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Congruences for the Number of Rational Points, Hodge Type and Motivic Conjectures for Fano Varieties

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ABSTRACT. A Fano variety is a smooth, geometrically connected variety over a field, for which the dualizing sheaf is anti-ample. For example the projective space, more generally flag varieties are Fano varieties, as well as hypersurfaces of degree $d \leq n$ in $\mathbb{P}^n$. We discuss the existence and number of rational points over a finite field, the Hodge type over the complex numbers, and the motivic conjectures which are controlling those invariants. We present a geometric version of it.

1 Congruence for the Number of Rational Points for a Variety Over a Finite Field

Let X be a smooth projective variety over a field k. If $k = \mathbb{F}_q$ is finite, it will be rarely the case that X has a rational point. Yet, if the variety is very *negative* in the sense of differential geometry, then X will have some, or even many rational points. The simplest example is the projective space $\mathbb{P}^n$. Since one description of the k-rational points $|\mathbb{P}^n(k)|$ is the quotient of the punctured vector space $k^{n+1} \setminus \{0\}$ by the diagonal action of the homotheties $k^\times$, one sees that

$$|\mathbb{P}^n(\mathbb{F}_q)| = \frac{q^{n+1}-1}{q-1} = 1 + q + \ldots + q^n. \tag{1.1}$$

One way to measure $|X(\mathbb{F}_q)|$ for a smooth projective variety $X \subset \mathbb{P}^n$ is to consider the congruence

$$|X(\mathbb{F}_q)| \equiv |\mathbb{P}^n(\mathbb{F}_q)| \bmod q^\kappa, \tag{1.2}$$

or equivalently

$$|U(\mathbb{F}_q)| \equiv 0 \bmod q^\kappa, \tag{1.3}$$

where $U = \mathbb{P}^n \setminus X$, for some natural number κ. Of course if $\kappa = 0$, then (1.2) says nothing, but if there is a $\kappa \geq 1$ which satisfies (1.2), then first of all, X has a rational point, and secondly the larger κ, the more rational points. One codes $|U(\mathbb{F}_{q^s})|$ for all finite extensions $\mathbb{F}_q \subset \mathbb{F}_{q^s}$ in the zeta function defined by its logarithmic derivative

$$\frac{\zeta'(U,t)}{\zeta(U,t)} = \sum_{s \geq 1} |U(\mathbb{F}_{q^s})| t^{s-1}. \tag{1.4}$$

Thus the existence of a $\kappa \in \mathbb{N} \setminus \{0\}$ as in (1.3) for all finite extensions of $\mathbb{F}_q$ is equivalent to $\zeta(U,t)$, as a power expansion in t, be in $\mathbb{Z}[[q^\kappa t]]$. On the other hand, the fundamental theorem of Dwork [12] asserts that $\zeta(U,t)$ is a rational function over the rational numbers

$$\zeta(U,t) \in \mathbb{Q}(t). \tag{1.5}$$

One concludes that writing

$$\zeta(U,t) = \frac{\prod_{i=1}^a (1-\alpha_i t)}{\prod_{j=1}^b (1-\beta_j t)}, \tag{1.6}$$

the reciprocal roots α_i and poles β_j of the $\zeta(U,t)$ are divisible by q^κ as algebraic integers, i.e. in $\bar{\mathbb{Z}} \subset \bar{\mathbb{Q}}$. On the other hand, the Grothendieck-Lefschetz fixed point trace formula [19] asserts

$$\zeta(U,t) = \prod_{i=0}^{2\dim(U)} \det(1 - F_i t)^{(-1)^{i+1}} \tag{1.7}$$

where F_i is the arithmetic Frobenius acting on the compactly supported ℓ-adic cohomology $H^i_c(\bar{U}, \mathbb{Q}_\ell)$, which is isomorphic to the primitive cohomology $H^{i-1}_{\text{prim}}(\bar{X}, \mathbb{Q}_\ell) = H^{i-1}(\bar{X}, \mathbb{Q}_\ell)/$

$H^{i-1}(\bar{\mathbb{P}^n}, \mathbb{Q}_\ell)$ of X for $(i-1) \leq 2\dim(X)$, and $= H^i(\bar{\mathbb{P}^n}, \mathbb{Q}_\ell)$ for $i \geq 2\dim(X)$. By the Weil conjectures proven by Deligne [9], the eigenvalues of F_i in any complex embedding $\mathbb{Q}_\ell \subset \mathbb{C}$ have absolute values $q^{\frac{(i-1)}{2}}$. Thus in (1.7) there can't be any cancellation between odd and even i's. Our condition (1.3) translates then exactly into the condition that the eigenvalues of F_i be all divisible by q^κ as algebraic integers.

