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Left invariant geometry of Lie groups

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ABSTRACT

In this article we investigate the geometry of a Lie group N with a left invariant metric, particularly in the case that N is 2-step nilpotent. Our primary interest will be in properties of the geodesic flow, but we describe a more general framework for studying left invariant functions and vector fields on the tangent bundle TN. Here we consider the natural left action λ of N on TN given by $\lambda_n(\xi) = (L_n)_*(\xi)$, where $L_n : N \longrightarrow N$ denotes left translation by n and $(L_n)_*$ denotes the differential map of L_n.

For convenience all manifolds in this article are assumed to be connected and C^∞ unless otherwise specified. Many of the assertions remain valid true for manifolds that are not connected and are C^k for a small integer k.

We assume that the reader has a familiarity with manifold theory and with the basic concepts of Lie groups and and their associated Lie algebras of left invariant vector fields.

Contents

1. Basic properties and examples of symplectic structures

We first recall some basic results of manifold theory that will be useful.

1.1 Lie derivative and exterior derivative of k-forms

Let X be a vector field on M with flow transformations $\{X^t\}$. By definition the integral curves of X are the curves $t \to X^t(m)$, where m is an arbitrary point of M. For each compact subset C of M there exists a positive number $\epsilon = \epsilon(C)$ such that the flow transformations $\{X^t\}$ are defined at every point of C on the interval $(-\epsilon, \epsilon)$. Moreover, $X^{t+s} = X^t \circ X^s = X^s \circ X^t$ at all points of M for which X^t, X^s and X^{t+s} are defined.

Every vector field X on M defines an interior product i_X that maps a k-form ω on M to a $(k-1)$-form $i_X\omega$ on M given by $i_X\omega(X_1, ...X_{k-1}) = \omega(X, X_1, ...X_{k-1})$, where $(X_1, ...X_{k-1})$ are arbitrary vector fields on M. Since the exterior derivative d maps a k-form on M to a $(k+1)$-form it follows that both $(d \circ i_X)(\omega)$ and $(i_X \circ d)(\omega)$ are k-forms for any vector field X and any k-form ω.

If ω is a k-form on M, then we define the Lie derivative $L_X\omega$ to be the k-form on M given by $L_X\omega = \frac{d}{dt}|_{t=0}(X^t) * (\omega)$. The Lie derivative is related to the interior product and exterior differentiation by the important formula

$$L_X\omega = (d \circ i_X)(\omega) + (i_X \circ d)(\omega) \qquad (1)$$

Since $d \circ d = 0$ it follows immediately that

$$d \circ L_X = L_X \circ d \qquad (2)$$

If ω is a 1-form on M, then $d\omega$ satisfies the basic and useful formula

$$d\omega(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y]) \tag{3}$$

where X and Y are vector fields on M and $[X,Y]$ denotes their Lie bracket. There is a generalization of this formula for $d\omega$, where ω is a k-form. See for example, [Hel, p. 21].

1.2 f-related vector fields

Let M and N be C^∞ manifolds, and let $f : M \to N$ be a C^∞ map. We say that vector fields X in M and Y in N are <u>f–related</u> if $f_*(X(m)) = Y(f(m))$ for all $m \in M$. The following result is standard.

Proposition Let M and N be C^∞ manifolds, and let $f : M \to N$ be a C^∞ map. Let X_1, X_2 and Y_1, Y_2 be vector fields in M and N such that X_1 is f-related to Y_1 and X_2 is f-related to Y_2. Then $[X_1, X_2]$ is f-related to $[Y_1, Y_2]$.

1.3 Flow box coordinates and commuting vector fields

The following result is often useful. For a proof see [S, Theorems 7 and 14, Chapter 5].

Proposition Let M be a C^∞ manifold of dimension n.

1) Let X be a C^∞ vector field on M, and let m be a point of M such that $X(m) \neq 0$. Then there exists a coordinate system $x = (x_1, x_2, ..., x_n)$ in a neighborhood U of m such that $X = \frac{\partial}{\partial x_1}$ in U.

2) Let $\{X_1, ..., X_k\}$ be linearly independent C^∞ vector fields that commute on some open subset U of M; that is, $[X_i, X_j] = 0$ on U for $1 \leq i, j \leq k$. Then for every point m of U there exists a coordinate system $x = (x_1, x_2, ..., x_n)$ in a neighborhood V of m such that $X_i = \frac{\partial}{\partial x_i}$ in V for $1 \leq i \leq k$

1.4 Definition of a symplectic structure

A C^∞ manifold M of dimension $2n$ is said to have a <u>symplectic</u> structure if there exists a nondegenerate 2- form Ω on M such that $d\Omega = 0$ and $\Omega \wedge \cdots \wedge \Omega$ (n times) is nonzero at every point of M. Globally symplectic manifolds may vary considerably, but locally the symplectic structure has a canonical form.

Proposition (Darboux) Let M be a manifold of dimension $2n$ with a symplectic structure given by a closed 2-form Ω. For every point m of M there there exists a coordinate neighborhood U of m and coordinate system
$x = (p_1, \cdots, p_n, q_1, \cdots, q_n) : U \to \mathbb{R}^{2n}$ such that $\Omega = dq_1 \wedge dp_1 + dq_2 \wedge dp_2 + \cdots + dq_n \wedge dp_n$ on U.

Example 1. Symplectic structure on the cotangent bundle [AM]
Let M be a C^∞ manifold with tangent bundle TM and cotangent bundle TM^*. The cotangent bundle admits a canonical 1-form θ defined by $\theta(\xi) = \omega(d\pi(\xi))$, where ξ is an element of $T_\omega(TM^*)$ and $\pi : TM^* \to M$ is the projection that assigns to an element ω of $(T_m M)^*$ the point m. The 1-form θ satisfies the following properties and is characterized by the first of these:
1) If $\beta : M \to TM^*$ is a smooth 1-form on M, then $\beta^*(\theta) = \beta$, where $\beta^*(\theta)$ denotes the pullback of θ by β.
2) If $f : M \to M$ is any diffeomorphism, then the natural extension $\bar{f} : TM^* \to TM^*$ given by $\bar{f}(\omega) = f^*(\omega)$ preserves θ; that is, $(\bar{f})^*(\theta) = \theta$.

The 2-form $\Omega = -d\theta$ is a symplectic form on TM^*; that is, Ω is nondegenerate at every point of TM^* and $\Omega_\wedge \ldots {}_\wedge\Omega$ (n times) is a nonzero $2n$-form at every point of TM^*. Since pullbacks commute with exterior differentiation the two properties above for θ have immediate analogues for Ω.
1) If $\beta : M \to TM^*$ is a smooth 1-form on M, then $\beta^*(\Omega) = -d\beta$.
2) If $f : M \to M$ is any diffeomorphism, then $\bar{f}^*(\Omega) = \Omega$.

Example 2. Coadjoint action of a Lie group on its Lie algebra
Let H be a connected Lie group, and let $\mathfrak{H}$ denote its Lie agebra. On the dual space $\mathfrak{H}^*$ we define a left action $\mathrm{Ad}^* : H \to GL(\mathfrak{H}^*)$ called the coadjoint action of H. Given $\omega \in \mathfrak{H}^*$ and $h \in H$ we define $\mathrm{Ad}^*(h)(\omega) = \omega \circ Ad(h^{-1})$, where $Ad : H \to GL(\mathfrak{H})$ denotes the usual adjoint action of H on $\mathfrak{H}$. It is routine to check that $\mathrm{Ad}^*(h_1h_2) = \mathrm{Ad}^*(h_1) \circ \mathrm{Ad}^*(h_2)$ for all $h_1, h_2 \in H$. Hence $\mathrm{Ad}^*(H)$ is a subgroup of $GL(\mathfrak{H}^*)$.

For each $X \in \mathfrak{H}$ define $\mathrm{ad}^*X \in \mathrm{End}(\mathfrak{H}^*)$ by $\mathrm{ad}^*X(\omega) = -\omega \circ \mathrm{ad}X$. It is easy to see that $e^{t\mathrm{ad}^*X} = \mathrm{Ad}^*(e^{tX})$ for all $t \in \mathbb{R}$ and all $X \in \mathfrak{H}$. Hence $\mathrm{ad}^*\mathfrak{H} = \{\mathrm{ad}^*X : X \in \mathfrak{H}\}$ is the Lie algebra of Ad^*H.

Next, we show that each orbit of Ad^*H in $\mathfrak{H}^*$ is a symplectic manifold. Given an element ω of $\mathfrak{H}^*$ the orbit $\mathrm{Ad}^*H(\omega)$ is naturally diffeomorphic to the coset manifold H/H_ω, where $H_\omega = \{h \in H : \mathrm{Ad}^*h(\omega) = \omega\}$. It suffices to define a symplectic structure on H/H_ω.

For each $\omega \in \mathfrak{H}^*$ define a skew symmetric bilinear form $B_\omega : \mathfrak{H} \times \mathfrak{H} \to \mathbb{R}$ by $B_\omega(X,Y) = \omega([X,Y])$. If $\mathfrak{H}_\omega$ denotes the Lie algebra of H_ω, then it is easy to see that $\mathfrak{H}_\omega = \{Y \in \mathfrak{H} : \mathrm{ad}^*Y(\omega) = 0\} = \{Y \in \mathfrak{H} : \omega([X,Y]) = 0 \text{ for all } X \in \mathfrak{H}\} = \{Y \in \mathfrak{H} : B_\omega(X,Y) = 0 \text{ for all } X \in \mathfrak{H}\} =$ nullity of B_ω. Hence B_ω defines a nondegenerate, skew symmetric bilinear form $\overline{B}_\omega$ on $\mathfrak{H}/\mathfrak{H}_\omega$ by $\overline{B}_\omega([X],[Y]) = B_\omega(X,Y)$, where $[X]$ denotes the element of $\mathfrak{H}/\mathfrak{H}_\omega$ determined by $X \in \mathfrak{H}$. In particular, $\mathfrak{H}/\mathfrak{H}_\omega$ has even dimension $2n_\omega$ for every $\omega \in \mathfrak{H}^*$.

Let $[e]$ be the identity coset in H/H_ω. The tangent space $T_{[e]}(H/H_\omega)$ may be naturally identified with $\mathfrak{H}/\mathfrak{H}_\omega$ by the isomorphism $[X] \to \alpha_{[X]}'(0)$, where $\alpha_{[X]}(t) = e^{tX}H_\omega$ for $X \in \mathfrak{H}$ and $t \in \mathbb{R}$. Now let Ω_ω be the nondegenerate, left invariant 2-form on H/H_ω such that $\Omega_\omega = \overline{B}_\omega$ on $T_{[e]}(H/H_\omega)$. It is routine to show that $\Omega_\omega \wedge \cdots \wedge \Omega_\omega$ (n_ω times) is nonzero at every point of H/H_ω. This completes the construction of a symplectic 2-form Ω_ω on each orbit $\mathrm{Ad}^* H(\omega)$ in $\mathfrak{H}^*$.

2. Symplectic structure on the tangent bundle of a pseudoRiemannian manifold

2.1 Definition of the symplectic 2-form

Let $m \to <,>_m$ be a smooth assignment of a nondegenerate symmetric bilinear form of fixed signature to each tangent space $T_m M$; that is, $<,>$ is a pseudoRiemannian structure on M. The bilinear form $<,>$ defines an isomorphism f_m between $T_m M$ and $T_m M^*$ for each point m of M by $f_m(v)(w) =< v, w >$ for all vectors v, w in $T_m M$. The resulting diffeomorphism $f = f_{<,>} : TM \to TM^*$ allows one to pull back the 1-form θ and the 2-form $\Omega = -d\theta$ on TM^*. These pullbacks will also be denoted θ and Ω. It is routine to show

$$\theta(\xi) =< v, d\pi(\xi) > \text{ for every } \xi \in T_v(TM) \tag{4}$$

where $\pi : TM \to M$ denotes the projection map that sends a vector $v \in T_m M$ to the point of attachment m. Since d commutes with pullbacks we also have

$$\Omega = -d\theta \tag{5}$$

An isometry of M relative to $<,>$ is a diffeomorphism $f : M \to M$ such that $< f_*(v), f_*(w) >=< v, w >$ for all vectors v, w in $T_m M$ at all points m of M. Again, $f_* : T_m M \to T_{f(m)} M$ denotes the differential map of f. From the definitions it is routine to show

$$f^*(\theta) = \theta \text{ and } f^*(\Omega) = \Omega \text{ for every isometry } f : M \to M \tag{6}$$

where f^* denotes the pullback of a differential form by f.

In this article we will always assume that $<,>$ is positive definite; that is, $<,>$ is a Riemannian structure on M. Moreover, we shall assume that M is a complete Riemannian manifold so that all geodesics are defined on $(-\infty, \infty)$. In particular, the geodesic flow $\{F^t\}$ is defined on TM for all t.

Remark:

Operating on the cotangent bundle TM^* has some obvious advantages over operating on the tangent bundle TM. The forms θ and Ω are intrinsically defined on TM^*, and they are preserved by the natural extension $\bar{f} : TM^* \to TM^*$ of any diffeomorphism $f : M \to M$. The corresponding forms θ and Ω on TM are not intrinsically defined but depend on a choice of nondegenerate bilinear form $<,>$ for the tangent spaces of M. Moreover, the forms θ and Ω are not left invariant by every diffeomorphism of M but only those diffeomorphisms that leave $<,>$ invariant; that is, the isometries of M. However, to define many interesting flows and vector fields on TN, such as the geodesic flow of Riemannian geometry, it is necessary to introduce an inner product $<,>$.

2.2 Geometry of the tangent bundle of a Riemannian manifold

Let M be a connected C^∞ manifold with a positive definite Riemannian structure $<,>$. We shall define a natural induced Riemannian structure $<<,>>$ on TM, usually called the Sasaki metric, and develop some of its basic properties. In particular we define a connection map $K : T(TM) \to M$ and use it to give an alternate definition of the symplectic 2-form Ω. We describe a natural almost complex structure J on TM that relates Ω and $<<,>>$. For further discussion see [E5] or [P].

2.2a Connection map

For each vector $v \in T_mM$ we define a linear map $K_v : T_v(TM) \to T_mM$ as follows. Let $\xi \in T_v(TM)$ be given and let $Z(t)$ be a smooth curve in TM with initial velocity ξ. If $\alpha(t) = \pi(Z(t))$, where $\pi : TM \to M$ is the projection, then we may regard $Z(t)$ as a vector field along the curve $\alpha(t)$ in M. Now define $K_v(\xi)$ to be $Z'(0)$, the covariant derivative at $t = 0$ of $Z(t)$ along $\alpha(t)$. By computing in local coordinates it is not difficult to show that $K_v(\xi)$ does not depend on the choice of curve $Z(t)$ in TM with initial velocity ξ. A formula for the connection map K in local coordinates may be found in [GKM], and to my knowledge this is the first discussion in the literature of the connection map.

2.2b Sasaki metric

It is not difficult to show that $\xi = 0$ in $T_v(TM) \Leftrightarrow d\pi(\xi) = 0$ and $K(\xi) = 0$. Hence if $H(v) = \ker K_v$ and $V(v) = \ker d\pi_v$, then $T_v(TM) = H(v) \oplus V(v)$, direct sum. We call $H(v)$ and $V(v)$ the horizontal and vertical subspaces of $T_v(TM)$ respectively.

Define the Sasaki metric $<<,>>$ on the tangent spaces of TM by

$$<<\xi,\eta>> = <d\pi(\xi), d\pi(\eta)> + <K(\xi), K(\eta)>$$

for $\xi, \eta \in T_v(TM)$ and $v \in TM$. Note that the vertical and horizontal subspaces are orthogonal relative to $<<,>>$.

Remark

If $f : M \to N$ is an isometry of Riemannian manifolds, then it is routine to show that the differential map $\tilde{f} : TM \to TN$ is an isometry relative to the associated Sasaki metrics.

2.2c Jacobi vector fields

The connection map allows one to define an explicit isomorphism between $T_v(TM)$ and the vector space of Jacobi vector fields $J(\gamma_v)$ along the geodesic γ_v with initial velocity v. Given a vector $\xi \in T_v(TM)$ we define $Y_\xi(t)$ to be the unique Jacobi vector field on γ_v such that $Y_\xi(0) = d\pi(\xi)$ and $Y_\xi'(0) = K(\xi)$, where $Y_\xi'(t)$ denotes the covariant derivative of $Y_\xi(t)$ along γ_v.

The map $\xi \to Y_\xi$ is a linear isomorphism of $T_v(TM)$ onto $J(\gamma_v)$. For any real number t, one may also show that $Y_\xi(t) = d\pi((\mathfrak{G}^t)_*\xi)$ and $Y_\xi'(t) = K((\mathfrak{G}^t)_*\xi)$, where $\{\mathfrak{G}^t\}$ denotes the geodesic flow in TM. See [E5] for further details.

2.2d Symplectic 2-form Ω

The symplectic 2-form Ω on TM has the following description in terms of the metric $<,>$ on$M$and the connection map K. See [P, p. 14] for details.

$$\Omega(\xi,\eta) = < d\pi(\xi), K(\eta) > - < K(\xi), d\pi(\eta) >$$

From this it follows that $\{(\mathfrak{G}^t)^*\Omega\}(\xi,\eta) = \Omega((\mathfrak{G}^t)_*\xi, (\mathfrak{G}^t)_*\eta) =< d\pi((\mathfrak{G}^t)_*\xi), K((\mathfrak{G}^t)_*\eta) > - < K((\mathfrak{G}^t)_*\xi), d\pi((\mathfrak{G}^t)_*\eta) >=< Y_\xi(t), Y_\eta'(t) > - < Y_\eta(t), Y_\xi'(t) >$. The Jacobi equation and curvature identities imply that the derivative of this function of t has derivative identically zero. This proves that $(\mathfrak{G}^t)^*\Omega = \Omega$ for all t.

Warning

It is not true that $(\mathfrak{G}^t)^*\theta = \theta$ for all t, where θ is the canonical 1-form on TM. In fact, if $\mathfrak{G}$ denotes the geodesic vector field with flow $\{\mathfrak{G}^t\}$, then $L_{\mathfrak{G}}\theta = \frac{d}{dt}|_{t=0}\{(\mathfrak{G}^t)^*\theta\} = dE$, where $E : TM \to \mathbb{R}$ is the energy function given by $E(v) = \frac{1}{2} < v, v >$ for $v \in TM$. However, if we restrict our attention and also θ to a hypersurface of constant energy, say the unit tangent bundle $SM = E^{-1}(1)$, then $(\mathfrak{G}^t)^*\theta = \theta$ for all t. See (5.1) for further details.

2.2e The almost complex structure J on TM

Given $v \in TM$ and a vector $\xi \in T_v(TM)$ we define $\xi_h = d\pi(\xi)$ and $\xi_v = K(\xi)$. We refer to ξ_h and ξ_v as the horizontal and vertical parts of ξ. We associate ξ with the pair $(\xi_h, \xi_v) \in T_mM \times T_mM$, where $m = \pi(v)$. Conversely, fix $v \in TM$ and let $m = \pi(v)$. Then for every pair (a,b) in $T_mM \times T_mM$ there exists a unique element $\xi \in T_v(TM)$ such that $(\xi_h, \xi_v) = (a,b)$.

Define a map $J = J_v : T_v(TM) \to T_v(TM)$ by requiring

$$d\pi(J(\xi)) = -K(\xi) \text{ and } K(J(\xi)) = d\pi(\xi) \text{ for all } \xi \in T_v(TM), v \in TM.$$

If we identify ξ with the pair (ξ_h, ξ_v) as above, then we may describe J as follows:

$$J(\xi)_h = -\xi_v \text{ and } J(\xi)_v = \xi_h \qquad \text{or equivalently}$$
$$J(\xi_h, \xi_v) = (-\xi_v, \xi_h)$$

From this description it is clear that $J^2 = -Id$ and J interchanges the horizontal and vertical subspaces $H(v)$ and $V(v)$ of $T_v(TM)$. Moreover, from the alternate description above of the symplectic form Ω in terms of K and $<,>$ it is routine to check that

$$\Omega(\xi,\eta) =<< J(\xi), \eta >> \text{ for all } \xi, \eta \in T_v(TM) \text{ and all } v \in TM.$$

It follows from the skew symmetry of Ω that J is skew symmetric relative to $<<,>>$. However, J is also a linear isometry relative to $<<,>>$ on each tangent space $T_v(TM)$

since $<< J(\xi), J(\eta) >>= - << J^2(\xi), \eta >>=<< \xi, \eta >>$ for all $\xi, \eta \in T_v(TM)$.

3. Poisson manifolds

3.1 Definition

A C^∞ manifold P is called a **Poisson manifold** *if there is a structure* $\{,\} : C^\infty(P) \times C^\infty(P) \to C^\infty(P)$ *that satisfies the following properties :*

1) (Skew symmetry) $-\{f, g\} = \{g, f\}$ *for all* $f, g \in C^\infty(P)$

2) (Bilinearity) $\{,\}$ *is* $\mathbb{R}$*-bilinear*

a) $\{af + bg, h\} = a\{f, h\} + b\{g, h\}$ for $a, b \in \mathbb{R}$ and $f, g, h \in C^\infty(P)$

b) $\{f, ag + bh\} = a\{f, g\} + b\{f, h\}$ for $a, b \in \mathbb{R}$ and $f, g, h \in C^\infty(P)$

3) (Leibniz) $\{f, g \cdot h\} = \{f, g\} \cdot h + g \cdot \{f, h\}$ for all $f, g, h \in C^\infty(P)$

4) (Jacobi Identity) $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$ for all $f, g, h \in C^\infty(P)$

If we are given a Poisson structure $\{,\}$ on $C^\infty(P)$, then for each $f \in C^\infty(P)$ we may define $X_f : C^\infty(P) \to C^\infty(P)$ by $X_f(g) = \{g, f\}$. The Leibniz property then becomes the statement $X_f(g \cdot h) = (X_f g) \cdot h + g \cdot (X_f h)$ for all $f, g, h \in C^\infty(P)$. In particular, each map X_f is a derivation on $C^\infty(P)$ and consequently X_f defines an element of $\mathfrak{X}(P)$, the C^∞ vector fields on P. The vector field X_f is called the Hamiltonian vector field on P determined by f.

Notation Let $\mathfrak{X}_H(P) = \{X_f : f \in C^\infty(P)\}$, the collection of Hamiltonian vector fields on P.

3.2 Reformulation of the Jacobi identity

The Jacobi identity may be restated in two equivalent ways:

1) $[X_f, X_g] = -X_{\{f,g\}}$ for all $f, g \in C^\infty(P)$, where $[X_f, X_g]$ is the usual Lie bracket in $\mathfrak{X}(P)$.

