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Bell Polynomials and Some of Their Applications

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ABSTRACT. We recall the most important properties and applications of the Bell polynomials, and in particular the possibility to represent symmetric functions of a countable set of numbers. The derivation of the so called Robert formulas for the reduction of the orthogonal invariants of a positive compact operator is also included.

1 Introduction

The Bell polynomials first appear as a mathematical tool for representing the nth derivative of a composite function.

Being related to partitions, the Bell polynomials often appear in Combinatorial Analysis [17]. They have been also applied in many different situations, such as the Blissard problem (see [17], p. 46), the representation of Lucas polynomials of the first and second kind

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[6], [8], the construction of recurrence relations for a class of Freud-type polynomials [5], etc., however, in our opinion, the most important of their applications is connected with the possibility to represent, by using such a powerful tool, the symmetric functions of a countable set of numbers. As a matter of fact, by using Bell polynomials, it is possible to deduce the relations which generalize the classical algebraic Newton-Girard formulas. Consequently, as it was recently shown [7], it is possible to find reduction formulas for the *orthogonal invariants* of a strictly Positive Compact Operator (shortly PCO), deriving in a simple way the so called Robert formulas [18].

In this article, after recalling the most important formulas related to Bell polynomials, we will show this last application, which can be used as a trace to follow in every problem involving symmetric functions of a countable set of numbers.

2 Recalling the Bell Polynomials

The problem of finding an explicit expression for the the nth derivative of a composite function was first solved by F. Faà di Bruno [9]. The relevant problem of finding an efficient computational method was solved by E.T. Bell, by means of the introduction of his polynomials [4], which can be computed recursively, whereas the Faà di Bruno formula is based on the partitions of the integer n, a set whose cardinality increases in an extremely fast way.

Consider $\Phi(t) := f(g(t))$, i.e. the composition of functions $x = g(t)$ and $y = f(x)$, defined in suitable intervals of the real axis, and suppose that $g(t)$ and $f(x)$ are n times differentiable with respect to the independent variables so that $\Phi(t)$ can be differentiated n times with respect to t, by using the differentiation rule of composite functions.

We use the following notations:

$$\Phi_m := D_t^m \Phi(t), \qquad f_h := D_x^h f(x)|_{x=g(t)}, \qquad g_k := D_t^k g(t).$$

Then the n-th derivative can be represented by

$$\Phi_n = Y_n(f_1, g_1; f_2, g_2; \dots; f_n, g_n),$$

where the Y_n are, by definition, the Bell polynomials.

For example one has:

$$\begin{aligned} Y_1(f_1, g_1) &= f_1 g_1 \qquad (2.1)\\ Y_2(f_1, g_1; f_2, g_2) &= f_1 g_2 + f_2 g_1^2 \\ Y_3(f_1, g_1; f_2, g_2; f_3, g_3) &= f_1 g_3 + f_2(3 g_2 g_1) + f_3 g_1^3. \end{aligned}$$

Further examples can be found in [17], p. 49.

Inductively, we can write:

$$Y_n(f_1, g_1; f_2, g_2; \ldots; f_n, g_n) = \sum_{k=1}^{n} A_{n,k}(g_1, g_2, \ldots, g_n) f_k, \tag{2.2}$$

where the coefficient $A_{n,k}$, for all $k = 1, \ldots, n$, is a polynomial in $g_1, g_2, \ldots, g_n$, homogeneous of degree k and *isobaric* of weight n (i.e. it is a linear combination of monomials $g_1^{k_1} g_2^{k_2} \cdots g_n^{k_n}$ whose weight is constantly given by $k_1 + 2k_2 + \ldots + nk_n = n$).

Since the coefficients $A_{n,k}$ are independent of f, their construction can be performed by choosing

$$f = e^{ax},$$

where a is an arbitrary constant.