On the other hand, the cohomology $H^m(\bar{X}, \mathbb{Q}_\ell)$ carries a *coniveau filtration* $\ldots \subset N^a H^m(\bar{X}, \mathbb{Q}_\ell) \subset N^{a-1} H^m(\bar{X}, \mathbb{Q}_\ell) \subset \ldots$, where N^a is the subgroup of classes which die after restriction outside of a codimension a subscheme. One checks by a dévissage as in [17], Lemma 2.1, that the eigenvalues of the Frobenius acting on $N^a H^m(\bar{X}, \mathbb{Q}_\ell)$ are divisible by q^a as algebraic integers. Thus if $N^\kappa H^m_{\text{prim}}(\bar{X}, \mathbb{Q}_\ell) = H^m_{\text{prim}}(\bar{X}, \mathbb{Q}_\ell)$, the congruence (1.3) holds. The Tate conjecture predicts the converse: if the eigenvalues of Frobenius acting on $H^m_{\text{prim}}(\bar{X}, \mathbb{Q}_\ell)$ are all divisible as algebraic integers by q^κ, then the primitive cohomology should be supported in codimension κ, that is in concrete terms, there should exist a codimension κ subscheme $Z \subset X$ such that the restriction map

$$H^m_{\text{prim}}(\bar{X}, \mathbb{Q}_\ell) \to H^m(\overline{X \setminus Z}, \mathbb{Q}_\ell)/H^m(\bar{\mathbb{P}^n}, \mathbb{Q}_\ell) \tag{1.8}$$

dies.

2 Hodge Type

The *Hodge type* of a projective variety $X \subset \mathbb{P}^n$ over a field k of characteristic 0 is defined to be the largest natural number κ such that the Hodge filtration $\ldots \subset F^a H^m_{DR,\text{prim}}(X) \subset F^{a-1} H^m_{DR,\text{prim}}(X) \subset \ldots$ fulfills

$$F^\kappa H^m_{DR,\text{prim}}(X) = H^m_{DR,\text{prim}}(X) \tag{2.1}$$

for all m. (One would define similarly the Hodge type of $H^m_{DR,\text{prim}}(X)$ for a given degree m). If X is projective smooth, then Hodge type $= \kappa$ means

$$H^q(X, \Omega^p_X) = \begin{cases} 0 \text{ for } q \neq p < \kappa \\ k \text{ for } q = p < \kappa. \end{cases}$$

If

$$N^\kappa H^m_{DR,\text{prim}}(X) = H^m_{DR,\text{prim}}(X), \tag{2.2}$$

with the same definition of the coniveau filtration for de Rham cohomology as for ℓ-adic cohomology, then one easily computes via the Gysin sequence that the Hodge type of the primitive cohomology is $\geq \kappa$. The Hodge conjecture predicts the converse: if the Hodge type is κ, then the primitive cohomology is supported in codimension κ.

3 Hodge Cohomology and Slopes

In this section, X is still assumed to be a smooth projective variety over a finite field $\mathbb{F}_q$, where $q = p^d$ for a prime number p. We are interested in conditions which force the reciprocal zeros α_i and poles β_j of the zeta function (1.6) to be divisible by q^κ as algebraic integers.

