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Semi-Classical Propagation of Singularities on Riemannian Manifolds without Boundary and Applications

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0 Introduction

The purpose of this survey article is to expose a semi-classical approach to several interesting problems concerning the behaviour of the resolvent of the Laplace-Beltrami operator on Riemannian manifolds without boundary as well as the energy decay properties of the solutions to the corresponding wave equation. In many situations this approach provides probably the most economical proof. My goal, however, will not be to give a self-contained exposition (and proof) of the results I am going to discuss. Nevertheless, I give the main ideas and proofs of the key relationships. In Section 1 I recall some well known basic facts about the semi-classical pseudodifferential operators (depending on a big parameter $\lambda \gg 1$) on compact manifolds without boundary or on $\mathbf{R}^n$, $n \geq 2$. Here I follow essentially the exposition in the Appendix of [5]. For a more complete and detailed exposition on this subject the reader is refered to Dimassi-Sjöstrand book [4]. In Section 2 I discuss some results (see Propositon 2.1 and Theorem 2.3) on propagation of inequalities along the bicharacteristic flow of solutions to the equation $(\Delta_g + \lambda^2)u = v$, where $\lambda \gg 1$, and Δ_g denotes the (negative) Laplace-Beltrami operator on a Riemannian manifold without boundary. These results admit a very simple proof, but have highly nontrivial consequences some of which will be discussed in the next sections. This approach also allows to get a semi-classical analogue (see Proposition 2.2) of propagation of singularities of solutions to the wave equation on manifolds without boundary (see Theorem 23.2.9 of [6]). Among other things, this kind of results turn out to be very useful in the study of the (semi-classical) wavefront set of quasimodes (e.g. see [12]).

The first application of the inequalities from Section 2 concerns the so called damped wave equation (which is, roughly speaking, the wave equation plus a lower order dissipative term) on a compact manifold without boundary and is presented in Section 3. Under some kind of a *nontrapping* condition (see (3.4)) it is shown that there exists a constant $\alpha > 0$ such that in $|\mathrm{Re}|\, z < \alpha$ there could be at most a finite number of eigenvalues, and as a consequence we have a uniform exponential energy decay of the solutions. This result was first proved by Rauch-Taylor [10] by using quite different methods. Later on Lebeau [7] extended their result to manifolds with boundary (and Dirichlet boundary conditions) using methods different from those in [10] or these presented here. He also gave an explicit formula for the optimal value of the constant α. This has been recently extended by Sjöstrand [11] who obtained quite precise and complete results on the distribution of the eigenvalues in this context.

In Section 4 I consider a Riemannian metric, g, on $\mathbf{R}^n$, which is a long-range perturbation of the Euclidean one (see (4.1)) and *nontrapping* (which roughly speaking means that every geodesic leaves any compact in a finite time - see (4.2) for a more precise definition). Under these assumptions, a high-frequency estimate of the norm of the resolvent of the Laplace-Beltrami operator on weighted L^2-spaces is obtained (see Theorem 4.1). The proof combines the inequalities from Section 2 (Theorem 2.3) together with an a priori estimate in weighted L^2-spaces of solutions to $(\Delta_g + \lambda^2)u = v$ outside a sufficently big compact (see Proposition 4.2) which has been recently obtained in [3]. Note that an analogue of Theorem 4.1 on asymptotically Euclidean spaces and in a semi-classical setting has been proved by

Vasy-Zworski [14] by using a different approach.

In Section 5 I consider a nontrapping Riemannian metric, g, on $\mathbf{R}^n$, which coincides with the Euclidean one outside a compact. In this case it is well known that the cutoff resolvent extends meromorphically (e.g. see [15]) with poles called *resonances*. Theorem 4.1 is used to show that there is a strip free of resonances (see Proposition 5.2), and as a consequence uniform estimates for the decay of the local energy of the solutions to the wave equation are proved (see Theorem 5.4). Moreover, using Vainberg's method (see [13]) it is shown that for nontrapping metrics a better free of resonances region of the form $\{|\mathrm{Im}\, z| \le N \log |z|, |z| \ge C_N\}$, $\forall N \gg 1$, exists (see Theorem 5.5).

In Section 6 I consider a Riemannian metric, g, on $\mathbf{R}^n$, which is a long-range perturbation of the Euclidean one. In this section some recent results on the behaviour of the resolvent of the Laplace-Beltrami operator on weighted L^2-spaces are reviewed. Since the nontrapping condition is no longer assumed, the inequalities from Section 2 no longer work in this case. Instead, we have Carleman type inequalities (see Proposition 6.2). The Carleman inequalities on manifolds without boundary are due to Hörmander (see Theorem 28.2.3 of [6]), while in case of non-empty boundary (and Dirichlet or Robin boundary conditions) they are due to Lebeau-Robbiano [8], [9]. Proposition 6.5 and Theorem 6.6 are due to Burq (see [1]) as well as a weaker version of Corollary 6.4 (see [2]). Theorem 6.1 in this generality is due to Cardoso-Vodev [3].

1 Semi-Classical Pseudo-Differential Operators on Manifolds without Boundary

Throughout this section X will denote either a n-dimensional compact manifold without boundary or the Euclidean space $\mathbf{R}^n$, where $n \ge 2$. Denote by $T^*X = \cup_{x\in X} T_x^*X$ the cotangential bundle of X, with a fiber at $x \in X$, $T_x^*X \simeq (\mathbf{R}^n)^*$. Denote $\tilde{T}^*X = T^*X \cup T^*S$, where $T^*S = \cup_{x\in X} T_x^*S$ is the cotangent sphere bundle, $T_x^*S \simeq \{\xi \in (\mathbf{R}^n)^* : |\xi| = 1\}$. The points from T^*X will be called finite, while those from T^*S will be called infinite. Fix a $\lambda \gg 1$ and given $m, k \in \mathbf{R}$, $\rho, \delta \ge 0$, $\rho + \delta < 1$, introduce the class of symbols $S^{m,k}_{\rho,\delta}(X)$ which consists of functions $a(x, \xi; \lambda) \in C^\infty(T^*X)$ satisfying

$$\left|\partial_x^\alpha \partial_\xi^\beta a(x, \xi; \lambda)\right| \le C_{\alpha,\beta,K} \lambda^{k+\rho|\beta|+\delta|\alpha|}(1 + |\xi|)^{m-|\beta|}, \quad \forall x \in K, \forall \xi \in T_x^*X, \tag{1.1}$$

for every multi-indeces α and β and every compact $K \subset X$. The semi-classical quantization, $a(x, D_x; \lambda)$, of a symbol $a(x, \xi; \lambda)$ is defined as follows

$$a(x, D_x; \lambda)u := \left(\frac{\lambda}{2\pi}\right)^n \int e^{i\lambda\langle x,\xi\rangle} a(x, \xi; \lambda)\hat{u}(\xi)d\xi$$

$$= \left(\frac{\lambda}{2\pi}\right)^n \int\int e^{i\lambda\langle x-y,\xi\rangle} a(x,\xi;\lambda)u(y)d\xi dy, \quad u \in C_0^\infty(X), \tag{1.2}$$

where $\mathcal{D}_x := (i\lambda)^{-1}\partial_x$, $\widehat{u}$ is the λ- Fourier transform of u, and the integrals with respect to ξ must be understood as oscilatory ones. The operator $A = a(x, \mathcal{D}_x; \lambda) : C_0^\infty(X) \to C^\infty(X)$ will be called a λ-pseudodifferential operator (or simply a $\lambda - \Psi DO$) with symbol $a(x,\xi;\lambda)$ (which will be also denoted by $\sigma(A)$). Denote by $L_{\rho,\delta}^{m,k}(X)$ the set of all $\lambda - \Psi DO$'s with symbols belonging to $S_{\rho,\delta}^{m,k}(X)$. In what follows, given $a_j \in S_{\rho,\delta}^{m-j,k-j}(X)$, $j = 0,1,...,$ the notation $a \sim \sum_{j=0}^{\infty} a_j$ will mean that a is a symbol satisfying $a - \sum_{j=0}^{N-1} a_j \in S_{\rho,\delta}^{m-N,k-N}(X)$ for every integer $N \geq 1$. Introduce the space $S_{cl}^m(X) \subset S_{0,0}^{m,0}(X)$ of symbols $a(x,\xi;\lambda)$ of the form

$$a(x,\xi;\lambda) \sim \sum_{j=0}^{\infty} \lambda^{-j} a_j(x,\xi), \tag{1.3}$$

where $a_j \in C^\infty(T^*X)$ do not depend on λ and satisfy

$$\left|\partial_x^\alpha \partial_\xi^\beta a_j(x,\xi)\right| \leq C_{\alpha,\beta,j,K}(1+|\xi|)^{m-j-|\beta|}, \quad \forall x \in K, \forall \xi \in T_x^*X, \tag{1.4}$$

for every multi-indeces α and β and every compact $K \subset X$. The function $a_0(x,\xi)$ will be called principal symbol of the operator $A = a(x,\mathcal{D}_x;\lambda)$ and will be denoted by $\sigma_p(A)$. Denote by $L_{cl}^m(X)$ the set of all $\lambda - \Psi DO$'s with symbols belonging to $S_{cl}^m(X)$. The $\lambda - \Psi DO$'s admit a calculus very similar to that of the classical pseudo-differential operators. The most important properties are given in the following

Proposition 1.1 *Let $A_j \in L_{\rho,\delta}^{m_j,k_j}(X)$, $j = 1,2$. Then the composition*

$$A_1 A_2 \in L_{\rho,\delta}^{m_1+m_2,k_1+k_2}(X)$$

with symbol

$$\sigma(A_1A_2) \sim \sum_\alpha \frac{(i\lambda)^{-|\alpha|}}{\alpha!} \partial_\xi^\alpha \sigma(A_1) \partial_x^\alpha \sigma(A_2). \tag{1.5}$$

Moreover, the formally adjoint, A^*, of a $\lambda - \Psi DO$, $A \in L_{\rho,\delta}^{m,k}(X)$, belongs also to $L_{\rho,\delta}^{m,k}(X)$ and its symbol is given by

$$\sigma(A^*) \sim \sum_\alpha \frac{(i\lambda)^{-|\alpha|}}{\alpha!} \partial_\xi^\alpha \partial_x^\alpha \overline{\sigma(A)}. \tag{1.6}$$

This proposition can be proved by using the stationary phase method in the same way as in the classical case. As an immediate consequence we have the following

