Cubo A Mathematical Journal
Vol.05/Nº03 - OCTOBER 2003

Substitutions of the Independent Variable in Linear Differential Equations

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1 Introduction

It is common in differential equations courses to obtain explicit solutions to a second-order or higher order linear differential equation only if the equation has constant coefficients or has fairly simple polynomial coefficients. We shall explore how the techniques of solutions of such problems can be extended to solve explicitly certain equations with fairly complicated coefficient functions. In particular, we characterize those second- and third-order linear equations which can be converted, via a 'nonsingular' transformation of the independent variable, into (1) an equation with constant coefficients, (2) an equation with polynomial coefficients, or (3) an equation with certain specified coefficients. Moreover, our results will show how to determine the necessary variable transformation. If an equation is not in the proper form for conversion to one of these forms, then an 'integrating factor' may be used to convert the given equation into the proper form, then the equation may be transformed via an appropriate variable substitution. We show how to find such integrating factors to convert an equation into one with constant coefficients. Alas, integrating factors for the other two

key words and phrases. linear differential equation, Euler equations, independent variable.

cases are not easy to come by, and so in these cases, integrating factors must be found on a case-by-case basis. Such results can then be used to recognize when a variable transformation will convert one problem into another. Once a problem is transformed into another, various conclusions may be drawn. For example, in some cases, the original equation may be solved explicitly, while in other cases, information about asymptotic behavior for one equation at infinity or near a singular point may be carried over to another equation, providing another tool for analysis of equations. Some results along these lines for second-order equations are presented in exercises in [1], but the general theory of independent variable transformations is not presented. Variable transformations are used extensively in [2], but, again, a general theory of variable transformations is not presented. Finally, much theory of Differential Equations is presented in [3], but we present our results in a form accessible to advanced undergraduates. These tools can be developed so that the appropriate transformations are obtained with a method similar in difficulty to finding integrating factors for first order linear equations, and so the mystery is removed from the process of finding a suitable variable transformation. Finally, the work of Ritt and Kaplansky implies results related to those presented here, in a more abstract algebraic form. We generate closed-form soluctions to equations in a class of equations that is not often explored in this manner. Our solutions can be used in the description of various physical phenomena, as we will discuss in the future.

2 Transforming Into a Constant Coefficient Equation

In this section we give necessary and sufficient conditions for an equation of the form

$$P(t)y'' + Q(t)y' + R(t)y = G(t) \tag{1}$$

to be transformable to a constant-coefficient equation via a 'nonsingular' transformation of the independent variable. Then we treat third order equations and indicate the general approach for higher order equations. Similar approaches can be taken for systems of equations, but for first-order systems, the results are not as promising. Let us begin by noting that one classical example of a variable transformation of the type we consider is the case of Euler equations: An equation of the form

$$t^2y'' + \alpha t y' + \beta y = g(t) \tag{2}$$

can be converted into a constant-coefficient equation by the transformation $t = e^x$, or equivalently, $x = \ln(t)$: In this case, (2) is transformed into the equation

$$\frac{d^2y}{dx^2} + (\alpha - 1)\frac{dy}{dx} + \beta y = g(e^x). \tag{3}$$

The solution of (3) is obtained by the usual methods for equations with constant coefficients, and the solution is tranformed back into a solution of (2) by replacing x with $\ln(t)$. We shall generalize this process, and show how to tell when an equation can be transformed into a constant coefficient equation. The fundamental theorem is the following.

Theorem 2.1 The equation (1) can be transformed into a constant coefficient equation iff it is equivalent to an equation of the form

$$a\phi' y'' + (b(\phi')^2 - a\phi'')y' + c(\phi')^3 y = g, \tag{4}$$

where ϕ is a twice-differentiable function with $\phi' \neq 0$, and where a, b and c are constants. Moreover, in this case, the variable transformation $x = \phi(t)$ converts equation (4) (or (1)) into the constant coefficient equation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = g(\phi^{-1}(x)). \tag{5}$$