In this case

$$f_k = a^k e^{ag(t)} = a^k e^{ag}, \qquad (g = g(t)),$$

so that eq. (2.2) becomes:

$$\Phi_n = \sum_{k=1}^{n} a^k e^{ag} A_{n,k}(g_1, g_2, \ldots, g_n),$$

i.e.

$$e^{-ag} D_t^n e^{ag} = \sum_{k=1}^{n} a^k A_{n,k}(g_1, g_2, \ldots, g_n). \tag{2.3}$$

The last equation characterize $A_{n,k}$ as the coefficient of a^k in the polynomial expansion of $e^{-ag} D_t^n e^{ag}$.

For example:

- for $n = 1$: $e^{-ag} D_t e^{ag} = ag_1$, so that: $A_{1,1} = g_1$.

- for $n = 2$: $e^{-ag} D_t^2 e^{ag} = ag_2 + a^2 g_1^2$, so that: $A_{2,1} = g_2$, $A_{2,2} = g_1^2$,

and so on.

It is easy to prove the following result:

Proposition 2.1 *The Bell polynomials satisfy the recurrence relation:*

$$\begin{cases} Y_0 := f_1; \\ Y_{n+1}(f_1, g_1; \ldots; f_n, g_n; f_{n+1}, g_{n+1}) = \\ = \sum_{k=0}^{n} \binom{n}{k} Y_{n-k}(f_2, g_1; f_3, g_2; \ldots; f_{n-k+1}, g_{n-k}) g_{k+1}. \end{cases} \tag{2.4}$$

Proof. First note that the above recursion, for $n = 0$ reduces to $Y_1 = Y_0 g_1$, so that, by comparison with eq. (2.1), we must assume $Y_0 := f_1$.

For proving the above recursion in the general case, note that by using Leibniz' rule we can write:

$$\begin{aligned}
\Phi_{n+1} &= Y_{n+1}(f_1, g_1; \dots; f_n, g_n; f_{n+1}, g_{n+1}) = \\
&= D_t^n D_t \Phi(t) = D_t^n (f_1 g_1) = \sum_{k=0}^{n} \binom{n}{k} D_t^{n-k} f_1 D_t^k g_1 = \\
&= \sum_{k=0}^{n} \binom{n}{k} D_t^{n-k} f_1 g_{k+1} = \\
&= Y_{n-k}(f_2, g_1; f_3, g_2; \dots; f_{n-k+1}, g_{n-k}) g_{k+1}.
\end{aligned}$$

As we recalled before, an explicit expression for the nth derivative of a composite function, i.e. for the Bell polynomials, is given by the Faà di Bruno formula:

$$\Phi_n = Y_n(f_1, g_1; f_2, g_2; \dots; f_n, g_n) = \sum_{\pi(n)} \frac{n!}{r_1! r_2! \dots r_n!} f_r \left[\frac{g_1}{1!}\right]^{r_1} \left[\frac{g_2}{2!}\right]^{r_2} \cdots \left[\frac{g_n}{n!}\right]^{r_n}, \quad (2.5)$$

where the sum runs over all partitions $\pi(n)$ of the integer n, r_i denotes the number of parts of size i, and $r = r_1 + r_2 + \dots + r_n$ denotes the number of parts of the considered partition.

A simple proof of the Faà di Bruno formula can be found in the Riordan book [17]. Another proof, based on the so called *umbral calculus*, can be found in a paper by S. Roman [19]. The *umbral calculus* is a classical tool tracing back to the operational calculus of O. Heaviside, but recovered, in modern form, by E.T. Bell and more recently by G.C. Rota (see [20] and the references therein).

Before ending this section, we recall that a generalization of the Bell polynomials suitable for the differentiation of multivariable composite functions can be found in [16].

3 Generalization of the Newton-Girard Formulas

We want to present now the most important application of the Bell polynomials, i.e. the generalization of the algebraic Newton-Girard formulas (see e.g. [2], [14]).