Corollary 1.2 *Let $A_j \in L_{cl}^m(X)$, $j = 1,2$. Then the commutator $A := i\lambda[A_1, A_2]$ belongs to $L_{cl}^{m-1}(X)$ with principal symbol given by*

$$\sigma_p(A) = \{\sigma_p(A_1), \sigma_p(A_2)\}. \tag{1.7}$$

where $\{\cdot,\cdot\}$ *are the Poisson brackets defined by*

$$\{f,g\} = \sum_{j=1}^{n} \left(\frac{\partial f}{\partial \xi_j} \frac{\partial g}{\partial x_j} - \frac{\partial f}{\partial x_j} \frac{\partial g}{\partial \xi_j} \right).$$

An operator $A \in L^{m,k}_{\rho,\delta}(X)$ will be called elliptic at a finite point $(x^0, \xi^0) \in T^*X$ if

$$|\sigma(A)(x,\xi;\lambda)| \geq C\lambda^k, \quad C > 0,$$

in a neighbourhood of (x^0, ξ^0), and it will be called elliptic at an infinite point $(x^0, \xi^0) \in T^*S$ if

$$|\sigma(A)(x,\xi;\lambda)| \geq C\lambda^k (1 + |\xi|)^m, \quad C > 0,$$

in a conic neighbourhood of (x^0, ξ^0). Clearly, an operator $A \in L^m_{cl}(X)$ is elliptic at a finite point $(x^0, \xi^0) \in T^*X$ if

$$|\sigma_p(A)(x,\xi;\lambda)| \geq C, \quad C > 0,$$

in a neighbourhood of (x^0, ξ^0), and it is elliptic at an infinite point $(x^0, \xi^0) \in T^*S$ if

$$|\sigma_p(A)(x,\xi;\lambda)| \geq C(1 + |\xi|)^m, \quad C > 0,$$

in a conic neighbourhood of (x^0, ξ^0). The semi-classical wave front, $\widetilde{WF}(u) \subset \widetilde{T}^*X$, of a distribution $u \in \mathcal{D}'(X)$ is defined as follows: $(x^0, \xi^0) \notin \widetilde{WF}(u)$ iff there exists an operator $A \in L^{0,0}_{\rho,\delta}(X)$ elliptic at (x^0, ξ^0) so that $Au \in C_0^\infty(X)$ and $\|Au\|_{L^2(X)} = O(\lambda^{-\infty})$. It is not hard to see that if (x^0, ξ^0) is a finite point, $(x^0, \xi^0) \notin \widetilde{WF}(u)$ iff there exists a function $\chi \in C_0^\infty(X)$, $\chi = 1$ in a neighbourhood of x^0 so that $|\widehat{\chi u}(\xi;\lambda)| = O(\lambda^{-\infty})$ for all ξ in a neighbourhood of ξ^0. If (x^0, ξ^0) is an infinite point, $(x^0, \xi^0) \notin \widetilde{WF}(u)$ iff there exists a function $\chi \in C_0^\infty(X)$, $\chi = 1$ in a neighbourhood of x^0 so that $|\widehat{\chi u}(\xi;\lambda)| = O((\lambda(1+|\xi|))^{-\infty})$ for all ξ in a conic neigbourhood of ξ^0.

We also have the following (see Proposition A.1.6 of [5])

Proposition 1.3 *The operators in* $L^{0,0}_{\rho,\delta}(X)$ *are bounded from* $L^2_{comp}(X)$ *to* $L^2_{loc}(X)$ *with norm* $O(1)$. *Moreover, if* X *is compact and if* $A \in L^{0,0}_{\rho,\delta}(X)$ *such that* $|\sigma(A)| < M$ *on* T^*X, *then* A *is bounded on* $L^2(X)$ *with norm* $\leq M + O_N(\lambda^{-N})$, $\forall N \gg 1$.

2 Semi-Classical Propagation of Singularities on Riemannian Manifolds without Boundary

Throughout this section we will equipe the manifold X with a Riemannian metric

$$g = \sum_{i,j=1}^{n} g_{ij}(x) dx_i dx_j, \quad g_{ij}(x) \in C^\infty(X).$$

Denote by Δ_g the corresponding (negative) Laplace-Beltrami operator, i.e.

$$\Delta_g := (\det g_{ij})^{-1/2} \sum_{i,j=1}^{n} \partial_{x_i} \left((\det g_{ij})^{1/2} g^{ij}(x) \partial_{x_j} \right),$$

where (g^{ij}) denotes the inverse matrix of (g_{ij}). The principal symbol of $-\Delta_g$ is given by

$$r(x,\xi) := \sum_{i,j=1}^{n} g^{ij}(x)\xi_i\xi_j \geq C|\xi|^2, \quad C > 0.$$

Fix a $\lambda \gg 1$. Then the operator $P := -\lambda^{-2}\Delta_g - 1$ can be considered as a $\lambda - \Psi DO$ (or more precisely a λ-differential operator) of class $L^2_{cl}(X)$ with principal symbol $r(x,\xi) - 1$.

The bicharactersitic flow $\Phi(t) : T^*X \to T^*X$, $t \in \mathbf{R}$, associated to the Hamiltonian $r(x,\xi)$ is defined by $\Phi(t)(x^0,\xi^0) := (x(t),\xi(t))$, where the pair $(x(t),\xi(t))$ solves the Hamilton equation

$$\frac{\partial x(t)}{\partial t} = \frac{\partial r(x,\xi)}{\partial \xi}, \ \frac{\partial \xi(t)}{\partial t} = -\frac{\partial r(x,\xi)}{\partial x}, \ x(0) = x^0, \ \xi(0) = \xi^0. \tag{2.1}$$

It is well known that the solutions to (2.1) exist for all t and depend smoothly on t and (x^0,ξ^0). The projection of the bicharactersitics on the x-space X are just the geodesics associated to the Riemannian metric g. Fix $\zeta^0 = (x^0,\xi^0) \in T^*X$ so that $r(x^0,\zeta^0) = 1$. Choose a real-valued function $p(x,\xi) \in C_0^\infty(T^*X)$, $0 \leq p \leq 1$, such that $p = 1$ in a neighbourhood of ζ^0 and $p = 0$ outside a biger neighbourhood. Given a $t \in \mathbf{R}$, define the function $p_t(x,\xi) \in C_0^\infty(T^*X)$ by $p_t(x,\xi) = p(\Phi(-t)(x,\xi))$. Throughout this section, $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ will denote the norm and the scalar product of the Hilbert space $L^2(X, d\mathrm{Vol}_g)$, where $d\mathrm{Vol}_g = (\det g_{ij})^{1/2}dx$. Also, given any domain $Y \subset X$, the Sobolev space $H^1(Y)$ will be equipped with the norm

$$\|u\|_{H^1(Y)} := \sum_{0\leq|\alpha|\leq 1} \|\mathcal{D}_x^\alpha u\|_{L^2(Y)}.$$

More generally, given any real k the semi-classical norm of the Sobolev space $H^k(Y)$ is defined by

$$\|u\|_{H^k(Y)} := \|(1-\lambda^{-2}\Delta_g)^{k/2}u\|_{L^2(Y)}.$$

Proposition 2.1 *For every $u \in H^2_{comp}(X)$ and $\forall T > 0$ the following estimate holds*

$$\|p(x,\mathcal{D}_x)u\| \leq \|p_t(x,\mathcal{D}_x)u\| + 2T\lambda\|Pu\| + C\lambda^{-1}\|u\| \quad \textit{for} \quad |t| \leq T, \ \lambda \geq \lambda_0, \tag{2.2}$$

with constants $C = C(T) > 0$, $\lambda_0 = \lambda_0(T) > 0$ independent of λ, t and u.

Proof. It follows from (2.1) that

$$\partial_t p_t + \{r, p_t\} = 0. \tag{2.3}$$

Therefore, it follows from (1.7) that

$$Q_t := \lambda\partial_t p_t(x,\mathcal{D}_x) + i\lambda^2[P, p_t(x,\mathcal{D}_x)] \in L^0_{cl}(X),$$

and hence by Proposition 1.3,

$$Q_t = O_t(1) : L^2_{comp}(X) \to L^2(X). \tag{2.4}$$

We have

$$\frac{1}{2}\frac{d}{dt}\|p_t(x,\mathcal{D}_x)u\|^2 = \mathrm{Re}\,\langle\partial_t p_t(x,\mathcal{D}_x)u, p_t(x,\mathcal{D}_x)u\rangle$$

$$= \lambda \mathrm{Im}\,\langle[P, p_t(x,\mathcal{D}_x)]u, p_t(x,\mathcal{D}_x)u\rangle + \lambda^{-1}\mathrm{Re}\,\langle Q_t u, p_t(x,\mathcal{D}_x)u\rangle$$

$$= -\lambda \mathrm{Im}\,\langle p_t(x,\mathcal{D}_x)Pu, p_t(x,\mathcal{D}_x)u\rangle + \lambda^{-1}\mathrm{Re}\,\langle Q_t u, p_t(x,\mathcal{D}_x)u\rangle. \tag{2.5}$$

Hence, in view of Proposition 1.3, we get

$$\left|\frac{d}{dt}\|p_t(x,\mathcal{D}_x)u\|\right| \le 2\lambda\|Pu\| + O(\lambda^{-1})\|u\|, \tag{2.6}$$

which implies

$$\|p(x,\mathcal{D}_x)u\| = \|p_t(x,\mathcal{D}_x)u\| - \int_0^t \frac{d}{d\tau}\|p_\tau(x,\mathcal{D}_x)u\|d\tau$$

$$\le \|p_t(x,\mathcal{D}_x)u\| + 2t\lambda\|Pu\| + O(\lambda^{-1})\|u\|.$$

□

Note that it is possible to improve (2.2) replacing $C\lambda^{-1}$ by $O(\lambda^{-\infty})$ and $p_t(x,\xi)$ by a symbol $\sim \sum_{j=0}^{\infty}\lambda^{-j}q_t^{(j)}(x,\xi)$, with $q_t^{(0)}(x,\xi) = p_t(x,\xi)$, and $q_t^{(j)}(x,\xi)$, $j = 1, 2, \ldots$, solving transport equations of the form

$$\partial_t q_t^{(j)} + \{r, q_t^{(j)}\} = r_t^{(j)}, \quad q_0^{(j)} = 0,$$

where $r_t^{(j)}$ is a function determined by $q_t^{(0)}, \ldots, q_t^{(j-1)}$ (e.g. see Lemma 4.1 of [12]). This enables to get a result on the propagation of the semi-classical singularities which can be considered as an analogue of the classical propagation of singularities. Namely, we have the following