Proof The transformation $x = \phi(t)$ yields the following:

$$\frac{dy}{dx} = \frac{dy}{dx}\frac{dx}{dt} = \frac{dy}{dx}\phi'$$

and

$$\begin{aligned}
\frac{d^2y}{dt^2} &= \frac{d}{dt}\frac{dy}{dx}\phi' + \frac{dy}{dx}\phi'' \\
&= \frac{d}{dx}\frac{dy}{dt}\phi' + \frac{dy}{dx}\phi'' \\
&= \frac{d}{dx}\frac{dy}{dx}\phi'\phi' + \frac{dy}{dx}\phi'' \\
&= \frac{d^2y}{dx^2}(\phi')^2 + \frac{dy}{dx}\phi''
\end{aligned}$$

Together, these results produce

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{dt}\frac{1}{\phi'} \\
\frac{d^2y}{dx^2} &= \frac{\frac{d^2y}{dt^2} - \frac{dy}{dx}\phi''}{(\phi')^2} \\
&= \frac{\frac{d^2y}{dt^2} - \frac{dy}{dt}\frac{\phi''}{\phi'}}{(\phi')^2}
\end{aligned}$$

so that solutions of (1) must satisfy

$$a\left[\frac{\frac{d^2y}{dt^2} - \frac{dy}{dx}\phi''}{(\phi')^2}\right] + b\left[\frac{dy}{dt}\frac{1}{\phi'}\right] + cy = g$$

which becomes (on multiplying by $(\phi')^3$)

$$a\left[\phi'\frac{d^2y}{dt^2} - \frac{dy}{dx}\phi''\right] + b\left[\frac{dy}{dt}(\phi')^2\right] + c(\phi')^3 y = g$$

and gives

$$a\phi'\frac{d^2y}{dt^2} + (b(\phi')^2 - a\phi'')\frac{dy}{dt} + c(\phi')^3 y = g$$

thus showing (1) is equivalent to (4).

To complete the proof, start with (4) and reverse the substitutions until (5) is obtained.

QED

One may note that the Euler equation (2) is not in the form (4). What is needed is an "integration factor". In particular, if we let $\mu(t) = t^{-3}$ and multiply (2) by μ, we see that the Euler equation is equivalent to the equation

$$t^{-1}y'' + \alpha t^{-2}y' + \beta t^{-3}y = t^{-3}g(t),$$

which is in the form (4) with $\phi(t) = \ln(t)$:

$$\phi' y'' + [(\alpha - 1)(\phi')^2 - \phi'']y' + \beta(\phi')^3 y = t^{-3}g(t).$$

Example 2.1 To solve the equation

$$\cos(t)y'' + \sin(t)y' - \cos^3(t)y = 0,$$

we let $x = \sin(t)$. Then

$$\frac{dy}{dt} = \cos(t)\frac{dy}{dx}$$

and

$$\frac{d^2y}{dt^2} = \cos^2(t)\frac{d^2y}{dx^2} - \sin(t)\frac{dy}{dx}$$

so the given equation is transformed into

$$\cos^3(t)\left[\frac{d^2y}{dx^2} - y\right] = 0,$$

or, equivalently, $\frac{d^2y}{dx^2} - y = 0$. Thus

$$y(t) = C_1 e^{\sin(t)} + C_2 e^{-\sin(t)}$$

is the general solution of the given equation.

Example 2.2 The equation

$$\cos(t)y'' - \cos^3(t)y = f(t)$$

cannot be transformed into a constant coefficient equation. Suppose, on the contrary, that $x = \phi(t)$ is a transformation which converts the given equation into one with constant coefficients. Then the given equation is equivalent to an equation of the form (4):

$$a\phi' y'' + (b(\phi')^2 - a\phi'')y' + c(\phi')^3 y = g.$$

Thus any solution of the given equation must satisfy

$$[b(\phi')^2 - a\phi''] \cos(t) y' + [c(\phi')^3 \cos(t) - a\phi' \cos^3(t)]y = \cos(t) g(t) - \phi' f(t).$$

It follows that $b(\phi')^2 - a\phi'' = c(\phi')^2 - a\cos^2(t) = 0$ and so

$$a\phi'' = b(\phi')^2,\ c(\phi')^2 = a\cos^2(t) \quad \text{and} \quad g(t) = \frac{a\phi'}{\cos(t)} f(t).$$

But these conditions are incompatible: The first two yield

$$\phi'(t) = \sqrt{\frac{a}{c}} \cos(t) \ \ \textit{and} \ \ \phi''(t) = \frac{b}{c}\cos^2(t)$$

which is impossible.