2) $X_f\{g, h\} = \{X_f g, h\} + \{g, X_f h\}$ for all $f, g, h \in C^\infty(P)$.

The first statement says that $f \to X_f$ is a Lie algebra antihomomorphism of $\{C^\infty(P), \{,\}\}$ into $\{\mathfrak{X}_H(P), [,]\}$. It also shows that $\mathfrak{X}_H(P)$ is a Lie subalgebra of $\mathfrak{X}(P)$ with respect to the Lie bracket of vector fields. The second statement says that X_f is a derivation of the Lie algebra $\{C^\infty(P), \{,\}\}$ for every $f \in C^\infty(P)$.

3.3 Examples of Poisson manifolds

We present some examples that are discussed in more detail below.

Example 1 Let P be a C^∞ manifold with a symplectic 2-form Ω. For every point x in P and every $\omega \in T_xP^*$ there is a unique vector $\xi \in T_xP$ such that $\omega(\xi^*) = \Omega(\xi, \xi^*)$

for all $\xi^* \in T_x P$; this follows from the fact that Ω is nondegenerate at every point of P. In particular, for every $f \in C^\infty(P)$ there exists a unique C^∞ vector field X_f such that $\Omega(X_f, \cdot) = df$. Now define $\{f, g\} = \Omega(X_f, X_g)$. We verify in example 2 of (3.7) that $\{,\}$ satisfies the Poisson axioms on P and X_f is the Hamiltonian vector field associated to f by the Poisson structure $\{,\}$.

Example 2 Let $\mathfrak{H}$ be a finite dimensional real Lie algebra. The Lie algebra $\mathfrak{H}$ may be regarded as the subspace of linear functions in $C^\infty(\mathfrak{H}^*)$ under the natural isomorphism between $\mathfrak{H}$ and $(\mathfrak{H}^*)^*$: given $A \in \mathfrak{H}$ and $\omega \in \mathfrak{H}^*$ define $A(\omega) = \omega(A)$. Define $\{,\}$ on $\mathfrak{H}^*$ by requiring that $\{A, B\} = [A, B]$ for all A, B in $\mathfrak{H}$. Then there is a unique extension of $\{,\}$ from $\mathfrak{H}$ to all of $C^\infty(\mathfrak{H}^*)$. Note that $\mathfrak{H}$ is a Lie subalgebra of $C^\infty(\mathfrak{H}^*)$. See example 3 of (3.7) for further discussion.

Example 3 Let $\mathfrak{H}$ be a finite dimensional real Lie algebra, and let $<,>$ be a nondegenerate, symmetric bilinear form on $\mathfrak{H}$. Let $^\# : \mathfrak{H} \to \mathfrak{H}^*$ be the isomorphism defined by $A^\#(B) =< A, B >$ for all $A, B \in \mathfrak{H}$. Define $\{,\}^\#$ on $\mathfrak{H}^* \subseteq C^\infty(H)$ by $\{A^\#, B^\#\} = [A, B]^\#$ for all A, B in $\mathfrak{H}$. Then $\{,\}^\#$ has a unique extension to a Poisson structure on $\mathfrak{H}$. As in the previous example, we note that $\mathfrak{H}^*$ is a Lie subalgebra of $C^\infty(\mathfrak{H})$. See example 4 of (3.7) for further details.

3.4 Symplectic Stratification

Example 2 shows that there are important Poisson structures that do not arise from a symplectic structure. Symplectic manifolds must have even dimension, but Lie algebras have no such restriction. However, the process of symplectic stratification allows one, in effect, to reduce to the case of a symplectic Poisson structure for many situations. We give a brief outline here and provide more details later. The interested reader should consult [O, Chapter 6] , [MR, Chapters 10-13] and the references in these books for proofs of the statements here.

The Hamiltonian foliation and its rank

For each point x of a Poisson manifold P let $\mathcal{H}(x) = \{Y(x) : Y \in \mathfrak{X}_H(P)\}$ and let rank(x) =dim $\mathcal{H}(x)$. Call $\mathcal{H}$ the Hamiltonian foliation in P even though the rank of $\mathcal{H}$ may not be constant in P. If the rank of $\mathcal{H}$ is constant in an open subset U of P, then by the Frobenius theorem the distribution $\mathcal{H}$ is integrable in U since $\mathfrak{X}_H(P)$ is closed under Lie brackets.

Symplectic leaves

An extension of the Frobenius theorem shows that if x is any point of P, then there exists a maximal integral manifold $L(x)$ of $\mathcal{H}$ that contains p and has dimension rank(x). Note that the restriction of any Hamiltonian vector field X_f to any leaf $L(x)$ is tangent to $L(x)$. In fact, the manifold $L(x)$ carries a symplectic 2-form Ω_x defined by $\Omega_x(X_f(x), X_g(x)) - \{f, g\}(x)$ for all $f, g \in C^\infty(P)$. In particular every leaf $L(x)$ has even dimension. The manifolds $L(x), x \in P$, are called the symplectic leaves of $\mathfrak{H}$

or P.

The discussion above shows that each Poisson manifold P can be decomposed into a disjoint union of immersed even dimensional submanifolds that carry a symplectic structure arising from the restriction of the Poisson structure of P. The symplectic leaves may not all have the same dimension since the foliation $\mathcal{H}$ may not have constant rank. We shall see later in (3.8a) that if $f : P_1 \to P_2$ is a C^∞ diffeomorphism that preserves Poisson structures then f maps each symplectic leaf in P_1 onto a symplectic leaf in P_2.

We consider left actions λ by a connected Lie group H on a Poisson manifold P such that the elements of H preserve the Poisson structure of P. It frequently occurs that H leaves each symplectic leaf invariant, which in principle allows one to study the dynamics of the action on each symplectic leaf. In this case each $X \in \mathfrak{H}$ defines a vector field $\lambda(X)$ on P that is tangent to each symplectic leaf and whose flow transformations are $\{\lambda_{e^{tX}}\}$. If in addition each vector field $\lambda(X)$ is Hamiltonian, then one obtains a momentum map $J : P \to \mathfrak{H}^*$ that is an important tool in analyzing the action of H. See (3.11e) for further discussion.

3.5 The Poisson structure in local coordinates

Before describing examples we exhibit formulas in local coordinates for the Poisson structure $\{,\}$ and the associated Hamiltonian vector fields. Let P be a Poisson manifold, and let $x = (x_1, \ldots x_n) : U \to \mathbb{R}^n$ be a local coordinate sytem defined on an open subset U of M. For C^∞ functions F, H on P the Poisson axioms yield the following:

$$(*) \qquad \{F, H\} = \sum_{i=1}^{n} \sum_{j=1}^{n} \{x_i, x_j\} \frac{\partial F}{\partial x_i} \frac{\partial H}{\partial x_j}$$

$$X_H = \sum_{i=1}^{n} \{\sum_{j=1}^{n} \{x_i, x_j\} \frac{\partial H}{\partial x_j}\} \frac{\partial}{\partial x_i}$$

The structure matrix $J(x)$

It is evident from these formulas that the Poisson structure is completely determined locally by the skew symmetric structure matrix $J(x) = J_{ij}(x) = \{x_i, x_j\}(x)$ for $x \in U$. The Jacobi identities $0 = \{x_i, \{x_j, x_k\}\} + \{x_j, \{x_k, x_i\}\} + \{x_k, \{x_i, x_j\}\}$ define a family of first order nonlinear partial differential equations for the structure matrix $J(x)$ that must be satisfied. Conversely, let (x, U) be a local coordinate system on P and let $J(x)$ be a skew symmetric matrix that satisfies these partial differential equations. If we define $\{x_i, x_j\}(x) = J_{ij}(x)$ and $\{f, g\}$ by the formula above in $(*)$ for functions f, g in $C^\infty(U)$, then $\{,\}$ defines a Poisson structure on $C^\infty(U)$. See [O, pp. 395-396] for a proof.

The observation above can be restated in another way. Let (x, U) be a local coordinate system in a manifold P. Suppose that $\{,\}$ is a bilinear pairing on $C^\infty(U)$ of the form $(*)$ above such that the Jacobi identities $0 = \{x_i, \{x_j, x_k\}\} + \{x_j, \{x_k, x_i\}\} + \{x_k, \{x_i, x_j\}\}$ are satisfied. Then $\{,\}$ defines a Poisson structure on $C^\infty(U)$.

This criterion is useful in the discussion below of the canonical Poisson structure on $P = \mathfrak{H}^*$, the dual space of a finite dimensional real Lie algebra $\mathfrak{H}$. This example can be dualized to a canonical Poisson structure on a Lie algebra $\mathfrak{H}$ with an inner product $<,>$, but in this case the Poisson structure depends on $<,>$.

Remarks

1. The rank of the Hamiltonian foliation $\mathcal{H}$ equals the rank of the structure matrix $J(x)$ in any local coordinate system. See [O, p. 399] for details.

2. If F, F^* and H, H^* are pairs of functions with the same derivative maps at x, then from the formulas above we see immediately that
   a) $\{F, H\}(x) = \{F^*, H^*\}(x)$.
   b) $X_{F^*}(x) = X_F(x)$.
   Conversely, if $X_{F^*}(x) = X_F(x)$ for two functions F and F^*, then F and F^* have the same differential maps at x. One may verify this either directly from the Poisson axioms or from the local coordinate representations of X_F and X_{F^*} above,

3. A vector subspace V of $C^\infty(P)$ will be called first order dense in $C^\infty(P)$ if for every point m of P and every element $\overline{f}$ of $C^\infty(P)$ there exists an element f of V such that f and $\overline{f}$ have the same differential map at m. For example, the linear functions W^* on a finite dimensional real vector space W are first order dense in $C^\infty(W)$.
   If V is first order dense in $C^\infty(P)$ and $\{,\} : V \times V \to V$ is a map that satisfies the Poisson axioms, then by 2) there is at most one extension of $\{,\}$ to a Poisson structure on $C^\infty(P)$.

3.6 Local coordinates of Lie - Weinstein (cf. [O, p.405], [MR, p. 348])

If the Poisson structure $\{,\}$ has constant rank $2n$ in some open set U of P, then we may hope to choose local coordinates cleverly so that the structure matrix $J(x)$ has the simplest possible form in U. To see what this simple form might be we consider a single skew symmetric $m \times m$ matrix A. By linear algebra A has rank $2n \leq m = 2n + \ell$ and there exists an element g of $O(m)$ such that $B = gAg^{-1}$ has the following canonical form:

(*) 1) $B_{ij} = 0$ for $i \geq 2n+1$ or $j \geq 2n+1$.
2) The upper $2n \times 2n$ block, namely $\{B_{ij} : 1 \leq i, j \leq 2n\}$ consists of n copies of the 2×2 matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ along the diagonal and zeros elsewhere.

We can now make our question more precise. If $\{,\}$ has constant rank $2n$ in some open set U of P, then can we find a local coordinate system around each point of U so that the structure matrix $J(x)$ has the canonical form described above in (*) ?

The answer is yes, and local coordinate systems with this property are called Lie - Weinstein coordinates.

We formulate this result in greater detail.

Canonical form for the structure matrix $J(x)$

Proposition Let P be a Poisson manifold. Suppose that the Poisson structure $\{,\}$ has constant rank $2n$ in some open subset U of P. Then for each point m of U there exists a coordinate system $x = (q_1, p_1, q_2, p_2, \ldots, q_n, p_n, z_1, \ldots, z_\ell)$ in an open set V with $p \in V \subseteq U$ such that

1. $\{p_i, p_j\} = \{q_i, q_j\} = 0$ for all i, j
   $\{q_i, p_j\} = \delta_{ij}$
   $\{p_i, z_r\} = \{q_i, z_r\} = \{z_s, z_r\} = 0$ for $1 \le i \le n$ and $1 \le r, s \le \ell$
2. $X_{p_i} = \frac{\partial}{\partial q_i}$ and $X_{q_i} = -\frac{\partial}{\partial p_i}$ for $1 \le i \le n$
   $X_{z_r} = 0$ for $1 \le r \le \ell$
3. A function $f : U \to \mathbb{R}$ is a Casimir function (cf. (3.9)) $\Leftrightarrow \frac{\partial f}{\partial p_i} = \frac{\partial f}{\partial q_j} = 0$ for $1 \le i, j \le n$. In this case $f = A(z_1, \ldots, z_\ell)$ for some function A of ℓ variables.

Remark The bracket relations in 1) are equivalent to the statement that the structure matrix J for this coordinate system has the canonical form in (*) above.

As a straight forward consequence of the result above we obtain

Corollary Let m be a point of P such that the rank of $\{,\}$ is $\equiv 2n$ in some neigborhood of m. Let P have dimension $2n+\ell$ for some integer $\ell \ge 0$. Then m lies in a coordinate neighborhood U with coordinate functions $x = (q_1, p_1, q_2, p_2, \ldots, q_n, p_n, z_1, \ldots, z_\ell)$ such that

a) The symplectic leaves in U are the slice submanifolds $(z_1, \ldots, z_\ell) = (c_1, \ldots, c_\ell)$, where $\{c_i\}$ are constants.

b) The Poisson bracket takes the form $\{F, H\} = \sum_{i=1}^{n} \{\frac{\partial F}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial H}{\partial q_i}\}$.

Proof of the Proposition The coordinates p, q, z of the proposition are constructed inductively, two at a time. The first step is the following

Lemma Let P be a Poisson manifold of dimension N, and let $p : P \to \mathbb{R}$ be a C^∞ function with Hamiltonian vector field X_p. Let m be a point of P such that $X_p(m) \neq 0$. Then there exists a neighborhood O of m and functions $q, y_3, \ldots, y_N : O \to \mathbb{R}$ with the following properties :

1) $z = (q, p, y_3, \ldots, y_N) : O \to \mathbb{R}^N$ is a coordinate system in O.

2) $\{q, p\} = 1, \{q, y_i\} = \{p, y_i\} = 0$ for $i \ge 3$.
   $X_p = \frac{\partial}{\partial q}, X_q = -\frac{\partial}{\partial p}$

3) $\frac{\partial}{\partial p}\{y_i, y_j\} = \frac{\partial}{\partial q}\{y_i, y_j\} = 0$ for all $i, j \ge 3$.

Proof By (1.3) there exists a coordinate system $x = (x_1, \ldots, x_N)$ in a neighborhood V of m such that $X_p = \frac{\partial}{\partial x_1}$. If $q = x_1$, then $\{q,p\} = X_p(q) = 1$, and it follows that $[X_p, X_q] = -X_{\{q,p\}} = 0$ since $X_c = 0$ for any constant function c. By (1.3) we may choose a different coordinate system $y = (y_1, \ldots, y_N)$ in a neighborhood W of m such that $W \subseteq V, -X_q = \frac{\partial}{\partial y_1}$ and $X_p = \frac{\partial}{\partial y_2}$ in W. Now define $z = (q, p, y_3, \ldots, y_N) : W \to \mathbb{R}^N$. We show that there exists a neighborhood O of m with $O \subseteq W$ such that $z = (q, p, y_3, \ldots, y_N) : O \to \mathbb{R}^N$ has the properties of the lemma.

To prove 1) it suffices to show that z is nonsingular at m. The determinant of the Jacobian matrix $\frac{\partial z}{\partial y}$ is $\frac{\partial q}{\partial y_1}\frac{\partial p}{\partial y_2} - \frac{\partial p}{\partial y_1}\frac{\partial q}{\partial y_2} = dq \wedge dp(\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}) = dq \wedge dp(-X_q, X_p) = dq \wedge dp(X_p, X_q) = dq(X_p)dp(X_q) - dq(X_q)dp(X_p) = \{q,p\}\{p,q\} - \{q,q\}\{p,p\} = -1$. Hence z is nonsingular at all points of W, which proves 1).

2) We observed already that $\{q,p\} = 1$. Now, $\{y_i, p\} = X_p(y_i) = \frac{\partial y_i}{\partial y_2} = 0$ for $i \geq 3$. Similarly, $\{y_i, q\} = X_q(y_i) = -\frac{\partial y_i}{\partial y_1} = 0$ for $i \geq 3$. To prove the remaining assertions of 2) we relabel the coordinates. Let $z_1 = q, z_2 = p$ and $z_k = y_k$ for $k \geq 3$. Then from the bracket relations above and the local coordinate form for a Hamiltonian vector field we obtain $X_p = \sum_{k=1}^{n}(\sum_{j=1}^{n}\{z_k, z_j\}\frac{\partial p}{\partial z_j})\frac{\partial}{\partial z_k} = \sum_{k=1}^{n}\{z_k, p\}\frac{\partial}{\partial z_k} = \frac{\partial}{\partial q}$. Similarly, $X_q = \sum_{k=1}^{n}(\sum_{j=1}^{n}\{z_k, z_j\}\frac{\partial q}{\partial z_j})\frac{\partial}{\partial z_k} = \sum_{k=1}^{n}\{z_k, q\}\frac{\partial}{\partial z_k} = -\frac{\partial}{\partial p}$.

3) $\frac{\partial}{\partial p}\{y_i, y_j\} = -X_q\{y_i, y_j\} = -\{\{y_i, y_j\}, q\} = \{\{y_j, q\}, y_i\} + \{\{q, y_i\}, y_j\} = 0$ if $i, j \geq 3$ by the Jacobi identity and the bracket relations in 2). A similar argument shows that $\frac{\partial}{\partial q}\{y_i, y_j\} = 0$ if $i, j \geq 3$. The proof of the lemma is complete. ∎

Proof of the Proposition Let $N = \dim\ P$. Let U be an open set of P such that $\{,\}$ has constant rank $2n \geq 2$ on U. Fix a point m of U and choose a function $p : U \to \mathbb{R}$ such that $X_p(m) \neq 0$. Choose an open set O with $m \in O \subseteq U$ such that the conditions of the lemma hold. The structure matrix J in the coordinates $z = (q, p, y_3, ..., y_N)$ takes the block diagonal form $J = \begin{pmatrix} A & 0 \\ 0 & J_1 \end{pmatrix}$, where $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and J_1 is an $(N-2) \times (N-2)$ matrix whose entries are $\{y_i, y_j\}, i, j \geq 3$. Clearly the rank of J_1 is $2n - 2$ in U since the rank of J is $2n$ in U.

If $2n - 2 = 0$, then J_1 is the zero matrix and $\{y_i, y_j\} \equiv 0$ in U for all $i, j \geq 3$. In this case we set $q_1 = q, p_1 = p$ and $z_i = y_{i+2}$ for $1 \leq i \leq N - 2$. It is easy to see that these coordinates in U satisfy the assertions of the proposition.

If $2n - 2 > 0$, then we repeat the method of the lemma. Let $q_1 = q$ and $p_1 = p$. Write $m = (m_1, \ldots, m_N)$ in the coordinates $(q_1, p_1, y_3, \ldots, y_N)$. Now consider the foliation of U into codimension 2 submanifolds U_{c_1,c_2} defined by setting $q_1 = c_1$ and $p_1 = c_2$. Each submanifold U_{c_1,c_2} has a coordinate system $y = (y_3, \ldots, y_N)$ around $m' = (m_3, \ldots, m_N)$ in which the entries $\{y_i, y_j\}, i, j \geq 3$, of the structure matrix J_1 depend only on $y_3, \ldots, y_N$ by 3) of the lemma. Hence each submanifold U_{c_1,c_2} inherits from P a Poisson structure of rank $2n - 2$ with structure matrix J_1. For one of the submanifolds U_{c_1,c_2} (it doesn't matter which) we repeat the method of the lemma above to obtain a new coordinate system $(q_2, p_2, z_5, \ldots, z_N)$ on U_{c_1,c_2}

in a neighborhood U' of m' such that the structure matrix of U_{c_1,c_2} has the form $J_1' = \begin{pmatrix} A & 0 \\ 0 & J_2 \end{pmatrix}$, where $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, J_2 has entries $\{z_i, z_j\}, i, j \geq 5$ and J_2 has rank $2n-4$ on U'. Making the original neighborhood U smaller if necessary we now have coordinates $(q_1, p_1, q_2, p_2, z_5, \ldots, z_N)$ on U so that the structure matrix J has the form $\begin{pmatrix} B & 0 \\ 0 & J_2 \end{pmatrix}$, where $B = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$, $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\frac{\partial}{\partial p_\alpha}\{z_i, z_j\} = \frac{\partial}{\partial q_\alpha}\{z_i, z_j\} = 0$ for $\alpha = 1, 2$ and $i, j \geq 5$. It is now clear how to repeat this process until the coordinates $(q_1, p_1, q_2, p_2, \ldots, q_n, p_n, z_1, \ldots, z_\ell)$ described in the statement of the proposition have been achieved. ■

3.7 Examples of Poisson structures

1. A simple example in $\mathbb{R}^{2n}$

Let $x_1, \ldots, x_{2n}$ be the standard coordinate functions and relabel them so that $p_i = x_i$ for $1 \leq i \leq n$ and $q_i = x_{n+i}$ for $1 \leq i \leq n$. Define $\{p_i, p_j\} = 0; \{q_i, q_j\} = 0$ and $\{q_i, p_j\} = \delta_{ij}$. Substituting these structure functions into the formula above yields

$$\{F, H\} = \sum_{i=1}^{n} \{\frac{\partial F}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial H}{\partial q_i}\}$$

$$X_H = \sum_{i=1}^{n} \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q_i} - \sum_{i=1}^{n} \frac{\partial H}{\partial q_i}\frac{\partial}{\partial p_i}$$

One may check directly that that $\{,\}$ satisfies the Poisson axioms, but we omit this computation. This example is a special case of the next one.

2. Symplectic manifolds

Let P be a $2n$-manifold with a symplectic structure arising from a symplectic 2-form Ω. Since Ω is nondegenerate at each point of P, for every $m \in P$ and every $\omega \in (T_m P)^*$ there exists a unique vector $\xi \in T_m P$ such that $\Omega(\xi, \xi') = \omega(\xi')$ for all $\xi' \in T_m P$. In particular, if $f : P \to \mathbb{R}$ is any C^∞ function, then there exists a unique vector field X_f on P such that $\Omega(X_f, Y) = df(Y)$ for all $Y \in \mathfrak{X}(P)$. In terms of the interior product we may express this relationship as

$$i_{X_f}\Omega = df \text{ for all } f \in C^\infty(P).$$

Given $f, g \in C^\infty(P)$ we now define

$$\{f, g\} = \Omega(X_f, X_g)$$

We verify the Poisson axioms. It is not difficult to show that

$$[X_f, X_g] = -X_{\{f,g\}} \text{ for all } f, g \in C^\infty(P).$$

It now follows from the Jacobi identity for [,] on $\mathfrak{X}(P)$ that $\{,\}$ satisfies the Jacobi identity on $C^\infty(P)$. The skew symmetry and bilinearity of $\{,\}$ are evident from the definition and the fact that the map $f \to X_f$ is $\mathbb{R}$-linear. Finally, since $\{f,g\} = \Omega(X_f, X_g) = df(X_g) = X_g(f)$ it is clear that the Leibnizian property of $\{,\}$ follows from the Leibnizian property of the vector field X_g for each $g \in C^\infty(P)$. This shows that $\{,\}$ satisfies the Poisson axioms. Moreover, for each $f \in C^\infty(P)$ the vector field X_f defined above by Ω is precisely the Hamiltonian vector field associated to f by the Poisson structure $\{,\}$.