Proposition 2.2 *Let $u \in H^2_{comp}(X)$ and suppose that $\cup_{\tau=0}^t\Phi(\tau)(x^0,\xi^0) \notin \widetilde{WF}(Pu)$. Then,*

$$(x^0,\xi^0) \notin \widetilde{WF}(u) \Rightarrow \Phi(\tau)(x^0,\xi^0) \notin \widetilde{WF}(u) \quad \textit{for} \quad 0 \le \tau \le t.$$

For our purposes, however, the weaker bound (2.2) will be sufficient. Fix a compact $K \subset X$ and suppose that

$$\text{there exist an open bounded domain } U \subset K \text{ and a } T > 0 \text{ so that } \forall x^0 \in K \setminus U$$
$$\text{and every geodesic } \gamma(\tau) \text{ with } \gamma(0) = x^0, \text{ we have } \gamma(t) \in U \text{ for some } 0 \leq t \leq T. \tag{2.7}$$

Using Proposition 2.1 we will prove the following

Theorem 2.3 *Under the assumption (2.7), for every $u \in H^2(X)$ with $\operatorname{supp} u \subset K$ the following estimate holds*

$$\|u\|_{H^1(X)} \leq C\|u\|_{L^2(U)} + C\lambda\|Pu\| \quad \text{for} \quad \lambda \geq \lambda_0, \tag{2.8}$$

with constants $C, \lambda_0 > 0$ independent of λ and u.

Proof. Fix a $\zeta^0 = (x^0, \xi^0) \in T^*X$ such that $r(x^0, \xi^0) = 1$ and $x^0 \in K$. By (2.7) there exists an open neighbourhood $V(\zeta^0)$ of ζ^0 and $0 \leq t \leq T$ so that $\Phi(t)(V(\zeta^0)) \subset U$. Let $p \in C_0^\infty(V(\zeta^0))$, $0 \leq p \leq 1$, $p = 1$ in a smaller neighbourhood of ζ^0. If p_t is as above, we clearly have $\operatorname{supp}_x p_t \subset U$. Therefore, by (2.2) we have

$$\|p(x, D_x)u\| \leq C\|u\|_{L^2(U)} + O_T(\lambda)\|Pu\| + O(\lambda^{-1})\|u\|. \tag{2.9}$$

Fix now a $\zeta^0 = (x^0, \xi^0) \in T^*X$ such that $r(x^0, \xi^0) \neq 1$ and $x^0 \in K$. Let us first suppose that $r(x^0, \xi^0) > 1$. Then, there exists an open neighbourhood (conic for $|\xi| \gg 1$) $W(\zeta^0)$ of ζ^0 such that $r > 1$ on $W(\zeta^0)$. Let $p_j \in C^\infty(W(\zeta^0))$, $\operatorname{supp} p_j \subset W(\zeta^0)$, $0 \leq p_j \leq 1$, $p_1 = 1$ in a smaller neighbourhood of ζ^0, $p_2 = 1$ on $\operatorname{supp} p_1$. Then, $p_2(x, D_x)P \in L^2_{cl}(X)$ is elliptic at ζ_0, and by Proposition 1.1 it is easy to see that $p_1(x, D_x) = A_1 P + \lambda^{-1} A_2$ with some operators $A_j \in L^{-2}_{cl}(X)$, $j = 1, 2$. Hence, in view of Proposition 1.3,

$$\|p_1(x, D_x)u\| \leq O(1)\|Pu\| + O(\lambda^{-1})\|u\|. \tag{2.10}$$

Clearly, the case of $r(x^0, \xi^0) < 1$ can be treated similarly. So, by a partition of the unity on T^*K one can get from (2.9) and (2.10),

$$\|u\|_{L^2(K)} \leq O(1)\|u\|_{L^2(U)} + O(\lambda)\|Pu\| + O(\lambda^{-1})\|u\|.$$

As $\operatorname{supp} u \subset K$, this estimate implies

$$\|u\| \leq O(1)\|u\|_{L^2(U)} + O(\lambda)\|Pu\|. \tag{2.11}$$

On the other hand, we have

$$\|\lambda^{-1}\nabla_g u\|^2 = \|u\|^2 + (Pu, u) \leq 2\|u\|^2 + \|Pu\|^2. \tag{2.12}$$

Clearly, (2.8) follows from (2.11) and (2.12).

3 Applications to the Damped Wave Equation on a Compact Riemannian Manifold without Boundary: Distribution of Eigenvalues and Energy Decay

Throughout this section (X, g) will be a compact Riemannian manifold without boundary. We will be interested in the behaviour of the energy of the solutions to the damped wave equation

$$\begin{cases} (\partial_t^2 - \Delta_g + a(x)\partial_t)u(t,x) = 0 & \text{in} \quad \mathbf{R}^+ \times X, \\ u(0,x) = f_1(x),\ \partial_t u(0,x) = f_2(x), \end{cases} \tag{3.1}$$

where $a(x) \in C^\infty(X)$ is a non-identically zero real-valued function, $a(x) \geq 0$, $\forall x \in X$. The energy of $u(t,x)$ is given by

$$E(t) = \frac{1}{2}\int_X \left(|\partial_t u(t,x)|^2 + |\nabla_g u(t,x)|^2\right) d\mathrm{Vol}_g.$$

We have

$$\frac{dE(t)}{dt} = -\int_X a(x)|\partial_t u(t,x)|^2 d\mathrm{Vol}_g < 0, \tag{3.2}$$

provided u is not a constant, so the energy decays as $t \to +\infty$. Denote $H = L^2(X, d\mathrm{Vol}_g)$ and let H_1 be the closure of $C^\infty(X)$ with respect to the norm $\|\nabla_g u\|_H$. Consider in the Hilbert space $\mathcal{H} = H \oplus H_1$ the operator

$$A = \begin{pmatrix} 0 & Id \\ \Delta_g & -a(x) \end{pmatrix}$$

with domain $D(A) = \{u \in \mathcal{H} : Au \in \mathcal{H}\}$. It is well known that A is a generator of a semi-group, e^{tA}, and the solutions of (3.1) are given by

$$\begin{pmatrix} u(t,x) \\ \partial_t u(t,x) \end{pmatrix} = e^{tA}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}.$$

Since the resolvent of A is compact, the spectrum of A is discrete, contained in the strip $-\sup_{x\in X} a(x) \leq \mathrm{Re}\, z \leq 0$. Moreover, the only eigenvalue of A on $\mathrm{Re}\, z = 0$ is $z = 0$ with a corresponding eigenfunction $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. It is easy to see that $i\lambda \in \mathrm{spec}\, A$ iff the following equation has a nontrivial solution:

$$(\Delta_g + \lambda^2 - i\lambda a)v = 0 \quad \text{in} \quad X. \tag{3.3}$$

We make the following assumption

$\exists T, \delta > 0$ so that for every geodesic $\gamma(\tau)$with $\gamma(0) \in X$ we have

$$\gamma(t) \in Y_\delta := \{x \in X : a(x) \geq \delta\} \text{ for some } 0 \leq t \leq T. \tag{3.4}$$

We have the following

Theorem 3.1 *Under the assumption (3.4), there exists a constant $\alpha > 0$ so that* spec $A \setminus \{0\} \subset \{z \in \mathbf{C} : \mathrm{Re}\, z < -\alpha\}$, *and the resolvent of A satisfies the bound*

$$\|(A-z)^{-1}\|_{\mathcal{L}(\mathcal{H})} \le Const \quad for \quad \mathrm{Re}\, z \ge -\alpha,\ |z| \ge 1. \tag{3.5}$$

Proof. Let $\lambda \gg 1$ be real. In view of (3.4) we can use Theorem 2.3 to obtain

$$\|u\|_{H^1(X)} \le C\|u\|_{L^2(Y_\delta)} + C\lambda^{-1}\|(\Delta_g + \lambda^2)u\|_{L^2(X)}. \tag{3.6}$$

On the other hand, we have

$$\|(\Delta_g + \lambda^2 - i\lambda a)u\|_{\mathcal{H}}^2 = \|(\Delta_g + \lambda^2)u\|_{\mathcal{H}}^2 + \|\lambda a u\|_{\mathcal{H}}^2 + \lambda \mathrm{Re}\,\langle[\Delta_g, a]u, u\rangle_{\mathcal{H}},$$

hence

$$\|(\Delta_g + \lambda^2)u\|_{L^2(X)} \le \|(\Delta_g + \lambda^2 - i\lambda a)u\|_{L^2(X)} + \mathcal{O}(1)\|u\|_{H^1(X)}. \tag{3.7}$$

By (3.6) and (3.7),

$$\|u\|_{H^1(X)} \le C\|u\|_{L^2(Y_\delta)} + C\lambda^{-1}\|(\Delta_g + \lambda^2 - i\lambda a)u\|_{L^2(X)}. \tag{3.8}$$

Using (3.8) we obtain

$$\lambda^{-1}\mathrm{Im}\,\langle(\Delta_g + \lambda^2 - i\lambda a)u, u\rangle_{\mathcal{H}} = \mathrm{Re}\,\langle au, u\rangle_{\mathcal{H}}$$

$$\ge \delta\|u\|_{L^2(Y_\delta)}^2 \ge C_1\|u\|_{L^2(X)}^2 - C_2\lambda^{-2}\|(\Delta_g + \lambda^2 - i\lambda a)u\|_{L^2(X)}^2,$$

with some constants $C_1, C_2 > 0$, which easily implies

$$\|u\|_{L^2(X)} \le C'\lambda^{-1}\|(\Delta_g + \lambda^2 - i\lambda a)u\|_{L^2(X)}, \tag{3.9}$$

with some constant $C' > 0$. This inequality is equivalent to the following one

$$\|(A - i\lambda)^{-1}\|_{\mathcal{L}(\mathcal{H})} \le Const, \tag{3.10}$$

for real $\lambda \gg 1$. If λ is complex, we have

$$(A - i\lambda)^{-1}\left(1 + \mathrm{Im}\,\lambda(A - i\mathrm{Re}\,\lambda)^{-1}\right) = (A - i\mathrm{Re}\,\lambda)^{-1}, \tag{3.11}$$

so by (3.10), $A - i\lambda$ is invertible provided $|\mathrm{Im}\,\lambda| \le C_1$ and $\mathrm{Re}\,\lambda \ge C_2$ for some constants $C_1, C_2 > 0$. □

As an immediate consequence of this theorem we have the following

Corollary 3.2 *Under the assumption (3.4),*

$$\|e^{tA}f\|_{\mathcal{H}} \le e^{-\alpha t}\|f\|_{\mathcal{H}}, \quad t \ge 1,\ \forall f \in \mathcal{H}' := \mathcal{H} \ominus \mathrm{Ker}\, A, \tag{3.12}$$

which in turn implies

$$E(t) \le e^{-\alpha t}E(0). \tag{3.13}$$

Without the assumption (3.4) we have the following weaker analogues of the above results which are due to Lebeau [7].