Write $W(\mathbb{F}_q)$ for the ring of Witt vectors over $\mathbb{F}_q$. It is a complete discrete valuation ring with residue field $\mathbb{F}_q$. For example, $W(\mathbb{F}_p) = \mathbb{Z}_p$, the p-adic integers. Let K be the quotient field of $W(\mathbb{F}_q)$. Frobenius on $\mathbb{F}_q$ induces automorphisms σ of $W(\mathbb{F}_q)$ and K satisfying $\sigma^d = \text{identity}$. The crystalline cohomology [3] $H^*_{\text{crys}}(X/W(\mathbb{F}_q)) \otimes K$ is a finite dimensional K-vector space with an endomorphism f (Frobenius) satisfying

$$f(wx) = \sigma(w)f(x); \quad w \in K,\ x \in H^*. \tag{3.1}$$

In particular, f^d is K-linear, and the basic theorem [3] is that

$$\zeta(X/\mathbb{F}_q, T) = \det(1 - f^d T|H^*)^{-1}, \tag{3.2}$$

where of course the determinant of the right is taken in the graded sense, i.e. characteristic polynomials coming from H^{odd} appear in the numerator of ζ.

In order to estimate divisibility for the eigenvalues, we can calculate crystalline cohomology using the de Rham-Witt complex [20]. This is a complex of pro-sheaves for the Zariski topology on X

$$W_\bullet\Omega^* := \{W_\bullet\mathcal{O} \xrightarrow{d} W_\bullet\Omega^1 \xrightarrow{d} W_\bullet\Omega^1 \to \cdots \xrightarrow{d} W_\bullet\Omega^{\dim X}\}. \tag{3.3}$$

In other words, each $W_\bullet\Omega^i$ is a projective system of Zariski sheaves on X

$$\ldots \twoheadrightarrow W_n\Omega^i \twoheadrightarrow W_{n-1}\Omega^i \twoheadrightarrow \ldots \twoheadrightarrow W_1\Omega^i := \Omega^i_X \tag{3.4}$$

where Ω^i_X is the sheaf of Kähler i-forms on X. Each $W_n\Omega^i$ has a finite filtration with graded pieces coherent, so the cohomology groups $H^j(X, W_n\Omega^i)$ have finite length. For $i = 0$, $W_n\mathcal{O}$ is the sheaf of Witt vectors of length n over the structure sheaf $\mathcal{O}_X$.

It is not true in general that the $H^j(X, W_\bullet\Omega^i) := \varprojlim_n H^j(X, W_n\Omega^i)$ are finitely generated $W(\mathbb{F}_q)$-modules (even for $i = 0$, [31]). However, the groups $H^j(X, W_\bullet\Omega^i)/(\text{torsion})$ are finitely generated. In particular, the $H^j(X, W_\bullet\Omega^i) \otimes K$ are finite K-vector spaces. The differentials in (3.3) come from differentials $W_n\Omega^i \xrightarrow{d} W_n\Omega^{i+1}$, and we define

$$\mathbb{H}^*(X, W_\bullet\Omega^*) := \varprojlim_n \mathbb{H}^*(X, W_n\Omega^*). \tag{3.5}$$

The de Rham-Witt complex plays the role of a sort of de Rham complex calculating crystalline cohomology. Namely, there is a canonical, functorial isomorphism

$$H^*_{\text{crys}}(X/W(\mathbb{F}_q)) \cong \mathbb{H}^*(X, W_\bullet\Omega^*). \tag{3.6}$$

Crucial for our purposes is that the frobenius f has a nice description on the de Rham-Witt cohomology. Namely, one has endomorphisms

$$f^{(i)}, v^{(i)} : W_\bullet\Omega^i \to W_\bullet\Omega^i \tag{3.7}$$

which satisfy $f^{(i)}v^{(i)} = v^{(i)}f^{(i)} = p$. The endomorphism $v^{(i)}$ is topologically nilpotent in the sense of the inverse system (3.4). One has $d \circ f^{(i)} = pf^{(i+1)} \circ d$, so in particular the $p^i f^{(i)}$ on $W_\bullet\Omega^i$ induce a map of complexes on $W_\bullet\Omega^*$. The resulting map on $\mathbb{H}^*(X, W_\bullet\Omega^*)$ coincides with the Frobenius f on crystalline cohomology under the isomorphism (3.6).