Consider the (finite or infinite) sequence of real or complex numbers $\mu_1, \mu_2, \mu_3, \dots$, and denote by

$$\sigma_1 = \sum_i \mu_i, \qquad \sigma_2 = \sum_{i<j} \mu_i \mu_j, \quad \dots, \quad \sigma_k = \sum_{i_1 < i_2 < \dots < i_k} \mu_{i_1} \mu_{i_2} \cdots \mu_{i_k}, \quad \dots, \qquad (3.1)$$

the relevant elementary symmetric functions, and by

$$s_1 = \sum_i \mu_i, \quad s_2 = \sum_i \mu_i^2, \quad \ldots, \quad s_k = \sum_i \mu_i^k, \quad \ldots, \tag{3.2}$$

the symmetric functions *power sums*.

Of course, in case of infinite sets of indices, if we are interested to non-formal results, we must assume that all the above considered expansions are convergent.

We premit a lemma, which is frequently used in Statistics [13].

Lemma 3.1 *The following expansion holds true:*

$$\begin{aligned}(1-\mu_1 x)(1-\mu_2 x)(1-\mu_3 x)\cdots &= 1 - \sigma_1 x + \sigma_2 x^2 - \ldots = \\ &= \exp\left(-s_1 x - s_2\frac{x^2}{2} - s_3\frac{x^3}{3} - \ldots\right),\end{aligned} \tag{3.3}$$

and, in equivalent form:

$$\begin{aligned}&\log\left((1-\mu_1 x)(1-\mu_2 x)(1-\mu_3 x)\cdots\right) = \\ &= \log\left(1 - \sigma_1 x + \sigma_2 x^2 - \ldots\right) = \\ &= -s_1 x - s_2\frac{x^2}{2} - s_3\frac{x^3}{3} - \ldots\end{aligned} \tag{3.4}$$

Proof. The above eqs. (3.3)-(3.4) are both equivalent to the eq. below

$$\frac{1}{(1-\mu_1 x)(1-\mu_2 x)(1-\mu_3 x)\cdots} = \exp\left(s_1 x + s_2\frac{x^2}{2} + s_3\frac{x^3}{3} + \ldots\right). \tag{3.5}$$

This last equation can be easily proved by *transfinite induction* with respect to the cardinality of the set of indices. In fact, the result is certainly true if this cardinality is $n = 1$, so that: $s_1^{(1)} = \mu_1$, $s_2^{(1)} = \mu_1^2, \ldots,$ $s_k^{(1)} = \mu_1^k, \ldots$. Then eq. (3.5) becomes:

$$\begin{aligned}&\frac{1}{(1-\mu_1 x)} = \exp\left(-\log(1-\mu_1 x)\right) = \\ &= \exp\left(\mu_1 x + \frac{(\mu_1 x)^2}{2} + \frac{(\mu_1 x)^3}{3} + \ldots\right) = \exp\left(s_1^{(1)} x + s_2^{(1)}\frac{x^2}{2} + s_3^{(1)}\frac{x^3}{3} + \ldots\right).\end{aligned}$$

Suppose that eq. (3.5) is valid for the above cardinality equal to $n-1$, then, noting that

$$s_1^{(n)} = s_1^{(n-1)} + \mu_n, \quad s_2^{(n)} = s_2^{(n-1)} + \mu_n^2, \quad \ldots, \quad s_k^{(n)} = s_k^{(n-1)} + \mu_n^k, \quad \ldots,$$

we can write:

$$\frac{1}{(1-\mu_1 x)\cdots(1-\mu_{n-1}x)(1-\mu_n x)} = \frac{1}{(1-\mu_1 x)\cdots(1-\mu_{n-1}x)} \times \frac{1}{(1-\mu_n x)} =$$
$$= \exp\left(s_1^{(n-1)}x + s_2^{(n-1)}\frac{x^2}{2} + \dots\right)\exp\left(\mu_n x + \frac{\mu_n x^2}{2} + \dots\right) =$$
$$= \exp\left(s_1^{(n)}x + s_2^{(n)}\frac{x^2}{2} + \dots\right),$$

so that eq. (3.5) holds true for cardinality equal to n.

The general result, relevant to infinite many indices, can consequently be obtained by a limit process, taking into account the hypothesis about the convergence of all the considered expansions.