Hamiltonian foliation of a symplectic manifold

In a symplectic manifold P the Hamiltonian foliation is trivial; that is, $\mathcal{H}(m) = T_mP$ for every $m \in P$. Given a point $m \in M$ and a vector $\xi \in T_mP$ recall that $i_\xi\Omega \in T_mP^*$ is defined by $i_\xi\Omega(\eta) = \Omega(\xi,\eta)$ for all $\eta \in T_mP$. Let $f \in C^\infty(P)$ be a function such that $df_m = i_\xi\Omega \in T_mP^*$. By definition, $i_{X_f(m)}\Omega = df_m$, and hence $X_f(m) = \xi$ by the nondegeneracy of Ω. This proves that $T_mP = \{X_f(m) : f \in C^\infty P)\} = \mathfrak{H}(m)$ for all m.

Special case

Let $M = \mathbb{R}^{2n}$ with coordinates $p_1, ..., p_n, q_1, ..., q_n$ and let $\Omega = \sum_{i=1}^{n} dq_i \wedge dp_i$. If X is any vector field on $\mathbb{R}^{2n}$, then it is easy to compute

$$i_X\Omega = \sum_{i=1}^{n} X(q_i)dp_i - \sum_{i=1}^{n} X(p_i)dq_i .$$

In particular, $i_{X_H}\Omega = dH$ satisfies the equations $X_H(q_i) = \frac{\partial H}{\partial p_i}$ and $X_H(p_i) = -\frac{\partial H}{\partial q_i}$ or equivalently $X_H = \sum_{i=1}^{n} \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q_i} - \sum_{i=1}^{n} \frac{\partial H}{\partial q_i}\frac{\partial}{\partial p_i}$. A comparison with the first example shows that these two Poisson structures are the same.

3. The canonical Poisson structure on $\mathfrak{H}^*$

Let $\mathfrak{H}$ be a finite dimensional real Lie algebra, and let $\mathfrak{H}^*$ denote its dual space. The Lie algebra structure [,] on $\mathfrak{H}$ defines a canonical Poisson structure on $\mathfrak{H}^*$. First, we observe that each element X of $\mathfrak{H}$ can be regarded as a linear function from $\mathfrak{H}^*$ to $\mathbb{R}$ by defining $X(\omega) = \omega(X)$ for every $\omega \in \mathfrak{H}^*$. Define $\{,\} : \mathfrak{H} \times \mathfrak{H} \to \mathfrak{H} \subseteq C^\infty(\mathfrak{H}^*)$ by $\{X,Y\} = [X,Y]$.

Next, we extend $\{,\}$ to a Poisson structure on $\mathfrak{H}^*$. Note that there is at most one extension by the remarks 2) and 3) of (3.5).

If $\{x_1, ..., x_n\}$ is a basis for $\mathfrak{H}$, then $x = (x_1, ..., x_n) : \mathfrak{H}^* \to \mathbb{R}^n$ defines a linear coordinate system on $\mathfrak{H}^*$. Let $\{C_{ij}^k\}$ be the structure constants defined by $[x_i, x_j] = \sum_{k=1}^{n} C_{ij}^k x_k$. If $A = \sum_{i=1}^{n} A_i x_i$ and $B = \sum_{j=1}^{n} B_j x_j$ are arbitrary elements of $\mathfrak{H}$, then $\{A, B\} = [A, B] = \sum_{k=1}^{n}\{\sum_{i,j=1}^{n} C_{ij}^k A_i B_j\} x_k$. Noting that $A_i = \frac{\partial A}{\partial x_i}$ and $B_j = \frac{\partial B}{\partial x_j}$ we guess that for arbitrary C^∞ functions $f, g : \mathfrak{H}^* \to \mathbb{R}$ the pairing

$$(^*)\{f, g\} = \sum_{k=1}^{n}\{\sum_{i,j=1}^{n} C_{ij}^k \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}\} x_k$$

with structure functions

$$\{x_i, x_j\} = \sum_{k=1}^{n} C_{ij}^k x_k$$

is a good candidate for a Poisson structure on $C^\infty(\mathfrak{H}^*)$. For $f \in C^\infty(\mathfrak{H}^*)$ the map $X_f : C^\infty(\mathfrak{H}^*) \to C^\infty(\mathfrak{H}^*)$ given by $X_f(g) = \{g, f\}$ has the local coordinate formula

$$X_f = \sum_{j=1}^{n}\{\sum_{i,k=1}^{n} C_{ji}^k \frac{\partial f}{\partial x_i} x_k\}\frac{\partial}{\partial x_j}$$

It is evident that X_f is a C^∞ vector field on $\mathfrak{H}^*$ for each $f \in C^\infty(\mathfrak{H}^*)$, and this property is equivalent to the Leibnizian property of $\{,\}$. The skew symmetry and $\mathbb{R}$-bilinearity of $\{,\}$ are obvious from the expression for $\{f, g\}$ in (*) and the fact that $C_{ij}^k = -C_{ji}^k$ for all i, j, k.

It remains only to check the Jacobi identity for $\{,\}$. By the discussion above in (3.5) it suffices to check the Jacobi identity on the coordinate functions $\{x_1, ..., x_n\}$ for $\mathfrak{H}^*$. However, since $\{x_i, x_j\} = [x_i, x_j]$ the Jacobi identity for $\{,\}$ follows immediately from the Jacobi identity for $\mathfrak{H}$.

Hamiltonian foliation in $\mathfrak{H}^*$

Let H be a connected Lie group with Lie algebra $\mathfrak{H}$. For all $\omega \in \mathfrak{H}$ we show that $\mathcal{H}(\omega) = \mathrm{ad}^*\mathfrak{H}(\omega) = T_\omega\, \mathrm{Ad}^*H(\omega)$, the tangent space at ω to the Ad^*H orbit of ω. This proves at the same time that the orbits of Ad^*H are the symplectic leaves of the canonical Poisson structure on $\mathfrak{H}^*$. See example 2 of (1.4) for a definition and discussion of $\mathrm{Ad}^*H \subseteq GL(\mathfrak{H}^*)$ and its Lie algebra $\mathrm{ad}^*\mathfrak{H} \subseteq \mathrm{End}(\mathfrak{H}^*)$

Given an element $\omega \in \mathfrak{H}^*$ we let η_ω denote the element of $T_\omega\mathfrak{H}^*$ that is the initial velocity of $t \to \omega + t\eta$. Regard $\mathfrak{H}$ as the vector space of linear functions on $\mathfrak{H}^*$ defined by $A(\omega) = \omega(A)$ for all $A \in \mathfrak{H}$ and all $\omega \in \mathfrak{H}^*$. It follows from the definitions that $\eta_\omega(B) = \eta(B)$ for all $B \in \mathfrak{H}$ and all $\eta, \omega \in \mathfrak{H}^*$.

Given $A, B \in \mathfrak{H}$ and $\omega \in \mathfrak{H}^*$ we compute $X_A(\omega)(B) = \{B, A\}(\omega) = [B, A]\,(\omega) = \omega([B, A]) = \mathrm{ad}^*A(\omega)(B)$. It follows that $X_A(\omega) = \mathrm{ad}^*A(\omega)_\omega$ for all $A \in \mathfrak{H}$ and all $\omega \in \mathfrak{H}^*$ since a vector field on $\mathfrak{H}^*$ is determined by its values on linear functions. Recall that $\mathfrak{H}$ is first order dense in $C^\infty(\mathfrak{H}^*)$.

If $f \in C^\infty(\mathfrak{H}^*)$ and $\omega \in \mathfrak{H}^*$ are given, then since $\mathfrak{H}$ is first order dense in $C^\infty(\mathfrak{H}^*)$ there exists $A \in \mathfrak{H}$ such that $X_f(\omega) = X_A(\omega) = \mathrm{ad}^*A(\omega)_\omega$. This proves that $\mathcal{H}(\omega) = \mathrm{ad}^*\mathfrak{H}(\omega)$, and it is an easy exercise to show that $\mathrm{ad}^*\mathfrak{H}(\omega) = T_\omega \mathrm{Ad}^*H(\omega)$.

4. The canonical Poisson structure on $\{\mathfrak{H}, <, >\}$

Now let $<, >$ be a positive definite inner product on a finite dimensional real Lie algebra $\mathfrak{G}$. We dualize the construction of the previous example. This construction of a Poisson structure $\{,\}$ depends on the choice of $<, >$, but it is important for the later discussion. The construction of $\{,\}$ actually works for any nondegenerate, symmetric bilinear form on $\mathfrak{H}$, but our interest is in the Riemannian case.

As before, we let $^{\#} : \mathfrak{H} \to \mathfrak{H}^*$ be the linear isomorphism defined by $A^{\#}(B) = < A, B >$ for all A, B in $\mathfrak{H}$. We regard $\mathfrak{H}^*$ as the subspace of $C^{\infty}(\mathfrak{H})$ consisting of linear functions.

Proposition For every positive definite inner product $<,>$ on $\mathfrak{H}$ there exists a unique Poisson structure $\{,\}$ on $\mathfrak{H}$ such that

1) If $f, g \in \mathfrak{H}^*$, then $\{f,g\} \in \mathfrak{H}^*$.
2) $\{A^{\#}, B^{\#}\} = [A,B]^{\#}$ for all $A, B \in \mathfrak{H}$.

Proof By remarks 2) and 3) in (3.5) there can be at most one Poisson structure $\{,\}$ that satisfies 1) and 2) since the subspace $\mathfrak{H}^*$ is first order dense in $C^{\infty}(\mathfrak{H})$.

To show that there exists a Poisson structure $\{,\}$ satisfying 1) and 2) let $\{E_1, ..., E_n\}$ be an orthonormal basis of $\mathfrak{H}$ with structure constants $\{C_{ij}^k\}$ defined by $[E_i, E_j] = \sum_{k=1}^{n} C_{ij}^k E_k$. The dual basis $\{x_1, ..., x_n\}$ in $\mathfrak{H}^*$ defines a linear coordinate system $x = (x_1, ..., x_n) : \mathfrak{H} \to \mathbb{R}^n$. It is evident that $x_i = E_i^{\#}$ and hence the condition 2) implies that

$$(^*) \qquad \{x_i, x_j\} = \sum_{k=1}^{n} C_{ij}^k x_k$$

Substituting these expressions into the local coordinate formula for a Poisson structure yields the following candidate for a Poisson structure :

$$\{f,g\} = \sum_{k=1}^{n} \{ \sum_{i,j=1}^{n} C_{ij}^k \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} \} x_k$$

$$X_f = \sum_{j=1}^{n} \{ \sum_{i,k=1}^{n} C_{ji}^k \frac{\partial f}{\partial x_i} x_k \frac{\partial}{\partial x_j} \} \frac{\partial}{\partial x_j}$$

Note: These formulas have the same appearance as those of the previous example. However, observe that in the previous example we used an arbitrary basis $\{x_i\}$ of $\mathfrak{H}$ to define a linear coordinate system $x = (x_1, ..., x_n) : \mathfrak{H}^* \to \mathbb{R}^n$. Here we need a basis $\{x_i\}$ of $\mathfrak{H}^*$ that is dual to an orthonormal basis of $\mathfrak{H}$. This should not be surprising since the Poisson structure $\{,\}$ on $\mathfrak{H}$ depends on the inner product $<,>$ on $\mathfrak{H}$.

In the formula above for $\{,\}$ it is immediately evident that 1) is satisfied. Condition 2) reduces to $(^*)$ above. By the discussion in (3.5) it remains only to check the Jacobi identity for the coordinate functions $\{x_i\}$ to show that $\{,\}$ is a Poisson structure on $\mathfrak{H}$. From 2) and the discussion above it follows that $\{x_j, x_k\} = \{E_j^{\#}, E_k^{\#}\} = [E_j, E_k]^{\#}$ and $\{x_i, \{x_j, x_k\}\} = \{E_i^{\#}, \{E_j^{\#}, E_k^{\#}\}\} = [E_i, [E_j, E_k]]^{\#}$. Hence the Jacobi identity for $\{x_i\}$ follows from the Jacobi identity on $\mathfrak{H}$. ■

Hamiltonian foliation in $\mathfrak{H}$

We define an odd looking left action λ of H on $\mathfrak{H}$ by $\lambda_h(A) = Ad(h^{-1})^t(A)$, where $Ad(h^{-1})^t$ denotes the metric transpose defined by $<,>$ of $Ad(h^{-1}) : \mathfrak{H} \to \mathfrak{H}$. It is easy to check that $\text{Ad}^*(h) \circ \# = \# \circ \lambda_h$ for all $h \in H$, where $^{\#} : \mathfrak{H} \to \mathfrak{H}^*$ is the isomorphism defined by $<,>$ and Ad^* denotes the coadjoint action of H on $\mathfrak{H}^*$. Since $^\# : \mathfrak{H} \to \mathfrak{H}^*$ is a Poisson map (details omitted) it follows that the symplectic leaves of the canonical Poisson structure on $\mathfrak{H}^*$ are the images under $^\#$ of the symplectic leaves on $\mathfrak{H}$ with the Poisson structure defined by $<,>$. See (3.8a). We saw earlier that the symplectic leaves on $\mathfrak{H}^*$ are the orbits of $\mathrm{Ad}^*(H)$. It follows that the symplectic leaves on $\mathfrak{H}$ are the orbits of $\lambda(H)$. Hence $\mathcal{H}(A) = T_A\lambda(H)(A)$ for every $A \in \mathfrak{H}$.

Invariant Hamiltonian formula for linear functions

For later use we give an invariant description of the Hamiltonian vector field X_f determined by a linear function $f : \mathfrak{H} \to \mathbb{R}$, (i.e. an element of $\mathfrak{H}^*$). For elements $\alpha, \xi, \eta \in \mathfrak{H}$ let $\xi_\alpha \in T_\alpha\mathfrak{H}$ denote the initial velocity of $t \to \alpha + t\xi$ and define $<,>_\alpha$ on $T_\alpha\mathfrak{H}$ in the usual way by $< \xi_\alpha, \eta_\alpha >_\alpha = < \xi, \eta >$. Then for elements α, A and $\xi \in \mathfrak{H}$ we have

$$< X_{A^\#}(\alpha), \xi_\alpha >_\alpha = - < \alpha, [A, \xi] > .$$

Proof For $\alpha, \xi \in \mathfrak{H}$ it is routine to show that $(\mathrm{grad}\ \xi^\#)(\alpha) = \xi_\alpha$ since $\xi^\# \in \mathfrak{H}^*$ is a linear function on $\mathfrak{H}$. It follows that $< X_{A^\#}(\alpha), \xi_\alpha >_\alpha = < X_{A^\#}(\alpha), (\mathrm{grad}\ \xi^\#)(\alpha) >_\alpha = (d\xi^\#(X_{A^\#}))(\alpha) = (X_{A^\#})(\xi^\#))(\alpha) = \{\xi^\#, A^\#\}(\alpha) = -\{A^\#, \xi^\#\}(\alpha) = -[A, \xi]^\#(\alpha) = - < \alpha, [A, \xi] >$. ■

Invariant Hamiltonian formula for arbitrary functions in $C^\infty(\mathfrak{H})$

Let $f \in C^\infty(H)$ be arbitrary, and let elements $\alpha, \xi \in \mathfrak{H}$ be given. Then

$$< X_f(\alpha), \xi_\alpha >_\alpha = - < \alpha, [\ \mathrm{grad} f(\alpha), \xi] >$$

Proof The meaning of the right hand side of the equality requires some explanation. Let $f \in C^\infty(\mathfrak{H})$ and $\alpha \in \mathfrak{H}$ be given. Let $A \in \mathfrak{H}$ be the unique element such that grad $f(\alpha) = A_\alpha$. Now define $[\ \mathrm{grad} f(\alpha), \xi] = [A, \xi]$.

Given $f \in C^\infty(\mathfrak{H})$ and $\alpha \in \mathfrak{H}$ we choose $A \in \mathfrak{H}$ so that grad $f(\alpha) = A_\alpha$. It follows easily that $df_\alpha = dA^\#_\alpha$. From the formula above for linear functions and remark 2) of (3.5) we see that $< X_f(\alpha), \xi_\alpha >_\alpha = < X_{A^\#}(\alpha), \xi_\alpha >_\alpha = - < \alpha, [A, \xi] > - < \alpha, [\ \mathrm{grad} f(\alpha), \xi] >$. ■

3.8 Poisson maps and automorphisms

Let P_1 and P_2 be Poisson manifolds with Poisson structures $\{,\}_1$ and $\{,\}_2$. A C^∞ map $\varphi : P_1 \to P_2$ is called a Poisson map if $\{f \circ \varphi, g \circ \varphi\}_1 = \{f, g\}_2 \circ \varphi$ for all $f, g \in C^\infty(P_2)$. If $P_1 = P_2$ and φ is a diffeomorphism, then φ is called a Poisson automorphism. The collection $\mathfrak{P}(P)$ of all Poisson automorphisms of a Poisson manifold P is a subgroup of the diffeomorphism group Diff(P) of P.

We call a C^∞ map $\varphi : P_1 \to P_2$ an anti Poisson map if $\{f \circ \varphi, g \circ \varphi\}_1 = -\{f, g\}_2 \circ \varphi$ for all $f, g \in C^\infty(P_2)$. If H is a connected Lie group with Lie algebra $\mathfrak{H}$, then the Gauss map $G : TH \to \mathfrak{H}$ is an anti Poisson map See example 6 of (3.8b) below for a precise statement and proof.

3.8a Basic properties of Poisson maps

Proposition Let $P_1, \{,\}_1$ and $P_2, \{,\}_2$ be Poisson manifolds, and let $f : P_1 \to P_2$ be a Poisson map. Then

1) For every $H \in C^\infty(P_2)$ the vector fields $X_{H\circ f}$ and X_H are f-related.

2) If f is a diffeomorphism, then f preserves the symplectic stratifications; that is, $f(L_1(x)) = (L_2(fx))$ for all $x \in P_1$, where $L_1(x)$ and $L_2(fx)$ denote the symplectic leaves of P_1 and P_2 containing x and $f(x)$ respectively.

Proof

1) Given $H, H' \in C^\infty(P_2)$ and $x \in P_1$ we compute $f_*(X_{H\circ f}(x))(H') = X_{H\circ f}(x)(H' \circ f) = \{H' \circ f, H \circ f\}(x) = \{H', H\}(fx) = X_H(fx)(H')$. Since $H' \in C^\infty(P_2)$ was arbitrary it follows that $f_*(X_{H\circ f}(x)) = X_H(fx)$.

2) Let $\mathcal{H}_1$ and $\mathcal{H}_2$ denote the foliations of P_1 and P_2 spanned by the Hamiltonian vector fields. The symplectic leaves of P_1 and P_2 are the arc connected integral manifolds of $\mathcal{H}_1$ and $\mathcal{H}_2$. If $\gamma(t)$ is any C^∞ curve in $L_1(x)$, then $(f \circ \gamma)(t)$ is everywhere tangent to $\mathcal{H}_2$ since $f_*(\mathcal{H}_1(x')) = \mathcal{H}_2(fx')$ for all $x' \in P_1$ by 1). Hence $(f \circ \gamma)(t)$ lies in a single leaf of $\mathcal{H}_2$, namely $\mathcal{H}_2(fx)$. This proves that $f(L_1(x)) \subseteq (L_2(fx))$ for all $x \in P_1$. We obtain equality by applying the same argument to the inverse of f. ■

3.8b Examples of Poisson maps

Example 1 Pullbacks on TP by diffeomorphisms on P

If Q is any C^∞ manifold, let $P = TQ^*$, the cotangent bundle of Q, and let $\{,\}$ be the Poisson structure on P that arises from the canonical symplectic structure Ω on P. Then every diffeomorphism f of Q extends to a Poisson automorphism f^* of TQ^*, where f^* denotes the pull back action of f on P.

The proof of this assertion is routine. We now present some other examples of Poisson maps and automorphisms that will be discussed later in greater detail.

Example 2 Musical isomorphism $^\# : \mathfrak{H} \to \mathfrak{H}^*$

Let $\mathfrak{H}$ be a finite dimensional real Lie algebra, and let $<,>$ be a nondegenerate, symmetric bilinear form on $\mathfrak{H}$. Let $^\# : \mathfrak{H} \to \mathfrak{H}^*$ be the isomorphism induced by $<,>$. Let $\{,\}^*$ denote the canonical Poisson structure on $\mathfrak{H}^*$, and let $\{,\}$ be the Poisson structure on $\mathfrak{H}$ determined canonically by $<,>$. Then $^\# : \mathfrak{H} \to \mathfrak{H}^*$ is a Poisson map.

Example 3 Infinitesimal Poisson automorphisms

Let P be any Poisson manifold with a Poisson structure $\{,\}$. Let $\mathfrak{X}_{\{,\}}(P)$ denote the set of all vector fields X on P such that $L_X\{,\} = 0$; that is, $L_X\{f, g\} = \{L_X(f), g\} + \{f, L_X(g)\}$ for all $f, g \in C^\infty(P)$. Then the flow transformations $\{X^t\}$ of every X in $\mathfrak{X}_{\{,\}}(P)$ are Poisson automorphisms. See [MR, p. 339] for a proof. The vector fields in $\mathfrak{X}_{\{,\}}(P)$ are called <u>infinitesimal Poisson automorphisms</u>. For further discussion see (3.9) below.

Example 4 Flows of Hamiltonian vector fields

Let P be any Poisson manifold with a Poisson structure $\{,\}$. Let $\mathfrak{X}_H(P) = \{X_f :$

$f \in C^\infty(P)\}$, the collection of Hamiltonian vector fields on P. Then $\mathfrak{X}_H(P) \subseteq \mathfrak{X}_{\{,\}}(P)$ by the definition of Poisson structure, so by example 2 the flow transformations $\{X^t\}$ of every X in $\mathfrak{X}_H(P)$ are Poisson automorphisms.