Theorem 3.3 *There exist constants $C, \beta > 0$ so that* $\operatorname{spec} A \setminus \{0\} \subset \{z \in \mathbf{C} : \operatorname{Re} z < -Ce^{-\beta|z|}\}$, *and the resolvent of A satisfies the bound*

$$\|(A-z)^{-1}\|_{\mathcal{L}(\mathcal{H})} \le C' e^{\beta|z|} \quad \textit{for} \quad \operatorname{Re} z \ge -Ce^{-\beta|z|}, \ |z| \ge 1. \tag{3.14}$$

Clearly, in this case the estimate (3.8) is no longer true, but we have the following analogue which follows from the Carleman estimates (see [7]):

$$\|u\|_{H^1(X)} \le C_1 e^{C_2\lambda} \left(\|u\|_{L^2(Y_a)} + \|(\Delta_g + \lambda^2 - i\lambda a)u\|_{L^2(X)} \right), \tag{3.15}$$

for real $\lambda \gg 1$, with some constants $C_1, C_2 > 0$ independent of λ. In the same way as above, (3.15) implies

$$\|(A - i\lambda)^{-1}\|_{\mathcal{L}(\mathcal{H})} \le \widetilde{C}_1 e^{\widetilde{C}_2\lambda} \tag{3.16}$$

for real $\lambda \gg 1$, with some constants $\widetilde{C}_1, \widetilde{C}_2 > 0$ independent of λ, which in turn implies the free of eigenvalues region as well as (3.14) in view of the resolvent identity (3.11).

In this case we also have the following analogue of (3.12).

Corollary 3.4 *For every integer $m \ge 0$, $\exists C_m > 0$ so that*

$$\|e^{itA} f\|_{\mathcal{H}} \le C_m (\log t)^{-m} \|f\|_{D(A^m)}, \quad t \ge 2, \forall f \in D(A^m) \cap \mathcal{H}', \tag{3.17}$$

where $\|f\|_{D(A^m)} := \|f\|_{\mathcal{H}} + \|A^m f\|_{\mathcal{H}}$.

4 Applications to the Scattering Theory for Long-Range Nontrapping Riemannian Metrics on $\mathbf{R}^N$: Uniform Limiting Absorption Principle and Decay of the Local Energy

Throughout this section the space $\mathbf{R}^n$, $n \ge 2$, will be equipped with a C^∞-smooth Riemannian metric $g = \sum g_{ij}(x) dx_i dx_j$ satisfying

$$|\partial_x^\alpha (g_{ij}(x) - \delta_{ij})| \le C_\alpha \langle x \rangle^{-\epsilon_0 - |\alpha|}, \quad \forall \alpha, \tag{4.1}$$

where $\langle x \rangle := (1 + |x|^2)^{1/2}$, $\epsilon_0 > 0$, and δ_{ij} is the Kronecker's symbol. In other words, the metric g is a long-range perturbation of the Euclidean metric $g_0 = \sum dx_j^2$. Then the (negative) Laplace-Beltrami operator Δ_g has a self-adjoint realization (which will be again denoted by Δ_g) on the Hilbert space $H = L^2(\mathbf{R}^n; d\mathrm{Vol}_g)$ with continuous spectrum, only. The metric g will be called *nontrapping* if

$$\text{for every } R_2 > R_1 \gg 1 \text{ there exists } T = T(R_1, R_2) > 0 \text{ so that for every geodesic } \gamma(\tau)$$
$$\text{with } \gamma(0) \in \{x \in \mathbf{R}^n : |x| \le R_1\} \text{ we have}$$
$$\gamma(t) \in \{x \in \mathbf{R}^n : |x| \ge R_2\} \text{ for some } 0 \le t \le T. \tag{4.2}$$

Using Theorem 2.3 we will prove the following

Theorem 4.1 *Under the assumptions (4.1) and (4.2), for* $\forall s > 1/2$, $\lambda \ge \lambda_0$ *the limit*

$$\lim_{\varepsilon \to 0^+} \langle x \rangle^{-s} (\Delta_g + \lambda^2 \pm i\varepsilon)^{-1} \langle x \rangle^{-s} : H \to H$$

exists and satisfies the estimates

$$\left\| \langle x \rangle^{-s} \lambda (\Delta_g + \lambda^2 \pm i0)^{-1} \langle x \rangle^{-s} \right\|_{\mathcal{L}(H)} \le C, \tag{4.3}$$

with constants $C, \lambda_0 > 0$ *independent of* λ.

The fact that the limit above exists is known as limiting absorption principle and holds without the assumption (4.2). The idea of the proof of (4.3) is to use the fact that, because of (4.1), there exists a global smooth change of variables, $(\rho, \theta) = (\rho(x), \theta(x))$, for $|x| \gg 1$, where $\rho \in [\rho_0, +\infty)$, $\rho_0 \gg 1$, $\theta \in S = \{y \in \mathbf{R}^n : |y| = 1\}$, which transforms the metric g in the form

$$g = d\rho^2 + \rho^2 \sum_{i,j} h_{ij}(\rho, \theta) d\theta_i d\theta_j,$$

where $h_{ij} \in C^\infty$ satisfy the inequalities

$$\left| \partial_\rho^\alpha \partial_\theta^\beta \left(h_{ij}(\rho, \theta) - h_{ij}^0(\theta) \right) \right| \le C_{\alpha,\beta} \rho^{-\epsilon_0 - \alpha}, \quad \forall \alpha, \beta, \tag{4.4}$$

where $\sum_{i,j} h_{ij}^0(\theta) d\theta_i d\theta_j$ is the metric on S induced by the Euclidean one. Denote $B_R = \{x \in \mathbf{R}^n : \rho(x) \le R\}$ for $R > \rho_0$, and choose a function $\chi \in C_0^\infty(B_{R+1})$, $\chi = 1$ on B_R. Clearly, (4.2) implies (2.7) with $K = B_{R+1}$ and $U = B_{R+1} \setminus B_R$. Therefore, given any $u \in H^2(\mathbf{R}^n)$, applying Theorem 2.3 to χu leads to the estimate

$$\|u\|_{H^1(B_R)} \le O(1) \|u\|_{H^1(B_{R+1} \setminus B_R)} + O(\lambda) \|Pu\|_{L^2(B_{R+1})}. \tag{4.5}$$

On the other hand, we have the following

Proposition 4.2 *Let* $R_1 > \rho_0$ *and let* $u \in H^2(\mathbf{R}^n \setminus B_{R_1})$ *be such that* $\rho^s(P + i\varepsilon)u \in L^2(\mathbf{R}^n \setminus B_{R_1})$ *for* $1/2 < s \le (1 + \epsilon_0)/2$, $0 < \varepsilon = O(\lambda^{-2})$. *Then, under the assumption (4.4),*

$\forall 0 < \delta \ll 1$ *there exist constants* $C_1, C_2, \lambda_0 > 0$ *(depending on* δ *but independent of* λ *and* ε*) so that for* $\lambda \geq \lambda_0$ *we have*

$$\|\rho^{-s}u\|^2_{H^1(\mathbf{R}^n\setminus B_{R_1+1})} \leq C_1\lambda^2\|\rho^s(P+i\varepsilon)u\|^2_{L^2(\mathbf{R}^n\setminus B_{R_1})}$$

$$+\delta\|u\|^2_{H^1(B_{R_1+1}\setminus B_{R_1})} - C_2\lambda^{-1}\mathrm{Im}\,\langle\partial_\rho u, u\rangle_{L^2(\partial B_{R_1})}. \tag{4.6}$$

For a proof of (4.6) we refer to [3] (Proposition 2.4). Now, combining (4.5) with (4.6) used with $R_1 = R-2$, we get

$$\|u\|^2_{H^1(B_R)} \leq O(1)\|\rho^{-s}u\|^2_{H^1(\mathbf{R}^n\setminus B_{R_1+1})} + O(\lambda^2)\|\rho^s(P+i\varepsilon)u\|^2_{L^2(\mathbf{R}^n)}$$

$$+O(\lambda^{-2})\|\rho^{-s}u\|^2_{L^2(\mathbf{R}^n)} \leq O(\lambda^2)\|\rho^s(P+i\varepsilon)u\|^2_{L^2(\mathbf{R}^n)} + \delta\|u\|^2_{H^1(B_R)}$$

$$+O(\lambda^{-2})\|\rho^{-s}u\|^2_{L^2(\mathbf{R}^n)} - C_2\lambda^{-1}\mathrm{Im}\,\langle\partial_\rho u, u\rangle_{L^2(\partial B_{R_1})}. \tag{4.7}$$

On the other hand, by Green's formula we have

$$-\lambda^{-1}\mathrm{Im}\,\langle\partial_\rho u, u\rangle_{L^2(\partial B_{R_1})} = \lambda\mathrm{Im}\,\langle Pu, u\rangle_{L^2(B_{R_1})} \leq \lambda\mathrm{Im}\,\langle(P+i\varepsilon)u, u\rangle_{L^2(B_{R_1})}$$

$$\leq O_\delta(\lambda^2)\|(P+i\varepsilon)u\|^2_{L^2(B_{R_1})} + \delta\|u\|^2_{L^2(B_{R_1})}. \tag{4.8}$$

By (4.7) and (4.8), taking $\delta > 0$ small enough one arrives at the estimate

$$\|u\|^2_{H^1(B_R)} \leq O(\lambda^2)\|\rho^s(P+i\varepsilon)u\|^2_{L^2(\mathbf{R}^n)} + O(\lambda^{-2})\|\rho^{-s}u\|^2_{L^2(\mathbf{R}^n)}. \tag{4.9}$$

Combining (4.9) with (4.6) and (4.8) leads to the estimate

$$\|\rho^{-s}u\|_{H^1(\mathbf{R}^n)} \leq C\lambda\|\rho^s(P+i\varepsilon)u\|_{L^2(\mathbf{R}^n)} \tag{4.10}$$

with a constant $C > 0$ independent of λ, ε and u, which in turn easily implies Theorem 4.1.

