $v \in V$, then v^θ indicates that vertex to which θ maps v. If an edge e_i has endpoints uv, then e_i^θ indicates the edge with endpoints $u^\theta v^\theta$. Similarly, G^θ indicates the graph with vertex set $V(G)$ and edge set $E(G)^\theta$. We will know from the context whether the superscript represents a mapping or a rotation system. In case G has multiple edges and/or loops, then G^θ will also have multiple edges and/or loops.

Definition 2.1 *Let G and H be graphs on n vertices with rotation systems t_1 and t_2, respectively. The embeddings G^{t_1} and H^{t_2} are said to be isomorphic if there exists a bijection $\theta : V(G) \to V(H)$ such that for every $v \in V(G)$, $v \to (e_1, e_2, \dots, e_k)$ in G^{t_1} if and only if $v^\theta \to (e_1^\theta, e_2^\theta, \dots, e_k^\theta)$ in H^{t_2}. We write $G^{t_1} \cong H^{t_2}$.*

This defines isomorphism of embeddings combinatorially. If the embeddings G^{t_1} and H^{t_2} are isomorphic, then it is clear that the graphs G and H are also isomorphic, as the embedding-isomorphism θ is also a graph isomophism. However, G may admit many distinct (pairwise non-isomorphic) embeddings on the torus. Note also that isomorphic embeddings are not always isotopic, that is, toroidal embeddings G^{t_1} and G^{t_2} may be isomorphic, but there may be no homeomorphism of the torus that maps G^{t_1} to G^{t_2} without cutting the torus. Such graphs G^{t_1} and G^{t_2} can look very different when drawn on the torus, even though they are considered isomorphic. An example is shown in Figure 2. The graph here consists of two vertices, each with a loop, and joined by 3 multiple edges. The embeddings are combinatorially isomorphic, but non-isotopic. In order to transform one embedding into the other, it is necessary to cut the torus along an essential cycle, creating a cyclinder. One end of the cylinder is then given one full twist, and the ends are then glued back together to create a torus. This is called a *Dehn twist* [1].

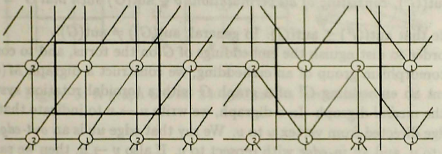


Figure 2: Two isomorphic, non-isotopic embeddings

2.1 The Torus as Oriented Surface

The torus is an oriented surface. We assume that an orientation has been given to the surface, and that one side is called the *outside* and the other the *inside*. When a graph G^t with a toroidal rotation system t is embedded on the torus, we embed it on the outside. If G^t is then viewed from the inside, the rotation system of G will appear to be reversed. Let t' be the rotation system obtained by reversing all cyclic adjacency lists of t. Clearly G^t and $G^{t'}$ are isomorphic as graphs. However they can be either isomorphic or non-isomorphic as embeddings. If $G^t \cong G^{t'}$ we say that G^t is *non-orientable*. Otherwise G^t is *orientable*. The emeddings of G on the torus can be divided into orientable and non-orientable embeddings. Each orientable embedding G^t will be paired with its *converse* $G^{t'}$.

2.2 Automorphism Group

Given a graph G, we denote the automorphism group of G by $\text{aut}(G)$. This consists of all permutations θ of $V(G)$ such that $E(G)^\theta = E(G)$. In case G has multiple edges and/or loops, multiple edges with the same endpoints, or loops on the same vertex are considered equivalent under this definition, that is, we are not permuting the edges within a set of multiple edges or a set of loops.

Given an embedding G^t and a permutation $\theta \in \mathrm{aut}(G)$, we can permute the vertices of G, and hence the cyclically ordered adjacency lists of t, by θ, to obtain an embedding $(G^t)^\theta$.

Definition 2.2 *Let G^t be an embedding of G. The automorphism group of G^t is $\mathrm{aut}(G^t)$, consisting of all permutations $\theta \in \mathrm{aut}(G)$ such that $G^t = (G^t)^\theta$.*

Note that $\mathrm{aut}(G^t) \leq \mathrm{aut}(G)$. In general, $\mathrm{aut}(G^t) \neq \mathrm{aut}(G)$.

In order to distinguish the embeddings of G on the torus, and to compute the automorphism group of an embedding, we construct a digraph $M(G^t)$ to represent an embedding G^t of a graph G with a toroidal rotation system t, called the *medial* digraph. In a digraph, we write $u \to v$ to indicate that there is an edge directed from vertex u to v. We say that edge uv is an *out-edge* with repsect to u, and an *in-edge* with repsect to v. If also $v \to u$, then we say that the edge uv is a *bi-edge*. Strictly speaking, there are two oppositely directed edges here, (u, v) and (v, u), but it is more convenient to refer to them as a single, undirected bi-edge uv.