Now consider a third-order equation

$$P(t)y''' + Q(t)y'' + R(t)y' + S(t)y = G(t). \tag{6}$$

We have a similar theorem to the second-order case. We leave the proof to the reader.

Theorem 2.2 Equation (6) may be transformed into a constant coefficient equation,

$$a_3 \frac{d^3y}{dx^3} + a_2 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_0 y = h(x),$$

via the transformation $x = \phi(t)$, iff it is equivalent to the equation

$$a_3\phi' y''' + [a_2(\phi')^2 - 3a_3\phi'']y'' + \left[a_1(\phi')^3 - a_3\phi''' - a_2\phi'\phi'' + 3a_3\frac{(\phi'')^2}{\phi'}\right] y' - a_0(\phi')^4 y = h \circ \phi.$$

For higher order equations, similar theorems may be developed by assuming a transformation to a constant coefficient equation exists, but it is not clear how to find a general pattern; that is, one must consider each order separately. A useful necessary condition may be stated, however, and this result is easily obtained by generalizing the proceeding results.

Theorem 2.3 If the equation

$$P_n(t)y^{(n)} + \cdots + P_1(t)y' + P_0(t)y = G(t) \tag{7}$$

can be transformed into one with constant coefficients via the transformation $x = \phi(t)$, then it is equivalent to an equation in which the lead coefficient is a constant multiple of ϕ' and the coefficient on y is a constant multiple of $(\phi')^{n+1}$.

Even though an n^{th} order differential equation can be transformed into a first-order system of n differential equations, there is no general result obtainable by our approach that gives necessary and sufficient conditions for all higher orders. The following theorem shows limits of our previous methods as applied to systems.

Theorem 2.4 Let $\mathbf{x}'(t) = A(t)\mathbf{x}(t) + \mathbf{b}(t)$ be a given system with continuous nonconstant coefficient matrix $A(t)$. This system can be transformed, via the transformation $\tau = \phi(t)$, into one with constant coefficient matrix B,

$$\frac{d(\mathbf{x}\circ\phi^{-1})(\tau)}{d\tau} = B\mathbf{x}(\phi^{-1}(\tau)) + \mathbf{c}(\tau), \tag{8}$$

iff it is equivalent to

$$\mathbf{x}'(t) = \phi'(t)B\mathbf{x}(t) + \phi'(t)\mathbf{c}(\phi(t)).$$

Moreover, in this case, $A(t) = \phi'(t)B$ and $\mathbf{b} = \phi'\mathbf{c}\circ\phi$.

Proof We leave to the reader the verification of sufficiency of the condition, and show only its necessity. Assume $\mathbf{x}'(t) = A(t)\mathbf{x}(t) + \mathbf{b}(t)$ can be transformed into the system (8) by the variable transformation $\tau = \phi(t)$. Then, since $\dfrac{d\mathbf{x}(t)}{dt} = \phi'(t)\dfrac{d(\mathbf{x}\circ\phi^{-1})(\tau)}{d\tau}$, it follows that

$$\mathbf{x}'(t) = \phi'(t)\mathbf{c}(\phi(t)) - \phi'(t)B\mathbf{x}(t),$$

for every solution $\mathbf{x}$ of the given system. To see that $A(t) = \phi'(t)B$ and $\mathbf{b} = \phi'\mathbf{c}\circ\phi$, subtract the two equivalent equations to see that every solution of the given system must satisfy

$$(A(t) - \phi'(t)B)\mathbf{x}(t) = (\phi'(t))\mathbf{c}(\phi(t)) - \mathbf{b}(t).$$

Now let $\mathbf{x} = \mathbf{x}_c + \mathbf{x}_p$ be the general solution of the original system, where $\mathbf{x}_c$ is the general solution of the corresponding homogeneous system, and $\mathbf{x}_p$ is a fixed solution of the given system. Then we have

$$(A(t) - \phi'(t)B)\mathbf{x}_c = \mathbf{b}(t) - (\phi'(t))\mathbf{c}(\phi(t)) - (A(t) - \phi'(t)B)\mathbf{x}_p.$$