Example 5 Momentum maps of Lie groups acting on Poisson manifolds

Let H be a Lie group with Lie algebra $\mathfrak{H}$, and let $\lambda : H \to \mathrm{Diff}(P)$ be a left action on a Poisson manifold P. Assume furthermore that for each $A \in \mathfrak{H}$ the elements $\{\lambda_{e^{tA}}\}$ are the flow transformations of a Hamiltonian vector field $\lambda(A)$ on P. Then there exists a momentum map $J : P \to \mathfrak{H}^*$ that under certain additional restrictions is a Poisson map relative to the given Poisson structure $\{,\}$ on P and the canonical Poisson structure $\{,\}^*$ on $\mathfrak{H}^*$. The momentum map is important in physics, and there are many situations in which it exists and is a Poisson map. We shall define and discuss the momentum map in greater detail in (3.11e).

In this article we shall primarily be concerned with the case that H acts by left translations on $P = TH$, and TH is equipped with the Poisson structure arising from the symplectic 2-form defined by a left invariant Riemannian metric on H.

Example 6 The Gauss map $G : TH \to \mathfrak{H}$

Let H be a Lie group with Lie algebra $\mathfrak{H}$ and tangent bundle TH. Define the left translation λ_h on H by $\lambda_h(h^*) = hh^*$ for $h, h^* \in H$. This defines a natural left action of H on itself. Identify $\mathfrak{H}$ with T_eH, the tangent space to H at the identity, and identify TH with $H \times \mathfrak{H}$ under the diffeomorphism $(h, X) \to d\lambda_h(X)$.

Define $G : TH = H \times \mathfrak{H} \to \mathfrak{H}$ by $G(h, X) = X$, projection onto the second factor. Geometrically, this amounts to the left translation of a vector $\xi \in TH$ back to the identity. In the case that H is the abelian Lie group $\mathbb{R}^3$ with vector addition as the group operation, then the Gauss map defined here is precisely the classical Gauss map used to study the geometry of surfaces in $\mathbb{R}^3$.

Now let H be a Lie group with Lie algebra $\mathfrak{H}$ and let $<,>$ be a nondegenerate, symmetric bilinear form on $\mathfrak{H}$. Let $^\# : \mathfrak{H} \to \mathfrak{H}^*$ be the natural isomorphism defined by $<,>$. Let Ω denote the symplectic 2-form on TH pulled back by $^\#$ from the canonical symplectic 2-form on TH^*, and let $\{,\}$ denote the corresponding Poisson structure on TH. Let $\{,\}^\#$ denote the Poisson structure on $\mathfrak{H}$ that is pulled back by $^\#$ from the canonical Poisson structure $\{,\}^*$ on $\mathfrak{H}^*$. Then $G : TH \to \mathfrak{H}$ is an anti Poisson map for every choice of a nondegenerate , symmetric bilinear form $<,>$ on $\mathfrak{H}$.

Remark The Gauss map $G : TH \to \mathfrak{H}$ does not depend on a choice of $<,>$ but it acts as Poisson map for every choice of $<,>$. The Gauss map is an important tool for the study of the geodesic flow on the unit tangent bundle SH of a Lie group H with a left invariant Riemannian metric arising from a choice of positive definite inner product $<,>$ on $\mathfrak{H}$.

In this context, the Gauss map is also a valuable resource for studying totally geodesic submanifolds of H with dimension ≥ 2 in the case that H is simply connected and 2-step nilpotent. See [E2] and (6.9) below for further discussion.

3.9 Poisson subalgebras

If $\{,\}$ is a Poisson structure on a C^∞ manifold P, then we say that a real vector subspace $\mathfrak{A}$ of $C^\infty(M)$ is a Poisson subalgebra of $C^\infty(P)$ if it contains 1 (and hence the constant functions) and is closed under the operations of addition, pointwise multiplication and Poisson brackets.

Example 1 Casimir functions $\mathfrak{C}(P)$

A function $f \in C^\infty(P)$ is called a Casimir function if $X_f = 0$, or equivalently if $\{f,g\} = 0$ for all $g \in C^\infty(P)$. One may also describe the Casimir functions as the center of the Poisson algebra.

It is obvious from the definition that $\mathfrak{C}(P)$ is not only a subalgebra of $C^\infty(P)$ but an ideal in $C^\infty(P)$; that is, if $f \in \mathfrak{C}(P)$ and $g \in C^\infty(P)$, then $\{f,g\} \in \mathfrak{C}(P)$.

The Casimir functions play an important role in Poisson mechanics. By their definition Casimir functions are constant along the integral curves of Hamiltonian vector fields X_g for every function $g \in C^\infty(P)$, and hence Casimir functions are constant on each symplectic leaf of P. Conversely, if f is constant on each symplectic leaf L, then $df(X_g) = 0$ on L since each Hamiltonian function X_g is tangent to each symplectic leaf of P. Hence $\{f,g\} = X_g(f) = df(X_g) = 0$ for all $g \in C^\infty(P)$, and we conclude that f is a Casimir function. This proves the following

Proposition Let P be a Poisson manifold. A function $f \in C^\infty(P)$ is Casimir $\Leftrightarrow$ f is constant on each symplectic leaf of P.

Local coordinate description of Casimir functions

Let $(q_1, p_1, q_2, p_2, ..., q_n, p_n, z_1, ..., z_\ell)$ be Lie -Weinstein local coordinates on an open set U (cf. (3.6)). The symplectic leaves in U are the slice submanifolds $(z_1, ..., z_\ell) = (c_1, ..., c_\ell)$, where $\{c_i\}$ are constants. By the result above a function $f : U \to \mathbb{R}$ is Casimir $\Leftrightarrow$ $\frac{\partial f}{\partial p_i} = 0$ and $\frac{\partial f}{\partial q_i} = 0$ for $1 \le i \le n$.

The symplectic case If the Poisson structure on P arises from a symplectic 2-form Ω on P, then by definition $i_{X_f}\Omega = df$ for every function $f \in C^\infty(P)$. Hence the Casimir functions are just the constant functions in this case.

However, it is not true that a Poisson structure whose Casimir functions are the constant functions must arise from a symplectic structure. If some Poisson manifold has a symplectic leaf that is dense in P, then every Casimir function f must be constant on P since f is constant on the closure of each symplectic leaf.

The Lie algebra case Let $\mathfrak{H}$ be a finite dimensional real Lie algebra with an inner product $<,>$, and let $\{,\}$ be the canonical Poisson structure determined by $<,>$. Let $^\# : \mathfrak{H} \to \mathfrak{H}^*$ be the isomorphism determined by $<,>$.

We give two examples of Casimir functions.

1) Let $f : \mathfrak{H} \to \mathbb{R}$ be a linear function and write $f = A^\#$ for a unique element A of $\mathfrak{H}$. Then f is a Casimir function $\Leftrightarrow$ $A \in \mathfrak{Z}$, the center of $\mathfrak{H}$.

Proof If $f = A^\#$ is a Casimir function, then for any $B \in \mathfrak{H}$ we have $0 = \{B^\#, A^\#\} = [B, A]^\#$. It follows that $A \in \mathfrak{Z}$. Conversely suppose that $f = A^\#$, where $A \in \mathfrak{Z}$.

Given $g \in C^\infty(\mathfrak{H})$ and $\alpha \in \mathfrak{H}$ choose $B \in \mathfrak{H}$ so that $dg_\alpha = dB^\#_\alpha$. Then $\{g, f\}(\alpha) = \{B^\#, A^\#\}(\alpha) = [B, A]^\#(\alpha) = 0$. Hence f is a Casimir function.

2) Let $g : \mathfrak{H} \to \mathbb{R}$ be a function that depends only on $\mathfrak{Z}$; that is, $g = f \circ \pi_3$, where $f : \mathfrak{Z} \to \mathbb{R}$ is any function and $\pi_3 : \mathfrak{H} \to \mathfrak{Z}$ is orthogonal projection onto $\mathfrak{Z}$. Then g is a Casimir function.

Remark If $\mathfrak{H}$ is an almost nonsingular 2-step nilpotent Lie algebra (cf. (6.4)), then we shall see in (6.6) that every Casimir function $g : \mathfrak{H} \to \mathbb{R}$ has the form above in 2).

Proof of 2) Let $\{E_1, E_2, ..., E_n\}$ be an orthonormal basis of $\mathfrak{H}$ such that $\{E_{q+1}, ...E_n\}$ is a basis of $\mathfrak{Z}$, where $q = \dim \mathfrak{Z}^\perp$ and $n - q = p = \dim \mathfrak{Z}$. Let $\{x_1, x_2, ..., x_n\}$ denote the dual basis of $\mathfrak{H}^*$, and let $x = (x_1, x_2, ..., x_n) : \mathfrak{H} \to \mathbb{R}$ denote the corresponding linear coordinate system for $\mathfrak{H}$. Let $\{C^k_{ij}\}$ be the structure constants defined by $C^k_{ij} = x_k([E_i, E_j])$ for all i, j, k. By the discussion of example 4 in (3.7) the Hamiltonian vector field X_f is given by $X_f = \sum_{j=1}^{n}\{\sum_{i,k=1}^{n} C^k_{ji}\frac{\partial f}{\partial x_i}x_k\}\frac{\partial}{\partial x_j}$. By hypothesis $\frac{\partial f}{\partial x_i} = 0$ for $1 \le i \le q = \dim \mathfrak{Z}^\perp$. However, $C^k_{ij} = x_k([E_i, E_j]) = 0$ for $i > q$ since $E_i \in \mathfrak{Z}$ for $i > q$. Hence $X_f = 0$ by the expression above. ∎

Example 2 First integrals for the flows of Hamiltonian vector fields

Let P be a Poisson manifold. For any C^∞ function $f : P \to \mathbb{R}$ we define $\mathfrak{F}(f) = \{g \in C^\infty(P) : dg(X_f) = 0\}$. Equivalently, $\mathfrak{F}(f) = \{g \in C^\infty(P) : \{g, f\} = 0\}$.

The elements of $\mathfrak{F}(f)$ are called first integrals for the flow of the Hamiltonian vector field X_f. They may also be characterized as those functions in $C^\infty(P)$ that are constant along the integral curves of X_f.

Proposition For any function $f \in C^\infty(P), \mathfrak{F}(f)$ is a Poisson subalgebra of $C^\infty(P)$ that contains f and the Casimir functions.

Proof Clearly $\mathfrak{F}(f)$ contains f and the Casimir functions. Note that $g \in \mathfrak{F}(f) \Leftrightarrow \{g, f\} = 0$ since $\{g, f\} = X_f(g) = dg(X_f)$. Since $d(g + h) = dg + dh$ and $d(gh) = gdh + hdg$ it is clear that $\mathfrak{F}(f)$ contains the constants and is closed under addition and pointwise multiplication. If g, h are elements of $\mathfrak{F}(f)$, then by the Jacobi identity $\{\{g, h\}, f\} = -\{\{h, f\}, g\} - \{\{f, g\}, h\} = 0$. Hence $\{g, h\} \in \mathfrak{F}(f)$ for all $g, h \in \mathfrak{F}(f)$. ∎

Remarks

1) If one chooses f at random, then $\mathfrak{F}(f)$ will consist only of the Poisson subalgebra generated by f and the Casimir functions However, if f has special significance, which occurs in cases of physical importance, then $\mathfrak{F}(f)$ is nontrivial and it is an interesting problem to determine it.

2) Let H be a Lie group acting by Poisson automorphisms on a manifold P with Poisson structure $\{,\}$. Let $f : P \to \mathbb{R}$ be any C^∞ function that is constant along each H-orbit in P. We shall see that the moment map $J : P \to \mathfrak{H}^*$, whenever J exists and is equivariant, defines an $\mathfrak{H}^*$-valued first integral for the flow $\{X^t_f\}$ of the Hamiltonian vector field X_f. See Proposition A of (3.11e) below.

3) Let P be a finite dimensional real Lie algebra $\mathfrak{H}$ equipped with an inner product

$<,>$. We saw in example 4 of (3.7) that $<,>$ and the Lie bracket $[,]$ of $\mathfrak{H}$ determine a Poisson structure $\{,\}$ on $\mathfrak{H}$ in a canonical way. Let $E : \mathfrak{H} \to \mathbb{R}$ be the energy function defined by $E(v) = \frac{1}{2} < v, v >$. The Hamiltonian vector vector field X_E defined by E is called the geodesic vector field, and we shall see that X_E corresponds in a natural way to the geodesic flow on TH by means of the Gauss map $G : TH \to \mathfrak{H}$. Here H is given the left invariant Riemannian metric determined by $<,>$. It is an interesting problem of Riemannian geometry to determine the Poisson algebra $\mathfrak{F}(E)$ of first integrals for the geodesic vector field X_E. We will give special attention later to the special case that $\mathfrak{H}$ is a 2-step nilpotent Lie algebra.

3.10 Infinitesimal Poisson automorphisms

Definition Let P be a Poisson manifold. We say that a vector field X on P is an infinitesimal Poisson automorphism if $L_X\{,\} = 0$; that is $X\{f,g\} = \{Xf,g\} + \{f,Xg\}$ for all $f,g \in C^\infty(P)$. Let $\mathfrak{X}_{\{,\}}(P)$ denote the collection of all infinitesimal Poisson automorphisms on P.

If X is an infinitesimal Poisson automorphism, then it is not difficult to show that the flow transformations $\{X^t\}$ are Poisson maps. Conversely, if X is a vector field on P whose flows transformations $\{X^t\}$ are Poisson maps, then X is an infinitesimal Poisson automorphism.

It is routine to check that $\mathfrak{X}_{\{,\}}(P)$ is a Lie subalgebra of $\mathfrak{X}(P)$ with respect to the Lie bracket. From the Poisson axioms it follows that every Hamiltonian vector field X_f is an infinitesimal Poisson automorphism, or equivalently $\mathfrak{X}_H(P) \subseteq \mathfrak{X}_{\{,\}}(P)$.

In general the inclusion $\mathfrak{X}_H(P) \subseteq \mathfrak{X}_{\{,\}}(P)$ is strict as the following example shows. Let x be a point of a Poisson manifold P such that the rank of $\{,\}$ is $\equiv 2n$ in some neigborhood of x. Let P have dimension $2n + \ell$ for some integer $\ell \geq 0$, and let U be a coordinate neighborhood equipped with Lie-Weinstein coordinates $(q_1, p_1, q_2, p_2, ..., q_n, p_n, z_1, ..., z_\ell)$ as in (3.6). Let A be a nonzero linear transformation of $\mathbb{R}^\ell$, and define a 1-parameter family of maps $A(t) : U \to U$ by $A(t)(q_1, p_1, q_2, p_2, ..., q_n, p_n, z_1, ..., z_\ell) = (q_1, p_1, q_2, p_2, ..., q_n, p_n, e^{tA}(z_1, ..., z_\ell))$. It is easy to check that the maps $A(t)$ are Poisson maps, and hence $A' = \frac{d}{dt}|_{t=0} A(t)$ is an infinitesimal Poisson automorphism on U. The vector field A' cannot be a Hamiltonian vector field since its flow transformations do not leave the symplectic leaves invariant.

3.11 Orbit structure of Lie group actions on Poisson manifolds

3.11a A left action $\lambda : H \to \mathrm{Diff}(P)$ and its differential $\lambda : \mathfrak{H} \to \mathfrak{X}(P)$

A left action of a Lie group H on a C^∞ manifold P is a homomorphism $\lambda : H \to \mathrm{Diff}(P)$; that is, λ_h is a diffeomorphism of P for every $h \in H$ and $\lambda(hh^*) = \lambda(h) \circ \lambda(h^*)$ for all elements $h, h^* \in H$. The left action is C^∞ if the map $\Lambda : H \times P \to P$ given by

$\Lambda(h,m) = \lambda_h(m)$ is C^∞. We shall consider left actions on a Poisson manifold P by a connected Lie group H whose elements act as Poisson automorphisms of P. Let $\mathfrak{H}$ denote the Lie algebra of H.

If $\lambda : H \to \text{Diff}(P)$ is a left action, then for every $X \in \mathfrak{H}$ let $\lambda(X)$ denote the vector field in P whose flow transformations are $\{\lambda_{e^{tX}}\}$. It is not too difficult to show that $\lambda : \mathfrak{H} \to \mathfrak{X}(P)$ is a Lie algebra antihomomorphism; that is , $[\lambda(X), \lambda(Y)] = -\lambda([X,Y])$. See [Br, p. 53]. This antihomomorphism is the derivative map of the homomorphism $\lambda : H \to \text{Diff}(P)$ if one regards Diff(P) as an infinite dimensional Lie group with Lie algebra $\mathfrak{X}(P)$.

3.11b Properties of actions

A left action $\lambda : H \to \text{Diff}(P)$ is effective if $\ker \lambda = \{1\}$, or equivalently, if $\lambda_h(x) = x$ for all $x \in P$ and some $h \in H$, then $h = \{1\}$. Any action $\lambda : H \to \text{Diff}(P)$ induces an effective action $\overline{\lambda} : \overline{H} = H/\ker\lambda \to \text{Diff}(P)$ by $\overline{\lambda}(h \ker \lambda) = \lambda(h)$.

For each $x \in P$ define $H_x = \{h \in H : \lambda_h(x) = x\}$ and $\mathfrak{H}_x = \{A \in \mathfrak{H} : \lambda(A)(x) = 0\}$. It is routine to show $\mathfrak{H}_x$ that is the Lie algebra of $\mathfrak{H}_x$. A left action $\lambda : H \to \text{Diff}(P)$ is free if $H_x = \{1\}$ for all $x \in P$ and it is almost free if $\mathfrak{H}_x = \{0\}$ for all $x \in P$.

Every free action is almost free, but the converse is false.

The next result will be useful.

Proposition Let $\lambda : H \to \text{Diff}(P)$ be a left action and let $\overline{\lambda} : \overline{H} = H/\ker\lambda \to \text{Diff}(P)$ be the effective induced action. Then

1) $\overline{H}_x = H_x/\ker\lambda$ and $\overline{\mathfrak{H}}_x = \mathfrak{H}_x/\ker\lambda$, where λ in the second equality refers to the antihomomorphism $\lambda : \mathfrak{H} \to \mathfrak{X}(P)$.

2) The action $\overline{\lambda} : \overline{H} \to \text{Diff}(P)$ is almost free $\Leftrightarrow$ $\mathfrak{H}_x = \ker\lambda$ for all $x \in P$, where $\lambda : \mathfrak{H} \to \mathfrak{X}(P)$ is the antihomomorphism.

Proof Assertion 2) follows immediately from 1) and the proof of 1) follows routinely from the definitions. ■

3.11c Relation to symplectic stratification

The symplectic stratification theorem gives an intrinsic decomposition of P into disjoint symplectic submanifolds. The elements of H permute these symplectic leaves by (3.8*a*) since H acts on P by Poisson automorphisms.

We consider the special case that H leaves each symplectic leaf invariant. In this situation each element X of $\mathfrak{H}$ defines an infinitesimal Poisson automorphism $\lambda(X)$ on each symplectic leaf, as we explain below. It is natural to ask under what conditions these vector fields $\lambda(X)$ are all Hamiltonian vector fields, an outcome that is not guaranteed in general. When this optimal situation does arise, then one may define a momentum map $J : P \to \mathfrak{H}^*$ with important properties. There turns out to be an obstruction to this optimal situation that can be described explicitly. The obstruction involves both the topological structure of the manifold P and the algebraic structure of the Lie algebra $\mathfrak{H}$. To explain, we first need to reformulate the problem.

Since we are assuming that H leaves each symplectic leaf invariant we may as well restrict our attention to an individual symplectic leaf. Therefore it suffices to consider the case that H acts on a symplectic manifold P and the elements of H leave invariant the symplectic 2-form Ω.

3.11d Hamiltonian actions on symplectic manifolds

Let $\lambda : H \to \mathrm{Diff}(P)$ be a left action by symplectic automorphisms on a symplectic manifold P. For every $A \in \mathfrak{H}$ it follows that $0 = L_{\lambda(A)}\Omega = d(i_{\lambda(A)}\Omega) + i_{\lambda(A)}(d\Omega) = d(i_{\lambda(A)})\Omega)$ since the flow transformations of $\lambda(A)$ leave Ω invariant. Hence $i_{\lambda(A)}\Omega$ is a closed 1-form for every $A \in \mathfrak{H}$.

We note that a vector field Z on P is Hamiltonian $\Leftrightarrow$ $i_Z\Omega$ is exact, that is $i_Z\Omega = df$ for some $f \in C^\infty(P)$. In this case $Z = X_f$ on P by the nondegeneracy of Ω. By the discussion above the restriction of $\lambda(A)$ to a simply connected open subset of P is exact for each $A \in \mathfrak{H}$ since closed forms and exact forms are the same on U.

We say that the action $\lambda : H \to \mathrm{Diff}(P)$ is Hamiltonian if $\lambda(A)$ is a Hamiltonian vector field on P for every $A \in \mathfrak{H}$. For a left action by symplectic automorphisms the problem then is to find criteria under which $i_{\lambda(A)}\Omega$ is exact for every $A \in \mathfrak{H}$. If every closed 1-form on P is exact; that is, the first de Rham cohomology group $H^1_{DR}(P, \mathbb{R})$ vanishes, then that is clearly sufficient. This criterion would be satisfied, for example, in the case that P is simply connected. However, one can do better. One defines a map $H_\lambda : \mathfrak{H} \to H^1_{DR}(P, \mathbb{R})$ by $H_\lambda(A) = [i_{\lambda(A)}\Omega]$ where $[,]$ denotes the equivalence class of a closed 1-form of P in $H^1_{DR}(P, \mathbb{R})$ = (closed 1-forms) / (exact 1-forms). It is easy to see that H_λ is a linear map. One can show that $[\lambda(X), \lambda(Y)]$ is the Hamiltonian vector field determined by $f = -\Omega(X, Y)$. See, for example [Br, p.104], but note that a different sign convention is used there for the definition of the Hamiltonian vector field, namely $i_{X_f}\Omega = -df$. It follows that the kernel of H_λ contains the commutator subalgebra $[\mathfrak{H}, \mathfrak{H}]$.

We summarize : Every vector field $\lambda(A), A \in \mathfrak{H}$, is Hamiltonian $\Leftrightarrow$ the linear map $H_\lambda : \mathfrak{H} \to H^1_{DR}(P, \mathbb{R})$ given by $H_\lambda(A) = [i_{\lambda(A)}\Omega]$ is identically zero. This happens in at least the following three cases :

1) $H^1_{DR}(P, \mathbb{R}) = \{0\}$; e.g. P is simply connected.

2) $\mathfrak{H} = [\mathfrak{H}, \mathfrak{H}]$

3) There exists a 1-form θ on P such that the elements of H leave θ invariant and $\Omega = d\theta$.

