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Proofs for the Limit of Ratios of Consecutive Terms in Fibonacci Sequence

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ABSTRACT. In the short note, six proofs for the limit of ratios of consecutive terms in Fibonacci sequence are provided, including using Weierstrass-Bolzano theorem, series method, by definition of limit, difference equation method, matrix method, algebraic method.

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1 Introduction

It is well-known that the Fibonacci sequence $\{F_n\}_{n=1}^{\infty}$ is defined by the following recurrence formula

$$\begin{cases} F_{n+2} = F_{n+1} + F_n, \\ F_1 = 1, \quad F_2 = 1. \end{cases} \tag{1}$$

We also call F_n the Fibonacci number. Its general term can be expressed as

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right] \tag{2}$$

for $n \in \mathbb{N}$.

The Fibonacci numbers give the number of pairs of rabbits n months after a single pair begins breeding (and newly born bunnies are assumed to begin breeding when they are two months old).

Define

$$x_n = \frac{F_{n+1}}{F_n}, \tag{3}$$

then the sequence $\{x_n\}_{n=1}^{\infty}$ converges, this limit

$$\phi = \lim_{n\to\infty} x_n \tag{4}$$

is called Golden Ratio.

The ratios of alternate Fibonacci numbers are given by the convergents to ϕ^{-2} and are said to measure the fraction of a turn between successive leaves on the stalk of a plant (Phyllotaxis): $\frac{1}{2}$ for elm and linden, $\frac{1}{3}$ for beech and hazel, $\frac{2}{5}$ for oak and apple, $\frac{3}{8}$ for poplar and rose, $\frac{5}{13}$ for willow and almond, etc.. The Fibonacci numbers are sometimes called Pine Cone Numbers. The role of the Fibonacci numbers in botany is sometimes called Ludwig's Law.

To prove the convergence and to solve its limit of the sequence $\{x_n\}_{n=1}^{\infty}$ is a standard exercise or example in calculus and mathematical analysis for graduate students.

In this paper, we will prove the convergence of the sequence $\{x_n\}_{n=1}^{\infty}$ and solve its limit using six approaches, including using Weierstrass-Bolzano theorem, series method, by definition of limit, difference equation method, matrix method, algebraic method.

2 Proofs of Convergence and Limit

The first proof: using Weierstrass-Bolzano Theorem. It is clear that

$$x_1 = \frac{F_2}{F_1} = 1, \quad x_{n+1} = \frac{F_{n+2}}{F_{n+1}} = \frac{F_{n+1} + F_n}{F_{n+1}} = 1 + \frac{F_n}{F_{n+1}} = 1 + \frac{1}{x_n} \tag{5}$$

for $n \in \mathbb{N}$. Since the sequence $\{F_n\}_{n=1}^{\infty}$ is an increasing positive sequence, then $1 \le x_{n+1} \le 2$ for $n \in \mathbb{N}$.

From (5), it follows that

$$x_{n+1} - x_n = \left(1 + \frac{1}{x_n}\right) - \left(1 + \frac{1}{x_{n-1}}\right) = -\frac{x_n - x_{n-1}}{x_n x_{n-1}}, \tag{6}$$

thus $\{x_n\}_{n=1}^{\infty}$ is not monotonic. Using formula (5) again gives us

$$x_{n+2} = 1 + \frac{1}{x_{n+1}} = 1 + \frac{1}{1 + \frac{1}{x_n}} = \frac{2x_n + 1}{1 + x_n}. \tag{7}$$

$$x_{n+2} - x_n = \frac{2x_n + 1}{1 + x_n} - \frac{2x_{n-2} + 1}{1 + x_{n-2}} = \frac{x_n - x_{n-2}}{(x_n + 1)(x_{n-2} + 1)}. \tag{8}$$

Since $(x_n + 1)(x_{n-2} + 1) > 0$, then $(x_{n+2} - x_n)(x_n - x_{n-2}) \ge 0$. This implies that the sequences $\{x_{2n-1}\}_{n=1}^{\infty}$ and $\{x_{2n}\}_{n=1}^{\infty}$ are all monotonic. In fact, by induction, we can prove that $\{x_{2n-1}\}_{n=1}^{\infty}$ is increasing and $\{x_{2n}\}_{n=1}^{\infty}$ is decreasing.

The subsequence $\{x_{2n-1}\}_{n=1}^{\infty}$ and $\{x_{2n}\}_{n=1}^{\infty}$ are all monotonic and bounded, so they are all convergent.