Definition 2.3 *Let G^t be an embedding of G on the torus. We define $M(G^t)$, the medial digraph of G^t. Given the edge multiset $E(G)$, define a multiset $E'(G)$ consisting of $E(G)$, plus a double (mate) $(v, v)'$ of every loop $(v, v) \in E(G)$.*

1. $V(M(G^t))$ consists of $V(G) \cup E'(G)$.
2. Let $v \to (e_1, e_2, \ldots, e_k)$ in G^t. Then in $M(G^t)$ there are edges
   a) $v \to e_i$ and $e_i \to v$, for all i; (ie, bi-edges ve_i)
   b) $e_i \to e_{i+1}$, for all i, where $k + 1$ is replaced by 1; (ie, out-edges)
   c) If e_i is a loop, with mate e_i' having the same endpoints, then $e_i \to e_i'$ and $e_i' \to e_i$.

An example of an embedding G and its medial digraph is shown in Figure 3. Here G consists of a graph with one vertex and two loops. $M(G)$ is shown drawn on the torus, but the definition of $M(G^t)$ does not define an embedding.

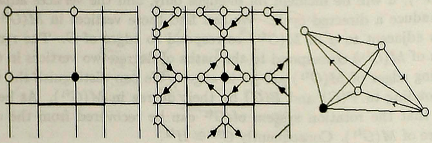


Figure 3: A graph G with 2 loops, and its medial digraph

Theorem 2.4 *Let G^{t_1} and H^{t_2} be graphs with toroidal rotation systems t_1 and t_2. Then $G^{t_1} \cong H^{t_2}$ if and only if $M(G^{t_1})$ and $M(H^{t_2})$ are isomorphic as digraphs.*

Proof. If $G^{t_1} \cong H^{t_2}$, with isomorphism θ, it is clear that θ extends to an isomorphism of $M(G^{t_1})$ and $M(H^{t_2})$. Conversely, assume that $M(G^{t_1})$ and $M(H^{t_2})$ are isomorphic as digraphs, with isomorphism $\theta : V(M(G^{t_1})) \to V(M(H^{t_2}))$. We show that G^{t_1} and H^{t_2} are isomorphic as embeddings. Suppose first that G has minimum degree at least three. The vertices of $M(G^{t_1})$ are $V(G) \cup E'(G)$. If $v \in V(G)$, then all incident edges in $M(G^{t_1})$ are bi-edges. If $e_i \in E'(G)$, then in $M(G^{t_1})$, e_i has at least one incident edge that is not a bi-edge. This distinguishes $V(G)$ and $E'(G)$ in $M(G^{t_1})$. Let $v \in V(G)$ be given. Its adjacent vertices in $M(G^{t_1})$ will be $k \geq 3$ vertices $e_1, e_e, \ldots, e_k \in E'(G)$. They must induce a directed cycle in $M(G^{t_1})$. An e_i will be a loop if and only if it is adjacent with a bi-edge to another e_j in the same directed cycle. Thus, given v, we can determine its incident edges in G, including loops. This identifies the cyclic adjacency list of v in G^{t_1}. It follows that the rotation system of G^{t_1} can be recovered from the digraph structure of $M(G^{t_1})$. Consequently $G^{t_1} \cong H^{t_2}$.

Otherwise suppose that G has one or more vertices of degree 2. Not every vertex can have degree 2, as a cycle has no 2-cell embedding. Consider a vertex v of degree two in $M(G^t)$. Then $v \in V(G)$. Let $v \to (e_1, e_2)$ be the cyclic adjacency list of v in G^t. Then e_1 and e_2 cannot be a loop, since G would then be a graph with a connected component consisting of a loop. Therefore v is part of a path of degree-two vertices; the endpoints of this path are vertices of degree 3 or more. Thus G consists of vertices of degree 3 or more, connected by paths consisting of degree-two vertices. Let u be any vertex of degree $k \geq 3$.

In $M(G^{t_1})$, u will be incident on bi-edges only, and the vertices adjacent to u will induce a directed cycle. We can find these vertices in $M(G^{t_1})$. The vertices adjacent to u in $M(G^{t_1})$ correspond to edges of G. The remaining vertices of $M(G^{t_1})$ correspond to the paths of degree-two vertices in G. The remaining edges of $M(G^{t_1})$ are all bi-edges. We can distinguish the vertices corresponding to $V(G)$ and $E(G)$ by their degree in $M(G^{t_1})$. As before, it follows that the rotation system of G^{t_1} can be recovered from the digraph structure of $M(G^{t_1})$. Consequently $G^{t_1} \cong H^{t_2}$.