This fact, along with the fact that any multiple of a solution of the corresponding homogeneous system is again a solution yields $\mathbf{b}(t) - (\phi'(t))\mathbf{c}(\phi(t)) - (A(t) - \phi'(t)B)\mathbf{x}_p = \mathbf{0}$. Then the fact that $(A(t) - \phi'(t)B)\mathbf{x}_c = \mathbf{0}$ yields $(A(t) - \phi'(t)B)X = 0$ for any fundamental matrix X of the corresponding homogeneous system. Since such fundamental matrices are invertible, it follows that $A(t) = \phi'(t)B$, as claimed. But then it follows that $\mathbf{b} = \phi'\mathbf{c}\circ\phi$, and so we are done. QED

Example 2.3 Consider the (2×2) system $\mathbf{x}'(t) = A(t)\mathbf{x}(t)$ where

$$A(t) = \begin{bmatrix} t & 1 \\ 1 & 1 \end{bmatrix}.$$

This system cannot be transformed into one with constant coefficients via a variable transformation of the type we have been considering because there is no pair (ϕ, B) where ϕ is a function of one variable, B is a constant matrix, and $A(t) = \phi'(t)B$. Note that it is possible for a system to be transformed into one with constant coefficients by methods other than our variable transformation. To see this, transform Example 2.1 into a system and realize, using Theorem 2.4, that no $\tau = \phi(t)$ can produce a constant coefficient system; instead, other methods are needed. This further demonstrates the limitations of applying our methods to systems.

3 Transforming Into a Polynomial Coefficient Equation

In this section, we generalize the results of the preceding section to the case that one wishes to convert equations to a given equation with polynomial coefficients. This may be desirable, because for equations with polynomial coefficients, power series solutions are readily available, and this will lead to series solutions, though not power series solutions, for equations which previously have not been solved. The main theorem of this section is the following.

Theorem 3.1 The equation (1) can be tranformed into an equation of the form

$$\left(\sum_{k=0}^{n} a_k x^k\right) y'' + \left(\sum_{k=0}^{n} b_k x^k\right) y' + \left(\sum_{k=0}^{n} c_k x^k\right) y = g(x), \tag{9}$$

via the variable transformation $x = \phi(t)$, iff (1) is equivalent to the equation

$$\left(\sum_{k=0}^{n} a_k\phi^k(t)\right)\phi'(t)y'' + \left(\sum_{k=0}^{n} b_k\phi^k(t)(\phi'(t))^2 - \sum_{k=0}^{n} a_k\phi^k(t)\phi''(t)\right)y'$$
$$+\left(\sum_{k=0}^{n} c_k\phi^k(t)(\phi'(t))^3\right)y = (\phi'(t))^3 g(\phi(t)). \tag{10}$$

Example 3.1 To solve the equation

$$(1+t^2)^2\arctan(t)y'' + (1+t^2)(1+2t\arctan(t))y' + \arctan(t)y = 0,$$

we first divide by $(1+t^2)^3$, then note that the resulting equation is equivalent to the equation

$$\phi(t)\phi'(t)y'' + [(\phi'(t))^2 - \phi(t)\phi''(t)]y' + \phi(t)(\phi'(t))^3 y = 0,$$

where $\phi(t) = \arctan(t)$. Thus this variable transformation yields the equation

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} + xy = 0.$$

which has the power series solution

$$y = \sum_{n=0}^{\infty} a_n (x-1)^n,$$

where the coefficients satisfy the recurrence relations

$$a_{n+2} = -\frac{(n+1)^2 a_{n+1} + a_n + a_{n-1}}{(n+1)(n+2)};$$

$$a_2 = -\frac{a_0 + a_1}{2}.$$

Thus the solution of the original equation is given by

$$y(t) = \sum_{n=0}^{\infty} a_n (\arctan(t) - 1)^n.$$

Example 3.2 The equation

$$y'' - y' + (e^{2t} - 1)y = 0,$$

when multiplied by e^{3t}, can be seen to be in the form (10), using $\phi(t) = e^t$:

$$(\phi(t))^2 \phi'(t) y'' - (\phi(t))^2 \phi'(t) y' + ((\phi(t))^2 - 1)(\phi'(t))^3 y = 0.$$