Case 3) occurs, for example if $P = TH^*$, the cotangent bundle of a connected Lie group H, and H acts on TH by the pullbacks of left translations on H. In this case θ is the canonical 1-form and $\Omega = -d\theta$ the canonical symplectic 2-form. We compute $0 = L_A\theta = d(i_{\lambda(A)}\theta) + i_{\lambda(A)}(d\theta) = d(\theta(\lambda A)) + i_{\lambda(A)}(\Omega)$, which shows that $i_{\lambda(A)}(\Omega)$ is exact for every $A \in \mathfrak{H}$.

Case 3) also occurs if H is a connected subgroup of isometries of a Riemannian manifold M and H acts on TM on the left by the differential maps of its elements. In this case θ is again the canonical 1-form and $\Omega = -d\theta$ the canonical symplectic 2-form. The proof that $i_{\lambda(A)}(\Omega)$ is exact for every $A \in \mathfrak{H}$ is the same as above.

3.11e The momentum map

Let P be a Poisson manifold, and let $\lambda : H \to \mathrm{Diff}(P)$ be a left action on P by a connected Lie group H. We say that the action λ is Hamiltonian if $\lambda(A)$ is a

Hamiltonian vector field on P for every $A \in \mathfrak{H}$. Henceforth we assume that the action λ of H on P is Hamiltonian.

The linear map $\hat{J} : \mathfrak{H} \to C^\infty(P)$

By hypothesis, for every $A \in \mathfrak{H}$ there exists a function $\hat{J}(A) \in C^\infty(P)$ such that $\lambda(A) = X_{\hat{J}(A)}$. Note that $\hat{J}(A)$ is not uniquely determined since we may always replace $\hat{J}(A)$ by $\hat{J}(A)+f$, where f is a Casimir function. Nevertheless, we may always choose the map $\hat{J} : \mathfrak{H} \to C^\infty(P)$ to be a linear map. To see this, let $\{X_1, ..., X_n\}$ be a basis of $\mathfrak{H}$, and let $\{H_1, ..., H_n\}$ be functions in $C^\infty(P)$ such that $X_{H_i} = \lambda(X_i)$ for $1 \le i \le n$. If we define $\hat{J}(\sum_{i=1}^{n} a_i X_i) = \sum_{i=1}^{n} a_i H_i$, then $\hat{J} : \mathfrak{H} \to C^\infty(P)$ is linear and $\lambda(A) = X_{\hat{J}(A)}$ for all $A \in \mathfrak{H}$.

We may now define a map $J : P \to \mathfrak{H}^*$ by $J(x)(A) = \hat{J}(A)(x)$ for all $x \in P$ and $A \in \mathfrak{H}$. Note that $J(x) \in \mathfrak{H}^*$ for every $x \in P$ since $\hat{J}$ is linear. Although the map $\hat{J}$ seems to have no name, the map $J : P \to \mathfrak{H}^*$ is called the momentum map.

The existence of $\hat{J}$ and J has the following immediate payoff :

Proposition A Let H be a connected Lie group with Lie algebra $\mathfrak{H}$. Let λ be a Hamiltonian action of H on a Poisson manifold P, and let $\hat{J} : \mathfrak{H} \to C^\infty(P)$ be a linear map such that $\lambda(A) = X_{\hat{J}(A)}$ for all $A \in \mathfrak{H}$. Let $f : P \to \mathbb{R}$ be a C^∞ function that is constant along each H orbit in P. Then $\hat{J}(A)$ is a first integral for the flow of X_f for all $A \in \mathfrak{H}$. Equivalently, the momentum map $J : P \to \mathfrak{H}^*$ is an $\mathfrak{H}^*$-valued first integral for the flow of X_f.

Proof Let $f : P \to \mathbb{R}$ be a C^∞ function that is constant along each H orbit in P. Let $A \in \mathfrak{H}$ be given. Then $\{f, \hat{J}(A)\} = X_{\hat{J}(A)}(f) = \lambda(A)(f) = 0$ since f is constant on the integral curves of $\lambda(A)$, which all have the form $x \to \lambda_{e^{tA}}(x)$. ■

Obstruction to $\hat{J}$ being a Lie algebra homomorphism

We investigate further the properties of $\hat{J} : \mathfrak{H} \to C^\infty(P)$ and $J : P \to \mathfrak{H}^*$.

Proposition B Let H be a connected Lie group with Lie algebra $\mathfrak{H}$. Let λ be a Hamiltonian action of H on a Poisson manifold P, and let $\hat{J} : \mathfrak{H} \to C^\infty(P)$ be a linear map such that $\lambda(A) = X_{\hat{J}(A)}$ for all $A \in \mathfrak{H}$. Then $X_{\hat{J}[A,B]} = X_{\{\hat{J}(A),\hat{J}(B)\}}$ for all $A, B \in \mathfrak{H}$.

Proof For $A, B \in \mathfrak{H}$ we have $X_{\hat{J}[A,B]\}} = \lambda([A,B]) = -[\lambda(A), \lambda(B)] = -\left[X_{\hat{J}(A)}, X_{\hat{J}(B)}\right] = X_{\{\hat{J}(A),\hat{J}(B)\}}$ by (3.2) and (3.11a). ■

The previous result does not quite imply that $\hat{J}$ is a Lie algebra homomorphism, but only that $c_{\hat{J}}(A,B) = \hat{J}([A,B]) - \{\hat{J}(A), \hat{J}(B)\}$ is a Casimir function for all $A, B \in \mathfrak{H}$. The map $c_{\hat{J}} : \mathfrak{H} \times \mathfrak{H} \to \mathfrak{C}(P)$ is clearly skew symmetric and bilinear, and the Jacobi identity implies that $c_{\hat{J}}([A,B], C) + c_{\hat{J}}([B,C], A) + c_{\hat{J}}([C,A], B) = 0$ for all $A, B, C \in \mathfrak{H}$.

We wish to choose $\hat{J}$, if possible, so that $c_{\hat{J}} = 0$, and this will imply that $\hat{J} : \mathfrak{H} \to C^\infty(P)$ is a Lie algebra homomorphism. The discussion above shows that $\hat{J}_1 : \mathfrak{H} \to C^\infty(P)$ and $\hat{J}_2 : \mathfrak{H} \to C^\infty(P)$ are linear maps with $X_{\hat{J}_1}(A) = X_{\hat{J}_2}(A) = \lambda(A)$ for all

$A \in \mathfrak{H} \Leftrightarrow \xi = \hat{J}_1 - \hat{J}_2$ is a linear map from $\mathfrak{H}$ to $\mathfrak{C}(P)$. If we replace our original linear map $\hat{J} : \mathfrak{H} \to C^\infty(P)$ by $\hat{J}^* = \hat{J} - \xi$, where $\xi : \mathfrak{H} \to \mathfrak{C}(P)$ is linear, then a computation shows that

$$c_{j^*}(A,B) = c_j(A,B) - \xi([A,B]) \text{ for all } A,B \in \mathfrak{H}.$$

We summarize the discussion above:

Proposition C Let H be a connected Lie group with Lie algebra $\mathfrak{H}$. Let λ be a Hamiltonian action of H on a Poisson manifold P, and let $\hat{J} : \mathfrak{H} \to C^\infty(P)$ be a linear map such that $\lambda(A) = X_{j(A)}$ for all $A \in \mathfrak{H}$. Then there exists a Lie algebra homomorphism $\hat{J}^* : \mathfrak{H} \to C^\infty(P)$ such that $\lambda(A) = X_{j^*(A)}$ for all $A \in \mathfrak{H} \Leftrightarrow$ there exists a linear map $\xi : \mathfrak{H} \to \mathfrak{C}(P)$ such that $c_j(A,B) = \xi([A,B])$ for all $A,B \in \mathfrak{H}$.

The obstruction for $\hat{J}$ to be a Lie algebra homomorphism vanishes in each of the following cases :

1) P is compact (cf. [Br, p.132])
2) H is compact (cf. [MR, p.380])
3) $[\mathfrak{H},\mathfrak{H}] = \mathfrak{H}$ (cf. [Br, p.132])
4) There exists a 1-form θ on P such that the elements of H leave θ invariant and $\Omega = d\theta$. In this case let $\hat{J}(A) = \theta(\lambda(A))$ for all $A \in \mathfrak{H}$.

Remark Example 4 is particularly important for Riemannian geometry. Let M be a Riemannian manifold, and let H be any closed, connected subgroup of the isometry group $I(M)$. It is a classical result that H is a Lie group [MS], but of course $I(M)$ may be the trivial group for a random Riemannian manifold M. In any case the tangent bundle $P = TM$ admits a symplectic structure $\Omega = -d\theta$ as described in (2.1). It is easy to check from the definition of θ that $(L_h)^*\theta = \theta$ for all $h \in H$. We may now invoke the assertion of example 4.

Because of its importance for Riemannian geometry we prove assertion 4).

Proposition D Let P be a connected C^∞ symplectic manifold, and let Ω denote the symplectic 2-form. Let λ be a left action on P of a connected Lie group H by Poisson automorphisms. Let $\lambda : \mathfrak{H} \to \mathfrak{X}(P)$ also denote the canonical anti homomorphism, where $\mathfrak{H}$ is the Lie algebra of H. Suppose that $\Omega = d\theta$, where θ is a 1-form on P such that $(L_h)^*\theta = \theta$ for all $h \in H$. If $\hat{J} : \mathfrak{H} \to C^\infty(P)$ is the linear map defined by $\hat{J}(A) = \theta(\lambda(A))$ for all $A \in \mathfrak{H}$, then $\hat{J}([A,B]) = \{\hat{J}(A), \hat{J}(B)\}$ for all $A,B \in \mathfrak{H}$.

Lemma $X_{j(A)} = \lambda(A)$ for all $A \in \mathfrak{H}$.

Proof of the Lemma The flow transformations of $\lambda(A)$ are $\{\lambda_{e^{tA}}\}$, which leave the canonical 1-form θ invariant. Hence $0 = L_{\lambda(A)}\theta = d \circ i_{\lambda(A)}\theta + i_{\lambda(A)}d\theta = d(\hat{J}(A)) - i_{\lambda(A)}\Omega = i_{X_{j(A)}}\Omega - i_{\lambda(A)}\Omega$. Hence $X_{j(A)} = \lambda(A)$ for all $A \in \mathfrak{H}$ since Ω is nondegenerate on TH. ∎

We show that $\hat{J}([A,B]) = \{\hat{J}(A), \hat{J}(B)\}$ for all $A,B \in \mathfrak{H}$ by computing $\Omega(\lambda(A),\lambda(B))$ in two different ways. On the one hand $\Omega(\lambda(A),\lambda(B)) = \Omega(X_{j(A)}, X_{j(B)}) = \{\hat{J}(A), \hat{J}(B)\}$ by the lemma. On the other hand $\Omega(\lambda(A),\lambda(B)) = -d\theta(\lambda(A),\lambda(B)) = -\lambda(A)(\theta(\lambda(B))) + \lambda(B)(\theta(\lambda(A))) + \theta([\lambda(A),\lambda(B)]) = -\lambda(A)(\hat{J}(B)) + \lambda(B)(\hat{J}(A)) - \theta(\lambda([A,B])) = -X_{j(A)}(\hat{J}(B)) + X_{j(B)}(\hat{J}(A)) - \hat{J}([A,B]) = -\{\hat{J}(B), \hat{J}(A)\} + \{\hat{J}(A), \hat{J}(B)\} - \hat{J}([A,B])$.

Comparing the two expressions for $\Omega(\lambda(A), \lambda(B))$ yields $\{\hat{J}(A), \hat{J}(B)\} = -\{\hat{J}(B), \hat{J}(A)\} + \{\hat{J}(A), \hat{J}(B)\} - \hat{J}([A,B])$, which shows that $\hat{J}([A,B]) = \{\hat{J}(A), \hat{J}(B)\}$ for all $A, B \in \mathfrak{h}$. ■

The consequences of $\hat{J}$ being a Lie algebra homomorphism

The ability to find a Lie algebra homomorphism $\hat{J} : \mathfrak{h} \to C^\infty(P)$ such that $\lambda(A) = X_{\hat{J}(A)}$ for all $A \in \mathfrak{h}$ has powerful equivalent formulations.

Proposition E Let H be a connected Lie group with Lie algebra $\mathfrak{h}$. Let λ be a Hamiltonian action of H on a Poisson manifold P, and let $\hat{J} : \mathfrak{h} \to C^\infty(P)$ be a linear map such that $\lambda(A) = X_{\hat{J}(A)}$ for all $A \in \mathfrak{h}$. Let $J : P \to \mathfrak{h}^*$ be the momentum map given by $J(x)(A) = \hat{J}(A)(x)$ for all $x \in P$ and all $A \in \mathfrak{h}$. Then the following are equivalent :

1)$\hat{J} : \mathfrak{h} \to C^\infty(P)$ is a Lie algebra homomorphism.

2)$J : P \to \mathfrak{h}^*$ is a Poisson map relative to the canonical Poisson structure on $\mathfrak{h}^*$.

3)$J : P \to \mathfrak{h}^*$ is an equivariant map; that is, for every $h \in H$ the following diagram commutes :

$$\begin{array}{ccc} P & \overset{J}{\dashrightarrow} & \mathfrak{h}^* \\ \lambda_h \downarrow & & \downarrow \mathrm{Ad}^*(h) \\ P & \overset{J}{\dashrightarrow} & \mathfrak{h}^* \end{array}$$

Proof Note that $\mathfrak{h} \subseteq C^\infty(\mathfrak{h}^*)$ by the discussion in example 3 of (3.7). Recall that $\{A, B\}^* = [A, B]$ for all $A, B \in \mathfrak{h}$ by the definition of the canonical Poisson structure $\{,\}^*$ on $\mathfrak{h}^*$. It follows that $A \circ J = \hat{J}(A)$ for all $A \in \mathfrak{h}$ the definitions of J and $\hat{J}$. Hence $\{A \circ J, B \circ J\} = \{\hat{J}(A), \hat{J}(B)\}$, and $\{A, B\}^* \circ J = [A, B] \circ J = \hat{J}([A, B])$. This proves

Lemma The map $\hat{J} : \mathfrak{h} \to C^\infty(P)$ is a Lie algebra homomorphism $\Leftrightarrow$ $\{A \circ J, B \circ J\} = \{A, B\}^* \circ J$ for all $A, B \in \mathfrak{h}$.

We prove that 2) $\Rightarrow$ 1). Suppose that $J : P \to \mathfrak{h}^*$ is a Poisson map. Since $\mathfrak{h} \subseteq C^\infty(\mathfrak{h}^*)$ it follows that $\{A \circ J, B \circ J\} = \{A, B\}^* \circ J$ for all $A, B \in \mathfrak{h}$, and by the lemma above we conclude that $\hat{J}$ is a Lie algebra homomorphism.

We prove 1) $\Rightarrow$ 2). Let $\hat{J} : \mathfrak{h} \to C^\infty(P)$ be a Lie algebra homomorphism. Let $f, g \in C^\infty(\mathfrak{h}^*)$ and $x \in P$ be given. Let $\omega = J(x) \in \mathfrak{h}^*$. Since $\mathfrak{h}$ is first order dense in $C^\infty(\mathfrak{h}^*)$ we can find $A, B \in \mathfrak{h}$ so that $df_\omega = dA_\omega$ and $dg_\omega = dB_\omega$. Then $d(f \circ J)_x = df_\omega \circ dJ_x = dA_\omega \circ dJ_x = d(A \circ J)_x$. We obtain

(*) $df_\omega = dA_\omega$, $dg_\omega = dB_\omega$

$d(f \circ J)_x = d(A \circ J)_x$, $d(g \circ J)_x = d(B \circ J)_x$

Since $\hat{J}$ is a Lie algebra homomorphism, the lemma above together with (*) and the discussion in (3.5) show that $\{f \circ J, g \circ J\}(x) = \{A \circ J, B \circ J\}(x) = \{A, B\}^*(\omega) = \{f, g\}^*(\omega) = (\{f, g\}^* \circ J)(x)$.

We prove 3) $\Rightarrow$ 1). Let $J : P \to \mathfrak{h}^*$ be an equivariant map. Let $x \in P$ and $A, B \in \mathfrak{h}$ be given. By hypothesis $\mathrm{Ad}^*(e^{tA}) \circ J(x)(B) = J(\lambda_{e^{tA}} x)(B)$ for all $t \in \mathbb{R}$. We may rewrite this equality as

(*) $J(x)(Ad(e^{-tA})(B) = \hat{J}(B)(\lambda_{e^{tA}} x)$ for all $t \in \mathbb{R}$.

Note that $t \to \lambda_{e^{tA}} x$ is the integral curve of $\lambda(A) = X_{\hat{J}(A)}$ that starts at x. Recall that $Ad(e^{-tA})(B) = e^{-t ad A}(B)$ for all $t \in \mathbb{R}$. Differentiating both sides of (*) at $t = 0$ we obtain $\hat{J}([B,A])(x) = J(x)([B,A]) = J(x)(-[A,B]) = \lambda(A)(\hat{J}(B))(x) = X_{\hat{J}(A)}(\hat{J}(B))(x) = \{\hat{J}(B), \hat{J}(A)\}(x)$. (The third equality comes from the differentiation at $t = 0$.).

Proof of 1) $\Rightarrow$ 3). This proof is somewhat more difficult, but a proof may be found in $[MR, p.402]$. See also $[B, p.133]$ in the case that $\{,\}$ arises from a symplectic structure. This completes the proof of the Proposition. ■

3.11f Reduced first integrals for H-invariant functions $\mathbf{f}: \mathbf{P} \to \mathbb{R}$

We observed in Proposition A of the previous section that $\hat{J}(A)$ is a first integral for X_f if $f: P \to \mathbb{R}$ is constant along the orbits of H. If $\hat{J}: \mathfrak{H} \to C^\infty(P)$ is a Lie algebra homomorphism, then $\hat{J}(\mathfrak{H})$ is a finite dimensional subalgebra of $C^\infty(P)$ relative to the Poisson structure $\{,\}$. If $\hat{J}(A)$ is a Casimir function, then it is a relatively uninteresting first integral so we consider the space $\hat{J}(\mathfrak{H})/\mathfrak{C}(P) \cap \hat{J}(\mathfrak{H})$ of reduced first integrals. Note that $\mathfrak{C}(P) \cap \hat{J}(\mathfrak{H})$ is an ideal of $\hat{J}(\mathfrak{H})$ that is contained in the center of $\hat{J}(\mathfrak{H})$ since $\mathfrak{C}(P)$ is the center of the Poisson algebra $C^\infty(P)$. Hence $\hat{J}(\mathfrak{H})/\mathfrak{C}(P) \cap \hat{J}(\mathfrak{H})$ has a quotient Lie algebra structure.

Proposition A The Lie algebra $\hat{J}(\mathfrak{H})/\mathfrak{C}(P) \cap \hat{J}(\mathfrak{H})$ is Lie algebra isomorphic to $\mathfrak{H}/\ker\lambda$, where $\lambda: \mathfrak{H} \to \mathfrak{X}(P)$ is the canonical anti-homomorphism.

Proof Since $\lambda(A) = X_{\hat{J}(A)}$ for all $A \in \mathfrak{H}$ it follows that $A \in \ker\lambda \Leftrightarrow \hat{J}(A) \in \mathfrak{C}(P) \cap \hat{J}(\mathfrak{H})$. If we define $\varphi: \mathfrak{H}/\ker\lambda \to \hat{J}(\mathfrak{H})/\mathfrak{C}(P) \cap \hat{J}(\mathfrak{H})$ by $\varphi(A + \ker\lambda) = \hat{J}(A) + \mathfrak{C}(P) \cap \hat{J}(\mathfrak{H})$, then it is easy to check that φ is well defined and is a Lie algebra isomorphism. ■

Functional independence of first integrals in $\hat{\mathbf{J}}(\mathfrak{H})$

Elements $\{f_1, f_2, ..., f_k\}$ in $C^\infty(P)$ are said to be functionally independent if their differentials $\{(df_1)_x, (df_2)_x, ..., (df_k)_x\}$ are linearly independent for all $x \in P$. If P is a symplectic manifold, then it follows from the definitions that $\{f_1, f_2, ..., f_k\}$ are functionally independent in $C^\infty(P) \Leftrightarrow$ the Hamiltonian vector fields $\{X_{f_1}, X_{f_2}, ..., X_{f_k}\}$ are linearly independent in P. See Lemma A in (3.12).

Proposition B Let $\lambda: H \to \mathrm{Diff}(P)$ be a symplectic left action of a connected Lie group H on a symplectic manifold P. Suppose that there exists a linear map $\hat{J}: \mathfrak{H} \to C^\infty(P)$ such that $X_{\hat{J}(A)} = \lambda(A)$ for all $A \in \mathfrak{H}$ and $\hat{J}([A,B]) = \{\hat{J}(A), \hat{J}(B)\}$ for all $A, B \in \mathfrak{H}$. Then

1) There exist at most $N = \dim(\mathfrak{H}/\ker\lambda)$ functionally independent elements of $\hat{J}(\mathfrak{H})$.

2) If the induced action $\overline{\lambda}: \overline{H} = H/\ker\lambda \to \mathrm{Diff}(P)$ is almost free, then there exist N functionally independent elements of $\hat{J}(\mathfrak{H})$.

Remark The main case of interest to us in this article is the natural left action of H on TH by the differentials of the elements of H. This action is a free action, and the hypotheses of the proposition are satisfied for the map $\hat{J} : \mathfrak{H} \to C^\infty(TH)$ given by $\hat{J}(A) = \theta(\lambda(A))$. Hence we obtain $N = \dim \mathfrak{H}$ functionally independent first integrals in $\hat{J}(\mathfrak{H})$.

Proof of Proposition B 1) Let $m > N$ and let $\{A_1, A_2, ..., A_m\}$ be any elements of $\mathfrak{H}$. Choose constants $\{c_i\}$, not all zero, so that $A = \sum_{i=1}^{m} c_i A_i \in \ker\lambda$. Then $d\hat{J}(A) = i_{\lambda(A)}\Omega = 0$ since $X_{\hat{J}(A)} = \lambda(A)$. Therefore $0 = d\hat{J}(A)_x = \sum_{i=1}^{m} c_i d\hat{J}(A_i)_x$ for every $x \in P$, which shows that $\{\hat{J}(A_1), ..., \hat{J}(A_m)\}$ are functionally dependent.