2.3 Automorphism Group

A consequence of Theorem 2.4 is that the automorphism group, $\mathrm{aut}(M(G^t))$, of the medial digraph describes the automorphisms of the embedding G^t on the torus. We restrict the action of $\mathrm{aut}(M(G^t))$ to $V(G)$, and obtain $\mathrm{aut}(G^t)$, the automorphism group of G^t, as the result. If G has no multiple edges or loops, then every permutation of $V(G)$ will induce a unique permutation of $E(G)$, so that $\mathrm{aut}(G^t)$ will be a faithful representation of $\mathrm{aut}(M(G^t))$. However, if G has multiple edges or loops, $|\mathrm{aut}(G^t)|$ may be smaller than $|\mathrm{aut}(M(G^t))|$.

3 Planar Graphs

If G is a graph with a planar rotation system p, then G^p indicates the planar embedding of G. We could embed G on the torus by choosing an arbitrary 2-cell (disk) D on the torus, view it as part of a planar surface, and embed G^p into D. However this would not be a 2-cell embedding of G. One way to construct a 2-cell toroidal embedding of G is as follows.

Definition 3.1 *A theta graph H is a graph consisting of 2 vertices of degree 3, connected by 3 paths whose internal vertices all have degree 2.*

Assume that G is 2-connected, and that G is not a cycle. Then G must contain a theta subgraph H. One way to choose a theta-subgraph would be to choose an edge uv of G and the boundaries of the two faces on either side of uv in the planar embedding. A theta-subgraph can also be constructed by a depth-first search, breadth-first search, or other methods. Let the two vertices of H of degree three be called A and B. Let the 3 paths connecting A and B be called P_1, P_2, and P_3. The theta subgraph divides the plane into 3 regions. Without loss of generality, assume that P_2 is contained inside the region bounded by the cycle formed by $P_1 \cup P_3$. Refer to Figure 4.

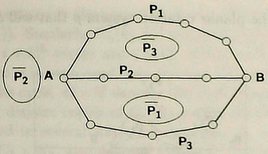


Figure 4: A planar embedding of a theta subgraph

The cycles $P_1 \cup P_2, P_2 \cup P_3$ and $P_3 \cup P_1$ divide the plane into regions. The region of the plane inside $P_1 \cup P_2$ contains a section of G, which we call $\overline{P_3}$. Similarly, the region inside $P_2 \cup P_3$ contains $\overline{P_1}$, and the region *outside* $P_3 \cup P_1$ contains $\overline{P_2}$. The rotation list for A can be described as $A \rightarrow (P_1, \overline{P_3}, P_2, \overline{P_1}, P_3, \overline{P_2})$, where P_1 indicates the first vertex of P_1 adjacent to A; $\overline{P_3}$ indicates the vertices of $\overline{P_3}$ adjacent to A, etc. Similarly the rotation list for B can be described as $B \rightarrow (P_1, \overline{P_2}, P_3, \overline{P_1}, P_2, \overline{P_3})$.

Theorem 3.2 *Let G^p be an embedding of G with a planar rotation system p, and a theta subgraph H, as described above. We construct a new rotation system t for G, as follows. The rotation list for A is changed to $A \rightarrow (P_2, \overline{P_3}, P_1, \overline{P_1}, P_3, \overline{P_2})$. All other vertices have the same rotation list as before. Then G^t is now a toroidal 2-cell embedding.*

Proof. Notice that we have altered the rotation list for A in only two vertices. The theta subgraph H has a 2-cell embedding on the torus, in which the 3 paths P_1, P_2 and P_3 cut the torus into a single 2-cell. The embedding of H can be chosen so that the rotation list of A in H is $A \rightarrow (P_1, P_3, P_2)$, and the list for B is $B \rightarrow (P_1, P_3, P_2)$. This is illustrated in Figure 5. In G^p, the rotation list for B is $B \rightarrow (P_1, \overline{P_2}, P_3, \overline{P_1}, P_2, \overline{P_3})$. We place the induced subgraphs $\overline{P_1}, \overline{P_2}$ and $\overline{P_3}$ in the torus with the same rotation system as in the plane. We need then only confirm that the connections to the vertices of H are in the same cyclic order in the torus as in the plane. This can be verified from the diagram. Since H is a 2-cell embedding, so is G^t.

This theorem provides a convenient way to convert a 2-cell embedding on the plane, to a 2-cell embedding on the torus. It requires changing the order of just two edges in the rotation list of one vertex. There are many other transformations of the planar rotation system p that will also give a toroidal rotation system.