Proposition 3.2 *For any integer k the following representation formulas hold true:*

$$\sigma_k = \frac{(-1)^k}{k!} Y_k\left(1, -s_1; 1, -s_2; 1, -2!s_3; \dots; 1, -(k-1)!s_k\right) \tag{3.6}$$

$$s_k = -\frac{1}{(k-1)!} Y_k\left(1, -\sigma_1; -1, 2!\sigma_2; \dots; (-1)^{k-1}(k-1)!, (-1)^k k!\sigma_k\right) \tag{3.7}$$

Proof. For proving eq. (3.6), we start from eq. (3.3) of Lemma 3.1. By using Taylor's expansion we can write, for any $k \geq 0$:

$$\sigma_k = \frac{(-1)^k}{k!} D_x^k \left[\exp\left(-s_1 x - s_2\frac{x^2}{2} - s_3\frac{x^3}{3} - \dots\right)\right]_{x=0} = \tag{3.8}$$
$$= Y_k(f_1, g_1; f_2, g_2; \dots; f_k, g_k),$$

where, putting $f(g(x)) = e^{g(x)}$, $y = g(x) = -s_1 x - s_2\frac{x^2}{2} - s_3\frac{x^3}{3} - \dots$, and observing that $x = 0$ corresponds to $y = 0$, we have to assume, for every $h = 1, 2, \dots, k$:

$$f_h = \left[D_y^h f(y)\right]_{y=0} = \left[D_y^h e^y\right]_{y=0} = 1,$$

and

$$g_h = \left[D_y^h g(x)\right]_{x=0} = h!\left(-\frac{s_h}{h}\right) = -(h-1)!s_h,$$

so that eq. (3.6) holds true.

For proving eq. (3.7), we start from eq. (3.4) of Lemma 3.1. By using Taylor's expansion we can write, for any $k \geq 1$:

$$-\frac{s_k}{k} = \frac{1}{k!} D_x^k \left[\log\left(1 + (-\sigma_1 x + \sigma_2 x^2 - \sigma_3 x^3 + \ldots)\right)\right]_{x=0} =$$
$$= \frac{1}{k!} D_x^k \left[\log\left(1 + Q(x)\right)\right]_{x=0},$$

so that

$$s_k = -\frac{1}{(k-1)!} Y_k(F_1, Q_1; F_2, Q_2; \ldots; F_k, Q_k),$$

where, putting $F(Q(x)) = \log(1 + Q(x))$, $y = Q(x) = -\sigma_1 x + \sigma_2 x^2 - \sigma_3 x^3 + \ldots$, and observing that $x = 0$ corresponds to $y = 0$, we have to assume, for every $h = 1, 2, \ldots, k$:

$$F_h = \left[D_y^h F(y)\right]_{y=0} = \left[D_y^h \log(1+y)\right]_{y=0} = (-1)^{h-1}(h-1)!,$$

and

$$Q_h = \left[D_x^h Q(x)\right]_{x=0} = (-1)^h h! \sigma_h,$$

so that eq. (3.7) holds true.

Note that the above formulas (3.6)-(3.7) constitute a generalization of the well known Newton-Girard formulas and their inverse, since we have, in particular:

$$\sigma_1 = s_1$$
$$\sigma_2 = \tfrac{1}{2}(s_1^2 - s_2)$$
$$\sigma_3 = \tfrac{1}{6}(s_1^3 - 3s_1 s_2 + 2s_3)$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$s_1 = \sigma_1$$
$$s_2 = \sigma_1^2 - 2\sigma_2$$
$$s_3 = \sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$

Before ending this section we want explicitly remark that the Gauss' theorem on symmetric functions can be stated also in the more general framework of symmetric functions of infinite many variables.

In fact, the following statement can be easily proved:

Proposition 3.3 *Every symmetric polynomial function (or absolutely convergent series expansion) of the variables $\mu_1, \mu_2, \mu_3, \ldots$, can be written as a polynomial (or absolutely convergent series expansion) of the variables $\sigma_1, \sigma_2, \sigma_3, \ldots$.*

Note that, by using Bell polynomials and eq. (3.6), the expansion derived from the above Proposition 3.3 can be transformed into a polynomial (or series expansion) of the variables *power sums*.