2) Suppose that $\overline{\lambda} : \overline{H} = H/\ker\lambda \to \mathrm{Diff}(P)$ is an almost free action. By (3.11b) this means that $\ker\lambda = \mathfrak{H}_x = \{A \in \mathfrak{H} : \lambda(A)(x) = 0\}$ for all $x \in P$. Equivalently, if $\lambda(A)$ vanishes at one point of P then $\lambda(A) = 0$ on P. Let $N = \dim(\mathfrak{H}/\ker\lambda)$ and let $\{A_1, A_2, ..., A_N\}$ be elements of $\mathfrak{H}$ such that $\{A_1 + \ker\lambda, A_2 + \ker\lambda, ..., A_N + \ker\lambda\}$ is a basis of $\mathfrak{H}/\ker\lambda$.

We assert that $\{\hat{J}(A_1), ..., \hat{J}(A_N)\}$ are functionally independent.

Suppose that $0 = \sum_{i=1}^{N} c_i d\hat{J}(A_i)_x$ for some $x \in P$. If $A = \sum_{i=1}^{N} c_i A_i$, then $d\hat{J}(A)_x = 0$. Hence $\lambda(A)(x) = 0$ since $d\hat{J}(A) = i_{\lambda(A)}\Omega$ and Ω is nondegenerate. By the discussion above we conclude that $\lambda(A) = 0$. Since $A = \sum_{i=1}^{N} c_i A_i \in \ker\lambda$ and $\{A_1 + \ker\lambda, A_2 + \ker\lambda, ..., A_N + \ker\lambda\}$ is a basis of $\mathfrak{H}/\ker\lambda$ it follows that $c_i = 0$ for every i. ■

3.12 Complete integrability on a symplectic manifold

Two functions f, g on a Poisson manifold P are said to be in involution if $\{f, g\} = 0$. It follows that the corresponding Hamiltonian vector fields commute since $[X_f, X_g] = -X_{\{f,g\}} = 0$. If X_f and X_g are linearly independent at some point x of P, then the result in (1.3) states that there exists a coordinate system on a neighborhood U of x such that X_f and X_g are the first two coordinate vector fields. Continuing in this vein, suppose that we can find functions $f_1, f_2, ..., f_k$ on P such that $\{f_i, f_j\} = 0$ for $1 \leq i, j \leq k$. The Hamiltonian vector fields $X_{f_1}, ... X_{f_k}$ commute by the argument above. If the vector fields are linearly independent at some point x of P, then the result of (1.3) says that there exists a coordinate system on a neighborhood U of x such that $X_{f_1}, ... X_{f_k}$ are the first k coordinate vector fields on U.

We now assume that P is a symplectic manifold of dimension $2n$. This is not really an additional condition since Hamiltonian vector fields on P are tangent to each symplectic leaf of P. The discussion above might lead one to conclude that there could be as many as $2n$ independent functions in involution if one were lucky. However, it turns out that there are at most $n = \frac{1}{2}\dim P$ independent functions in involution on P. If the maximum number n is achieved, then one says that the symplectic structure on P is completely integrable.

We now give some detail to the verbal description above.

Proposition Let P be a symplectic manifold of dimension $2n$ with symplectic 2-form Ω. Let $\{p_1, p_2, ..., p_n\}$ be functions such that $\{p_i, p_j\} = 0$ for $1 \leq i, j \leq n$ and the differentials $\{dp_1, dp_2, ..., dp_n\}$ are linearly independent at every point of P. Then for every point x of P there exists a neighborhood U of x and functions $q_i : U \to \mathbb{R}, 1 \leq i \leq n$, with the following properties:

1) $\Omega = \frac{1}{2}\sum_{i=1}^{n} dq_i \wedge dp_i$ in U.

2) $x = (q_1, p_1, q_2, p_2, ..., q_n, p_n)$ is a coordinate system on U such that $\{p_i, p_j\} = 0$ for all i, j and $\{q_i, p_j\} = \delta_{ij}$. Moreover

a) $[X_{p_i}, X_{p_j}] = [X_{p_i}, X_{q_j}] = 0$ for all i, j.

b) For $1 \leq i \leq n, X_{p_i} = \frac{\partial}{\partial q_i}$ and $X_{q_i} = -\frac{\partial}{\partial p_i} + \sum_{k=1}^{n}\{q_k, q_i\}\frac{\partial}{\partial q_k}$

3) If $H : U \to \mathbb{R}$ is a function such that $\{H, p_i\} = 0$ for $1 \leq i \leq n$, then $\frac{\partial H}{\partial q_i} = 0$ for $1 \leq i \leq n$. In particular, $H = \varphi(p_1, p - 2, ..., p_n)$, where φ is some function of n variables.

Lemma A The Hamiltonian vector fields $\{X_{p_1}, ..., X_{p_n}\}$ are linearly independent at every point of M.

Proof Suppose that $0 = \sum_{i=1}^{n} a_i X_{p_i}(x)$ for some point x of P and some real numbers $\{a_i\}$. Then $0 = i(\sum_{i=1}^{n} a_i X_{p_i}(x))\Omega = \sum_{i=1}^{n} a_i i_{X_{p_i}}(x)\Omega = \sum_{i=1}^{n} a_i (d_{p_i})_x$. Hence $a_i = 0$ for all i since the differentials $\{dp_1, dp_2, ..., dp_n\}$ are linearly independent at x. ■

As we explained above, the Hamiltonian vector fields $\{X_{p_1}, ..., X_{p_n}\}$ commute since $\{p_i, p_j\} = 0$ for $1 \leq i, j \leq n$. It now follows from Lemma A and the result in (1.3) that for every point p of P there exists a coordinate system $x = (x_1, x_2, ..., x_{2n})$ in a neighborhood V of p such that $X_{p_i} = \frac{\partial}{\partial x_i}$ in V for $1 \leq i \leq n$

Lemma B Fix a point p of P and choose a coordinate system $x = (x_1, x_2, ..., x_{2n})$ in a neighborhood V of p such that $X_{p_i} = \frac{\partial}{\partial x_i}$ in V for $1 \leq i \leq n$. Let $q_i = x_i$ for $1 \leq i \leq n$. Then $\{X_{p_1}(p'), ..., X_{p_n}(p'), X_{q_1}(p'), ..., X_{q_n}(p')\}$ is a basis for $T_{p'}P$ for every $p' \in V$.

Proof We begin by observing

$$(^*)\{p_i, p_j\} = 0 \text{ and } \{q_i, p_j\} = \delta_{ij} \text{ for all } i, j.$$

The first equations are the hypotheses and the second equations follow since $q_i = x_i$ and $X_{p_i} = \frac{\partial}{\partial x_i}$. Let a_i, b_j be real numbers such that $0 = \sum_{i=1}^{n} a_i X_{p_i}(p) + \sum_{i=1}^{n} b_i X_{q_i}(p) = X_{f+g}(p)$, where $f = \sum_{i=1}^{n} a_i p_i$ and $g = \sum_{j=1}^{n} b_j q_j$. Since $X_{f+g}(p) = 0$ we obtain

$0 = \{p_k, f+g\}(p) = \sum_{i=1}^{n} a_i\{p_k, p_i\} + \sum_{j=1}^{n} b_j\{p_k, q_j\} = -b_k$ by (*). Similarly, $0 = \{q_k, f+g\}(p) = \sum_{i=1}^{n} a_i\{q_k, p_i\} + \sum_{j=1}^{n} b_j\{q_k, q_j\} = a_k$ by (*) and the fact that $b_j = 0$ for all j. ■

Proof of the Proposition Let V be the neighborhood of p constructed in Lemma B. Let $q_i = x_i$ for $1 \le i \le n$. We prove that 1) holds in V. By the conclusion of Lemma B it suffices to show that the equation in 1) holds for all pairs of vector fields of the form a) (X_{p_i}, X_{p_j}) b) (X_{q_i}, X_{p_j}) or c) (X_{q_i}, X_{q_j}). The verification in each of these cases follows routinely from (*) in the proof of Lemma B and from the definition of the Hamiltonian vector fields X_{p_i} and X_{q_j} in terms of $\{,\}$. We omit the details.

2) Define a map $x : V \to \mathbb{R}^{2n}$ by $x = (q_1, p_1, q_2, p_2, ..., q_n, p_n)$. From 1) it follows that $\Omega^n = c_n(dq_1 \wedge dp_1 \wedge dq_2 \wedge dp_2 \wedge ... \wedge dq_n \wedge dp_n)$, where c_n is a constant that depends only on n. Since Ω^n is nonzero by the definition of symplectic 2-form it follows that $dq_1 \wedge dp_1 \wedge dq_2 \wedge dp_2 \wedge ... \wedge dq_n \wedge dp_n$ is nonzero in V, or equivalently, that x is nonsingular in V. If we replace V by a possibly smaller open neighborhood U of p, then x is a diffeomorphism on U and hence a coordinate system on U.

The assertion $\{p_i, p_j\} = 0$ for all i, j is the hypothesis of the proposition, and the assertion $\{q_i, p_j\} = \delta_{ij}$ was noted in (*) of the proof of Lemma B. Hence $[X_{p_i}, X_{p_j}] = -X_{\{p_i, p_j\}} = 0$, and $[X_{q_i}, X_{p_j}] = -X_{\{q_i, p_j\}} = 0$ since $X_c = 0$ for any constant c.

To compute X_{p_i} and X_{q_i} we relabel the coordinates for convenience, say $x_{2i} = p_i$ and $x_{2i-1} = q_i$ for $1 \le i \le n$. Then $X_{p_i} = \sum_{k=1}^{2n}(\sum_{j=1}^{2n}\{x_k, x_j\}\frac{\partial p_i}{\partial x_j})\frac{\partial}{\partial x_k} = \sum_{k=1}^{2n}\{x_k, p_i\}\frac{\partial}{\partial x_k} = \frac{\partial}{\partial q_i}$. Similarly, $X_{q_i} = \sum_{k=1}^{2n}(\sum_{j=1}^{2n}\{x_k, x_j\}\frac{\partial q_i}{\partial x_j})\frac{\partial}{\partial x_k} = \sum_{k=1}^{2n}\{x_k, q_i\}\frac{\partial}{\partial x_k} = -\frac{\partial}{\partial p_i} + \sum_{k=1}^{n}\{q_k, q_i\}\frac{\partial}{\partial q_k}$.

3) Let $H : U \to \mathbb{R}$ be a function such that $\{H, p_i\} = 0$ for $1 \le i \le n$. Then $\frac{\partial H}{\partial q_i} = X_{p_i}(H) = \{H, p_i\} = 0$ by 2). ■

4. Geometry of Lie groups with a left invariant metric

For each $g \in G$ let L_g and R_g denote the diffeomorphisms of G given by $L_g(h) = gh$ and $R_g(h) = hg$ respectively for all $h \in G$. Fix an inner product $<,>$ on the Lie algebra $\mathfrak{H}$. Then there is a unique extension of $<,>$, also denoted $<,>$, to the tangent spaces of H such that the left translations $\{L_g : g \in G\}$ are isometries of

$\{H, <,>\}$. Such a metric is called the left invariant metric determined by $<,>$. One defines the right invariant metric determined by $<,>$ in the same way by using the right translations $R_g : g \in G\}$. In this article we consider only Lie groups with a left invariant metric.

4.1 Optimal left invariant metrics on H

Although we shall consider the geometry of an arbitrary left invariant metric on a connected Lie group H there is an optimal left invariant metric on H in the case that H is simply connected. If H is nilpotent in addition, then we can say even more. See (6.2d).

Maximal compact subgroups of Aut(H)

Let H be a simply connected Lie group with Lie algebra $\mathfrak{H}$. Let $T : \mathrm{Aut}(H) \to \mathrm{Aut}(\mathfrak{H})$ be the continuous homomorphism defined by $T(\varphi) = (d\varphi)_e : T_eH = \mathfrak{H} \to \mathfrak{H}$. The homomorphism T is injective since H is connected, and T is surjective since H is simply connected. Hence T is an isomorphism of Lie groups. In particular K is a maximal compact subgroup of $\mathrm{Aut}(H) \Leftrightarrow K' = T(K)$ is a maximal compact subgroup of $\mathrm{Aut}(\mathfrak{H})$.

Now let $<,>$ denote an inner product on $\mathfrak{H}$ and also the corresponding left invariant metric on H. If $\varphi \in \mathrm{Aut}(H)$ and $T(\varphi) = (d\varphi)_e$ is a linear isometry of $\mathfrak{H}$, then $\varphi \in I(H)$, the isometry group of H, since $\varphi \circ L_h = L_{\varphi(h)} \circ \varphi$ for all $h \in H$. Hence the isomorphism $T : \mathrm{Aut}(H) \to \mathrm{Aut}(\mathfrak{H})$ maps $K = \mathrm{Aut}(H) \cap I(H)$ isomorphically onto $K' = \mathrm{Aut}(\mathfrak{H}) \cap O(\mathfrak{H})$, where $O(\mathfrak{H})$ denotes the orthogonal group of $\{\mathfrak{H}, <,>\}$. Both K and K' are compact Lie groups since K' is a closed subgroup of the compact group $O(\mathfrak{H})$.

Proposition Let H be a simply connected Lie group. Then there exists a left invariant metric $<,>_o$ on H such that $K_o = \mathrm{Aut}(H) \cap I(H, <,>_o)$ is a maximal compact subgroup of $\mathrm{Aut}(H)$. If $<,>$ is any left invariant metric on H and $K = \mathrm{Aut}(H) \cap I(H, <,>)$, then $\dim K \leq \dim K_o$ with equality $\Leftrightarrow$ K and $\varphi K_o \varphi^{-1}$ have the same identity component for some $\varphi \in \mathrm{Aut}(H)$.

Remark Although $<,>_o$ is not uniquely determined in the result above we refer to $<,>_o$ as an optimal left metric on H for reasons that are evident from the statement of the proposition.

Lemma Let K be any compact subgroup of $\mathrm{Aut}(H)$. Then there exists a left invariant metric $<,>$ on H such that $K \subseteq \mathrm{Aut}(H) \cap I(H, <,>)$.

Proof of the lemma Let $K' = T(K) \subseteq \mathrm{Aut}(\mathfrak{H}) \subseteq GL(\mathfrak{H})$. Since K' is a compact Lie group there exists an inner product $<,>$ on $\mathfrak{H}$ such that $K' \subseteq \mathrm{Aut}(\mathfrak{H}) \cap O(\mathfrak{H})$. It follows from the discussion above that $K \subseteq \mathrm{Aut}(H) \cap I(H, <,>)$. ■

Proof of the proposition Let K_o be a maximal compact subgroup of $\mathrm{Aut}(H)$. By the lemma there exists a left invariant metric $<,>_o$ on H such that $K_o \subseteq \mathrm{Aut}(H) \cap I(H, <,>_o)$. Equality holds by the maximality of K_o since $\mathrm{Aut}(H) \cap I(H, <,>_o)$ is a compact subgroup of $\mathrm{Aut}(H)$ by the discussion above.

Now let $<,>$ be any left invariant metric on H, and let $K = \mathrm{Aut}(H) \cap I(H, <,>)$.

If K^* is any maximal compact subgroup of $\mathrm{Aut}(H)$ that contains K, then $K^* = \varphi K_o \varphi^{-1}$ for some $\varphi \in \mathrm{Aut}(H)$ ([I], [Mal 2]). Hence $\dim K \leq \dim K^* = \dim K_o$. Moreover, it is clear that if equality holds then K and K^* have the same identity component. ■

4.2 Basic left invariant metric structure

4.2a The Levi Civita connection of a left invariant metric

Let H be a connected Lie group with Lie algebra $\mathfrak{H}$. Let $<,>$ denote an inner product on $\mathfrak{H}$ and also its extension to a left invariant metric on H. The Levi Civita connection for a Riemannian manifold takes a particularly simple form in this case.

Recall that the Levi Civita connection of a Riemannian manifold M can be described as the map $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ that is uniquely determined by the following properties :

1)$(X, Y) \to \nabla_X Y$ is $\mathbb{R}$-bilinear.

2)$\nabla_{fX} Y = f \nabla_X Y$ for all $X, Y \in \mathfrak{X}(M)$ and all $f \in C^\infty(M)$.

3)$\nabla_X fY = f \nabla_X Y + X(f) Y$ for all $X, Y \in \mathfrak{X}(M)$ and all $f \in C^\infty(M)$.

4)$\nabla_X Y - \nabla_Y X = [X, Y]$ for all $X, Y \in \mathfrak{X}(M)$

5)$X < Y, Z > = < \nabla_X Y, Z > + < Y, \nabla_X Z >$ for all $X, Y, Z \in \mathfrak{X}(M)$

If H is a connected Lie group with a left invariant metric, then $< X, Y >$ is a constant function on H for any two left invariant vector fields X, Y on H; that is, for any two elements $X, Y \in \mathfrak{H}$. In particular $X < Y, Z > = 0$ for all $X, Y, Z \in \mathfrak{H}$, and 5) now says that for each $X \in \mathfrak{H}$ the map $Y \to \nabla_X Y$ is skew symmetric. These observations lead to a considerable simplification of the usual formula for $\nabla_X Y$, which can be found, for example, in [He, p.48]. We obtain

Proposition Let H be a connected Lie group with Lie algebra $\mathfrak{H}$, and let $<,>$ be a left invariant metric on H. If $X, Y \in \mathfrak{H}$, then $\nabla_X Y \in \mathfrak{H}$ and satisfies

$$\nabla_X Y = \tfrac{1}{2}\{[X, Y] - (adX)^t Y - (adY)^t X\}$$

where $(adX)^t$ denotes the transpose of adX relative to $<,>$.

The Riemannian curvature tensor on a Riemannian manifold M is given by the formula

$$R(X, Y)Z = -\nabla_{[X,Y]} Z + \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) \text{ for } X, Y, Z \in \mathfrak{X}(M).$$

If H is a connected Lie group with a left invariant metric , then $R(X, Y)Z \in \mathfrak{H}$ if $X, Y, Z \in \mathfrak{H}$ since $\nabla_U V \in \mathfrak{H}$ for all $U, V \in \mathfrak{H}$.

The discussion above illustrates the principle that much of the geometry of a Lie group H with a left invariant metric can be computed from the algebraic structure of the Lie algebra $\mathfrak{H}$.

4.2b The tangent bundle TH as a Lie group

Let H be a connected Lie group with Lie algebra $\mathfrak{H} = T_e H$. We may identify the tangent bundle TH with $H \times \mathfrak{H}$ by means of the diffeomorphism that sends (h, X) to $(L_h)_*(X) \in T_h H$. We regard $H \times \mathfrak{H}$ as a product Lie group, where $\mathfrak{H}$ is a simply connected abelian Lie group. In this manner TH becomes a Lie group. The Lie algebra of $TH = H \times \mathfrak{H}$ may be identified with $\mathfrak{H} \times \mathfrak{H}$, and the exponential map

$\exp_{TH} : \mathfrak{H} \times \mathfrak{H} \to H \times \mathfrak{H}$ is defined by $\exp_{TH}(X, Y) = (e^X, Y)$ for all $X, Y \in \mathfrak{H}$, where $e : \mathfrak{H} \to H$ is the exponential map of H.

In the sequel we regard TH as the Lie group $H \times \mathfrak{H}$, and we consider the standard left action λ of TH on itself given by $\lambda_{(h,Z)}(h^*, Z^*) = (hh^*, Z + Z^*)$ for all $(h, Z), (h^*, Z^*) \in H \times \mathfrak{H}$. The restriction of this action λ to the subgroup H is also important since $H \times \mathfrak{H} = P$ will soon be equipped with a symplectic 2-form Ω. The restriction $\lambda : H \to \mathrm{Diff}(H \times \mathfrak{H})$ is clearly given by $\lambda_h(h^*, Z^*) = (hh^*, Z^*)$ for all $(h^*, Z^*) \in H \times \mathfrak{H}$.

Each element (X, Y) of $\mathfrak{H} \times \mathfrak{H}$ defines a left invariant vector field on $H \times \mathfrak{H}$ whose flow transformations $(X, Y)^t$ are given by $(X, Y)^t(h, Z) = (he^{tX}, Z + tY)$ for all t. (Recall that every left invariant vector field on a Lie group has flow transformations that are right translations by elements of the Lie group.) Since $\mathfrak{H} \times \mathfrak{H}$ is the Lie algebra of the product group $H \times \mathfrak{H}$ and $\mathfrak{H}$ is an abelian factor of $H \times \mathfrak{H}$ we obtain

1) $(X, Y) + (X', Y') = (X + X', Y + Y')$ for all $X, X', Y, Y' \in \mathfrak{H}$.

$c(X, Y) = (cX, Y) = (X, cY)$ for all $X, Y \in \mathfrak{H}, c \in \mathbb{R}$.

2) $[(X, Y), (X', Y')] = ([X, X'], 0)$ for all $X, Y \in \mathfrak{H}$.

3) Given $(h, X) \in H \times \mathfrak{H}$ and $\xi \in T_{(h,X)}(H \times \mathfrak{H})$ there exist unique vectors $Y, Z \in \mathfrak{H}$ such that $(Y, Z)(h, X) = \xi$.

4.2c The connection map in TH

We compute the values of the connection map $K : TTH \to TH$ on the left invariant vector fields $\{(Z, Z^*) : Z, Z^* \in \mathfrak{H}\}$. We identify TH with $H \times \mathfrak{H}$ as above.

Proposition Let H be a connected Lie group with Lie algebra $\mathfrak{H}$ and a left invariant metric $<,>$. Let ∇ denote the Levi Civita connection on H with corresponding connection map $K : TTH \to TH$. Let $\pi : TH \to H$ be the base point projection map. Then

$\pi_*(Z, Z^*)(h, \alpha) = (h, Z)$ for all $h \in H$ and all $Z \in \mathfrak{H}$.

$K((Z, Z^*)(h, \alpha)) = (h, \nabla_Z \alpha + Z^*)$ for all $(h, \alpha) \in TH$ and all $Z, Z^* \in \mathfrak{H}$

Proof The proof of the assertion for π_* is routine. For the proof of the assertion for K the following elementary result is useful. We leave the proof as an exercise.

Lemma Let $\beta(t)$ be a C^1 curve in a Riemannian manifold M, and let $\xi(t)$ be a C^1 vector field on β. Let φ be an isometry of M, and define $\beta^*(t) = \varphi(\beta(t))$ and $\xi^*(t) = \varphi_*(\xi(t))$. Then $\xi'(0) = (\varphi^{-1})_* \xi^{*\prime}(0)$, where $\xi'(0)$ and $\xi^{*\prime}(0)$ denote the covariant derivatives at $t = 0$ of ξ and ξ^* along β and β^*.