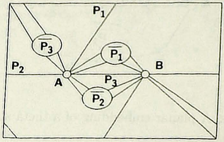


Figure 5: Converting a planar map to a toroidal map

4 Embeddings of Transitive Graphs

In this section we provide tables of the number of 2-cell embeddings of some small, 2-connected, vertex-transitive graphs on the torus, from 4 vertices up to 12 vertices, as well as some miscellaneous graphs. The transitive graphs are those published by G. Royle [9] and B.D. McKay [8]. We include drawings of some of the more interesting embeddings. Interesting embeddings will tend to be those with more automorphisms, or those that are self-dual, or have other special properties. The drawings are produced by the drawing algorithms of [6]. The tables indicate whether an embedding is orientable or non-orientable (eg., or., n.o.), and lists the automorphism group order of its embedding (eg., $g = 24$), and some information about the dual. Some embeddings are self-dual, that is, an embedding is isomorphic to its dual. This is indicated in the table (eg., self(2) means 2 self-dual embeddings). For many of the embeddings, the dual embedding has multiple edges or loops, or is not transitive. In these cases, the entry in the table will be blank.

4.1 Notation

Most of the vertex-transitive graphs up to 12 vertices can be described by a simple notation. K_n represents the complete graph on n vertices, $K_{m,n}$ a complete bipartite graph. C_n is a cycle on n vertices. If G is a graph, its complement is denoted by $\overline{G}$. If n is even, C_n^+ denotes a cycle C_n in which each vertex is also joined to its diametrically opposite vertex. $C_n(k)$ indicates a cycle in which each vertex i is also joined to the $i + k^{\text{th}}$ vertex on the cycle (therefore $C_n^+ = C_n(\frac{n}{2})$). Similarly, $C_n(k_1, k_2)$ is a cycle in which each vertex i is also joined to the $i + k_1^{\text{th}}$ vertex and $i + k_2^{\text{th}}$ vertex on the cycle. $C_n(k^+)$ indicates a cycle where n is even, and only the even vertices i are joined to the $i + k^{\text{th}}$ vertex on the cycle. $G \times H$ denotes the direct product of G and H. kG indicates k vertex disjoint copies of G. Q_k stands for the k-cube. Some graphs can be specified in several ways, for example, $Q_3 = C_4 \times K_2$. The k-prism is the graph $C_k \times K_2$, also denoted Prism(k). Given a planar graph G, we can form the *truncation* of G, denoted trunc(G), by replacing every vertex v having cyclically incident edges $(e_1, e_2, \ldots, e_k)$ by a cycle formed by k new vertices $e_1, e_2, \ldots, e_k$. That is, we subdivide every edge with two new vertices, and create cycles to replace the original vertices of G.

A *double cover* of a graph G on n vertices and ε edges is a graph H with $2n$ vertices and 2ε edges, together with a two-to-one mapping $\theta : V(H) \to V(G)$ such that there is an induced mapping $\theta : E(H) \to E(G)$ which is also two-to-one.

An $(n,3)$-*configuration* is a geometric configuration consisting of n points and n lines such that every line is incident on 3 points, and every point lies on 3 lines. The incidence graph (points versus lines) of an $(n,3)$-configuration is a 3-regular graph on $2n$ vertices. Often the duals of toroidal embeddings will be incidence graphs of $(n,3)$-configurations. For example, the incidence graphs of the Fano configuration, the Pappus configuration, and others appear as duals of toroidal embeddings. There are unique $(n,3)$-configurations when $n = 7, 8$; there are three configurations when $n = 9$; 10 when $n = 10$; 31 when $n = 11$; and 228 when $n = 12$. See Sturmfels and White [10] for more information on configurations.

A toroidal embedding G is a *triangulation* if every face is a triangle. Every embedding can be completed to a triangulation by adding diagonals across faces. Since every face in a triangulation has degree 3, it satisfies $3f = 2\varepsilon$. Combining this with Euler's formula $n + f - \varepsilon = 0$ gives $2n = f$, or $6n = 2\varepsilon$ for a triangulation. Given an arbitrary triangulation, let there be n_i vertices of degree i, where $i \geq 3$. Then summing the vertices and their degrees gives the equations

$$n_3 + n_4 + n_5 + \ldots = n$$

$$3n_3 + 4n_4 + 5n_5 + \ldots = 2\varepsilon$$

Multiply the first equation by 6 and subtract the second to obtain

$$3n_3 + 2n_4 + n_5 = n_7 + 2n_8 + 3n_9 + \ldots$$

Suppose that G is a k-regular toroidal embedding whose dual G^* is l-regular. Then $kn = 2\varepsilon$ and $lf = 2\varepsilon$. Combining this with Euler's formula $n + f - \varepsilon = 0$ gives