As a consequence of these facts, one deduces that the spectral sequence

$$E_1^{ab} = H^b(X, W_\bullet\Omega^a) \otimes K \Rightarrow H^{a+b}(X, W_\bullet\Omega^*) \otimes K \tag{3.8}$$

degenerates at E_1 ([4]), and that the eigenvalues of the K-linear endomorphism $q^a f^{(a)d}$: $H^*(X, W_\bullet\Omega^a) \to H^*(X, W_\bullet\Omega^a)$ coincide with the eigenvalues α_i and β_j appearing in $\zeta(X/\mathbb{F}_q, T)$ which are divisible by q^a but not by q^{a+1}. (This is because $q^a v^{(a)d} f^{(a)d} = q^a p^d = q^{a+1}$, and $v^{(a)}$ is topologically nilpotent.)

Of course, there is a lot of mathematics here, and it is not possible to give the details in a survey such as this. Note, however, that there are two deep global results, (3.2) and (3.6). The rest involves the definition and local structure of $W_\bullet\Omega^*$.

As a corollary of the above, we deduce

Corollary 3.1 *Let $\kappa \geq 1$ be a given integer. Then all the reciprocal zeroes and poles α_i and β_j of $\zeta(X/\mathbb{F}_q, T)$ are divisible by q^κ if and only if $H^*(X, W_\bullet\Omega^a) \otimes K = (0)$ for $a < \kappa$.*

We would like a criterion in terms of the Hodge groups $H^b(X, \Omega^a_X)$ which will insure the de Rham-Witt groups vanish as in corollary 3.1. The following is deduced from a purely local calculation using the structure of the sheaves $W_n\Omega^i$. We use the notation "Hodge type" as in section 2, even though the ground field is finite.

Proposition 3.2 *With notation as above, if X has Hodge type $\kappa \geq 1$, then $H^*(X, W_\bullet\Omega^a) = (0)$ for $a < \kappa$. In particular, all the reciprocal zeroes and poles of $\zeta(X/\mathbb{F}_q, T)$ are divisible by q^κ.*

Proof. We will use some results about the structure of $W_\bullet\Omega^*$ from [21]. The first point is that $f^{(i)}, v^{(i)}, p$ are injective on pro-objects. Topological nilpotence for $v^{(i)}$ means

$$v^{(i)n} W\Omega^i \subset \ker(W_\bullet\Omega^i \to W_n\Omega^i) \tag{3.9}$$

[21], (2.2.1). Further, by op. cit. (2.5.2), there is an exact sequence

$$0 \to W_\bullet\Omega^{i-1}/f^{(i-1)} \xrightarrow{dv^{(i-1)}} W_\bullet\Omega^i/v^{(i)} \to \Omega^i_X \to 0. \tag{3.10}$$

(For $i = 0$ this says $W/v^{(0)} \cong \mathcal{O}_X$.) By induction on i we see that $H^\bullet(X, W_\bullet\Omega^i/v^{(i)}) = (0)$ for $i < k$, so

$$v^{(i)n} : H^\bullet(X, W\Omega^i) \cong H^\bullet(X, W\Omega^i) = \varprojlim_n H^\bullet(X, W_n\Omega^i) \subset \prod_n H^\bullet(X, W_n\Omega^i). \tag{3.11}$$

By (3.9), we deduce that $H^\bullet(X, W_\bullet\Omega^i) = (0)$ for $i < k$. □

Remarks 3.3 *i) Proposition 3.2 is a special case of a more general theorem asserting that the Newton polygon of the F-crystal* $H^m(X/W)/(\text{torsion})$ *lies above the Hodge polygon defined by slope i with multiplicity* $\dim H^{m-i}(X, \Omega^i_X)$ *(see [25], [27]). We have seen that only the local properties of* $W_\bullet\Omega^\bullet$ *are relevant for Proposition 3.2.*

ii) It is of course very easy to compute the Hodge cohomology $H^j(X, \Omega^i)$ *for smooth complete intersections* $X \subset \mathbb{P}^n$ *defined by r equations of degrees* $d_1 \geq d_2 \geq \ldots \geq d_r$. *It is* $\kappa = [\frac{n-d_2-\ldots-d_r}{d_1}]$. *Here* $[z]$ *is the integral part of the rational number z. In other words, Proposition 3.2 is an easy proof of the theorem of Ax and Katz ([23]) asserting* (1.2) *or equivalently* (1.3) *in this case.*

4 From $\mathbb{F}_q$ to $\mathbb{C}$ and Vice-versa

Let us think now that our smooth variety over a finite field is coming via reduction modulo p from a variety defined in characteritic 0, over a ring of finite type over the integers. Then, via the comparison between ℓ-adic and de Rham cohomologies, the coniveau in which (primitive) de Rham cohomology is carried is the same as the coniveau in which (primitive) ℓ-adic is carried. In conclusion, one sees that the coincidence of the κ stemming from the ζ function with the κ from the Hodge type is a test both for the Tate and the Hodge conjectures.