Let $Z, Z^* \in \mathfrak{H}$ and $(h, \alpha) \in H \times \mathfrak{H} = TH$ be given. By definition $(Z, Z^*)(h, \alpha)$ is the initial velocity of $t \to (he^{tZ}, \alpha + tZ^*)$. Hence by definition $K((Z, Z^*)(h, \alpha)) = \xi'(0)$, where $\xi(t) = (\alpha + tZ^*)(he^{tZ})$.

We now apply the lemma above to $\beta(t) = he^{tZ}$ and $\varphi = L_{h^{-1}}$. Then $\beta^*(t) = e^{tZ}$ and $\xi^*(t) = (\alpha + tZ^*)(e^{tZ})$. We compute $\xi^{*\prime}(t) = \nabla_{\beta^{*\prime}(t)}\alpha + Z^*(\beta^*(t)) + t\nabla_{\beta^{*\prime}(t)}Z^*$, and hence $\xi^{*\prime}(0) = (\nabla_Z \alpha + Z^*)(e)$, where e denotes the identity of H. The lemma says that $K((Z, Z^*)(h, \alpha)) = L_{h*}\xi^{*\prime}(0) = L_{h*}(\nabla_Z \alpha + Z^*)(e) = (h, \nabla_Z \alpha + Z^*)$. ∎

4.2d The Sasaki metric on TH

The result above and the definition of the Sasaki metric in TM for an arbitrary

Riemannian manifold M yield the following

Proposition Let H be a connected Lie group with Lie algebra $\mathfrak{H}$ and a left invariant metric $<,>$. Let ∇ denote the Levi Civita connection on H with corresponding connection map $K : TTH \to TH$. Let $<<,>>$ denote the Sasaki metric in TH defined by $<,>$. Then for all $Z, Z^* \in \mathfrak{H}$

$$<<(Z_1, Z_1^*)(h,\alpha),(Z_2,Z_2^*)(h,\alpha)>> = <Z_1,Z_2> + <\nabla_{Z_1}\alpha + Z_1^*, \nabla_{Z_2}\alpha + Z_2^*>$$

4.3 The symplectic structure of TH = H $\times$ $\mathfrak{H}$

The left invariant inner product $<,>$ on H induces a diffeomorphism between the cotangent bundle TH^* and the tangent bundle TH. We use this diffeomorphism to transport the canonical 1-form θ and the symplectic 2-form $\Omega = -d\theta$ from TH^* to TH. We then study TH using both the geometric structure of TN with the Sasaki metric $<<,>>$ induced from $<,>$ and the symplectic structure from TH^*.

We first identify TH with $H \times \mathfrak{H}$ and construct some canonical vector fields on TH that will be useful.

Basic properties

Let H be a finite dimensional, connected real Lie group, and let $\mathfrak{H}$ denote its Lie algebra. Let $<,>$ be a positive definite inner product on $\mathfrak{H}$, and let $<,>$ also denote the corresponding left invariant inner product on H. The inner product $<,>$ defines a diffeomorphism $f : TH \to TH^*$ given by $f(\xi)(v) = <v,\xi>$ for $\xi, v \in T_hH$ and $h \in H$. Note that $f(T_hH) = T_hH^*$ for all $h \in H$. Let θ and $\Omega = -d\theta$ be the canonical 1-form and symplectic 2-form on TH^* that were defined earlier in (2.1). Let θ and $\Omega = -d\theta$ also denote the pullbacks to TH by f. It is easy to check the following assertions from the definitions :

Proposition A 1) $\theta(\xi) = <\pi_*(\xi), v>$ for $v \in TH$ and $\xi \in T_v(TH)$, where $\pi : TH \to H$ is the basepoint projection map.

2) The forms θ and $\Omega = -d\theta$ are H-invariant; that is, $(\lambda_h)^*\theta = \theta$ and $(\lambda_h)^*\Omega = \Omega$ for all $h \in H$.

Next, we evaluate θ on the left invariant vector fields $\{(Z, Z^*) : Z, Z^* \in \mathfrak{H}\}$ on TH that we defined in (4.2b).

Proposition B Let (Z, Z^*) and (U, U^*) be any left invariant vector fields on $TH = H \times H$, and let (n, β) be any point of $H \times \mathfrak{H}$. Then

$$\Omega((Z,Z^*),(U,U^*))(n,\beta) = -<U,Z^*> + <U^*,Z> + <[Z,U],\beta>$$

Proof We need two preliminary results whose proofs we leave as an exercise.

Lemma 1) $\theta(Z,Z^*)(n,\beta) = <Z,\beta>$.

2) $(Z,Z^*)\{\theta(U,U^*)\}(n,\beta) = <U,Z^*>$

Proof of Proposition B By the lemma above and the formula for the exterior derivative of a 1-form (cf. (1.1)) we obtain
$\Omega((Z,Z^*),(U,U^*))(n,\beta) = -d\theta((Z,Z^*),(U,U^*))(n,\beta) = -(Z,Z^*)\{\theta(U,U^*)\}(n,\beta) + (U,U^*)\{\theta(Z,Z^*)\}(n,\beta) + \theta([(Z,Z^*),(U,U^*)])(n,\beta) = -<U,Z^*> + <Z,U^*> + \theta([Z,U],0)(n,\beta) = -<U,Z^*> + <Z,U^*> + <[Z,U],\beta>$. ■

4.4 The Poisson structures on TH and $\mathfrak{H}$

The manifold TH has a Poisson structure arising from the symplectic 2-form Ω defined by $<,>$. The Lie algebra $\mathfrak{H}$ has a canonical Poisson structure determined by $<,>$. We now define a map $G : TH \to \mathfrak{H}$ that will turn out to intertwine these two Poisson structures; more precisely, we show that G is an anti Poisson map. Although the Poisson structures on TH and $\mathfrak{H}$ depend on the choice of $<,>$, the map G depends only on H and $\mathfrak{H}$.

4.4a The Gauss map $G : TH \to \mathfrak{H}$

We define the Gauss map $G : TH = H \times \mathfrak{H} \to \mathfrak{H}$ by $G(h, Z) = Z$ for all $Z \in \mathfrak{H}$. In view of the identifications we have made, the map G translates a vector in $T_h H$ back to $T_e H = \mathfrak{H}$ by the differential map of ${\lambda_h}^{-1}$. In the case that H is the abelian group $\mathbb{R}^3$ the Gauss map is the same as the Gauss map used in the classical study of surfaces in $\mathbb{R}^3$.

We note that $G : H \times \mathfrak{H} \to \mathfrak{H}$ is a Lie group homomorphism since addition is the group operation in $\mathfrak{H}$.

We introduce some definitions that will be useful in the proof that G is an anti Poisson map.

4.4b H-invariant functions on TH

A function $f : TH \to \mathbb{R}$ will be called H-invariant if f is constant on all H-orbits in $TH = H \times \mathfrak{H}$; that is , $f(h, X) = f(e, X)$ for all $h \in H$ and all $X \in \mathfrak{H}$.

The H-invariant functions on TH may be identified with the functions on $\mathfrak{H}$. If $f : TH \to \mathbb{R}$ is an H-invariant function, then we may define $\bar{f} : \mathfrak{H} \to \mathbb{R}$ by $\bar{f}(X) = f(e, X)$, where e denotes the identity in H. Conversely, if we are given a function $\bar{f} : \mathfrak{H} \to \mathbb{R}$, then $\bar{f}$ arises from an H-invariant function $f : TH \to \mathbb{R}$ given by $f(h, X) = \bar{f}(X)$ for all $(h, X) \in H \times \mathfrak{H}$. Note that $f = \bar{f} \circ G$.

4.4c H-invariant vector fields $\mathfrak{X}'(TH)$ on TH

A C^∞ vector field Y on TH will be called H-invariant if $(\lambda_h)_* Y(h', X') = Y(hh', X')$ for all $h, h' \in H$ and all $X' \in \mathfrak{H}$. Equivalently, Y is H-invariant $\Leftrightarrow$ Y is λ_h-related to itself for all $h \in H$ (cf. (1.2)). We let $\mathfrak{X}'(TH)$ denote the collection of H-invariant vector fields on TH. By (1.2)$\mathfrak{X}'(TH)$ is closed under Lie brackets.

The next result gives a useful characterization of H-invariant vector fields on TH.

Proposition The following assertions are equivalent for a vector field Y on $TH = H \times \mathfrak{H}$:

1)Y is H-invariant.

2) If $\{Y^t\}$ are the flow transformations of Y on TH, then $Y^t \circ \lambda_h = \lambda_h \circ Y^t$ for all $h \in H$ and all $t \in \mathbb{R}$.

3)$Y(h, X) = (f_1(X), f_2(X))(h, X)$ for C^∞ functions $f_1, f_2 : \mathfrak{H} \to \mathfrak{H}$.

The right hand side of 3) denotes the value of the left invariant vector field $(f_1(X), f_2(X))$ at (h, X); i.e., the initial velocity of $t \to (he^{tf_1(X)}, X + tf_2(X))$.

Remark Assertion 3) says that there is a one-one correspondence between H-invariant vector fields on TH and pairs of C^∞ functions $f_1, f_2 : \mathfrak{H} \to \mathfrak{H}$.

Proof of the Proposition The equivalence of the first two assertions is routine, and we omit the details. Given a pair of C^∞ functions $f_1, f_2 : \mathfrak{H} \to \mathfrak{H}$, the vector field $Y(h,X) = (f_1(X), f_2(X))(h,X)$ is easily seen to be H-invariant. Conversely, let Y be an H-invariant vector field on TH. If we are given $X \in \mathfrak{H}$, then by 3) of (4.2b) above there exist unique elements $f_1(X), f_2(X)$ of $\mathfrak{H}$ such that $Y(e,X) = (f_1(X), f_2(X))(e,X)$. The functions $f_1(X)$ and $f_2(X)$ define an H-invariant vector field on $H \times \mathfrak{H}$ whose values on $\{e\} \times \mathfrak{H}$ are the same as Y. Since an H-invariant vector field is determined by its values on $\{e\} \times \mathfrak{H}$, it follows that $Y(h,X) = (f_1(X), f_2(X))(h,X)$ for all $(h,X) \in H \times \mathfrak{H}$. ■

4.4d The Lie algebra homomorphism $\tilde{G} : \mathfrak{X}'(TH) \to \mathfrak{X}(\mathfrak{H})$

If Y is an H-invariant vector field on TH, then we may define a vector field $\bar{Y}$ on $\mathfrak{H}$ by $\bar{Y}(X) = G_*(Y(e,X))$. Note that Y and $\bar{Y}$ are G-related vector fields since $G_*(Y(h,X)) = G_*(\lambda_{h*} Y(e,X)) = (G \circ \lambda_h)_* Y(e,X) = (G)_* Y(e,X) = \bar{Y}(X) = \bar{Y}(G(h,X))$ for all $(h,X) \in H \times \mathfrak{H}$.

If we write $Y(h,X) = (f_1(X), f_2(X))(h,X)$ for suitable C^∞ functions $f_1, f_2 : \mathfrak{H} \to \mathfrak{H}$ as in 3) above, then it is easy to check that $\bar{Y}(X) = (f_2(X))_X \in T_X\mathfrak{H}$ for all $X \in \mathfrak{H}$.

Conversely, every vector field $\bar{Y}$ on $\mathfrak{H}$ arises in this fashion from an H-invariant vector field Y on TH. To see this, let $f : \mathfrak{H} \to \mathfrak{H}$ be the function such that $\bar{Y}(X) = f(X)_X \in T_X\mathfrak{H}$ for all $X \in \mathfrak{H}$. Then define $Y(h,X) = (0, f(X))(h,X)$ for all $(h,X) \in H \times \mathfrak{H}$.

By (1.2) we may summarize the discussion above as follows :

Proposition The map $\tilde{G} : Y \to \bar{Y}$ is a surjective Lie algebra homomorphism of $\mathfrak{X}'(TH)$ onto $\mathfrak{X}(H)$. The kernel of $\tilde{G}$ is the set of vector fields Y in $\mathfrak{X}'(TH)$ of the form $Y(h,X) = (f(X), 0)(h,X)$, where $f : \mathfrak{H} \to \mathfrak{H}$ is an arbitrary C^∞ function.

4.4e Hamiltonian vector fields of H-invariant functions

If $f : TH \to \mathbb{R}$ is an H-invariant function, then f defines a function $\bar{f} : \mathfrak{H} \to \mathbb{R}$ by $\bar{f}(X) = f(e,X)$, where $H \times \mathfrak{H} = TH$. Conversely, we observed above that every function $\bar{f} : \mathfrak{H} \to \mathbb{R}$ defines an H-invariant function $f : TH \to \mathbb{R}$ by $f(h,X) = \bar{f}(X)$ for all $(h,X) \in H \times \mathfrak{H}$. The next result relates the Hamiltonian vector fields determined by f and $\bar{f}$ relative to the Poisson structures on TH and $\mathfrak{H}$.

Proposition Let H be a connected Lie group with Lie algebra $\mathfrak{H}$ and a left invariant metric $<,>$. Let $f : TH \to \mathbb{R}$ be an H-invariant function with companion function $\bar{f} : \mathfrak{H} \to \mathbb{R}$. Let X_f and $X_{\bar{f}}$ denote the Hamiltonian vector fields defined by the Poisson structures on TH and $\mathfrak{H}$. Then

(*) $X_f(h,\alpha) = (\text{grad}\bar{f}(\alpha), -X_{\bar{f}}(\alpha))(h,\alpha)$ for all $(h,\alpha) \in H \times \mathfrak{H}$.

In particular, the vector fields X_f and $-X_{\bar{f}}$ are G-related, where $G : TH \to \mathfrak{H}$ is the Gauss map.

Remark The meaning of the right hand side of the expression above needs some explanation. Given $\alpha \in \mathfrak{H}$ let ξ, ξ^* be those elements of $\mathfrak{H}$ such that grad $\bar{f}(\alpha) = \xi_\alpha \in T_\alpha\mathfrak{H}$ and $X_{\bar{f}}(\alpha) = \xi^*_\alpha \in T_\alpha\mathfrak{H}$. Then $(\text{grad}\bar{f}(\alpha), -X_{\bar{f}}(\alpha))(h,\alpha)$ is defined to be $(\xi, -\xi^*)(h,\alpha)$.

Proof It is easy to see from the definition of the Gauss map that $G_*(Z, Z^*)(h,\alpha) =$

$Z^*_\alpha \in T_\alpha\mathfrak{H}$ for all $Z, Z^*, \alpha \in \mathfrak{H}$. The fact that X_f and $-X_{\bar{f}}$ are G-related now follows immediately from (*).

We prove (*). We show first that X_f is an H-invariant vector field on TH. We note that $(\lambda_h)^*\Omega = \Omega$ for all $h \in H$, and $(\lambda_h)^*df = df$ since f is H-invariant. It follows that X_f is H-invariant since $i_{X_f}\Omega = df$.

By (4.4c) there exist functions $f_1, f_2 : \mathfrak{H} \to \mathfrak{H}$ such that $X_f(h,\alpha) = (f_1(\alpha), f_2(\alpha))(h,\alpha)$ for all $(h,\alpha) \in H \times \mathfrak{H} = TH$. Our task is to show that for all $\alpha \in \mathfrak{H}$

$i) f_1(\alpha)_\alpha = \mathrm{grad}\bar{f}(\alpha)$ and $ii) f_2(\alpha)_\alpha = -X_{\bar{f}}(\alpha)$.

For vectors ξ, ξ^* we compute $(i_{X_f}\Omega)((\xi,\xi^*)(h,\alpha))$ in two different ways and compare the expressions. From the definition of X_f we compute $(i_{X_f}\Omega)((\xi,\xi^*)(h,\alpha)) = df((\xi,\xi^*)(h,\alpha)) = df((\xi,0)(h,\alpha)) + df((0,\xi^*)(h,\alpha)) = df((0,\xi^*)(h,\alpha)) = (d\bar{f})_\alpha(\xi^*)_\alpha$ by the H- invariance of f and the definitions of the vector fields $(\xi, 0)$ and $(0,\xi^*)$ on TH. On the other hand $(i_{X_f}\Omega)((\xi,\xi^*)(h,\alpha)) = \Omega(X_f(h,\alpha),(\xi,\xi^*)(h,\alpha)) = \Omega((f_1(\alpha), f_2(\alpha))(h,\alpha),(\xi,\xi^*)(h,\alpha)) = -< f_2(\alpha),\xi > + < f_1(\alpha),\xi^* > + < \alpha,[f_1(\alpha),\xi] >$ by (4.3). Comparing the two expressions for $(i_{X_f}\Omega)((\xi,\xi^*)(h,\alpha))$ yields

$(\#) (d\bar{f})_\alpha(\xi^*)_\alpha = -< f_2(\alpha),\xi > + < f_1(\alpha),\xi^* > + < \alpha,[f_1(\alpha),\xi] >$

Note that the left hand side of (#) does not depend on ξ. This shows

$a) (d\bar{f})_\alpha(\xi^*) = < f_1(\alpha),\xi^* >$ for all $\alpha, \xi^* \in \mathfrak{H}$

$b) < \alpha,[f_1(\alpha),\xi] > - < f_2(\alpha),\xi > = 0$ for all $\alpha,\xi \in \mathfrak{H}$

Assertion $a)$ proves $i)$ above. Substituting $i)$ into $b)$ yields $< f_2(\alpha),\xi > = < \alpha, [\mathrm{grad}\bar{f}(\alpha),\xi] > = - < X_{\bar{f}}(\alpha),\xi_\alpha >_\alpha$ by 4) of (3.7). This proves $ii)$ above. ■

4.4f Poisson maps associated to TH

Let H be a connected Lie group with Lie algebra $\mathfrak{H}$ and left invariant metric $<,>$. We show that $G : TH \to \mathfrak{H}$ is an anti Poisson map and $J : TH \to \mathfrak{H}^*$ is a Poisson map with respect to the canonical Poisson structure on $\mathfrak{H}^*$ and the Poisson structures on TH and $\mathfrak{H}$ defined by $<,>$. Here J denotes the moment map of the H-action, which exists in this setting as we will show. We then relate G and J.

The Gauss map $G : TH \to \mathfrak{H}$

Proposition A The map $G : TH \to \mathfrak{H}$ is an anti Poisson map relative to the Poisson structures on TH and $\mathfrak{H}$ defined by $<,>$.

Proof Let $\bar{f}, \bar{g}$: be C^∞ functions, and let $f, g : TH \to \mathbb{R}$ be the corresponding H-invariant functions defined in (4.4b). It follows from the H-invariance of f and g that $f = \bar{f} \circ G$ and $g = \bar{g} \circ G$. Let $\{,\}_\mathfrak{H}$ and $\{,\}_{TH}$ denote the Poisson structures defined on $\mathfrak{H}$ and TH by the inner product $<,>$. We must show

(*) $(\{\bar{f},\bar{g}\}_\mathfrak{H} \circ G)(h,\alpha) = -(\{f,g\}_{TH})(h,\alpha)$ for all $(h,\alpha) \in H \times \mathfrak{H}$

By the proposition in (4.4e) we know that $G_*(X_g(h,\alpha)) = -X_{\bar{g}}(\alpha)$ for all $(h,\alpha) \in H \times \mathfrak{H}$. Hence $(\{\bar{f},\bar{g}\} \circ G)_\mathfrak{H}(h,\alpha) = \{\bar{f},\bar{g}\}_\mathfrak{H}(\alpha) = (X_{\bar{g}}(\bar{f}))(\alpha) = (X_{\bar{g}})(\alpha)(\bar{f}) = -G_*(X_g(h,\alpha))(\bar{f}) = -(X_g(h,\alpha))(\bar{f} \circ G) = -\{\bar{f} \circ G, g\}(h,\alpha) = -\{f,g\}(h,\alpha)$. ■

The momentum map J : TH → $\mathfrak{H}^*$

The next result follows from Proposition D and the remark preceding it in (3.11e).

Proposition B A momentum map $J : TH \to \mathfrak{H}$ exists and is a Poisson map relative to the Poisson structures on TH and $\mathfrak{H}$ defined by $<,>$.

Relationship between J and G

Proposition C Let H be a connected Lie group, and let $G : TH \to \mathfrak{H}$ and $J : TH \to \mathfrak{H}^*$ denote the corresponding Gauss map and momentum map. Let $\pi : TH \to H$ denote the base point projection map. Then $J(\xi)(A) =< G(\xi), Ad(\pi\xi)^{-1}A >$ for all $\xi \in TH$ and all $A \in \mathfrak{H}$.

Proof We identify TH with $H \times \mathfrak{H}$ as in (4.2b). The projection map $\pi : TH = H \times \mathfrak{H}$ now has the form $\pi(n, \beta) = n$ for all $(n, \beta) \in H \times \mathfrak{H}$.

Lemma Let $(n, \beta) \in H \times \mathfrak{H}$ and $A \in \mathfrak{H}$. Let $\lambda : \mathfrak{H} \to \mathfrak{X}(TH)$ be the canonical anti homomorphism. Then

1)$\lambda(A)(n, \beta) = (Ad(n^{-1})A, 0)(n, \beta)$ in the terminology of (4.2b).

2)$\hat{J}(A)(n, \beta) =< \beta, Ad(n^{-1})A >$

Proof of the of the lemma

1) By definition $\lambda(A)(n, \beta)$ is the initial velocity of $t \to \lambda_{e^{tA}}(n, \beta) = (e^{tA}n, \beta) = (nn^{-1}e^{tA}n, \beta) = (ne^{tAd(n^{-1})A}, \beta)$.

2) We apply Proposition D of (3.11e). By the statement of that result , 1) above and the definition of θ we obtain $\hat{J}(A)(n, \beta) = \theta(\lambda(A)(n, \beta)) = \theta((Ad(n^{-1})A, 0)(n, \beta))$ $=< (n, \beta), \pi_*((Ad(n^{-1})A, 0)(n, \beta)) >=< (n, \beta), (Ad(n^{-1})A)(n) >=< \beta, Ad(n^{-1})A >$. ■

We now complete the proof of Proposition B. Let $\xi = (n, \beta) \in H \times \mathfrak{H} = TH$ and $A \in \mathfrak{H}$ be given. Then $J(\xi)(A) = \hat{J}(A)(n, \beta) =< \beta, Ad(n^{-1})A >=$ $< G(\xi), Ad(\pi\xi)^{-1}A >$ by 2) of the lemma. ■

5. The geodesic flow in TH and $\mathfrak{H}$

5.1 Geodesic flow in TM,M a Riemannian manifold

We begin by defining the geodesic flow in two equivalent ways for an arbitrary Riemannian manifold.

Definition 1 Let M be a complete Riemannian manifold with Riemannian metric $<,>$ and tangent bundle TM. For each $v \in TM$ and $t \in \mathbb{R}$ define $\mathfrak{G}^t(v) = \gamma_v{}'(t)$, the velocity at time t of the unique geodesic with initial velocity v.