$$\frac{2}{k} + \frac{2}{l} = 1.$$

The only integral solutions are $k = 3, l = 6$; $k = 4, l = 4$; and $k = 6, l = 3$. The embeddings of transitive graphs are naturally grouped into these families. The first and third solutions are just duals of each other, so that there are three main families of transitive embeddings: 3-regular (honeycomb pattern), 4-regular (rectangular pattern), and 6-regular (triangular pattern). The 6-regular graphs are always triangulations whose duals are the honeycomb graphs. The 4-regular graphs are often self-dual. These patterns are evident in many of the diagrams. For example, K_5 shown in Figure 1 has a self-dual rectangular embedding. K_7 (Figure 12) has a triangular embedding whose dual is a honeycomb embedding of the Heawood graph.

There are only 5 regular planar graphs whose planar duals are also regular – they are the platonic solids. On the torus, there are infinite families of graphs with these properties.

4.2 The 4-Vertex Graphs

There is only one transitive graph on 4 vertices with a 2-cell embedding, namely K_4. Its two embeddings are shown in Figure 6.

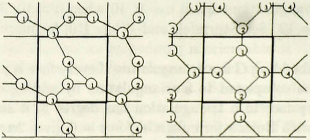


Figure 6: The 2 embeddings of K_4

4.3 The 5-Vertex Graphs

There is only one transitive graph on 5 vertices with a 2-cell embedding, namely K_5. Its 6 embeddings are shown in Figure 7.

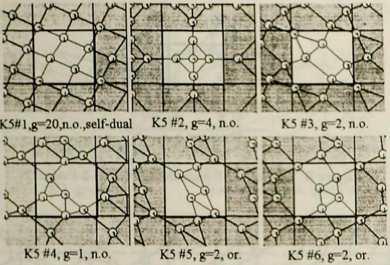
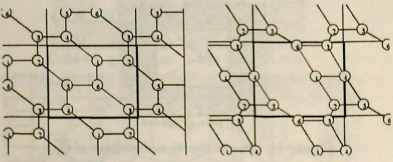


Figure 7: The 6 embeddings of K_5

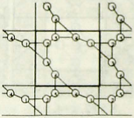
4.4 The 6-Vertex Graphs

There are four 2-connected transitive graphs on 6 vertices with 2-cell embeddings, namely $K_{3,3}$, the 3-prism, the Octahedron, and K_6. $K_{3,3}$ has two embeddings, shown in Figure 8. The 3-prism has 6 embeddings, all non-orientable. The octahedron has 17 embeddings, of which 4 are orientable. Three of them are shown in Figure 10. K_6 has 4 embeddings, 2 orientable and 2 non-orientable.



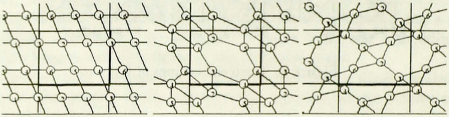
K(3,3) #1, g=18, n.o. K(3,3) #2, g=2, n.o.

Figure 8: The 2 embeddings of $K_{3,3}$



3-Prism, g=6, n.o.

Figure 9: One of the 6 embeddings of the 3-Prism

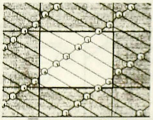


Octa#1,g=12,or.,self-dual Octa#2, g=6, n.o. Octa#3, g=4, n.o.

Figure 10: Three of the 17 embeddings of the Octahedron

4.5 The 7-Vertex Graphs

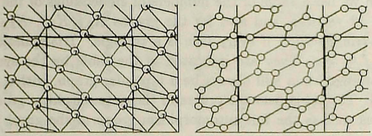
There are two 2-connected transitive graphs on 7 vertices with 2-cell embeddings, namely $\overline{C_7}$, which is the complement of the 7-cycle, and K_7. $\overline{C_7}$ has 46 embeddings, of which one is shown in Figure 11.



$\overline{C_7}$, g=14, or., self-dual

Figure 11: One of the 46 embeddings of $\overline{C_7}$

K_7 has one embedding, shown in Figure 12. The dual of K_7 on the torus is the Heawood graph, which is the incidence graph of the Fano plane, the 7-point finite projective plane. The dual of a 6-regular graph is always a 3-regular, bipartite graph. These dual graphs are often incidence graphs of projective configurations.

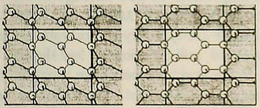


K7, g=42 or. Heawood graph, g=42 or.