We have two tests at disposal. First *smooth complete intersections*. Let $X \subset \mathbb{P}^n$ be a smooth complete intersection defined by r equations of degree $d_1 \geq d_2 \ldots \geq d_r$. We define $\kappa = [\frac{n-d_2-\ldots-d_r}{d_1}]$ as in Remark 3.3. Then, as already mentioned, the theorem of Ax and Katz [23] asserts (1.3) while Deligne's theorem [8] asserts (2.1). Moreover, this bound is sharp both on the ζ and on the Hodge sides.

The smooth complete intersections just discussed with $\kappa \geq 1$ are special Fano varieties. We consider now our second example: *Fano varieties* which are abstractly defined. In characteristic 0, Kodaira vanishing applied to the ample invertible sheaf ω^{-1} yields $H^q(X, \mathcal{O}_X) = 0$

for all $q > 0$. Thus the Hodge theoritic κ is at least 1. Over a finite field, [17], Corollary 1.3 asserts that $|X(\mathbb{F}_q)| \equiv 1 \bmod q$. Thus the κ of the ζ function is at least 1 as well. This is our second test. However, we observe that the test is not complete. It might well be that the Hodge type of X is ≥ 2. Yet the proof given in [17] does not give a better the congruence for the number of rational points over a finite field, unless we know the Chow groups of X. This is the subject of the next section.

5 Motivic Conjectures

The Beilinson-Bloch conjectures ([5], [1], [2]) predict that the Chow groups of a smooth projective variety defined over the complex numbers should be controlled by its Hodge theory. More precisely, it predicts that if the Hodge type of the primitive cohomology of a smooth complex projective variety X of dimension d is κ, that is (2.1) holds true, then one has

$$CH_i(X) \otimes \mathbb{Q} = H^{2(d-i)}(X_{\mathrm{an}}, \mathbb{Q}) \text{ for } 0 \leq i \leq (\kappa - 1). \tag{5.1}$$

Applying the splitting of the diagonal as initiated in [5], Appendix to Lecture 1, and then refined as in [22], [29], [15], one sees that (5.1) is equivalent to saying that there is a nontrivial natural number N, there are dimension i and codimension i cycles α_i and $\beta^i \subset X$, a codimension κ cycle $Z \subset X$, a codimension d cycle $\Gamma \subset X \times Z$ with

$$N\Delta \equiv \alpha_0 \times X + \alpha_1 \times \beta^1 + \ldots + \alpha_{\kappa-1} \times \beta^{\kappa-1} + \Gamma \tag{5.2}$$

where $\equiv$ means the equivalence in $CH^d(X \times X)$, and Δ is the diagonal. It is easily seen that such a decomposition (5.2) of the diagonal, up to torsion, implies (2.2). Over the complex numbers, it implies a fortiori (2.1) while over a finite field, it implies (1.3), by showing again that the eigenvalues of the Frobenius are divisible by q^κ as algebraic numbers.

In other words, the Beilinson-Bloch conjectures on the Chow groups imply here the Hodge conjecture. Thus we expect that our Fano varieties with Hodge type κ will fulfill (5.1) and (1.3) as well.

Let us look at our two classes of examples. The smooth complete intersections as in section 3 with $\kappa \geq 1$ fulfill $CH_0(X) = \mathbb{Z}$ by a theorem of Roitman [30]. However, we do not know in general whether $CH_i(X) \otimes \mathbb{Q} = \mathbb{Q}$ for $i < \kappa$. We have very few simple examples where the bound is achieved ([32], [28]), and the general results we have yield bad bounds ([16]). We observe nevertheless that Roitman's theorem (loc. cit.) yields a whole class of examples for which the Beilinson-Bloch conjecture is true and sharp. Let $Y \subset \mathbb{P}^{\kappa-1} \times \mathbb{P}^n$