Proof. The proof is obtained in the same way of the classical proof of Gauss' theorem (see e.g. [2] , pp. 210-217), by decomposing the given symmetric function into symmetric *multiple sums* (also called Σ sums, [3] , p. 44), and then proceeding by induction with respect to the *height* (i.e the difference between the degree and the number of variables appearing) of such *multiple sum*. Convergence, in case of infinite expansions, is guaranteed by the hypothesis of absolute convergence of the starting function.

4 Representation of Orthogonal Invariants for a PCO

It is well known that the eigenvalues μ_k of a positive compact operator T in a complex Hilbert space $\mathcal{H}$ can be ordered in a sequence

$$0 \leq \ldots \leq \mu_3 \leq \mu_2 \leq \mu_1, \tag{4.1}$$

s.t. when infinite many eigenvalues exist, they have the zero as an accumulation point.

A classical example of eigenvalue problem for such an operator (which is strictly positive) is given by

$$T\phi = \mathcal{K}\phi := \int_A K(x,y)\phi(y)dy = \mu\phi(x), \tag{4.2}$$

where the kernel $K(x,y)$ of the second kind Fredholm operator $\mathcal{K}$ belongs to $L^2(A \times A)$, and is such that $K(x,y) = \overline{K(y,x)}$, $(\mathcal{K}\phi, \phi) > 0$ if $\phi \neq 0 \in L^2(A)$ (see S.G. Mikhlin [15]).

The numerical computation of the eigenvalues of T is usually performed by using the Rayleigh-Ritz method [15] for obtaining lower bounds, and the orthogonal invariants method (see G. Fichera [10] - [11] - [12]) for upper bounds. A short description of such methods was given in [7]. In a recent paper [1], an iterative method for computing the above mentioned eigenvalues has been shown.

The orthogonal invariants are, by definition, symmetric functions of the eigenvalues of T:

$$\mathcal{I}_s^n(T) = \sum_{k_1<k_2<\ldots<k_s} [\mu_{k_1}\mu_{k_2}\cdots\mu_{k_s}]^n, \tag{4.3}$$

so that it is natural to expect that (as in the algebraic case) connections with $\mathcal{I}_s^1(T)$ ($s = 1,2,\ldots,n$), or $\mathcal{I}_1^n(T)$ ($n = 1,2,\ldots,s$) hold true.

5 Robert's Formulas

In the above cited article [18], D. Robert has found the following formulas

$$\mathcal{I}_s^n(T) = \frac{1}{s}\sum_{q=1}^{s}(-1)^{q-1}\mathcal{I}_1^{qn}(T)\mathcal{I}_{s-q}^n(T) \tag{5.1}$$

$$\mathcal{I}_s^n(T) = (-1)^s \sum_{k=1}^{s}\frac{(-1)^k}{k!}\sum_{\substack{q_1+\ldots+q_k=s \\ 1\le q_i\le s}}\frac{\mathcal{I}_1^{nq_1}(T)\cdots\mathcal{I}_1^{nq_k}(T)}{q_1\cdots q_k}, \tag{5.2}$$

which allow to reduce the orthogonal invariant $\mathcal{I}_s^n(T)$ to $\mathcal{I}_1^h(T)$ ($\forall h = 1, 2, \ldots, ns$).

Since the eigenvalues of T^n are given by μ_i^n, if μ_i are the eigenvalues of T, in the following, denoting by n the smallest integer such that $\mathcal{I}_s^n(T) < \infty$, we will put $T := T^n$, so that $\mathcal{I}_s^1(T) = \mathcal{I}_s^1(T^n) = \mathcal{I}_s^n(T)$, and the above eq. (5.1), (5.2) become:

$$\mathcal{I}_s^1(T) = \frac{1}{s}\sum_{q=1}^{s}(-1)^{q-1}\mathcal{I}_1^{q}(T)\mathcal{I}_{s-q}^1(T) \tag{5.3}$$

$$\mathcal{I}_s^1(T) = (-1)^s \sum_{k=1}^{s}\frac{(-1)^k}{k!}\sum_{\substack{q_1+\ldots+q_k=s \\ 1\le q_i\le s}}\frac{\mathcal{I}_1^{q_1}(T)\cdots\mathcal{I}_1^{q_k}(T)}{q_1\cdots q_k}. \tag{5.4}$$