It is worth noticing that (4.3) implies estimates for the resolvent on the Sobolev spaces $H^k(\mathbf{R}^n)$ still equipped with the semi-classical norms introduced in Section 2. Indeed, using the ellipticity of the operator Δ_g and (4.3) one can show, under the assumptions of Theorem 4.1,

$$\left\|\langle x\rangle^{-s}\lambda(\Delta_g + \lambda^2 \pm i0)^{-1}\langle x\rangle^{-s}\right\|_{\mathcal{L}(L^2(\mathbf{R}^n),H^k(\mathbf{R}^n))} \leq C, \quad \lambda \geq \lambda_0, \quad 0 \leq k \leq 2. \tag{4.11}$$

In what follows we will derive from the above theorem the following

Corollary 4.3 *Under the assumptions (4.1) and (4.2),* $\forall f \in L^2(\mathbf{R}^n)$, $\forall s > 1/2$ *the following inequalities hold:*

$$\int_{-\infty}^{\infty}\left\|\langle x\rangle^{-s}\cos\left(t\sqrt{-\Delta_g}\right)\langle x\rangle^{-s}f\right\|^2_{L^2(\mathbf{R}^n)}dt \leq C\|f\|^2_{L^2(\mathbf{R}^n)}, \tag{4.12}$$

$$\int_{-\infty}^{\infty}\left\|\langle x\rangle^{-s}\partial_{x_j}\frac{\sin\left(t\sqrt{-\Delta_g}\right)}{\sqrt{-\Delta_g}}\langle x\rangle^{-s}f\right\|^2_{L^2(\mathbf{R}^n)}dt\le C\|f\|^2_{L^2(\mathbf{R}^n)},\tag{4.13}$$

with a constant $C>0$ independent of f.

Proof. Let $V(t)\in L^1_{loc}(\mathbf{R}^+;\mathcal{L}(H))$ be a familly of bounded operators such that $\|V(t)\|=O(t^N)$, $t\gg 1$, for some $N\ge 0$. Then the Fourier-Laplace transform

$$\mathcal{F}_{t\to\lambda}V(t):=\int_0^\infty e^{-it\lambda}V(t)dt$$

is well defined as a bounded operator for $\operatorname{Im}\lambda<0$. The following formulae hold:

$$\mathcal{F}_{t\to\lambda}\cos\left(t\sqrt{-\Delta_g}\right)=i\lambda(\Delta_g+\lambda^2)^{-1},\quad \mathcal{F}_{t\to\lambda}\frac{\sin\left(t\sqrt{-\Delta_g}\right)}{\sqrt{-\Delta_g}}=(\Delta_g+\lambda^2)^{-1}.$$

Choose a function $\varphi(t)\in C^\infty(\mathbf{R})$, $\varphi(t)=0$ for $t\le 1$, $\varphi(t)=1$ for $t\ge 2$. We have

$$(\partial_t^2-\Delta_g)\left(\varphi(t)\cos\left(t\sqrt{-\Delta_g}\right)\right)=\varphi''(t)\cos\left(t\sqrt{-\Delta_g}\right)-2\varphi'(t)\sqrt{-\Delta_g}\sin\left(t\sqrt{-\Delta_g}\right),$$

and hence, taking the Fourier-Laplace transform of this identity, for $\operatorname{Im}\lambda<0$, we get

$$\mathcal{F}_{t\to\lambda}\left(\varphi(t)\cos\left(t\sqrt{-\Delta_g}\right)\right)=-i\lambda(\Delta_g+\lambda^2)^{-1}\mathcal{F}_{t\to\lambda}\left(\varphi'(t)\cos\left(t\sqrt{-\Delta_g}\right)\right)$$
$$-\sqrt{-\Delta_g}(\Delta_g+\lambda^2)^{-1}\mathcal{F}_{t\to\lambda}\left(\varphi'(t)\sin\left(t\sqrt{-\Delta_g}\right)\right).\tag{4.14}$$

Taking the limit $\operatorname{Im}\lambda\to 0^-$, in view of Theorem 4.1, we get, for real λ, with some $1/2<s_1<\min\{1,s\}$,

$$\mathcal{F}_{t\to\lambda}\left(\varphi(t)\langle x\rangle^{-s}\cos\left(t\sqrt{-\Delta_g}\right)\langle x\rangle^{-s}\right)f$$
$$=-\langle x\rangle^{-s}i\lambda(\Delta_g+\lambda^2-i0)^{-1}\langle x\rangle^{-s_1}\mathcal{F}_{t\to\lambda}\left(\varphi'(t)\langle x\rangle^{s_1}\cos\left(t\sqrt{-\Delta_g}\right)\langle x\rangle^{-s}\right)f$$
$$-\langle x\rangle^{-s}\sqrt{-\Delta_g}(\Delta_g+\lambda^2-i0)^{-1}\langle x\rangle^{-s_1}\mathcal{F}_{t\to\lambda}\left(\varphi'(t)\langle x\rangle^{s_1}\sin\left(t\sqrt{-\Delta_g}\right)\langle x\rangle^{-s}\right)f.\tag{4.15}$$

Using Plancherel's identity, (4.3), (4.11) and (6.4) below, one can easily derive from (4.15) the following estimate

$$\int_{-\infty}^{\infty}\left\|\varphi(t)\langle x\rangle^{-s}\cos\left(t\sqrt{-\Delta_g}\right)\langle x\rangle^{-s}f\right\|^2_{L^2(\mathbf{R}^n)}dt$$
$$\le C\int_{-\infty}^{\infty}\varphi'(t)^2\left(\left\|\langle x\rangle^{s_1}\cos\left(t\sqrt{-\Delta_g}\right)\langle x\rangle^{-s}f\right\|^2+\left\|\langle x\rangle^{s_1}\sin\left(t\sqrt{-\Delta_g}\right)\langle x\rangle^{-s}f\right\|^2\right)dt.\tag{4.16}$$

On the other hand, we have

$$(\partial_t^2-\Delta_g)\left(\langle x\rangle^{s_1}\cos\left(t\sqrt{-\Delta_g}\right)\langle x\rangle^{-s}\right)f=-[\Delta_g,\langle x\rangle^{s_1}]\cos\left(t\sqrt{-\Delta_g}\right)\langle x\rangle^{-s}f,$$

so by Duhamel's formula we obtain

$$\langle x\rangle^{s_1}\cos\left(t\sqrt{-\Delta_g}\right)\langle x\rangle^{-s}f=\cos\left(t\sqrt{-\Delta_g}\right)\langle x\rangle^{s_1-s}f$$

$$-\int_0^t\frac{\sin\left((t-\tau)\sqrt{-\Delta_g}\right)}{\sqrt{-\Delta_g}}[\Delta_g,\langle x\rangle^{s_1}]\cos\left(\tau\sqrt{-\Delta_g}\right)\langle x\rangle^{-s}fd\tau. \tag{4.17}$$

Since

$$\left(1+\sqrt{-\Delta_g}\right)^{-1}[\Delta_g,\langle x\rangle^{s_1}]=O\left(\langle x\rangle^{s_1-1}\right),$$

by (4.17) we get

$$\|\langle x\rangle^{s_1}\cos\left(t\sqrt{-\Delta_g}\right)\langle x\rangle^{-s}f\|_{L^2(\mathbf{R}^n)}\le\|f\|_{L^2(\mathbf{R}^n)}$$

$$+C\int_0^t\|\langle x\rangle^{s_1-1}\cos\left(\tau\sqrt{-\Delta_g}\right)\langle x\rangle^{-s}f\|_{L^2(\mathbf{R}^n)}d\tau\le(1+Ct)\|f\|_{L^2(\mathbf{R}^n)}, \tag{4.18}$$

and similarly for sin. It is easy to see that (4.12) follows from (4.16) and (4.18). The estimate (4.13) can be proved in the same way.

5 Distribution of Resonances and Decay of the Local Energy for Compactly Supported Nontrapping Riemannian Metrics on $\mathbf{R}^N$

Throughout this section the space $\mathbf{R}^n$, $n\ge 2$, will be equipped with a C^∞-smooth Riemannian metric g which coincides with the Euclidean metric g_0 outside some compact, say for $|x|\ge\rho_0\gg 1$. By Δ_g and Δ_0 we will denote the corresponding Laplace-Beltrami operators as well as their self-adjoint realizations on the Hilbert spaces $H=L^2(\mathbf{R}^n,d\mathrm{Vol}_g)$ and $H_0=L^2(\mathbf{R}^n)$, respectively. Let $\chi\in C_0^\infty(\mathbf{R}^n)$, $\chi(x)=1$ for $|x|\le\rho_0+1$. We have the following

Proposition 5.1 *The cutoff free resolvent*

$$\chi(\Delta_0+\lambda^2)^{-1}\chi:H_0\to H_0,\quad \mathrm{Im}\,\lambda<0,$$

extends to an entire function on $\mathbf{C}$ *if* $n\ge 3$ *is odd, and on the Riemann surface of* $\log\lambda$, $\Lambda:=\{-\infty<\arg\lambda<+\infty\}$, *if* $n\ge 2$ *is even. Moreover, modulo an entire function,*

$$\lambda\chi(\Delta_0+\lambda^2)^{-1}\chi=A(\lambda)\lambda^{n-1}\log\lambda, \tag{5.1}$$

where $A(\lambda)$ *is a finite rank (*0 *if* n *is odd and* 1 *if* n *is even) operator depending analytically on* λ, *and* $\log\lambda$ *takes its principal branch on* $-i\mathbf{R}$. *The same is true for the operator* $\chi\partial_{x_j}(\Delta_0+\lambda^2)^{-1}\chi$.