According to the theorem, then, the variable transformation $x = e^t = \phi(t)$ converts the given equation into a Bessel equation of order 1:

$$x^2 \frac{d^2y}{dx^2} + (x^2 - 1)y = 0.$$

Let J_1 and Y_1 be the Bessel functions of the first and second kinds, respectively. Then the general solution of the given equation is given by

$$y(t) = C_1 J_1(e^t) + C_2 Y_1(e^t).$$

4 Transforming Into an Equation With Certain Specified Coefficients

In this section, we assume the solutions of a given equation,

$$p(x)y'' + q(x)y' + r(x)y = g(x), \tag{11}$$

are known, and characterize those equations which may be transformed into (11) via a nonsingular variable transformation. (If the solutions of (11) are not known, but are better understood than solutions of some other equations, our approach will show how to study properties of less well-understood equations.)

Theorem 4.1 Equation (1) can be transformed into equation (11) via the variable transformation $x = \phi(t)$ iff it is equivalent to the equation

$$(p\circ\phi)\phi' y'' + [(q\circ\phi)(\phi')^2 - (p\circ\phi)\phi'']y' + (r\circ\phi)(\phi')^3 y = (\phi')^3 g\circ\phi. \qquad (12)$$

Proof We prove that the function ϕ and its inverse convert between equations (11) and (12). Then the theorem follows immediately. If y satisfies (11), then the variable transformation $x = \phi(t)$ yields

$$p(\phi(t))\phi'(t)\frac{d^2y}{dt^2} + [q(\phi(t))(\phi'(t))^2 - p(\phi(t))\phi''(t)]\frac{dy}{dt} + r(\phi(t))(\phi'(t))^3 y$$

$$= p(x)\left[\phi'(t)\frac{d^2y}{dt^2} - \phi''(t)\frac{dy}{dt}\right] + q(x)(\phi'(t))^2\frac{dy}{dt} + r(x)(\phi'(t))^3 y$$

$$= p(x)\left[\phi'(t)\left((\phi'(t))^3\frac{d^2y}{dx^2} + \phi''(t)\frac{dy}{dx}\right) - \phi''(t)\phi'(t)\frac{dy}{dx}\right] + q(x)(\phi'(t))^3\frac{dy}{dx} + r(x)(\phi'(t))^3 y$$

$$= (\phi'(t))^3\left[p(x)\frac{d^2y}{dx^2} + q(x)\frac{dy}{dx} + r(x)y\right]$$

$$= (\phi'(t))^3 g(x)$$

$$= (\phi'(t))^3 g(\phi(t)).$$

On the other hand, if a function y satisfies (12), then we let $x = \phi(t)$ again, to see that

$$p(x)\frac{d^2y}{dx^2} + q(x)\frac{dy}{dx} + r(x)y$$

$$= \frac{p(\phi(t))}{(\phi'(t))^2}\left[\frac{d^2y}{dt^2} - \frac{\phi''(t)}{\phi'(t)}\frac{dy}{dt}\right] + \frac{q(\phi(t))}{\phi'(t)}\frac{dy}{dt} + r(\phi(t))y$$

$$= \frac{1}{(\phi'(t))^3}\left[p(\phi(t))\phi'(t)\frac{d^2y}{dt^2} + (q(\phi(t))(\phi'(t))^2 - p(\phi(t))\phi''(t))\frac{dy}{dt} + r(\phi(t))(\phi'(t))^3 y\right]$$

$$= \frac{1}{(\phi'(t))^3}\left((\phi'(t))^3 g(\phi(t))\right)$$

$$= g(x).$$

Thus the function y also satisfies (11), as claimed. QED

Example 4.1 Consider the equation

$$-\frac{1}{t^2}\frac{d^2y}{dt^2}+\left[\frac{\sin\left(\frac{1}{t}\right)}{t}-2\right]\left(\frac{1}{t^3}\right)\frac{dy}{dt}-\frac{1}{t^6}\cos\left(\frac{1}{t}\right)y=0.$$

Letting $x=\phi(t)=\frac{1}{t}$, we see that the given equation is equivalent to

$$\frac{d^2y}{dx^2}+\sin(x)\frac{dy}{dx}+\cos(x)y=0,$$

which has analytic coefficients, and so has as its general solution

$$y=c_1y_1(x)+c_2y_2(x),$$

where y_1 and y_2 are analytic. Thus the general solution of the given equation is given by

$$y(t)=c_1y_1\left(\frac{1}{t}\right)+c_2y_2\left(\frac{1}{t}\right),$$

and is analytic except at zero.

