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A New Application for Room Squares: Tournaments with Internal Referees

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ABSTRACT: The purpose of this note is to present a new "real world" application of Room squares. Almost no theory is presented and some will note that our treatment is rather naive. In particular not even the idea of starters, which would make the presentation more elegant, is explicitly defined. We choose this way to hold the attention and arouse the curiosity of those who never heard of this combinatorial structure. An interesting theory, however, arises from the problem in this article. This will be developed elsewhere.

1 Introduction

Suppose we have a round-robin tournament among $2n+1$ individual competitors. (Round-robin means that any two competitors play precisely once). Each game involves the two competitors and an extra person, which is also a competitor, to be the referee and/or to perform bookkeeping functions relative to the game. The tournament is divided into $2n+1$ *rounds* of n games each. Each round is subdivided into a number of *phases*. The games of a phase are to be played at the same time. The restriction that the referee is also a competitor implies that the number of phases is at least two for each round, because a person cannot play while refereeing. The games of a phase of a round are played simultaneously in a fixed number of different "boards". The conditions of a board are supposed to be slightly different. Thus, we have the first requirement:

(rq1) Each competitor plays the same number of times in each board.

The condition about an internal referee, which motivated the present scheduling problem, is a natural one for amateur tournaments. In these tournaments it is not desirable to pay for external referees. Being a referee is considered a boring task (the price to pay for participation), hence we have the next two requirements:

(rq2) Each competitor is referee at most once for round. If he is referee in a round he plays in another phase of that round.

(rq3) Each competitor is the referee of precisely n games.

The last requirement (which we imposed after solving the problem) for certain n's is a particularly pleasing one:

(rq4) Given any ordered pair of competitors (p, q), there exists precisely one game in which p is the referee of a game involving q.

2 Room Squares: A way to Schedule the Games

A *Room square* of order $2n$, is an arrangement of the $n(2n-1)$ distinct pairs of $2n$ objects in a square array of side $2n-1$, such that each cell of the square is either empty or contains one pair and each of the $2n$ objects appears exactly once in each row and column.

The symbol set for the objects is $\{1, 2, \ldots, 2n-1, \infty\}$, and we permute rows and columns of the square array such as to position the pair $\{\infty, i\}$ in cell (i, i). A Room square is called *cyclic* if $\{p, q\}$ in cell (i, j) implies $\{p+1, q+1\}$ in cell $(i+1, j+1)$ where addition is mod $2n-1$ and $\infty + n = \infty$ for arbitrary integer n.

We show that the existence of a cyclic Room square of order $2n+2$ implies a solution for the schedule problem which satisfies (rq1)$\cdots$(rq4). Initially we give an example of a solution for the specific case that originated the question in 1975. We were playing a special kind of game called "futebol de mesa" or loosely translating "table soccer". For this game a referee and bookeeper is highly desirable. Also the boards are distinguishable since they offer different conditions of illumination smoothness, etc. Hence, the need for (rq1), (rq2), (rq3). Condition (rq4) was highly welcome after the solution using Room squares was obtained. There were 13 people and 3 boards available. In Table 1, below we present the scheduling:

ref A				K D 1 γ			F I 2 γ	L C 2 β		M B 1 β	G H 2 α	E J 1 α
F K 1 α	ref B				L E 1 γ			G J 2 γ	M D 2 β		A C 1 β	H I 2 α
I J 2 α	G L 1 α	ref C				M F 1 γ			H K 2 γ	A E 2 β		B D 1 β
C E 1 β	J K 2 α	H M 1 α	ref D				A G 1 γ			I L 2 γ	B F 2 β	
	D F 1 β	K L 2 α	I A 1 α	ref E				B H 1 γ			J M 2 γ	C G 2 β
D H 2 β		E G 1 β	L M 2 α	J B 1 α	ref F				C I 1 γ			K A 2 γ
L B 2 γ	E I 2 β		F H 1 β	M A 2 α	K C 1 α	ref G				D J 1 γ		
	M C 2 γ	F J 2 β		G I 1 β	A B 2 α	L D 1 α	ref H				E K 1 γ	
		A D 2 γ	G K 2 β		H J 1 β	B C α α	M E 1 α	ref I				F L 1 γ
G M 1 γ			B E 2 γ	H L 2 β		I K 1β	C D 2 α	A F 1 α	ref J			
	H A 1 γ			C F 2 γ	I M 2 β		J L 1 β	D E 2 α	B G 1 α	ref K		
		I B 1 γ			D G 2 γ	J A 2 β		K M 1 β	E F 2 α	C H 1 α	ref L	
			J C 1 γ			E H 2 γ	K B 2 β		L A 1 β	F G 2 α	D I 1 α	ref M