The cutoff resolvent

$$\chi(\Delta_g + \lambda^2)^{-1}\chi : H \to H, \quad \operatorname{Im}\lambda < 0,$$

extends to a meromorphic function on $\mathbf{C}$ *if* $n \geq 3$ *is odd, and on* Λ *if* $n \geq 2$ *is even. Moreover, modulo a polynomial function (of order* $n-2$*),*

$$\lambda\chi(\Delta_g + \lambda^2)^{-1}\chi = B\lambda^{n-1}\log\lambda + O(\lambda^{n-1}), \quad \lambda \to 0, \tag{5.2}$$

where B *is a finite rank (0 if* n *is odd and 1 if* n *is even) operator independent of* λ*. The same is true for the operator* $\chi\partial_{x_j}(\Delta_g + \lambda^2)^{-1}\chi$*.*

These properties are more or less well known (e.g. see [15], [16]). In particular, those concerning the free resolvent follow from the fact that its kernel can be expressed in terms of the Hankel functions and the properties ot these functions. The properties concerning the cutoff resolvent of the perturbed operator can be obtained by using the Fredholm theory in the following way. Fix a $\lambda_0 \in \mathbf{C}$ with $\operatorname{Im}\lambda_0 < 0$ and let $\chi_j \in C_0^\infty(\mathbf{R}^n)$, $j = 1, 2$, $\chi_1 = 1$ on $|x| \leq \rho_0$, $\chi_2 = 1$ on $\operatorname{supp}\chi_1$, $\chi = 1$ on $\operatorname{supp}\chi_2$. For $\operatorname{Im}\lambda < 0$, we have

$$(\Delta_g + \lambda^2)(1 - \chi_1)(\Delta_0 + \lambda^2)^{-1} = 1 - \chi_1 - [\Delta_0, \chi_1](\Delta_0 + \lambda^2)^{-1},$$

and hence

$$(\Delta_g + \lambda^2)^{-1}(1 - \chi_1) = (1 - \chi_1)(\Delta_0 + \lambda^2)^{-1} + (\Delta_g + \lambda^2)^{-1}[\Delta_0, \chi_1](\Delta_0 + \lambda^2)^{-1}. \tag{5.3}$$

Similarly,

$$(1 - \chi_2)(\Delta_g + \lambda^2)^{-1} = (\Delta_0 + \lambda^2)^{-1}(1 - \chi_2) + (\Delta_0 + \lambda^2)^{-1}[\Delta_0, \chi_2](\Delta_g + \lambda^2)^{-1}. \tag{5.4}$$

Since $1 - \chi_2 = (1 - \chi_1)(1 - \chi_2)$, using (5.3) and (5.4) one can write

$$\begin{aligned}
\chi(\Delta_g + \lambda^2)^{-1}\chi - \chi(\Delta_g + \lambda_0^2)^{-1}\chi &= (\lambda^2 - \lambda_0^2)\chi(\Delta_g + \lambda^2)^{-1}(\Delta_g + \lambda_0^2)^{-1}\chi \\
= (\lambda^2 - \lambda_0^2)\chi(\Delta_g + \lambda^2)^{-1}\chi_2(\Delta_g + \lambda_0^2)^{-1}\chi &+ \left(1 - \chi_1 + \chi(\Delta_g + \lambda^2)^{-1}\chi[\Delta_0, \chi_1]\right) \\
\left(\chi(\Delta_0 + \lambda^2)^{-1}\chi - \chi(\Delta_0 + \lambda_0^2)^{-1}\chi\right) &\left(1 - \chi_2 + [\Delta_0, \chi_2]\chi(\Delta_g + \lambda_0^2)^{-1}\chi\right).
\end{aligned}$$

Thus we get

$$\chi(\Delta_g + \lambda^2)^{-1}\chi(1 + K(\lambda)) = K_1(\lambda), \tag{5.5}$$

where

$$\begin{aligned}
K(\lambda) &= (\lambda^2 - \lambda_0^2)\chi_2(\Delta_g + \lambda_0^2)^{-1}\chi \\
+ \left([\Delta_0, \chi_1](\Delta_0 + \lambda^2)^{-1}\chi - [\Delta_0, \chi_1](\Delta_0 + \lambda_0^2)^{-1}\chi\right) &\left(1 - \chi_2 + [\Delta_0, \chi_2]\chi(\Delta_g + \lambda_0^2)^{-1}\chi\right), \\
K_1(\lambda) &= \chi(\Delta_g + \lambda_0^2)^{-1}\chi
\end{aligned}$$

$$+(1-\chi_1)\left(\chi(\Delta_0+\lambda^2)^{-1}\chi-\chi(\Delta_0+\lambda_0^2)^{-1}\chi\right)\left(1-\chi_2+[\Delta_0,\chi_2]\chi(\Delta_g+\lambda_0^2)^{-1}\chi\right).$$

Clearly, $K(\lambda)$ and $K_1(\lambda)$ are analytic functions with values in the compact operators in H. Therefore, since $K(\lambda_0)=0$, the meromorphic continuation of the cutoff resolvent follows from (5.5) and the Fredholm theorem. The property (5.2) can be easily derived (e.g. see Proposition 3.1 of [16]) from (5.1), (5.5) and (6.4) below.

The poles of the meromorphic continuation of the cutoff resolvent are called *resonances* and form a discrete set with no finite points of accumulation. In what follows we will suppose that the metric g is nontrapping, i.e. the assumption (4.2) is fulfilled. We will first show that Theorem 4.1 implies the following

Proposition 5.2 *Under the assumption (4.2), the cutoff resolvent $\chi(\Delta_g+\lambda^2)^{-1}\chi$ extends analytically to $\{\lambda\in\mathbf{C}:\operatorname{Im}\lambda\le C_1,|\operatorname{Re}\lambda|\ge C_2\}$ for some constants $C_1,C_2>0$, and satisfies in this region the estimate*

$$\|\chi\lambda(\Delta_g+\lambda^2)^{-1}\chi\|_{\mathcal{L}(H)}\le C, \tag{5.6}$$

for some constant $C>0$.

Proof. Clearly, the identity (5.5) extends also meromorphically in λ_0. So, the desired result follows easily from (5.5) with $\lambda_0=\operatorname{Re}\lambda$ and the following

Lemma 5.3 *For $\lambda\in\mathbf{C}$, $\forall\chi\in C_0^\infty(\mathbf{R}^n)$, $0\le k\le 2$, $\ell=0,1,\dots$, we have*

$$\|\partial_\lambda^\ell\chi(\Delta_0+\lambda^2)^{-1}\chi\|_{\mathcal{L}(L^2(\mathbf{R}^n),H^k(\mathbf{R}^n))}\le C_\ell|\lambda|^{-1}e^{C\operatorname{Im}\lambda},\quad |\lambda|\ge 1, \tag{5.7}$$

with constants $C_\ell, C>0$ independent of λ.

By (5.7) with $k=1$, $\ell=1$, one obtains, for $|\operatorname{Im}\lambda|\le 1$,

$$\|K(\lambda)\|_{\mathcal{L}(H)}\le C_1|\operatorname{Im}\lambda|\left(\|\lambda\chi(\Delta_g+\lambda_0^2)^{-1}\chi\|_{\mathcal{L}(H)}+\|[\Delta_0,\chi_1](\Delta_g+\lambda_0^2)^{-1}\chi\|_{\mathcal{L}(H)}\right). \tag{5.8}$$

Thus, by (4.3), (4.11) and (5.8) one concludes that

$$\|K(\lambda)\|_{\mathcal{L}(H)}\le C_2|\operatorname{Im}\lambda|\le 1/2$$

provided $|\operatorname{Im}\lambda|\le C=\min\{1,(2C_2)^{-1}\}$, $|\operatorname{Re}\lambda|\ge C_3>0$. Hence, by (5.5), $\chi(\Delta_g+\lambda^2)^{-1}\chi$ is analytic in this region and satisfies (5.6) as so does the operator $K_1(\lambda)$. □

The most important consequence of the above proposition is the following uniform local energy estimates to the solutions of the wave equation.

Theorem 5.4 *Under the assumption (4.2), for* $t \gg 1$,

$$\left\|\chi\cos\left(t\sqrt{-\Delta_g}\right)\chi\right\|_{\mathcal{L}(H)}+\left\|\chi\partial_{x_j}\frac{\sin\left(t\sqrt{-\Delta_g}\right)}{\sqrt{-\Delta_g}}\chi\right\|_{\mathcal{L}(H)}\leq\begin{cases}Ce^{-\alpha t}, & \text{if } n \text{ is odd},\\ Ct^{-n}, & \text{if } n \text{ is even},\end{cases}\tag{5.9}$$

with some constants $C, \alpha > 0$.

Proof. We are going to take advantage of the identity (4.14). Note that by the finite speed of propagation of the solutions to the wave equation we have that

$$(1-\eta(x))\varphi^{(k)}(t)\cos\left(t\sqrt{-\Delta_g}\right)\chi(x),\quad k=1,2,$$

are identically zero for some function $\eta \in C_0^\infty(\mathbf{R}^n)$. Therefore, we have the identity

$$S(\lambda)f := \mathcal{F}_{t\to\lambda}\left(\varphi(t)\chi\cos\left(t\sqrt{-\Delta_g}\right)\chi\right)f$$

$$=\chi(\Delta_g+\lambda^2)^{-1}\eta\mathcal{F}_{t\to\lambda}\left(\left(\varphi''(t)-2i\lambda\varphi'(t)\right)\cos\left(t\sqrt{-\Delta_g}\right)\chi\right)f,\tag{5.10}$$

which, in view of Proposition 5.2, extends analytically to $\{0\leq \operatorname{Im}\lambda\leq\beta,\ |\operatorname{Re}\lambda|\geq\varepsilon\}, \forall\varepsilon>0$, for some constant $\beta>0$. In view of (5.6), it is easy to see that

$$\lim_{|R|\to+\infty}\|S(R+i\sigma)f\|_H\to 0$$

uniformly in σ for $\sigma\leq\beta$. Therefore, one can change the contour of integration in the formula

$$\varphi(t)\chi\cos\left(t\sqrt{-\Delta_g}\right)\chi f=(2\pi)^{-1}\int_{\operatorname{Im}\lambda=-\varepsilon}e^{it\lambda}S(\lambda)fd\lambda$$

$$=(2\pi)^{-1}e^{-\beta t}\int_{-\infty}^{\infty}e^{it\lambda}S(\lambda+i\beta)fd\lambda+\lim_{\varepsilon\to0^+}(2\pi)^{-1}\int_{\gamma(\varepsilon)}e^{it\lambda}S(\lambda)fd\lambda$$