5 Integrating Factors

In this section, we tell how to find an integrating factor, if one exists, which will convert an equation of the form (1) (or more generally, (7)) into a canonical form for transformation into constant coefficients.

Theorem 5.1 Given an equation of the form (1), there exists a function μ which, when multiplied by (1) yields an equation of the form (4), with $a=1$ and $c\neq 0$, iff there are constants b and c such that

$$\sqrt{\frac{R}{cP^3}}(Q+P')=\frac{bR}{cP}-\frac{P}{2}\sqrt{\frac{cP^3}{R}}\left(\frac{R'}{cP^3}-\frac{3RP'}{cP^4}\right)$$

In this case, the integrating factor μ is given by

$$\mu=\sqrt{\frac{R}{cP^3}}$$

Proof It is easy to see that a function μ is an integrating factor for (1) iff for some function ϕ, and for some constants b and c, we have

$$\mu P=\phi',\quad \mu Q=b(\phi')^2-\phi''\qquad\text{and}\qquad \mu R=c(\phi')^3$$

Substituting μP for ϕ' in the second two of these equations gives us relationships between μ, P and Q:

$$\mu Q = b\mu^2 P^2 - \mu' P - \mu P' \qquad \text{and} \qquad \mu R = c\mu^3 P^3$$

The second of these equations may be solved for μ, and substitution into the first yields the conditions required for existence of μ. QED

The condition given here is tedious to check, so the best way to use this theorem is to compute μ and multiply by it, to see if it works, i.e., does the resulting equation transform into one with constant coefficients by the appropriate transformation? If so then the transformation to (5) is of course possible, and otherwise, it is impossible.

The following corollary generalizes the preceding theorem to the higher order case, but again does not give a characterization.

Corollary 5.1 If an integrating factor for (7) exists, it is given by

$$\mu = \sqrt[n]{\frac{P_0}{cP_n^{n+1}}},$$

where c is some constant.

Example 5.1 To solve the equation

$$y'' + \left(12t + 8 - \frac{1}{t + \frac{4}{6}}\right) y' + \left(36t^2 + 48t + 16\right) y = 0,$$

we let

$$\mu(t) = \sqrt{\frac{36t^2 + 48t + 16}{c}} = \frac{1}{\sqrt{c}}(6t + 4).$$

We assume $c = 1$ for convenience, so we multiply the original equation by $\mu(t) = 6t + 4$, to obtain the equation

$$(6t + 4)y'' + \left(2(6t + 4)^2 - 6\right) y' + (6t + 4)^3 y = 0.$$

We could check to see that this is in the form (4), but we just proceed as if the appropriate substitution will work. If it does, then we are done, and otherwise, nothing will work. Thus we let $x = 3t^2 + 4t$, so that

$$\frac{dy}{dt} = (6t + 4)\frac{dy}{dx}, \qquad \text{and} \qquad \frac{d^2y}{dt^2} = (6t + 4)^2\frac{d^2y}{dx^2} + 6\frac{dy}{dx}.$$

Substituting into the original equation yields

$$(6t + 4)^3 \left[\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y\right] = 0,$$

or, equivalently,

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0.$$

Thus the general solution of the original equation is given by

$$y(t) = C_1 e^{-3t^2-4t} + C_2\left(3t^2 + 4t\right)e^{-3t^2-4t}.$$

Example 5.2 Consider the equation

$$\frac{d^2y}{dt^2} + (8t-1)\frac{dy}{dt} + 16t^2 y = 0.$$

We would like to find an integrating factor for this equation, so we let $\mu(t) = \sqrt{\frac{16t^2}{c}} = \frac{4t}{\sqrt{c}}$, where c is to be determined. Multiplication of the given equation by μ yields