off the main diagonal, each cell of the above array is either empty or contains four entries: two capital latin letters in $\{A, B, \ldots, M\}$, a number 1 or 2, and a greek letter α, β, or γ. The latin letters mean the competitors; the numbers represent the phase of a round, which is represented by the columns; the greek letters represent the boards. Observe that restricting ourselves to the latin letters and replacing the entries "ref" by "∞" we obtain a cyclic Room square of side 13. This Room square was obtained from [SM]. The referees are represented by the rows. As two instances of complete assignments we have: J plays A in the 2nd. phase of round 7, the referee is L, and the board is β. C plays I in the 1st. phase of round 10, the referee is F, and the board is γ.

Note that the set of non-empty entries of a cyclic Room square can be partitioned into $n+1$ (cyclic) diagonals one of which is the main diagonal and plays no important role here other than indexing. Each other diagonal is labeled by a pair consisting of a phase and a board. In the case of Table 1 we have diagonals: 1α, 1β, 1γ, 2α, 2β, 2γ. Observe that a board is never iddle. So, in general for cyclic Room squares, we want to have a pair (phase, board)$=(p, b)$ associated with one diagonal. Since each diagonal have each element appearing twice (one as first element, one as second) requirement (rq1) is automatically satisfied in an even stronger form: each competitor plays twice in a given board and phase.

In some tournaments it might be useful to consider two "sides" for the boards; for instance, white and black in chess. In this case for each pair (p, b) each competitor plays once in each side.

Condition (rq2) is clearly met by the solutions given by Room squares, because the person which does not play in a round does not have to be a referee either in that round. Condition (rq3) is trivial from the properties of Room squares, because there are n off diagonal non-empty cells in a given row. Condition (rq4) is also clear, because each element other than i appears once in row i of a Room square.

3 Schedule for Internally – Referee Tournaments

In this section we define an *irt*-schedule (which is merely a new name of the class of structures that give a solution for the original scheduling problem) and show that no more than three phases by round are sufficient to schedule a round. An ***irt-schedule*** is a cyclic Room square with an ordered pair of labels from disjoint sets P and B attached to each off main diagonal entry satisfying the following conditions:

(irt1) Two distinct entries not in the main diagonal have the same pairs of labels if and if they are in the same (cyclic) diagonal.

(irt2) If XY is an entry which has label (p, b) and is in row Z, then the entry UZ(or ZU) in the same column as XY has first component of its label distinct from p.

Condition (irt2) is necessary to impose, otherwise Z would be a referee in phase p, but also supposed to play at the same time, which is not permitted.

Condition (irt1) is natural to impose because to determine how many phases and boards we need is sufficient to examine one column (round).

We show that given a cyclic Room square we can attach labels from sets P, B with $|P| = 2$ or 3 to have an *irt*-schedule. To this end we define the ***position digraph*** G of a cyclic Room square R. Consider any column c of R. The vertices of G are elements of the form $(\{x, y\}, r)$ where xy is an entry in row r, column c, with $c \neq r$. We have dart from $(\{x_1, y_1\}, c_1)$ to $(\{x_2, y_2\}, c_2)$ if $c_1 \in \{x_2, y_2\}$. This completes the definition of G. Observe that since R is cyclic G is abstractly the same digraph independently of the column used. Note also that the minimum number of phases required is precisely the minimum number of colors in which we can paint the vertices of G such that adjacent vertices have different colors (chromatic number, $\chi(G)$, of G): just let the vertices of the same color be games to be played in the same phase.

Our problem is then to establish that the chromatic number of a position digraph of a cyclic Room square is two or three. Actually the direction of the edges in G is irrelevant for the definition of chromatic number, but as we see, fundamental for the proof. The only property of G that we use is the following: the number of darts going away from (outvalency of) each vertex is one. This property follows from the fact that each element appears once in a column of a Room square.

The fact that the outvalency of each vertex of G is equal to one implies: (a) Each component of G contains a directed circuits. (b) No component of G contain two circuits.