$$:=e^{-\beta t}I_1(t)f+I_2(t)f,\tag{5.11}$$

where $\gamma(\varepsilon)=\{\operatorname{Im}\lambda<0,\ |\lambda|=\varepsilon\}\cup\{\operatorname{Re}\lambda=-\varepsilon,\ 0\leq\operatorname{Im}\lambda\leq\beta\}\cup\{\operatorname{Re}\lambda=\varepsilon,\ 0\leq\operatorname{Im}\lambda\leq\beta\}\cup\{\operatorname{Im}\lambda=\beta,\ -\varepsilon\leq\operatorname{Re}\lambda\leq\varepsilon\}$. It follows from Proposition 5.1 that when n is odd, $S(\lambda)$ is analytic at $\lambda=0$, so in this case $I_2(t)=0$. When n is even, by (5.2) it is easy to see that

$$\|I_2(t)f\|_H\leq Ct^{-n}\|f\|_H.\tag{5.12}$$

To estimate the norm of $I_1(t)$, set

$$J(t)=(\partial_t^2-\Delta_g)I_1(t),$$

and observe that

$$\mathcal{F}_{t\to\lambda}J(t)f=-(\Delta_g+\lambda^2)S(\lambda+i\beta)f$$

$$= -\left([\Delta_g,\chi](\Delta_g+(\lambda+i\beta)^2)^{-1}\eta + ((\lambda+i\beta)^2-\lambda^2))\chi(\Delta_g+(\lambda+i\beta)^2)^{-1}\eta+\chi\eta\right)$$
$$\mathcal{F}_{t\to\lambda}\left((\varphi''(t)-2i(\lambda+i\beta)\varphi'(t))e^{t\beta}\cos\left(t\sqrt{-\Delta_g}\right)\chi\right)f.$$

By (5.6), one can easily obtain, for real λ,

$$\left\|\left(1+\sqrt{-\Delta_g}\right)^{-1}\mathcal{F}_{t\to\lambda}J(t)f\right\|_H \le C\sum_{k=1}^{2}\left\|\mathcal{F}_{t\to\lambda}\left(e^{t\beta}\varphi^{(k)}(t)\cos\left(t\sqrt{-\Delta_g}\right)\chi\right)f\right\|_H, \quad (5.13)$$

with a constant $C>0$ independent of λ. By Plancherel's identity and (5.13) one can easily conclude

$$\int_{-\infty}^{\infty}\left\|\left(1+\sqrt{-\Delta_g}\right)^{-1}J(t)f\right\|_H^2 dt$$
$$\le C_1\int_{-\infty}^{\infty}e^{2t\beta}\left(\varphi'(t)^2+\varphi''(t)^2\right)\left\|\cos\left(t\sqrt{-\Delta_g}\right)\chi f\right\|_H^2 dt \le C_2\|f\|_H^2. \quad (5.14)$$

On the other hand, by Duhamel's formula

$$I_1(t)f=\int_0^t \frac{\sin((t-\tau)\sqrt{-\Delta_g})}{\sqrt{-\Delta_g}}J(\tau)f d\tau,$$

and hence, in view of (5.14),

$$\|I_1(t)f\|_H \le \int_0^t (t-\tau)\left\|\left(1+\sqrt{-\Delta_g}\right)^{-1}J(\tau)f\right\|_H d\tau$$
$$\le \left(\int_0^t (t-\tau)^2 d\tau\right)^{1/2}\left(\int_0^t \left\|\left(1+\sqrt{-\Delta_g}\right)^{-1}J(\tau)f\right\|_H^2 d\tau\right)^{1/2}$$
$$\le t^{3/2}\left(\int_0^\infty \left\|\left(1+\sqrt{-\Delta_g}\right)^{-1}J(\tau)f\right\|_H^2 d\tau\right)^{1/2} \le Ct^{3/2}\|f\|_H. \quad (5.15)$$

Now the first term in the LHS of (5.9) can be estimated by combining (5.11), (5.12) and (5.15). The second term is treated similarly. □

For nontrapping compactly supported perturbations of Δ_0 we have a better free of resonances region, which however does not follow anymore from Theorem 4.1. Namely, we have the following extension of Lemma 5.3 to more general nontrapping operators which is due to Vainberg [13].

Theorem 5.5 *Under the assumption (4.2), the cutoff resolvent* $\chi(\Delta_g+\lambda^2)^{-1}\chi$ *extends analytically to* $\{\lambda\in\mathbf{C}: \operatorname{Im}\lambda \le N\log|\lambda|, |\operatorname{Re}\lambda|\ge C_N\}$, $\forall N\gg 1$ *with a constant* $C_N>0$, *and satisfies in this region the estimate*

$$\|\partial_\lambda^\ell \chi(\Delta_g+\lambda^2)^{-1}\chi\|_{\mathcal{L}(L^2(\mathbf{R}^n),H^k(\mathbf{R}^n))} \le C_\ell|\lambda|^{-1}e^{C\operatorname{Im}\lambda} \quad (5.16)$$

for $0\le k\le 2$, $\ell=0,1,2,\ldots$, *with some constants* $C_\ell, C>0$.

Proof. We will derive (5.16) from the following result (known also as generalized Huyghens principle) which in turn can be deduced from the classical propagation of C^∞ singularities (e.g. see Theorem 23.2.9 of [6]). Let $R > 0$ be such that $\operatorname{supp}\chi \subset \{|x| \le R\}$.

Proposition 5.6 *Under the assumption (4.2), the kernel $U(t,x,y)$ of the operator $\frac{\sin(t\sqrt{-\Delta_g})}{\sqrt{-\Delta_g}}\chi$ is C^∞-smooth in (t,x,y) for $|x| \le ct - R$ with some constant $c > 0$.*

Choose a function $\zeta \in C^\infty(\mathbf{R}^{n+1})$ such that $\zeta(x,t) = 0$ in $\mathbf{R}^{n+1} \setminus \{|x| \le ct - R\}$, $\zeta(x,t) = 1$ in $\{|x| \le ct/2 - R\}$. Let $\chi_j \in C_0^\infty(\mathbf{R}^n)$, $\chi_1 = 1$ on $\operatorname{supp}\chi$, $\chi_2 = 1$ on $\operatorname{supp}\chi_1$. Setting $Q(t) = [\partial_t^2 - \Delta_g, \chi_1(x)\zeta(x,t)]$, in view of Duhamel's formula we have

$$\chi_1\zeta\frac{\sin(t\sqrt{-\Delta_g})}{\sqrt{-\Delta_g}}\chi = \frac{\sin(t\sqrt{-\Delta_g})}{\sqrt{-\Delta_g}}\chi + \int_0^t \frac{\sin((t-\tau)\sqrt{-\Delta_g})}{\sqrt{-\Delta_g}}Q(\tau)\frac{\sin(\tau\sqrt{-\Delta_g})}{\sqrt{-\Delta_g}}\chi d\tau.$$

Taking the Fourier-Laplace transform of this identity, we get, for $\operatorname{Im}\lambda < 0$,

$$-\mathcal{F}_{t\to\lambda}\left(\chi_1\zeta\frac{\sin(t\sqrt{-\Delta_g})}{\sqrt{-\Delta_g}}\chi\right) = \chi(\Delta_g+\lambda^2)^{-1}\chi + \chi(\Delta_g+\lambda^2)^{-1}\chi_2\mathcal{F}_{t\to\lambda}\left(Q(t)\frac{\sin(t\sqrt{-\Delta_g})}{\sqrt{-\Delta_g}}\chi\right).$$

Taking into account that $\chi = 1$ on $\operatorname{supp}(\Delta_g - \Delta_0)$, one can easily obtain from this identity the following one

$$\chi(\Delta_g + \lambda^2)^{-1}\chi(1 + \mathcal{K}(\lambda)) = \mathcal{K}_1(\lambda), \tag{5.17}$$

where

$$\mathcal{K}(\lambda) = (\Delta_g - \Delta_0)(\Delta_0 + \lambda^2)^{-1}\chi_2\mathcal{F}_{t\to\lambda}\left(Q(t)\frac{\sin(t\sqrt{-\Delta_g})}{\sqrt{-\Delta_g}}\chi\right),$$

$$\mathcal{K}_1(\lambda) = -\mathcal{F}_{t\to\lambda}\left(\chi\zeta\frac{\sin(t\sqrt{-\Delta_g})}{\sqrt{-\Delta_g}}\chi\right) - \chi(\Delta_0 + \lambda^2)^{-1}\chi_2\mathcal{F}_{t\to\lambda}\left(Q(t)\frac{\sin(t\sqrt{-\Delta_g})}{\sqrt{-\Delta_g}}\chi\right).$$

Since $\chi\zeta$ and $Q(t)$ are identically zero for $t \ge T$ for some constant $T > 0$, the operators $\mathcal{K}(\lambda)$ and $\mathcal{K}_1(\lambda)$ extend to entire functions in λ. Moreover, by Proposition 5.6 we have $Q(t)U(t,x,y) \in C_0^\infty$ with respect to (t,x,y). Therefore, for every integers $N \gg 1$, $\ell \ge 0$,

$$\left\|\lambda^N\mathcal{F}_{t\to\lambda}\left(Q(t)\frac{\sin(t\sqrt{-\Delta_g})}{\sqrt{-\Delta_g}}\chi\right)\right\| = \left\|\mathcal{F}_{t\to\lambda}\left(\partial_t^N Q(t)\frac{\sin(t\sqrt{-\Delta_g})}{\sqrt{-\Delta_g}}\chi\right)\right\| \le C_N e^{T\operatorname{Im}\lambda}, \tag{5.18}$$

$$\left\|\partial_\lambda^\ell\mathcal{F}_{t\to\lambda}\left(\chi\zeta\frac{\sin(t\sqrt{-\Delta_g})}{\sqrt{-\Delta_g}}\chi\right)\right\| = \left\|\mathcal{F}_{t\to\lambda}\left(t^\ell\chi\zeta\frac{\sin(t\sqrt{-\Delta_g})}{\sqrt{-\Delta_g}}\chi\right)\right\| \le C_\ell e^{T\operatorname{Im}\lambda}, \tag{5.19}$$

where $\|\cdot\|$ denotes the norm in $\mathcal{L}(H)$. Now (5.16) with $k = 0$ follows easily from (5.17)-(5.19) combined with (5.7). The bound (5.16) with $k = 2$ follows from (5.16) with $k = 0$ and the fact that Δ_g is elliptic. So, by an interpolation argument, (5.16) follows for every $0 \le k \le 2$.