be a smooth complete intersection of bidegree $(1,d)$. Then one easily computes that

$$H^q(Y,\Omega^p) = 0 \text{ for } q \neq p, p \leq (\kappa - 1) \tag{5.3}$$

$$\text{and } H^q(Y,\Omega^p) \neq 0 \text{ for some } p \geq \kappa, p \neq q$$

$$\text{if and only if } \kappa d \leq n.$$

Let us write $y_1 f_1 + \ldots + y_\kappa f_\kappa$ for the equation of Y, where y_i are the homogeneous coordinates in $\mathbb{P}^{\kappa-1}$ and f_i are homogeneous polynomials of degree d. Then Y is smooth if and only if the codimension κ subvariety $X \subset \mathbb{P}^n$ defined by $f_1 = \ldots = f_\kappa = 0$ is smooth. Moreover, Y is the blow-up of X in $\mathbb{P}^n$. Consequently, one has $CH_i(Y) = CH_{i-\kappa+1}(X) \oplus CH^{n+\kappa-2-i}(\mathbb{P}^n)$. Thus $CH_i(Y) \otimes \mathbb{Q} = \mathbb{Q}$ for $i \leq (\kappa - 2)$ and $CH_0(X) = \mathbb{Z}$ implies $CH_{\kappa-1}(Y) \otimes \mathbb{Q} = \mathbb{Q} \oplus \mathbb{Q}$.

Roitman's theorem is a special case of a deeper theorem for abstractly defined Fano varieties, due to Campana, and Kollár-Miyaoka-Mori ([24]). It is stemming from geometry. If X is a Fano variety, then it is rationally connected, that is any two closed points are linked by a chain of rational curves. This implies $CH_0(X) = \mathbb{Z}$, but is stronger than this. For example, in characteristic 0, surfaces with $H^m(X,\mathcal{O}_X) = 0, m = 1,2$, but with nonnegative Kodaira dimension ≤ 1 have $CH_0(X) = \mathbb{Z}$ ([6]), and certainly they are not rationally connected. As recalled in the abstract, Fano means that $\omega_X^\vee$ is ample. Thus this strong negativity condition on the top differential forms implies rational connectivity. On the other hand, strong negativity on the 1-forms implies rationality: by the fundamental theorem of Mori ([26]), $(\Omega_X^1)^\vee$ ample is equivalent to X being isomorphic to the projective space. Thus in this case $CH_0(X) = \ldots = CH_d(X) = \mathbb{Z}$ and $\kappa = (d+1)$.

In consequence, it is tempting to think that the condition (5.1) might result from a strong negativity condition on $d, (d-1), \ldots, (d-\kappa+1)$ differential forms. We have at disposal Demailly's positivity notion [11]. A vector bundle E on a smooth projective complex variety is s-positive if its hermitian curvature form, seen on $T_X \otimes E$, is positive on all tensors of length $\leq s$. Demailly's vanishing theorem says that if E is s positive, then $H^q(X, \omega_X \otimes E) = 0$ for $q \geq d - s + 1$. In particular, let us assume that

$$(\Omega^{n-p})^\vee \text{ is } (n-p) - \text{positive for } 0 \leq \mathrm{p} \leq (\kappa - 1). \tag{5.4}$$

Then $H^q(X, \Omega_X^p) = 0$ for $q > p, 0 \leq p \leq (\kappa - 1)$. But by Hodge duality, this implies $H^q(X, \Omega_X^p) = 0$ for $q < p, 0 \leq p \leq (\kappa - 1)$ as well. Thus under the assumption (5.4), the Beilinson-Bloch conjectures predict (5.1). One may hope that this geometric formulation yields more information, as we discussed for $p = 0$ and $p = (d-1)$ above. It is further to be remarked that, while applied to smooth complete intersections, Demailly's positivity is stronger than what would be needed to prove exactly (5.1), which is coming from the Hodge type. It is then likely that a positivity notion, a bit weaker than Demailly's one, will force Demailly's vanishing, and would be such that while applied to smooth complete intersections, it would yield, via the Beilinson-Bloch conjectures, exactly the right predicted statement on Chow groups.