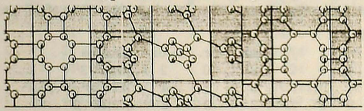
Figure 12: K_7 and its dual, the Heawood graph

4.6 The 8-Vertex Graphs

There are 8 2-connected transitive graphs on 8 vertices with 2-cell embeddings. They are: C_8^+, Q_3, $K_{4,4}$, $\overline{C_8^+}$, $\overline{Q_3}$, $\overline{C_8}$, $\overline{2C_4}$, $\overline{4K_2}$. Here C_8^+ stands for C_8 with main diagonals, and Q_3 stands for the graph of the cube. The embeddings of the cube (Figure 13) are interesting, as they show very different looking structures, but are all the same graph. Q_3 can also be written $C_4 \times K_2$. The patterns of embeddings of the cube extend to other graphs in the family $C_k \times K_2$.



Cube #1, g=24, n.o. Cube #2, g=8, n.o.



Cube #3, g=8, n.o. Cube #4, g=3, n.o. Cube #5, g=2, n.o.

Figure 13: The five embeddings of the cube

4.7 Tables of Embeddings

In this section we list a table of the numbers of embeddings of the transitive graphs up to 12 vertices, and also for the graphs $K_{3,4}, K_{3,5}$, $K_{3,6}$, and Q_4. A graph is a *triangulation* if every face is a triangle. Triangulations of the torus satisfy $f = 2n$. They are indicated in the table by the symbol (Δ). A transitive triangulation will always be a 6-regular graph, with a 3-regular, bipartite dual. The table lists the orders of the automorphism groups of the embeddings, as well as the full automorphism group of the graph in square brackets.

This table of embeddings was found as follows. For each graph (except $K_{3,5}$ and $K_{3,6}$), a spanning theta-subgraph H was chosen. Any toroidal embedding of G must contain a toroidal embedding of H. There are three ways to embed a theta-subgraph on the torus: 2-cell, flat, or cylindrical. There is a unique 2-cell embedding. To describe the non-2-cell embeddings of H, notice that H has just three cycles. There are three embeddings in which all cycles of H are flat, as one of the three cycles must be the outer face in a flat embedding. A non-2-cell embedding which is not flat must contain an essential cycle. As H has three cycles, there are three possible such embeddings, which we term cylindrical.

In order to find all 2-cell embeddings of G, we choose a spanning theta-subgraph H, and take each of its embeddings in turn. We then recursively add each remaining edge of G to the embedding, until either a 2-cell embedding is found, or until it is impossible to add an edge. For each embedding G^t found, we construct its medial digraph $M(G^t)$, and write these to a file. For a graph on 12 vertices, there are typically several hundred embeddings found, ocassionally several thousand, and occasionally less than 10. The file of medial digraphs is then input to the graph isomorphism program of Kocay [5], which produces a file of distinct graphs as output. The corresponding embeddings are then saved, and their converses are computed in order to distinguish orientable and non-orientable embeddings. The drawing algorithm of [6] produces the diagrams.

graph	n	ε	f	emb.	or.	n-or.	groups	duals
K_4	4	6	2	2	0	2	$[24]4^1, 3^1$	
K_5	5	10	5	6	2	4	$[120]20^1, 4^1, 2^3, 1^1$	self(1)
$K_{3,3}$	6	9	3	2	0	2	$[72]18^1, 2^1$	
3-Prism	6	9	3	5	0	3	$[12]6^1, 2^2, 1^2$	
Octa=$C_6(2)$	6	12	6	17	4	13	$[48]12^1, 6^1, 4^3, 3^1, 2^6, 1^5$	self(1)
K_6	6	15	9	4	2	2	$[720]6^2, 2^1, 1^1$	
$K_{3,4}$	7	12	5	3	0	3	$[144]4^1, 3^1, 2^1$	

