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Three proofs of an identity involving derivatives of a positive definite matrix and its determinant

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ABSTRACT

In the paper, three proofs for an identity involving derivatives of a positive definite matrix and its determinant are given using technique of linear algebra. The identity is basic in differential geometry.

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1 Introduction

Let M be an n-dimensional, $n \leq 1$, connected, C^∞, Riemannian manifold. For definition of manifold, please refer to standard texts [1, 4]. The Riemannian metric on M associates to each $p \in M$ an inner product on M_p, which we denote by $\langle\ ,\ \rangle$. The associated norm will be denoted by $|\ |$. The Riemannian metric is C^∞ in the sense that if X, Y are C^∞ vector fields on M, then $\langle X, Y\rangle$ is a C^∞ real-valued function on M.

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Let U be an open set in M, and $x : U \to \mathbb{R}^n$ a diffeomorphism of U into $\mathbb{R}^n$, that is, a chart on M. Then associated to the chart are n coordinate vector fields, written as $\partial/\partial x^j$ or as ∂_j, $j = 1, \dots, n$.

For the given Riemannian metric, define

$$g_{jk} = \langle \partial_j, \partial_k \rangle, \qquad G = (g_{jk})_{1 \le j,k \le n},$$
$$g = \det G, \qquad G^{-1} = (g^{jk})_{1 \le j,k \le n},$$

where $j, k = 1, \dots, n$, $\det G$ and G^{-1} denote the determinant and the inverse of G respectively. It is well-known that G is a positive definite matrix. See [2, pp. 3–7].

The following identity involving derivatives of a positive definite matrix and its determinant is fundamental in differential geometry.

Theorem 1 *For $1 \le j \le n$, we have*

$$\operatorname{tr}(G^{-1}\partial_j G) = \partial_j(\ln g). \tag{1}$$

In this short note, we will give three proofs of the identity (1) using different technique of linear algebra. For concepts of linear algebra, please refer to [3].

2 Three proofs of identity (1)

First proof. Since the metric matrix $G = (g_{ij})$ is a positive definite matrix, then we can assume its eigenvalues of G are $\lambda_i > 0$, $i = 1, \dots, n$. From theory of linear algebra, we have

$$g = \det G = |G| = \prod_{i=1}^{n} \lambda_i, \tag{2}$$

$$\ln g = \sum_{i=1}^{n} \ln \lambda_i, \tag{3}$$

$$\partial_j(\ln g) = \sum_{i=1}^{n} \frac{\partial_j \lambda_i}{\lambda_i}, \tag{4}$$

where $j = 1, \dots, n$.

Further, there is an orthogonal matrix P such that

$$P^{-1}GP = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} = \Lambda, \tag{5}$$

therefore, we have $G = P\Lambda P^{-1}$, $G^{-1} = P\Lambda^{-1}P^{-1}$, and

$$\begin{aligned}\partial_j G &= \partial_j(P\Lambda P^{-1}) \\ &= (\partial_j P)\Lambda P^{-1} + P(\partial_j\Lambda)P^{-1} + P\Lambda(\partial_j(P^{-1})),\end{aligned} \tag{6}$$

$$\begin{aligned}G^{-1}(\partial_j G) &= (P\Lambda^{-1}P^{-1})(\partial_j P)\Lambda P^{-1} + (P\Lambda^{-1}P^{-1})(P(\partial_j\Lambda)P^{-1}) \\ &\quad + (P\Lambda^{-1}P^{-1})(P\Lambda(\partial_j(P^{-1}))) \\ &= P\Lambda^{-1}(P^{-1}\partial_j P)\Lambda P^{-1} + P(\Lambda^{-1}\partial_j\Lambda)P^{-1} + P\partial_j(P^{-1}).\end{aligned} \tag{7}$$

From $P^{-1}P = E$, it follows that $(\partial_j(P^{-1}))P + P^{-1}(\partial_j P) = 0$, thus

$$\begin{aligned}G^{-1}(\partial_j G) &= -(P\Lambda^{-1})[(\partial_j(P^{-1}))P](P\Lambda^{-1})^{-1} + P(\Lambda^{-1}\partial_j\Lambda)P^{-1} + P\partial_j(P^{-1}) \\ &= -(P\Lambda^{-1}P^{-1})P[(\partial_j(P^{-1}))P]P^{-1}(P\Lambda^{-1}P^{-1})^{-1} \\ &\quad + P(\Lambda^{-1}\partial_j\Lambda)P^{-1} + P\partial_j(P^{-1}) \\ &= -G(P\partial_j(P^{-1}))G^{-1} + P(\Lambda^{-1}\partial_j\Lambda)P^{-1} + P\partial_j(P^{-1}).\end{aligned} \tag{8}$$

Using the formulae $\operatorname{tr}(AB) = \operatorname{tr}(BA)$, $\operatorname{tr}(P^{-1}AP) = \operatorname{tr} A$, and $\operatorname{tr}(A+B) = \operatorname{tr} A + \operatorname{tr} B$, we obtain

$$\begin{aligned}\operatorname{tr}[G^{-1}(\partial_j G)] &= \operatorname{tr}(P\partial_j(P^{-1})) + \operatorname{tr}[P(\Lambda^{-1}\partial_j\Lambda)P^{-1}] - \operatorname{tr}[G(P\partial_j(P^{-1}))G^{-1}] \\ &= \operatorname{tr}(\Lambda^{-1}\partial_j\Lambda) \\ &= \sum_{i=1}^{n}\frac{\partial_j\lambda_i}{\lambda_i} \\ &= \partial_j(\ln g).\end{aligned} \tag{9}$$