6 Singular Projective Varieties

In this section, $X \subset \mathbb{P}^n$ is still projective, but no longer necessarily smooth.

Let us assume first that X is defined over a finite field $\mathbb{F}_q$. We still have that the divisibility of the eigenvalues of Frobenius acting on $H^i_c(\bar{U}, \mathbb{Q}_\ell)$ implies the divisibility of the reciprocal roots and poles of the ζ function via (1.6) and (1.7). But the converse is not a priori clear. The problem is that the absolute values of the eigenvalues of F_i are no longer determined by i, there might be some cancellation in (1.7).

Next we think of $H^i_c(\bar{U}, \mathbb{Q}_\ell)$ as being no longer a semisimple Galois representation, and we consider its associated graded semisimple Galois representation. We apply the mechanism explained in the smooth case to predict that the Hodge type of X in characteric 0 should be the same as the κ such that q^κ divides the eigenvalues of the Frobenius acting on $H^i_c(\bar{U}, \mathbb{Q}_\ell)$. Now we have only one class of examples: projective varieties X defined by r equations of degrees $d_1 \geq d_2 \geq \ldots \geq d_r$. We don't require smoothness, not even that the dimension of X be $(n-r)$. If $\frac{n-d_2-\ldots-d_r}{d_1} \geq 0$ we define κ as in (2). Then the theorem of Ax and Katz ([23]) asserts that (1.3) is true, while [10], [13], [14] show that the Hodge type of X is κ. Those bounds are sharp. In particular, we see that the coincidence of the κ coming from the ζ function and the one coming from Hodge theory in this mixed case predicts that the eigenvalues of Frobenius will be divisible by q^κ, which we don't know so far.

In the same range of ideas, if we require now that X be a (nonsmooth) complete intersection, by [33] one has higher divisibility of the reciprocal poles of ζ (suitably normalized), while by [18] one has a better Hodge type for all the primitive cohomology beyond the middle dimension. Here one would also expect that the better divisibility is not only for the reciprocal poles of the ζ function, but also for the eigenvalues of Frobenius acting on the corresponding cohomology $H^i_c(\bar{U}, \mathbb{Q}_\ell)$ for all i modulo 2 corresponding to the poles. But in addition, in light of the Hodge type computation, one would expect the better divisibility beyond the middle dimension.

The Beilinson-Bloch conjectures are not formulated for projective nonsmooth varieties. It is tempting to think that motivic cohomology in some good sense will control both the Hodge type and the congruence (in particular the existence) of rational points over a finite field. However, in absence of a clear view of what would correspond to the easy implication in the smooth case (trivial Chow groups implies congruence for points and nontrivial

Hodge type), it is hard to forsee a good formulation of what would be the Beilinson-Bloch conjectures in the projective singular case.

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References

[1] BEILINSON, A., *Higher regulators and values of L-functions*, J. Soviet Math. **30** (1985), 2036-2070, transl. from Sovr. Probl. mat. **24** (1984), Moscow VINITI, 181–238.

[2] BEILINSON, A., *Notes on absolute Hodge cohomology.* in *Applications of Algebraic K-Theory to Algebraic Geometry and Number Theory*, Contemporary Mathematis **55** (1986), part 1, Providence AMS, 35–68.

[3] BERTHELOT, P., *Cohomologie cristalline des schémas de carastéristique $p > 0$*, Lect. Notes in Math. **407** (1974), Springer Verlag.

[4] BLOCH, S., *Algebraic K-theory and crystalline cohomology*, Publ. Math. IHES **47** (1978), 187–268.

[5] BLOCH, S., *Lecture on Algebraic cycles*, Duke University Mathematics Series, IV, 1980.

[6] BLOCH, S., KAS, A. AND LIEBERMAN, D., *Zero cycles on surfaces with $p_g = 0$.*, Compositio Math. **33** (1976), no. 2, 135–145.

[7] DELIGNE, P., *Théorie de Hodge II*, Publ. Math. IHES **40** (1972), 5–57.

[8] DELIGNE, P., *Cohomologie des intersections complètes*, in SGA 7 XI, Lect. Notes Math. vol. **340**, 39–61, Berlin Heidelberg New York Springer, 1973.