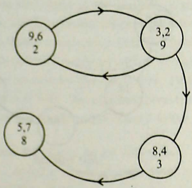
From (a) and (b) we obtain that each component of G is tree plus one chord. If for every component the number of edges in its only circuit is even, then $\chi(G) = 2$, otherwise $\chi(G) = 3$. This concludes the proof.

To get the boards assigned is convenient that the sets of vertices of G with the same color be approximately of the same cardinality. This is so to avoid idle boards. If they are all of the same cardinality, as in the case of Table 1, then there are never idle boards, that is, $n = |P| \times |B|$. The more interesting case is when $|P| = 2$. If that is so, and moreover there exists a bicoloring of G such that the two classes in the bipartition have the same cardinality, then we call the *irt*-schedule a *perfect schedule*. Of course, n has to be even in this case.

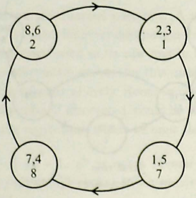
4 Some Small Perfect Schedules

The cyclic Room squares of [SM] produce perfect schedules for 9, 13, 17,21. The next case, Room square of side 25, fails because the position digraphs based in a column and in a row of this square are non-bipartite. We can obtain a solution for 25 based on a "strong starter" for $\mathbb{Z}_5 \times \mathbb{Z}_5$ obtained from $GF(25)$, see [MN]. This schedule does not have a cyclid structure and is not considered here. Below we present the first row and last column of Room squares which produce perfect schedules for 9,13,17,21 participants. The triples are the two entries and their column position (row position) in the first row (last column)

Side 9



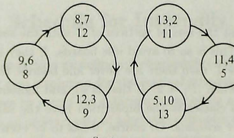
first row



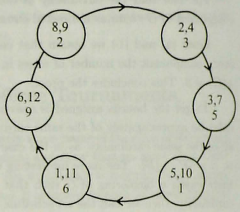
last column

Observe that the position digraphs based in the row or in the column give a perfect schedule.

Side 13



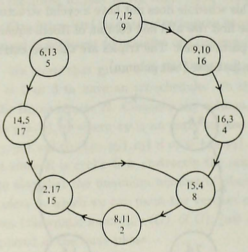
first row



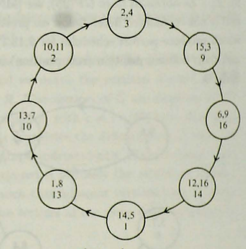
last column

The above Room square is the same given in Table 1. Only the position digraph based in the column gives a perfect schedule.

Side 17



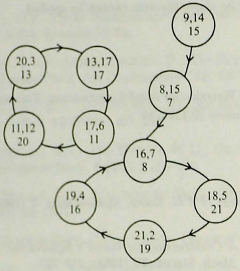
first row



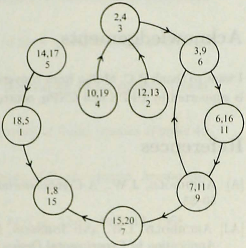
last column

The same comment relative to side 13 is applicable here.

Side 21



first row



last column

In this case only the position digraph based in the row produces a perfect schedule. Therefore our convention regarding rounds and referees (columns and rows) must be interchanged.

5 Conclusion

In view of the known freedom to produce Room squares of a given order we conjecture that there are perfect schedules for every integer of the form $4k+1$, $k>1$.

Note that some of the position digraphs above are directed polygons. A more strong question which arises naturally is the following: Are there for every odd integer $2n+1$, $n>2$ a cyclic Room square such that the position digraphs based in its row and in its columns are both directed polygons with n vertices? An affirmative answer for this question is also a positive answer for the conjecture. Perhaps this subclass of cyclic Room squares might be easier to deal with. An algebraic structure for the class, if discovered, could be very elegant. A first step towards such a theory is the following result from which we omit the proof.

Theorem 5.1 *Let p be an odd prime. If $p^n = 4k+1$, p^n is not a Fermat prime and $p^{n-1}+1$ is not a power of 2 the there exists a (not always cyclic) Room square of size p^n, whose position digraph is a collection of even polygons.*

A proof of this theorem involves an strengthening of the main result of [MN], and will appear elsewhere. The smallest prime power for which we do not know at present a perfect schedule is 49. Since $7^{2-1}+1$ is a power of 2, the above theorem cannot be applied.