The proof is complete. ■

Remark 1 *In fact, we have obtained the following*

$$\operatorname{tr}(G^{-1}\partial_j G) = \operatorname{tr}[(\partial_j G)G^{-1}] = \partial_j(\ln|G|) = \partial_j(\ln g). \tag{10}$$

Second proof. We partition the matrix G by columns, that is

$$G = (\alpha_1, \ldots, \alpha_n), \tag{11}$$

$$\alpha_i = \begin{pmatrix} g_{1i} \\ \vdots \\ g_{ni} \end{pmatrix}, \tag{12}$$

where $1 \le i \le n$. Then we have

$$\partial_j(\ln g) = \partial_j \ln |G| = \frac{\partial_j |G|}{|G|}, \tag{13}$$

where

$$\partial_j |G| = \partial_j |\alpha_1, \dots, \alpha_n| = \sum_{i=1}^{n} |\alpha_1, \dots, \alpha_{i-1}, \partial_j \alpha_i, \alpha_{i+1}, \dots, \alpha_n|, \tag{14}$$

$$\partial_j \alpha_i = \begin{pmatrix} \partial_j g_{1i} \\ \vdots \\ \partial_j g_{ni} \end{pmatrix}, \quad i = 1, 2, \dots, n. \tag{15}$$

The Laplace expansion yields

$$\partial_j |G| = \sum_{i=1}^{n} \sum_{k=1}^{n} (\partial_j g_{ki}) G_{ki}, \tag{16}$$

where $G_{ki} = G_{ik}$ is the cofactor of the element $g_{ik} = g_{ki}$ in symmetric matrix $G^T = G$. Hence

$$\partial_j(\ln g) = \frac{1}{|G|} \sum_{i,k=1}^{n} (\partial_j g_{ki}) G_{ki}. \tag{17}$$

Moreover, since $\partial_j G = (\partial_j g_{ik})$ and $G^{-1} = \frac{G^*}{|G|} = \frac{(G_{ik})}{|G|}$, where G^* denotes the adjoint of G, we have

$$\begin{aligned} \operatorname{tr}(G^{-1} \partial_j G) &= \operatorname{tr} \frac{(G_{ik})(\partial_j g_{ik})}{|G|} \\ &= \frac{1}{|G|} \sum_{i=1}^{n} (G_{i1}, \dots, G_{in}) \begin{pmatrix} \partial_j g_{1i} \\ \vdots \\ \partial_j g_{ni} \end{pmatrix} \\ &= \frac{1}{|G|} \sum_{i,k=1}^{n} G_{ik}(\partial_j g_{ki}), \end{aligned} \tag{18}$$

the identity $\operatorname{tr}(G^{-1}\partial_j G) = \partial_j(\ln |G|)$ follows. ■

Remark 2 *For arbitrary square matrix A of order n, if $|A| > 0$, its element a_{ij} is a function of x, then*

$$\frac{\mathrm{d}(\ln |A|)}{\mathrm{d}x} = \operatorname{tr}\left[A^{-1} \frac{\mathrm{d}A}{\mathrm{d}x}\right] = \operatorname{tr}\left(\frac{\mathrm{d}A}{\mathrm{d}x} A^{-1}\right). \tag{19}$$

Remark 3 *Let $A(t)$ is an invertible differentiable matrix, then*

$$(\det A)' = (\det A)\,\mathrm{tr}(A^{-1}A'), \tag{20}$$

where A' denotes the derivative of matrix A with respect to t.

Third proof. Let $G^* = (G_{ij})$ denote the adjoint of the positive definite matrix G, then $G_{ij} = G_{ji}$, and

$$\mathrm{tr}(G^{-1}\partial_j G) = \mathrm{tr}\,\frac{G^*\partial_j G}{|G|} = \frac{\mathrm{tr}(G^*\partial_j G)}{|G|} = \frac{1}{g}\sum_{i,k=1}^{n} G_{ik}(\partial_j g_{ki}), \tag{21}$$

$$\partial_j(\ln g) = \frac{\partial_j g}{g} = \frac{\partial_j |G|}{g} = \frac{1}{g}\partial_j \sum_{\ell=1}^{n} g_{1\ell}G_{1\ell} = \frac{1}{g}\sum_{\ell=1}^{n}\left[(\partial_j g_{1\ell})G_{1\ell} + g_{1\ell}\partial_j G_{1\ell}\right]. \tag{22}$$

The proof reduces to prove that

$$\sum_{\ell=1}^{n} g_{1\ell}(\partial_j G_{1\ell}) = \sum_{i=2}^{n}\sum_{\ell=1}^{n}(\partial_j g_{i\ell})G_{i\ell}. \tag{23}$$

In fact, we have

$$\sum_{\ell=1}^{n} g_{k\ell}(\partial_j G_{k\ell}) = \sum_{i\neq k}^{n}\sum_{\ell=1}^{n}(\partial_j g_{i\ell})G_{i\ell}, \quad k = 1, 2, \ldots, n. \tag{24}$$

This completes the proof. ∎

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