Remark 5.1 *It is worth to note that there exist PCO not satisfying the above mentioned condition which requires the existence of an integer n such that $\mathcal{I}_s^n(T) < \infty$ [12], however this condition is satisfied by all the PCO occurring in applications.*

6 Orthogonal Invariants' Reduction Formulas

Writing the representation formulas of Proposition 3.2 in terms of orthogonal invariants, we obtain:

$$\mathcal{I}_k^1(T) = \frac{(-1)^k}{k!}Y_k\left(1, -\mathcal{I}_1^1(T); 1, -\mathcal{I}_1^2(T); 1, -2!\mathcal{I}_1^3(T); \ldots; 1, -(k-1)!\mathcal{I}_1^k(T)\right) \tag{6.1}$$

$$\mathcal{I}_1^k(T) = -\frac{1}{(k-1)!}Y_k\left(1, -\mathcal{I}_1^1(T); -1, 2!\mathcal{I}_2^1(T); \ldots; (-1)^{k-1}(k-1)!, (-1)^k k!\mathcal{I}_k^1(T)\right). \tag{6.2}$$

6.1 A simple proof of Robert's formulas

Proposition 6.1 *The first Robert formula is equivalent to the recurrence relation of the Bell polynomials.*

Proof. We start from the recurrence relation (2.4) written in the form:

$$Y_s(f_1,g_1;\ldots;f_{s-1},g_{s-1};f_s,g_s)=\sum_{q=1}^{s}\binom{s-1}{q-1}Y_{s-q}(f_2,g_1;f_3,g_2;\ldots;f_{s-q+1},g_{s-q})g_q. \quad (6.3)$$

Then, by (6.1), it follows

$$\begin{aligned}\mathcal{I}_s^1(\mathcal{T})&=\frac{(-1)^s}{s!}Y_s(1,-\mathcal{I}_1^1(\mathcal{T});1,-\mathcal{I}_1^2(\mathcal{T});1,-2!\mathcal{I}_1^3(\mathcal{T});\ldots;1,-(s-1)!\mathcal{I}_1^s(\mathcal{T}))=\\&=\frac{(-1)^s}{s!}\sum_{q=1}^{s}\binom{s-1}{q-1}Y_{s-q}(1,-\mathcal{I}_1^1(\mathcal{T});1,-\mathcal{I}_1^2(\mathcal{T});\ldots\\&\qquad\ldots;1,-(s-q-1)!\mathcal{I}_{s-q}^1(\mathcal{T}))(-(q-1)!)\mathcal{I}_1^q(\mathcal{T})),\end{aligned}$$

and consequently:

$$\begin{aligned}\mathcal{I}_s^1(\mathcal{T})&=\frac{(-1)^s}{s!}\sum_{q=1}^{s}\binom{s-1}{q-1}(-1)^{s-q}(s-q)!\mathcal{I}_{s-q}^1(\mathcal{T})(-(q-1)!)\mathcal{I}_1^q(\mathcal{T})=\\&=\frac{1}{s}\sum_{q=1}^{s}(-1)^{q-1}\mathcal{I}_{s-q}^1(\mathcal{T})\mathcal{I}_1^q(\mathcal{T}),\end{aligned} \quad (6.4)$$

which is the first Robert formula.

Proposition 6.2 *The second Robert formula is equivalent to the Faà di Bruno representation formula for the Bell polynomials.*

Proof. We start from the representation of $\mathcal{I}_s^1(\mathcal{T})$, $s>1$, by means of the $\mathcal{I}_1^k(\mathcal{T})$, $\forall k=1,2,\ldots,s$.