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References

[A] ARCHBOLD, J.W., *A Combinatorial Problem of T.G. Room*, Mathematika, **7** (1960), 50–55.

[AJ] ARCHBOLD, J.W. AND JOHNSON, N.L., *A Construction for Room''s Squares and an Application in Experimental Design*, Ann. Math. Statist. **29** (1958), 219–225.

[B_r] BRUCK, R.H., *What is a Loop?*, (in Studies in Modern Algebra, ed. A.A. Albert), Prentice-hall, Englewood Cliffs, 1963.

[B_e] BEAMAN, I.R. AND WALLIS, W.D., *A Skew Room Square of Side Nine*, Utilitas Mathematica, **8** (1975), p.382.

[CC] CHONG, B.C. AND CHAN, K.M., *On the existence of normalized Room squares*, Nanta Math., **7** (1974), 8–17.

[DM] DILLON, J.F. AND MORRIS, R.A., *A skew Room square of side 257*. Utilitas Mathematica, **4** (1973), 187–192.

[G_1] GROSS, K.B., *Some new classes of strong starters*, Discrete Math. **12** (1975), 225–243.

[G_2] GROSS, K.B., MULLIN, R.C. AND WALLIS, W.D., *The number of pairwise orthogonal symmetric latin squares*, Utilitas Mathematica, **4** (1973), 239–251.

[G_3] GROSS, K.B., *Equivalent of Room Designs. I*, Journal of Combinatorial Theory, **16** N.02, (1974), 264–265.

[G_4] GROSS, K.B., *Equivalence of Room Designs. II*. Journal of Combinatorial Theory, **17** N.03, (1974), 299–316.

[H_1] HORTON, J.D., *Quintuplication of Room squares*, Aequationes Math., **7** (1971), 243–245.

[H_2] HORTON, J.D., MULLIN, R.C. AND STANTON, R.C., *A recursive construction for Room designs*, Aequationes Math., **6** (1971), 39–45.

[H_3] HORTON, J.D., *Room designs and one-factorization*, Aequationes Math., **22** (1981), 56–83.

[Hs] HSU, D.F., *Cyclic Neofields and Combinatorial designs*, Lectures Notes in Mathematics, N.824, Springer-Verlag.

[LS] LINS, S. AND SCHELLEMBERG, P., *The Existence of Skew Strong Starters in $\mathbb{Z}_{16k^2+1}$: A Simpler Proof*, Ars Combinatoria, **11**, 1981.

[MN] MULLIN, R.C. AND NEMETH, E., *An existence theorem for Room squares*, Canad. Math. Bull. **12** (1969), 493–497.

[MW1] MULLIN, R.C. AND WALLIS, W.D., *On the existence of Room squares of order 4n*, Aequationes Math. **6** (1971), 306–309.

[MW2] MULLIN, R.C. AND WALLIS, W.D., *The existence of Room squares*, Aequationes Math. **13**, 1975.

[O] O'SHAUGHNESSY, C.D., *A Room Design of Order 14*, Can. Math. Bull. **11**, 1968.

[R] ROOM, T.G., *A New Type of Magic Square*, Math. Gazette, **39** (1955), p. 307.

[SM] STANTON, R.G. AND MULLIN, R.C., *Construction of Room squares*, Ann. Math. Stats. **39** (1968), 1540–1548.

[SH] STANTON, R.G. AND HORTON, J.D., *Composition of Room squares*, Preprint Univ. Toronto, 1969.

[We] WEISNER, L., *A Room Design of Order 10*, Can. Math. Bull. **7**, 1964.

[W_1] WALLIS, W.D., *Duplication of Room Squares*, J. Austral. Math. Soc., **14** (1972), 75–81.

[W_2] WALLIS, W.D., *A Doubling Construction for Room Squares*, Discrete Mathematics, **3** (1972), 397–399.

[WSW] WALLIS, W.D., STREET, A.P. AND WALLIS, J.S., *Combinatorics: Room squares, sum-free sets, Hadamard matrices*, Lecture Notes in Mathematics, **292** (1972), Springer-Verlar, Berlin.

[W_3] WALLIS, W.D., *Solution of the Room Square Existence Problem*, Journal of Combinatorial Theory, **17** N.03, (1974), 379–383.