Let $\lim_{n\to\infty} x_{2n-1} = A$ and $\lim_{n\to\infty} x_{2n} = B$. By direct calculation, from (7), we have

$$A = \lim_{k\to\infty} x_{2k+1} = \lim_{k\to\infty} \frac{2x_{2k-1} + 1}{x_{2k-1} + 1} = \frac{2A + 1}{A + 1}, \tag{9}$$

$$B = \lim_{k\to\infty} x_{2k+2} = \lim_{k\to\infty} \frac{2x_{2k} + 1}{x_{2k} + 1} = \frac{2B + 1}{B + 1}. \tag{10}$$

On the interval $[1, 2]$, we obtain $A = B = \frac{1+\sqrt{5}}{2}$ by solving the equations (9) and (10). Therefore we have $\lim_{n\to 0} x_n = \frac{1+\sqrt{5}}{2}$. □

The second proof: series method. Since $1 \le x_n \le 2$ for $n \in \mathbb{N}$, from (8), it follows that

$$|x_{n+2} - x_n| = \frac{|x_n - x_{n-2}|}{(x_n + 1)(x_{n-2} + 1)} \le \frac{1}{4}|x_n - x_{n-2}|. \tag{11}$$

and then

$$|x_{2k+2} - x_{2k}| \leq \frac{1}{4^{k-1}} |x_4 - x_2|, \tag{12}$$

$$|x_{2k+1} - x_{2k-1}| \leq \frac{1}{4^{k-1}} |x_3 - x_1|, \tag{13}$$

where $k \in \mathbb{N}$. By properties of series with positive terms, it is deduced that the series $\sum_{k=1}^{\infty}(x_{2k+2} - x_{2k})$ and $\sum_{k=1}^{\infty}(x_{2k+1} - x_{2k-1})$ are all absolutely convergent. Since

$$x_{2k} = x_2 + \sum_{i=2}^{k}(x_{2i} - x_{2(i-1)}), \tag{14}$$

$$x_{2k-1} = x_1 + \sum_{i=2}^{k}(x_{2i-1} - x_{2i-3}), \tag{15}$$

the sequences $\{x_{2k}\}_{k=1}^{\infty}$ and $\{x_{2k-1}\}_{k=1}^{\infty}$ converge.

The rest is same as that in the first proof. □

The third proof: by definition of limit. Note that the equation $x = 1 + \frac{1}{x}$ corresponding to $x_{n+1} = 1 + \frac{1}{x_n}$ for $n \in \mathbb{N}$ has unique positive root $\frac{1+\sqrt{5}}{2}$.

By definition of limit, using the result $1 \leq x_n \leq 2$ for $n \in \mathbb{N}$, we have

$$\begin{aligned} \left|x_n - \frac{1+\sqrt{5}}{2}\right| &= \left|1 + \frac{1}{x_{n-1}} - \frac{1+\sqrt{5}}{2}\right| \\ &= \left|\frac{1}{x_{n-1}} + \frac{1-\sqrt{5}}{2}\right| = \left|\frac{1-\sqrt{5}}{2x_{n-1}}\right| \cdot \left|\frac{2}{1-\sqrt{5}} + x_{n-1}\right| \\ &= \frac{\sqrt{5}-1}{2x_{n-1}}\left|x_{n-1} - \frac{1+\sqrt{5}}{2}\right| \leq \frac{\sqrt{5}-1}{2}\left|x_{n-1} - \frac{1+\sqrt{5}}{2}\right|, \end{aligned} \tag{16}$$

then

$$\left|x_n - \frac{1+\sqrt{5}}{2}\right| \leq \left(\frac{\sqrt{5}-1}{2}\right)^{n-1}\left|x_1 - \frac{1+\sqrt{5}}{2}\right| = \left(\frac{\sqrt{5}-1}{2}\right)^{n} \to 0 \tag{17}$$

as $n \to \infty$. Therefore $\lim_{n\to\infty} x_n = \frac{1+\sqrt{5}}{2}$. □

The fourth proof: difference equation method. From $F_{n+2} = F_{n+1} + F_n$, we obtain the linear difference equation of second order with constant coefficients:

$$F_{n+2} - F_{n+1} - F_n = 0. \tag{18}$$

Its eigenequation is $\lambda^2 - \lambda - 1 = 0$, its eigenvalues are

$$\lambda_1 = \frac{1+\sqrt{5}}{2}, \quad \lambda_2 = \frac{1-\sqrt{5}}{2}.$$

The general solution of the difference equation is

$$F_n = C_1\left(\frac{1+\sqrt{5}}{2}\right)^n + C_2\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

By the initial conditions $F_1 = 1$ and $F_2 = 1$, we obtain

$$C_1 = \frac{1}{\sqrt{5}}, \quad C_2 = -\frac{1}{\sqrt{5}}.$$

Hence

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

and

$$x_n = \frac{F_{n+1}}{F_n} = \frac{\left[\frac{1+\sqrt{5}}{2}\right]^{n+1} - \left[\frac{1-\sqrt{5}}{2}\right]^{n+1}}{\left[\frac{1+\sqrt{5}}{2}\right]^{n} - \left[\frac{1-\sqrt{5}}{2}\right]^{n}} \to \frac{1+\sqrt{5}}{2}$$

as $n \to \infty$. □

The fifth proof: matrix method. It is easy to see that

$$\begin{pmatrix} F_{n+2} \\ F_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n\begin{pmatrix} F_2 \\ F_1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{19}$$

for $n \in \mathbb{N}$.