graph	n	ε	f	emb.	or.	n-or.	groups	duals
$\overline{C_7} = C_7(2)$	7	14	7	24	22	2	$[14]14^1, 2^{14}, 1^9$	self(1)
K_7 (Δ)	7	21	14	1	0	1	$[5040]42^1$	Heawood
$K_{3,5}$	8	15	7	1	0	1	$[720]3^1$	
Q_3 (Cube)	8	12	4	5	0	5	$[48]24^1, 8^2, 3^1, 2^1$	Dbld(K_4)
C_8^+	8	12	4	4	1	3	$[16]2^3, 1^1$	
$K_{4,4}$	8	16	8	2	0	2	$[1152]32^1, 16^1$	self(2)
$\overline{C_8^+} = C_8(2)$	8	16	8	32	20	12	$[16]16^1, 4^3, 2^{13}, 1^{15}$	self(1)
$\overline{Q_3}$	8	16	8	8	4	4	$[48]4^2, 2^5, 1^1$	
$\overline{C_8} = C_8(2,4)$	8	20	12	7	7	0	$[16]2^5, 1^2$	
$\overline{2C_4} = C_8(3,4)$	8	20	12	5	2	3	$[128]8^1, 4^2, 2^2$	
$\overline{4K_2}$ (Δ)	8	24	16	1	0	1	$[384]16^1$	DblCvr(Q_3)
$K_{3,6}$	9	18	9	1	0	1	$[4320]18^1$	Paley(9)
$C_9(2)$	9	18	9	(71)	?	?	$[18]18^1, 6^1, 2, 1$	self(1)
$C_9(3)$	9	18	9	5	4	1	$[18]18^1, 2^3, 1^1$	self(1)
$K_3 \times K_3$	9	18	9	7	3	4	$[72]36^1, 18^1, 4^1, 2^3, 1^1$	$K_{3,6}$, self(1)
$\overline{3K_3}$ (Δ)	9	27	18	1	0	1	$[1296]54^1$	Pappus
$\overline{C_9}$ (Δ)	9	27	18	1	1	0	$[18]18^1$	(9,3)-config
Petersen	10	15	5	1	0	1	$[120]3^1$	
C_{10}^+	10	15	5	6	1	5	$[20]10^1, 2^4, 1^1$	
$C_5 \times K_2$	10	15	5	5	0	5	$[20]2^3, 1^2$	
$C_{10}(2)$	10	20	10	(98)	?	?	$[20]20^1, 4, 2, 1$	self(1)
$C_{10}(4)$	10	20	10	1	1	0	$[20]20^1$	self(1)
$\overline{K_5 \times K_2}$	10	20	10	1	1	0	$[240]40^1$	self(1)
$\overline{C_{10}(2)}$	10	25	15	1	0	1	$[20]10^1$	
$\overline{C_{10}(4)}$	10	25	15	4	4	0	$[20]10^1, 2^3$	
$\overline{C_5 \times K_2}$ (Δ)	10	30	20	1	0	1	$[20]20^1$	(10,3)-config
$C_{11}(2)$	11	22	11	(147)	?	?	$[22]22^1, 2, 1$	self(1)
$C_{11}(3)$	11	22	11	1	1	0	$[22]22^1$	self(1)
$\overline{C_{11}(3)}$ (Δ)	11	33	22	1	1	0	$[22]22^1$	(11,3)-config
$C_{12}(5^+)$	12	18	6	2	1	1	$[48]12^1, 2^1$	
$C_6 \times K_2$	12	18	6	9	0	9	$[24]12^1, 4^2, 2^4, 1^2$	
C_{12}^+	12	18	6	7	1	6	$[24]6^1, 2^4, 1^2$	
trunc(K_4)	12	18	6	9	0	9	$[24]4^1, 3^1, 2^2, 1^5$	
$C_{12}(3^+,6)$	12	24	12	1	0	1	$[48]24^1$	self(1)
$C_{12}(2)$	12	24	12	(244)	?	?	$[24]24^1, 8, 6, 4, 2, 1$	self(1)
$C_{12}(3)$	12	24	12	1	1	0	$[24]24^1$	self(1)
$C_{12}(4)$	12	24	12	2	1	1	$[24]24^1, 4^1$	self(1)
$C_{12}(5)$	12	24	12	2	2	0	$[768]24^2$	self(2)
$C_{12}(5^+,6)$	12	24	12	10	10	0	$[48]6^2, 2^6, 1^2$	
L(Cube)	12	24	12	14	1	13	$[48]8^2, 3^1, 2^3, 1^8$	Figure 14
$C_4 \times C_3$	12	24	12	2	0	2	$[48]24^1, 2^1$	self(1)
$C_4 \bullet C_3$	12	24	12	1	0	1	$[24]3^1$	
Icosahedron	12	30	18	12	5	7	$[120]3^1, 2^4, 1^7$	

graph	n	ε	f	emb.	or.	n-or.	groups	duals
$K_2 \times$ Octa	12	30	18	1	0	1	$[96]12^1$	
$C_{12}(5,6)$	12	30	18	8	5	3	$[768]12^1,6^1,4^1,2^4,1^1$	
$C_{12}(2,5^+)$	12	30	18	1	1	0	$[12]12^1$	
$C_{12}(4,5^+)$	12	30	18	1	1	0	$[12]12^1$	
$C_{12}(2,3)$ (Δ)	12	36	24	1	1	0	$[24]24^1$	(12,3)-config
$C_{12}(2,5)$ (Δ)	12	36	24	1	0	1	$[144]72^1$	(12,3)-config
$C_{12}(3,4)$ (Δ)	12	36	24	1	1	0	$[24]24^1$	(12,3)-config
$C_{12}(4,5)$ (Δ)	12	36	24	1	0	1	$[48]24^1$	(12,3)-config
$Q_4 = C_4 \times C_4$	16	32	16	1	0	1	$[384]64^1$	self(1)