6 Uniform Limiting Absorption Principle and Decay of the Local Energy for General Long-Range Riemannian Metrics on $\mathbf{R}^N$

Throughout this section the space $\mathbf{R}^n$, $n \geq 2$, will be equipped with a C^∞-smooth Riemannian metric g satisfying (4.1), but we will no longer suppose that (4.2) is fulfilled. Of course, the results desribed in Sections 4 and 5 do not hold anymore. Nevetheless, there are weaker analogues of these results in this general setting and that is what we are going to discuss in the present section. Given an $a \gg 1$, choose a function $\eta_a \in C^\infty(\mathbf{R}^n)$ such that $\eta_a(x) = 0$ for $|x| \leq a$, $\eta_a(x) = 1$ for $|x| \geq a+1$.

Theorem 6.1 *Under the assumption (4.1), for $\forall s > 1/2$, $\lambda \geq \lambda_0$ the limit*

$$\lim_{\varepsilon \to 0^+} \langle x\rangle^{-s}(\Delta_g + \lambda^2 \pm i\varepsilon)^{-1}\langle x\rangle^{-s} : H \to H$$

exists and satisfies the estimates

$$\|\langle x\rangle^{-s}\lambda(\Delta_g + \lambda^2 \pm i0)^{-1}\langle x\rangle^{-s}\|_{\mathcal{L}(H)} \leq Ce^{C_1\lambda}, \tag{6.1}$$

$$\|\eta_a(x)\langle x\rangle^{-s}\lambda(\Delta_g + \lambda^2 \pm i0)^{-1}\langle x\rangle^{-s}\eta_a(x)\|_{\mathcal{L}(H)} \leq C, \tag{6.2}$$

with some constants $C, C_1, \lambda_0, a > 0$ independent of λ.

Clearly, in this more general situation the estimate (4.5) is no longer true. Instead, we have the following analogue

Proposition 6.2 *Let $u \in H^2(\mathbf{R}^n)$. Under the assumption (4.1), for $R \gg \rho_0$ and for $0 < \varepsilon \leq 1$, $\lambda \geq \lambda_0$, the following estimate holds*

$$e^{-c_1\lambda}\|u\|_{H^1(B_R)} + \|u\|_{H^1(B_{R+1}\setminus B_R)} \leq Ce^{c_2\lambda}\|(\Delta_g + \lambda^2 - i\varepsilon)u\|_{L^2(B_R)}$$
$$+C\lambda^{-1}\|(\Delta_g + \lambda^2 - i\varepsilon)u\|_{L^2(B_{R+2}\setminus B_R)} + C\|u\|_{H^1(B_{R+2}\setminus B_{R+1})}, \tag{6.3}$$

with some constants $c_1, c_2, C, \lambda_0 > 0$ independent of λ, ε and u.

This estimate is proved in [3] by using the Carleman inequalities in B_R. Theorem 6.1 follows from combining the estimates (4.6) and (6.3) in a way similar to that one carried out in Section 4. At low frequency we have the following

Proposition 6.3 *Under the assumption (4.1), for $0 < \lambda \leq \lambda_0$, $s > 1/2$, we have*

$$\|\langle x\rangle^{-s}\lambda(\Delta_g + \lambda^2 \pm i0)^{-1}\langle x\rangle^{-s}\|_{\mathcal{L}(H)} + \|\langle x\rangle^{-s}\partial_{x_j}(\Delta_g + \lambda^2 \pm i0)^{-1}\langle x\rangle^{-s}\|_{\mathcal{L}(H)}$$
$$+\|\langle x\rangle^{-s}\sqrt{-\Delta_g}(\Delta_g + \lambda^2 \pm i0)^{-1}\langle x\rangle^{-s}\|_{\mathcal{L}(H)} \leq Const. \tag{6.4}$$

Note that to prove (6.4) one can proceed as in the proof of Theorem 4.1 using instead of (4.5) the Carleman inequalities (for example, (6.3)) which hold at low frequency as well, together with a low frequency analogue of (4.6).

Using (6.2), in the same way as in Section 4 one can prove the following

Corollary 6.4 *Under the assumption (4.1), $\forall f \in L^2(\mathbf{R}^n)$, $\forall s > 1/2$ the following inequalities hold:*

$$\int_{-\infty}^{\infty} \left\| \eta_a(x)\langle x\rangle^{-s} \cos\left(t\sqrt{-\Delta_g}\right) \langle x\rangle^{-s}\eta_a(x) f\right\|^2_{L^2(\mathbf{R}^n)} dt \leq C\|f\|^2_{L^2(\mathbf{R}^n)}, \tag{6.5}$$

$$\int_{-\infty}^{\infty} \left\| \eta_a(x)\langle x\rangle^{-s} \partial_{x_j} \frac{\sin\left(t\sqrt{-\Delta_g}\right)}{\sqrt{-\Delta_g}} \langle x\rangle^{-s}\eta_a(x) f\right\|^2_{L^2(\mathbf{R}^n)} dt \leq C\|f\|^2_{L^2(\mathbf{R}^n)}, \tag{6.6}$$

with a constant $C > 0$ independent of f.

In what follows we will assume that $g = g_0$ outside a bounded domain. The following proposition is an easy consequence of the estimates (6.1) and (5.8) and can be considered as an analogue of the results in Section 5 in this general setting.

Proposition 6.5 *The cutoff resolvent $\chi(\Delta_g + \lambda^2)^{-1}\chi$ extends analytically to $\{\lambda \in \mathbf{C} : \operatorname{Im}\lambda \leq C_2\exp(-C_1|\lambda|),\ \pm\operatorname{Re}\lambda > 0\}$, where $C_1 > 0$ is the constant in (6.1), $C_2 > 0$, and satisfies in this region the estimate*

$$\|\chi\lambda(\Delta_g + \lambda^2)^{-1}\chi\|_{\mathcal{L}(H)} \leq Ce^{C_1|\lambda|} \tag{6.7}$$

for some constant $C > 0$.

We will derive from this proposition the following

Theorem 6.6 *For $m \geq 0$, $t \gg 1$, we have*

$$\left\|\chi\left(1-\Delta_g\right)^{-m/2} \cos\left(t\sqrt{-\Delta_g}\right)\chi\right\|_{\mathcal{L}(H)} \leq C_m(\log t)^{-m}, \tag{6.8}$$

$$\left\|\chi\left(1-\Delta_g\right)^{-m/2} \sin\left(t\sqrt{-\Delta_g}\right)\chi\right\|_{\mathcal{L}(H)} \leq C_m(\log t)^{-m}, \tag{6.9}$$

with a constant $C_m > 0$.

Proof. Let $A \gg 1$ and let $\psi_A(\sigma) = 1$ for $\sigma \leq A$, $\psi_A(\sigma) = 0$ for $\sigma > A$. By the spectral theorem we have

$$\left\|\left(1-\Delta_g\right)^{-m/2}\left(1-\psi_A\left(\sqrt{-\Delta_g}\right)\right)\cos\left(t\sqrt{-\Delta_g}\right)\right\|_{\mathcal{L}(H)} \leq A^{-m}, \tag{6.10}$$

and

$$\chi (1-\Delta_g)^{-m/2}\psi_A\left(\sqrt{-\Delta_g}\right)\cos\left(t\sqrt{-\Delta_g}\right)\chi$$

$$= (2\pi i)^{-1}\int_0^A (1+\lambda^2)^{-m/2}\cos(t\lambda)\chi\left((\Delta_g+\lambda^2+i0)^{-1}-(\Delta_g+\lambda^2-i0)^{-1}\right)\chi 2\lambda d\lambda$$

$$= (2\pi i)^{-1}\sum_{\nu,\mu=1}^{2}(-1)^\mu\int_0^A (1+\lambda^2)^{-m/2}e^{(-1)^\nu it\lambda}\chi(\Delta_g+\lambda^2+(-1)^\mu i0)^{-1}\chi\lambda d\lambda. \quad (6.11)$$

In view of Proposition 6.5 and the Cauchy theorem one can write

$$E(t) := (2\pi i)^{-1}\int_0^A (1+\lambda^2)^{-m/2}e^{it\lambda}\chi(\Delta_g+\lambda^2-i0)^{-1}\chi\lambda d\lambda$$

$$= (2\pi i)^{-1}\int_{\Gamma_\delta}(1+\lambda^2)^{-m/2}e^{it\lambda}\chi(\Delta_g+\lambda^2)^{-1}\chi\lambda d\lambda$$

$$+(2\pi i)^{-1}\int_{\mathrm{Im}\lambda=\delta,\delta<\mathrm{Re}\lambda<A}(1+\lambda^2)^{-m/2}e^{it\lambda}\chi(\Delta_g+\lambda^2)^{-1}\chi\lambda d\lambda$$

$$+(2\pi i)^{-1}\int_{0<\mathrm{Im}\lambda<\delta,\mathrm{Re}\lambda=A}(1+\lambda^2)^{-m/2}e^{it\lambda}\chi(\Delta_g+\lambda^2)^{-1}\chi\lambda d\lambda := E_1(t)+E_2(t)+E_3(t),$$

where $0<\delta = O\left(e^{-C_1A}\right)$ is such that $\chi(\Delta_g+\lambda^2-i0)^{-1}\chi$ extends analytically to $0<\mathrm{Im}\,\lambda\leq\delta$, and $\Gamma_\delta=\{\mathrm{Im}\,\lambda=0,\ 0\leq\mathrm{Re}\,\lambda\leq\delta\}\cup\{\mathrm{Re}\,\lambda=\delta,\ 0\leq\mathrm{Im}\,\lambda\leq\delta\}$. In view of (6.7), one has

$$\|E_1(t)\|_{\mathcal{L}(H)}\leq C\delta,\quad \|E_2(t)\|_{\mathcal{L}(H)}\leq CAe^{C_1A-\delta t},\quad \|E_3(t)\|_{\mathcal{L}(H)}\leq Ct^{-1}e^{C_1A}. \quad (6.12)$$

Take $A=(2C_1)^{-1}\log t$ so that $\delta=O(t^{-1/2})$. By (6.12),

$$\|E(t)\|_{\mathcal{L}(H)}\leq Ct^{-1/2}. \quad (6.13)$$

Clearly, the other terms in the sum in (6.11) can be treated similarly. This together with (6.10) imply (6.8). The estimate (6.9) is treated similarly.

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