[9] DELIGNE, P., *La conjecture de Weil*, I. Publ. Math. IHES **43** (1974), 273–307.

[10] DELIGNE, P. AND DIMCA, A., *Filtrations de Hodge et par l'ordre du pôle pour les hypersurfaces singulières*, Ann. Sci. Éc. Norm. Supér. (4) **23** (1990), 645–656.

[11] DEMAILLY, J.-P., *Estimations L^2 pour l'opérateur $\bar\partial$ d'un fibré vectoriel holomorphe semi-positif au-dessous d'une variété kählérienne complète*, Ann. Sci. Éc. Norm. Sup. **15** 4-ième série (1982), 457–511.

[12] Dwork, B., *On the rationality of the zeta function of an algebraic variety*, Amer. J. Math. **82** (1960), 631–648.

[13] Esnault, H., *Hodge type of subvarieties of* $\mathbb{P}^n$ *of small degrees*, Math. Ann. **288** (1990), no. 3, 549–551.

[14] Esnault, H., Nori, M. and Srinivas, V., *Hodge type of projective varieties of low degree*, Math. Ann. **293** (1992), no. 1, 1–6.

[15] Esnault, H. and Levine, M., *Surjectivity of cycle maps*, in Journées de géométrie algébrique d'Orsay, Juillet 1992, Astérisque **218** (1993), 203–226.

[16] Esnault, H., Levine, M. and Viehweg, E., *Chow groups of projective varieties of very small degree*, Duke Math. J. **87** (1997), no.1, 29–58.

[17] Esnault, H., *Varieties over a finite field with trivial Chow group of 0-cycles have a rational point*, Invent. math. **151** (2003), 187–191.

[18] Esnault, H. and Wan, D., *Hodge type of the exotic cohomology of complete intersections*, preprint (2002), 7 pages, appears in the C. R. Aca. Sc. Paris.

[19] Grothendieck, A., *Formule de Lefschetz et rationalité des fonctions* L, Séminaire Bourbaki **279**, 17-ième année (1964/1965), 1–15.

[20] Illusie, L., *Complexe de de Rham-Witt et cohomologie cristalline*, Ann. Sci. École Norm. Sup. (4) **12** (1979), 501–661.

[21] Illusie, L., *Complexe de De Rham-Witt*, Astérisque **63** (1979), 83–112.

[22] Jannsen, U., *Motivic sheaves and filtrations on Chow groups, in Motives*, Proceedings of Symposia in Pure Mathematics, AMS, **55** (1994), part I, 245–302.

[23] Katz, N., *On a theorem of Ax*, Amer. J. Math. **93** (1971), 485–499.

[24] Kollár, J., *Rational curves on algebraic varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge, **32** (1996), Springer-Verlag, Berlin, 1996.

[25] Mazur, B., *Frobenius and the Hodge filtration, estimates*, Annals of Math. **98** (1973), 58–95.

[26] Mori, S., *Projective manifolds with ample tangent bundles*, Ann. of Math. (2) **110** (1979), no. 3, 593–606.

[27] Ogus, A., *Frobenius and the Hodge filtration* in *Notes on crystalline cohomology*, by Berthelot, P. and Ogus, A., Mathematical Notes **21** (1978), Princeton Press.

[28] OTWINOWSKA, A., *Remarques sur les groupes de Chow des hypersurfaces de petit degré*, C. R. Acad. Sci. Paris Sér. I Math. **329** (1999), no. 1, 51–56.

[29] PARANJAPE, K., *Cohomological and cycle-theoritic connectivity*, Annals of Maths. (2) **139** (1994), no.3, 641–660.

[30] ROITMAN, A., *Rational equivalence of zero-dimensional cycles*, Mat. Zametki **28** (1980), no. 1, 85–90.

[31] SERRE, J.-P., *Sur la topologie des variétés algébriques en caractéristique p*, Symposium Internacional de Topologia Algebraica, Mexico (1958), 24–53.

[32] VOISIN, C., *Sur les groupes de Chow de certaines hypersurfaces*, C.R. Acad. sci. Paris Sér. I Math. **322** (1996),no.1, 73–76.

[33] WAN, D., *Poles of zeta functions of complete intersections*, Chinese Ann. Math., **21B** (2000), no. 2, 187–200.