In the above table, there are several graphs which require additional comments. The graph in the duals column named Dbld(K_4) is the complete graph K_4 in which each edge has been doubled. The Heawood graph is the incidence graph of the unique $(7,3)$-configuration. The incidence graph of the $(8,3)$-configuration is a double cover of the cube [7]. The Pappus graph is the incidence graph of the Pappus configuration, which is one of three $(9,3)$-configurations. Of the remaining two $(9,3)$-configurations, one can be embedded on the torus, and the other cannot. The incidence graph of the Desargues configuration cannot be embedded on the torus. The Paley graph on 9 vertices is a self-complementary quadratic residue graph. Paley(9) is isomorphic to $K_3 \times K_3$. It can also be described as the line graph of $K_{3,3}$. The line graph of G is denoted by $L(G)$. The graph called $C_4 \bullet C_3$ is similar to $C_4 \times C_3$. It consists of three disjoint copies of C_4, such that 4 triangles C_3 are also created by taking one corner of each C_4, in groups of three. Several of the entries have a question mark (?), and a total number of embeddings (N) in brackets (eg. (244) for $C_{12}(2)$). For these, the exact number of embeddings was not found. If there are x orientable embeddings and y non-orientable, then the relation $2x + y = N$ holds. Therefore the actual number of embeddings $x + y$ satisfies $\lceil \frac{N}{2} \rceil \le x + y \le N$.

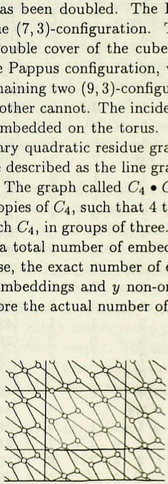


Figure 14: Dual(L(Cube))

Shown above is the dual of an embedding of L(Cube). The interesting thing about it is the regular cubic tiling of the plane that it creates.

4.8 Graphs without Embeddings

A transitive graph can have degree at most 6 in order to be embeddable on the torus. Some graphs with degree at most 6 do not have embeddings. They are listed here. $K_{3,7}$ and $K_{4,5}$ are also included.

graph	n	ε	f	group
$K_{4,5}$	9	20	11	2880
$K_{3,7}$	10	21	11	30240
$K_5 \times K_2$	10	20	10	240
$C_{10}(3,5) = K_{5,5}$	10	25	15	28800
Petersen	10	30	20	120
$C_{10}(2,4)$	10	30	20	20
$C_{11}(2,5)$	11	33	22	22
$K_{3,3} \times K_2$	12	24	12	144
$C_{12}(2,6)$	12	24	12	24
$C_{12}(3,6)$	12	30	18	24
$C_{12}(4,6)$	12	30	18	24
$K_3 \times K_4$	12	30	18	144
$C_{12}(3^+,5^+,6)$	12	30	18	48
$L(\text{Cube})^+$	12	30	18	48
$K_6 \times K_2$	12	30	18	1440
$C_{12}(3,5) = K_{6,6}$	12	36	24	1036800
Octa $\times K_2$	12	36	24	96
Icosahedron	12	36	24	120
$L(\text{Cube})^+$	12	36	24	48
$C_{12}(2,4)$	12	36	24	24
$C_{12}(2,5^+,6)$	12	36	24	12
$C_{12}(3^+,4,6)$	12	36	24	768
$C_{12}(4,5^+,6)$	12	36	24	12
$K_6 \times K_2$	12	36	24	1440

The graph denoted $L(\text{Cube})^+$ is based on the line graph of the cube, $L(Q_3)$, by adding several edges. $L(Q_3)$ has the property that every vertex has a unique opposite vertex at distance three. By adding the edges joining each vertex to its diametrically opposite vertex, we obtain $L(\text{Cube})^+$.

4.9 Questions

1. It appears from the table that the graphs with the greatest number of embeddings are of the form $C_n(2)$. Can this be proved?

2. How can isomorphic, non-isotopic embeddings be detected algorithmically?

3. A symmetric toroidal embedding creates a tiling of the plane. For example, the rectangular embeddings, honeycomb embeddings, and triangular embeddings are all symmetric tilings of the plane. Different sections of these tilings create different graphs. $K_{3,3}$, the Cube, the Heawood graph, the Pappus graph, etc, are all sections of the honeycomb tiling. K_5, $\overline{C_7}$, Q_4, and most of the self-dual 4-regular graphs are all sections of the rectangular tiling. K_7 and many others are sections of the triangular tiling. Can these graph families be classified?

4. Whitney's theorem [12] states that a 3-connected graph has a unique planar embedding. What is the corresponding theorem for the torus?

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