Let $A = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, then the eigenvalues of the square matrix A are $\lambda_1 = \frac{1+\sqrt{5}}{2}$ and $\lambda_2 = \frac{1-\sqrt{5}}{2}$, their corresponding eigenvectors are

$$\alpha_1 = \begin{pmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{pmatrix}. \tag{20}$$

Taking

$$P = \begin{pmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{pmatrix}, \tag{21}$$

then

$$P^{-1}AP = \begin{pmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{pmatrix}, \tag{22}$$

this implies

$$\begin{aligned} A^n &= P \begin{pmatrix} \left[\frac{1+\sqrt{5}}{2}\right]^n & 0 \\ 0 & \left[\frac{1-\sqrt{5}}{2}\right]^n \end{pmatrix} P^{-1} \\ &= \frac{1}{\sqrt{5}} \begin{pmatrix} \left[\frac{1+\sqrt{5}}{2}\right]^{n+1} - \left[\frac{1-\sqrt{5}}{2}\right]^{n+1} & \frac{1+\sqrt{5}}{2}\left[\frac{1-\sqrt{5}}{2}\right]^{n+1} - \frac{1-\sqrt{5}}{2}\left[\frac{1+\sqrt{5}}{2}\right]^{n+1} \\ \left[\frac{1+\sqrt{5}}{2}\right]^{n} - \left[\frac{1-\sqrt{5}}{2}\right]^{n} & \frac{1+\sqrt{5}}{2}\left[\frac{1-\sqrt{5}}{2}\right]^{n} - \frac{1-\sqrt{5}}{2}\left[\frac{1+\sqrt{5}}{2}\right]^{n} \end{pmatrix}, \end{aligned} \tag{23}$$

and

$$\begin{pmatrix} F_{n+2} \\ F_{n+1} \end{pmatrix} = \frac{1}{\sqrt{5}} \times$$

$$\begin{pmatrix} \left[\frac{1+\sqrt{5}}{2}\right]^{n+1} - \left[\frac{1-\sqrt{5}}{2}\right]^{n+1} & \frac{1+\sqrt{5}}{2}\left[\frac{1-\sqrt{5}}{2}\right]^{n+1} - \frac{1-\sqrt{5}}{2}\left[\frac{1+\sqrt{5}}{2}\right]^{n+1} \\ \left[\frac{1+\sqrt{5}}{2}\right]^{n} - \left[\frac{1-\sqrt{5}}{2}\right]^{n} & \frac{1+\sqrt{5}}{2}\left[\frac{1-\sqrt{5}}{2}\right]^{n} - \frac{1-\sqrt{5}}{2}\left[\frac{1+\sqrt{5}}{2}\right]^{n} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{24}$$

Thus

$$F_{n+1} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right], \tag{25}$$

that is

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n} - \left(\frac{1-\sqrt{5}}{2}\right)^{n}\right] \tag{26}$$

for $n \in \mathbb{N}$.

The rest is same as that in the third proof. □

The sixth proof: algebraic method. Let us rewrite $F_{n+2} = F_{n+1} + F_n$ as

$$F_{n+2} - pF_{n+1} = q(F_{n+1} - pF_n), \tag{27}$$

that is

$$F_{n+2} = (p+q)F_{n+1} - pqF_n. \tag{28}$$

Let

$$\begin{cases} p+q=1, \\ pq=-1, \end{cases} \tag{29}$$

then we obtain solutions of (29) as follows

$$\begin{cases} p=\frac{1+\sqrt{5}}{2}, \\ q=\frac{1-\sqrt{5}}{2}, \end{cases} \tag{30}$$

or

$$\begin{cases} p=\frac{1-\sqrt{5}}{2}, \\ q=\frac{1+\sqrt{5}}{2}. \end{cases} \tag{31}$$

Therefore the recurrence formula $F_{n+2}=F_{n+1}+F_n$ can be rewritten as

$$F_{n+2}-\frac{1+\sqrt{5}}{2}F_{n+1}=\frac{1-\sqrt{5}}{2}\left(F_{n+1}-\frac{1+\sqrt{5}}{2}F_n\right) \tag{32}$$

and

$$F_{n+2}-\frac{1-\sqrt{5}}{2}F_{n+1}=\frac{1+\sqrt{5}}{2}\left(F_{n+1}-\frac{1-\sqrt{5}}{2}F_n\right). \tag{33}$$

From (32), we obtain

$$F_{n+2}-\frac{1+\sqrt{5}}{2}F_{n+1}=\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}, \tag{34}$$

From (33), we obtain

$$F_{n+2}-\frac{1-\sqrt{5}}{2}F_{n+1}=\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}. \tag{35}$$

From system of the equations (34) and (35), we have

$$F_{n+2}=\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+2}-\left(\frac{1-\sqrt{5}}{2}\right)^{n+2}\right], \tag{36}$$

and

$$F_n=\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n}-\left(\frac{1-\sqrt{5}}{2}\right)^{n}\right]. \tag{37}$$

The rest is same as that in the third proof. The proof is complete. □

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