In fact, for any integer $s\geq 1$, the orthogonal invariant $\mathcal{I}_s^1(\mathcal{T})$, is expressed in terms of the $\mathcal{I}_1^k(\mathcal{T})$, $\forall k=1,2,\ldots,s$ by the Faà di Bruno formula:

$$\mathcal{I}_s^1(\mathcal{T})=(-1)^s\sum_{\pi(s)}\frac{(-1)^k}{r_1!r_2!\cdots r_s!}\left[\frac{\mathcal{I}_1^1(\mathcal{T})}{1}\right]^{r_1}\left[\frac{\mathcal{I}_1^2(\mathcal{T})}{2}\right]^{r_2}\cdots\left[\frac{\mathcal{I}_1^s(\mathcal{T})}{s}\right]^{r_s}, \quad (6.5)$$

where $\pi(s)$ denotes the sum running on all partitions of $s=r_1+2r_2+\ldots+sr_s$ and $k=r_1+r_2+\ldots+r_s$.

This formula is equivalent to the second Robert formula (5.4). In fact in eq. (5.4) the indices $q_1,\ldots,q_k$ are not all distinct. Denoting by r_1 the number of times that 1 appears, by r_2 the number of times that 2 appears, ..., by r_s the number of times that s appears, then $s = r_1 + 2r_2 + \ldots + sr_s$, and the second summation symbol in eq. (5.4) is extended to all partitions of the integer s composed of k parts. Since the first summation symbol in eq. (5.4) runs from 1 to s, this means that the sum is extended to all partitions of the integer s, and taking into account that every term appears $\frac{k!}{r_1! r_2! \cdots r_s!}$ times, eq. (5.4) is transformed into eq. (6.5).

In particular, from the above eq. (6.5) we find:

$$\mathcal{I}_2^1(\mathcal{T}) = \tfrac{1}{2}((\mathcal{I}_1^1(\mathcal{T}))^2 - \mathcal{I}_1^2(\mathcal{T}))$$
$$\mathcal{I}_3^1(\mathcal{T}) = \tfrac{1}{6}((\mathcal{I}_1^1(\mathcal{T}))^3 - 3\mathcal{I}_1^1(\mathcal{T})\mathcal{I}_1^2(\mathcal{T}) + 2\mathcal{I}_1^3(\mathcal{T})).$$

Lastly, in a similar way, by using eq. (6.2) and the Faà di Bruno formula, $\mathcal{I}_1^k(\mathcal{T})$, $k > 1$ can be represented by means of $\mathcal{I}_s^1(\mathcal{T})$, $\forall s = 1, 2, \ldots, k$:

Proposition 6.3 *For any integer* $k \geq 1$, *the orthogonal invariant* $\mathcal{I}_1^k(\mathcal{T})$, *is expressed in terms of* $\mathcal{I}_h^1(\mathcal{T})$, $h = 1, 2, \ldots, s$ *by*

$$\mathcal{I}_1^k(\mathcal{T}) = \sum_{\pi(k)} (-1)^{k+s} \frac{k\,(s-1)!}{r_1! r_2! \cdots r_k!} (\mathcal{I}_1^1(\mathcal{T}))^{r_1} (\mathcal{I}_2^1(\mathcal{T}))^{r_2} \cdots (\mathcal{I}_k^1(\mathcal{T}))^{r_k}, \qquad (6.6)$$

where $\pi(k)$ *denotes the sum running on all partitions of* $k = r_1 + 2r_2 + \ldots + kr_k$ *and* $s = r_1 + r_2 + \ldots + r_k$.

Note that the last eq. (6.6) is not included in Robert's formulas.

In particular, we have:

$$\mathcal{I}_1^2(\mathcal{T}) = (\mathcal{I}_1^1(\mathcal{T}))^2 - 2\mathcal{I}_2^1(\mathcal{T})$$
$$\mathcal{I}_1^3(\mathcal{T}) = (\mathcal{I}_1^1(\mathcal{T}))^3 - 3\mathcal{I}_1^1(\mathcal{T})\mathcal{I}_2^1(\mathcal{T}) + 3\mathcal{I}_3^1(\mathcal{T}).$$

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