$$\frac{4t}{\sqrt{c}}\frac{d^2y}{dt^2} + \left(\frac{32t^2}{\sqrt{c}} - \frac{4t}{\sqrt{c}}\right)\frac{dy}{dt} + \frac{(4t)^3}{\sqrt{c}}y = 0,$$

so the only possible transformation ϕ to convert it into a constant coefficient equation satisfies $\phi'(t) = \frac{4t}{\sqrt{c}}$. In this case, we choose $x = \frac{2}{\sqrt{c}}t^2$, so that the given equation transforms into

$$0 = \frac{4t}{\sqrt{c}}\left[\frac{(4t)62}{c}\frac{d^2y}{dx^2} + \frac{4}{\sqrt{c}}\right] + \left[\frac{32t^2}{\sqrt{c}} - \frac{4t}{\sqrt{c}}\right]\left(\frac{4t}{\sqrt{c}}\right) + \frac{(4t)^3}{\sqrt{c}}y$$

$$= \left(\frac{4t}{\sqrt{c}}\right)^3\left[\frac{d^2y}{dx^2} + \sqrt{c}\left[\frac{1}{16t^2} + 2 - \frac{1}{16t}\right]\frac{dy}{dx} + cy\right],$$

so that it is clear that the given equation cannot be transformed into one with constant coefficients. Note that if we modify the equation as follows, then the procedure above converts it into a constant coefficient equation:

$$\frac{d^2y}{dt^2} + \left(8t - \frac{1}{t}\right)\frac{dy}{dt} + 16t^2 y = 0.$$

6 Asymptotic Behavior of Solutions

We can use the transformations developed in the preceding sections to analyse the behavior of solutions at infinity or near a singular point. In this section, we give some examples of how this can be done by transforming an equation into one with constant coefficients or into one with polynomial coefficients. To begin with, consider an equation of the form (4), where the numbers a, b and c are all positive, and $\lim_{t\to\infty} \phi(t) = \infty$. The solutions of such an equation tend to zero as $t \to \infty$, because it may be transformed into an equation with all positive constant coefficients, and (see [1]) such a constant coefficient equation has the property that all its solutions tend to zero as $x \to \infty$.

Example 6.1 The solutions of the equation

$$y'' + (2e^t - 1)y' + 3e^{2t}y = 0$$

all approach zero as $t \to \infty$, because multiplication by e^t yields an equivalent equation of the form (4) with positive a, b and c (namely $a = 1$, $b = 2$ and $c = 3$), and $\phi(t) = e^t$.

Now recall that if a given equation is of the form

$$x^2\frac{d^2y}{dx^2} + xp(x)\frac{dy}{dx} + q(x)y = 0$$

where the functions p and q are analytic, then the roots of the corresponding *indicial equation* describe the qualitative behavior of the solutions near zero. We use this in the following example.

Example 6.2 The equation

$$\sin^2(t)\cos(t)y'' + \sin^2(t)[1 + \sin(t) - \sin^2(t)]y' + \frac{2}{9}\cos^3(t)y = 0$$

can, by our methods, be transformed into the equation

$$x^2\frac{d^2y}{dx^2} + x^2\frac{dy}{dx} + \frac{2}{9}y = 0,$$

using the transformation $x = \sin(t)$. The corresponding indicial equation is

$$r^2 - r + \frac{2}{9} = 0,$$

and its roots are $\frac{1}{3}$ and $\frac{2}{3}$. Thus, near zeros of the sine function, the solutions of the given equation are of the form

$$y(t) = C_1\sqrt[3]{\sin(t)}\,[1 + a_1(\sin(t))] + C_2\sqrt[3]{\sin^2(t)}\,[1 + a_2(\sin(t))],$$

where a_1 and a_2 are analytic functions which are zero at the origin. Thus near zeros of the sine function, the solutions asymptotically approach a linear combination of the cube root of the sine function and its square.

Acknowledgements

This work began while the second author was a graduate student at the University of Missouri-Rolla and was completed while the second author was working at the University of Missouri-St Louis. We would like to thank the University of Missouri-St Louis for use of computing facilities and other support. We would also like to thank Theodore Burton, who refereed this paper, for pointing us towards areas in need of clarification as well as in the direction of the fascinating results of Ritt and Kaplansky. This paper was typset by the authors using LaTex.

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