The fact that M is complete implies that the geodesics of M are defined on $\mathbb{R}$. Hence $\mathfrak{G}^t$ is defined on M for all $t \in \mathbb{R}$. It is easy to check that $\mathfrak{G}^t \circ \mathfrak{G}^s = \mathfrak{G}^s \circ \mathfrak{G}^t = \mathfrak{G}^{t+s}$ for all $s, t \in \mathbb{R}$. We define the geodesic vector field $\mathfrak{G}$ to be the vector field on TM whose flow transformations are $\{\mathfrak{G}^t\}$.

Definition 2 Let M be a complete Riemannian manifold with Riemannian metric $<,>$ and tangent bundle TM. Let θ and $\Omega = -d\theta$ denote the canonical 1-form and symplectic 2-form defined on TM by $<,>$. Let $E : TM \to \mathbb{R}$ be the energy function defined by $E(v) = \frac{1}{2} < v, v >$ for all $v \in TM$. Let $\mathfrak{G} = X_E$, the Hamiltonian vector field determined by E. Now let $\{\mathfrak{G}^t\}$ denote the flow of $\mathfrak{G}$.
Remark If φ is any isometry of H, then $E((\varphi_* v)) = E(v)$ for all $v \in TM$. Hence if H is any connected subgroup of $I(M)$, then E is constant along H-orbits in $P = TM$, where $\lambda_h(v) = (L_h)_*(v)$ for all $h \in H$ and $v \in TM$. If $\lambda : \mathfrak{H} \to \mathfrak{X}(TM)$ is the anti homomorphism defined by the action of H, then it follows from Propositions A and D in (3.11e) that $\hat{J}(A) = \theta(\lambda(A))$ is a first integral for the geodesic flow.

We now show that these two definitions of the geodesic flow are equivalent. We need two preliminary results.
Lemma 1 $L_{\mathfrak{G}}\theta = dE$.
Proof By definition of the Lie derivative $L_{\mathfrak{G}}\theta = \frac{d}{dt}|_{t=0}\{(\mathfrak{G}^t)^*\theta\}$. Let $\xi \in T_v(TM)$ be given, and let $Y_\xi(t)$ denote the Jacobi vector vector along the geodesic γ_v such that $Y_\xi(0) = d\pi(\xi)$ and $Y_\xi'(0) = K(\xi)$, where K denotes the connection map ($cf.$(2.2a)). For any $t \in \mathbb{R}, \mathfrak{G}^t(\xi) \in T_{\mathfrak{G}^t v}(TM)$, and we compute $\{(\mathfrak{G}^t)^*\theta\}(\xi) = \theta((\mathfrak{G}^t)_*(\xi)) = < \mathfrak{G}^t v, \pi_*(\mathfrak{G}^t)_*(\xi) > = < \gamma_v'(t), Y_\xi(t) >$ by Proposition 1.7 of [E5]. Hence $L_{\mathfrak{G}}\theta(\xi) = \frac{d}{dt}|_{t=0} < \gamma_v'(t), Y_\xi(t) > = < v, Y_\xi'(0) > = < v, K(\xi) >$.

On the other hand, write $\xi = \frac{dZ}{dt}|_{t=0}$, where $Z(t)$ is a curve in TM with $Z(0) = v$. We regard $Z(t)$ as a vector field on the curve $\sigma(t) = \pi(Z(t))$. Then $K(\xi) = Z'(0)$ by the definition of K. If $E(t) = E(Z(t)) = \frac{1}{2} < Z(t), Z(t) >$, then $dE(\xi) = E'(0) = < Z'(0), Z(0) > = < K(\xi), v > = L_{\mathfrak{G}}\theta(\xi)$. ∎
Lemma 2 $\theta(\mathfrak{G}) = 2E$.
Proof We note that $\pi_*\mathfrak{G}(v) = v$ for all $v \in TM$. Hence $\theta(\mathfrak{G})(v) = < v, \pi_*\mathfrak{G}(v) > = < v, v > = 2E(v)$. ∎

We now complete the proof that the two definitions of the geodesic flow given above are equivalent. From Lemmas 1 and 2 and standard facts about the Lie derivative ($cf.$(1.1)) we obtain $dE = L_{\mathfrak{G}}\theta = (d \circ i_{\mathfrak{G}} + i_{\mathfrak{G}} \circ d)\theta = d(\theta(\mathfrak{G})) - i_{\mathfrak{G}}\Omega = 2dE - i_{\mathfrak{G}}\Omega$. This proves that $i_{\mathfrak{G}}\Omega = dE$, and it follows that $\mathfrak{G} = X_E$. ∎

5.2 Computation of the geodesic vector field $\mathfrak{G}$ on TH

We use the proposition in (4.4e) for the computation. Let $E : TH \to \mathbb{R}$ denote the H-invariant energy function $E(v) = \frac{1}{2} < v, v >$ and let $\bar{E} : \mathfrak{H} \to \mathbb{R}$ denote the restriction of E to $\mathfrak{H} = T_eH$. It is easy to check that (grad $\bar{E}$)$(\alpha) = \alpha_\alpha$ for all $\alpha \in \mathfrak{H}$; that is, grad $\bar{E}$ is the position vector field in $\mathfrak{H}$. The Hamiltonian vector field $X_{\bar{E}}$ is called the geodesic vector field in $\mathfrak{H}$ and is denoted $\bar{\mathfrak{G}}$.
The geodesic vector field $\bar{\mathfrak{G}}$ on $\mathfrak{H}$

We first relate the flow transformations of $\mathfrak{G}$ in TH to those of $\bar{\mathfrak{G}}$ in $\mathfrak{H}$.
Proposition A Let H be a connected Lie group with Lie algebra $\mathfrak{H}$, and let $<,>$ be a left invariant metric on H. Let $\{\mathfrak{G}^t\}$ and $\{\bar{\mathfrak{G}}^t\}$ denote the flow transformations of

the geodesic vector fields $\mathfrak{G}$ on TH and $\bar{\mathfrak{G}}$ on $\mathfrak{H}$. Let $G : TH \to \mathfrak{H}$ be the Gauss map. Then $G \circ \mathfrak{G}^{-t} = \bar{\mathfrak{G}}^t \circ G$ for all $t \in \mathbb{R}$.
Proof By the proposition in (4.4e) applied to the energy functions E and $\bar{E}$ we conclude that $\mathfrak{G}$ and $-\bar{\mathfrak{G}}$ are G-related; that is, $G_*(\mathfrak{G}(\xi)) = -\bar{\mathfrak{G}}(G(\xi))$ for all $\xi \in TH$. The assertion of the proposition now follows immediately. ■

$\bar{\mathfrak{G}}(\alpha) = \nabla_\alpha \alpha$

Proposition B Let H be a connected Lie group with Lie algebra $\mathfrak{H}$, and let $<,>$ be a left invariant metric on H. Then the geodesic vector field $\bar{\mathfrak{G}}$ on $\mathfrak{H}$ is given by $\bar{\mathfrak{G}}(\alpha) = \nabla_\alpha \alpha$.
Proof From the invariant formula for $X_{\bar{E}}(\alpha)$ in example 4 of (3.7) we obtain $< X_{\bar{E}}(\alpha), \xi_\alpha >_\alpha = - < \alpha, [(\text{grad } \bar{E})(\alpha), \xi] > = - < \alpha, [\alpha, \xi] >$ for all $\alpha, \xi \in \mathfrak{H}$. From properties of the Levi Civita connection and the discussion in (4.2a) we obtain $- < \alpha, [\alpha,\xi] > = - < \alpha, \nabla_\alpha \xi - \nabla_\xi \alpha > = - < \alpha, \nabla_\alpha \xi > + < \alpha, \nabla_\xi \alpha > = < \nabla_\alpha \alpha, \xi >$ $+ \frac{1}{2}\xi < \alpha, \alpha > = < \nabla_\alpha \alpha, \xi >$. We conclude that $X_{\bar{E}}(\alpha) = (\nabla_\alpha \alpha)_\alpha$ for all $\alpha \in \mathfrak{H}$. ■

The geodesic vector field $\mathfrak{G}$ on TH

Proposition C Let H be a connected Lie group with Lie algebra $\mathfrak{H}$, and let $<,>$ be a left invariant metric on H. Then the geodesic vector field $\mathfrak{G}$ on $TH = H \times \mathfrak{H}$ is given by

$\mathfrak{G}(h,\alpha) = (\alpha, -\nabla_\alpha \alpha)(h,\alpha)$ for all $(h,\alpha) \in H \times \mathfrak{H}$

where the meaning of this notation is defined in (4.2b).
Proof By the Proposition in (4.4e) and the facts that $\bar{\mathfrak{G}} = X_{\bar{E}}$ and $(\text{grad } \bar{E})(\alpha) = \alpha_\alpha$ for all $\alpha \in \mathfrak{H}$ we see that $\mathfrak{G}(h,\alpha) = (\alpha, -X_{\bar{E}})(h,\alpha)$ for all $(h,\alpha) \in H \times \mathfrak{H}$. It remains only to show that $X_{\bar{E}}(\alpha) = (\nabla_\alpha \alpha)_\alpha$ for all $\alpha \in \mathfrak{H}$. This is the content of proposition A since $\bar{\mathfrak{G}} = X_{\bar{E}}$. ■

5.3 First integrals for the geodesic flow in TH

5.3a Universal first integrals for H-invariant functions

We begin by recalling the first integrals arising from the momentum map. In fact, these are first integrals for any Hamiltonian vector field X_f, where $f : TH \to \mathbb{R}$ is any H-invariant function.
Proposition A Let H be a connected Lie group with Lie algebra $\mathfrak{H}$, and let $<,>$ be a left invariant metric on H. Let λ be the left action of H on TH given by $\lambda_h(v) = (L_h)^*(v)$ for all $v \in TH$, and let $\lambda : \mathfrak{H} \to \mathfrak{X}(TH)$ be the corresponding anti homomorphism. Then $f_A = \theta(\lambda(A))$ is a first integral for the geodesic flow on TH.
Proof We showed earlier that $\mathfrak{G} = X_E$, and we observed that E is H-invariant. The result now follows from Propositions A and D in (3.11b). ■
Remark The functions $f_A = \theta(\lambda(A)) : TH \to \mathbb{R}$ are not H-invariant first integrals in general unless A lies in the center of $\mathfrak{H}$. In the next result we use the same notation as in Proposition A and we identify TH with $H \times \mathfrak{H}$.

Proposition B Let $A \in \mathfrak{H}$ be given. Then

1) $f_A(h,\alpha) = < \alpha, \mathrm{Ad}(h^{-1})A >$ for all $(h,\alpha) \in H \times \mathfrak{H}$.

2) $f_A(h'h,\alpha) = f_A(h, \mathrm{Ad}(h'^{-1})\alpha)$ for all $h, h' \in H$ and $\alpha \in \mathfrak{H}$

Proof Assertion 2) follows immediately from assertion 1). By definition $f_A(h,\alpha) = \theta(\lambda(A))(h,\alpha) = <(h,\alpha), \pi_* \lambda(A)(h,\alpha)> = < (h,\alpha), (h, \mathrm{Ad}(h^{-1})A)> = < \alpha, \mathrm{Ad}(h^{-1})A >$ by the left invariance of the inner product. We use also the fact that $\lambda(A)(h,\alpha)$ is the initial velocity of $t \to (e^{tA}h, \alpha) = (hh^{-1}e^{tA}h, \alpha) = (he^{t\mathrm{Ad}(h^{-1})A}, \alpha)$. ∎

5.3b H-invariant first integrals for the geodesic flow

Equivalence with first integrals for the geodesic flow $\mathfrak{G}$ in $\mathfrak{H}$

Let $f : TH = H \times \mathfrak{H} \to \mathbb{R}$ be an H-invariant function, and let $\bar{f} : \mathfrak{H} \to \mathbb{R}$ denote its restriction to $\mathfrak{H}$. By (4.4f) we know that $\{f, E\} = \{\bar{f} \circ G, \bar{E} \circ G\} = -\{\bar{f}, \bar{E}\} \circ G$, where $G : TH \to \mathfrak{H}$ denotes the Gauss map and $\bar{E} : \mathfrak{H} \to \mathbb{R}$ denotes the restriction of E to $\mathfrak{H}$. The function f is a first integral for the geodesic flow in $TH \Leftrightarrow \{f, E\} = 0 \Leftrightarrow \{\bar{f}, \bar{E}\} = 0$. Hence finding H-invariant first integrals for the geodesic flow in TH is equivalent to the problem of finding functions $\bar{f} : \mathfrak{H} \to \mathbb{R}$ such that $\{\bar{f}, \bar{E}\} = 0$.

Examples of first integrals for $\bar{\mathfrak{G}}$

Proposition A A function $\bar{f} : \mathfrak{H} \to \mathbb{R}$ satisfies $\{\bar{f}, \bar{E}\} = 0 \Leftrightarrow < (\mathrm{grad}\ \bar{f})(\alpha), \nabla_\alpha \alpha > = 0$ for all $\alpha \in \mathfrak{H}$.

Proof The meaning of the expression $< (\mathrm{grad}\ \bar{f})(\alpha), \nabla_\alpha \alpha >$ is defined as in the discussion of example 4 of (3.7).

By the definitions and Proposition A of (5.2) we obtain $\{\bar{f}, \bar{E}\}(\alpha) = X_{\bar{E}}(\alpha)(\bar{f}) = (d\bar{f})_\alpha X_{\bar{E}}(\alpha) = < (\mathrm{grad}\ \bar{f})(\alpha), X_{\bar{E}}(\alpha) > = < (\mathrm{grad}\ \bar{f})(\alpha), \nabla_\alpha \alpha >$. The result follows. ∎

Example 1 $\bar{\mathfrak{G}} \equiv 0$

The simplest situation in which to find first integrals for $X_{\bar{E}}$ is when the energy function $\bar{E}$ is Casimir, which occurs precisely when $\nabla_\alpha \alpha = 0$ for all $\alpha \in \mathfrak{H}$ by the discussion in (5.2). This condition is equivalent to requiring that the inner product $<,>$ be ad-invariant; that is, ad $X : \mathfrak{H} \to \mathfrak{H}$ is skew symmetric for all $X \in \mathfrak{H}$. The equivalence of these conditions is easily verified from the formula for $\nabla_X Y$ in (4.2a), and we leave the details as an exercise.

Examples in which $<,>$ is ad-invariant include the case that $\mathfrak{H}$ is abelian and more generally the case that $\mathfrak{H}$ is the Lie algebra of a compact connected Lie group H. In this case by averaging any inner product $<,>^*$ on $\mathfrak{H}$ over $Ad(H)$ one obtains an inner product $<,>$ that is preserved by the elements of $Ad(H)$. By differentiating the identity $Ad(e^{tX}) = e^{t\,ad\,X}$ at $t = 0$ it follows that $<,>$ is ad-invariant. If $\mathfrak{H}$ has trivial center, then one may always choose $<,> = -B$, where B is the Killing form given by $B(X,Y) = \mathrm{trace}\ (adX \circ adY)$ for X, Y in $\mathfrak{H}$. For the group $H = SO(n, \mathbb{R})$ the Killing form is a constant multiple (depending on n) of the natural trace form $: (X,Y) \to \mathrm{trace}\ XY$.

Conversely, one can show that if $\mathfrak{H}$ admits an ad-invariant inner product, then there exists a compact connected Lie group H with Lie algebra $\mathfrak{H}$ [Mi, Corollary 21.6].

Example 2 Polynomial first integrals

If $\bar{f}, \bar{g} : \mathfrak{H} \to \mathbb{R}$ are polynomial functions of degree m, n respectively, then $\{\bar{f}, \bar{g}\} : \mathfrak{H} \to \mathbb{R}$ is a polynomial function of degree $m + n - 1$ by the local coordinate expression for $\{\bar{f}, \bar{g}\}$ that appears in the discussion of example 4 of (3.7). Since polynomial functions are the simplest smooth functions it makes sense to consider polynomial first integrals for the geodesic flow; that is, polynomial functions $\bar{f} : \mathfrak{H} \to \mathbb{R}$ such that $\{\bar{f}, \bar{E}\} = 0$.

Reduction to the homogeneous case

Let $\mathfrak{H}$ be equipped with linear coordinates $x = (x_1, x_2, ..., x_n)$ arising from an orthonormal basis $\{E_1, E_2, ..., E_n\}$ of $\mathfrak{H}$ and its dual basis $\{x_1, x_2, ..., x_n\}$ of $\mathfrak{H}^*$. Let $\bar{f} : \mathfrak{H} \to \mathbb{R}$ be a polynomial function of degree n and write $\bar{f} = c + \bar{f}_1 + ... + \bar{f}_i + ... + \bar{f}_n$, where c is a constant and each $\bar{f}_i$ is a homogeneous polynomial of degree $i, 1 \le i \le n$. Hence $\{\bar{f}, \bar{E}\} = \sum_{i=1}^{n} \{\bar{f}_i, \bar{E}\}$, where $\{\bar{f}_i, \bar{E}\}$ is a homogeneous polynomial of degree $i + 1$. Since each of the terms $\{\bar{f}_i, \bar{E}\}$ has a different degree of homogeneity it follows that $\{\bar{f}, \bar{E}\} = 0 \Leftrightarrow \{\bar{f}_i, \bar{E}\} = 0$ for each $i, 1 \le i \le n$. Therefore, in looking for polynomial first integrals $\bar{f}$ for the geodesic flow $X_{\bar{E}}$ it sufffices to consider the case that $\bar{f}$ is homogeneous.

For general use we prove the following

Proposition B Let $\bar{f} : \mathfrak{H} \to \mathbb{R}$ be any function. Then $\{\bar{f}, \bar{E}\} = 0 \Leftrightarrow$ $< \alpha, [\text{grad } \bar{f}(\alpha), \alpha] >= 0$ for all $\alpha \in \mathfrak{H}$.

Proof Since the linear functions are first order dense in $C^\infty(\mathfrak{H})$ it suffices by remark 2) in (3.5) to prove this result for a linear function $\bar{f} = A^\# : \mathfrak{H} \to \mathbb{R}$, where $^\# : \mathfrak{H} \to \mathfrak{H}^*$ is the isomorphism defined by the inner product $<,>$. In this case grad $\bar{f}(\alpha) = A_\alpha$ for all α, so it suffices to prove the next result.

Linear first integrals

Proposition C Let $\bar{f} = A^\# : \mathfrak{H} \to \mathbb{R}$ be a linear function. Then $\{\bar{f}, \bar{E}\} = 0 \Leftrightarrow$ $< \alpha, [A, \alpha] >= 0$ for all $\alpha \in \mathfrak{H}$.

Remark If $A \in \mathfrak{Z}$, the center of $\mathfrak{H}$, then the criterion above is clearly satisfied, but we know from the discussion in (3.9) that $\bar{f}$ is a Casimir function in this case. We shall see later in (6.8b) that if $\mathfrak{H}$ is a 2-step nilpotent Lie algebra, then every linear function $\bar{f} = A^\#$ that is a first integral for $X_{\bar{E}}$ must be a Casimir function with $A \in \mathfrak{Z}$.

Proof of the Proposition If $\bar{f} = A^\#$, then $\{\bar{f}, \bar{E}\}(\alpha) =< (\text{grad } \bar{f})(\alpha), X_{\bar{E}}(\alpha) >=$ $< A, \nabla_\alpha \alpha >= - < \nabla_\alpha A, \alpha >= - < \nabla_A \alpha + [\alpha, A], \alpha >= -\frac{1}{2} A < \alpha, \alpha > -$ $< [\alpha, A], \alpha >= - < [\alpha, A], \alpha >=< [A, \alpha], \alpha >$. The result follows. ∎

Quadratic first integrals

Proposition D Let $\bar{f}(x) =< S(x), x >$ be a homogeneous polynomial of degree 2, where $S : \mathfrak{H} \to \mathfrak{H}$ is a symmetric linear transformation. Then $\{\bar{f}, \bar{E}\} = 0 \Leftrightarrow <$

$S(\alpha), \nabla_\alpha \alpha >= 0$ for all $\alpha \in \mathfrak{H}$.

Remarks 1) It is easy to see that every homogeneous second order polynomial $\bar{f} : \mathfrak{H} \to \mathbb{R}$ can be written $\bar{f}(x) =< S(x), x >$ as above for a suitable S.

2) We shall see later in (6.8*b*) that if $\mathfrak{H}$ is an almost nonsingular 2-step nilpotent Lie algebra, then $\bar{f}(x) =< S(x), x >$ is a first integral for the geodesic flow $X_E \Leftrightarrow S(\mathfrak{Z}) \subseteq \mathfrak{Z}, S(\mathfrak{Z}^\perp) \subseteq \mathfrak{Z}^\perp$ and $[S(A), B] = [A, S(B)]$ for all $A, B \in \mathfrak{H}$. Here $\mathfrak{Z}$ denotes the center of $\mathfrak{H}$, which is always nontrivial if $\mathfrak{H}$ is nilpotent, and $\mathfrak{Z}^\perp$ denotes the orthogonal complement of $\mathfrak{Z}$.

Proof of Proposition D Since $X_E(\alpha) = (\nabla_\alpha \alpha)_\alpha$ by the discussion in (5.2) it suffices to show that (grad $\bar{f})(\alpha) = (2S(\alpha))_\alpha$ for all $\alpha \in \mathfrak{H}$. We leave this statement as an exercise. ■

5.4 Closed geodesics in $\Gamma \backslash H$

We describe some general results about closed geodesics in a coset manifold $\Gamma \backslash H$, where H is a connected Lie group with a left invariant metric and Γ is a discrete subgroup of H that acts on H by left multiplications. In section 6 we will obtain more specialized results for the case that H is a simply connected, 2-step nilpotent Lie group and Γ is a lattice in H; that is, $\Gamma \backslash H$ is compact.

Basic definitions and notation

Definition A geodesic $\sigma(t)$ in a Riemannian manifold M is said to be closed with period $\omega > 0$ if $\sigma(t + \omega) = \sigma(t)$ for all $t \in \mathbb{R}$.

Equivalently, a geodesic $\sigma(t)$ is closed with period ω if $\sigma'(\omega) = \sigma'(0)$. To check the equivalence merely observe that the geodesics $\sigma^*(t) = \sigma(t + \omega)$ and $\sigma(t)$ have the same initial velocity $\Leftrightarrow \sigma'(\omega) = \sigma'(0)$.

Notation We let SM denote the unit tangent bundle of a Riemannian manifold M.

Definition Let M be a complete Riemannian manifold. A vector $v \in SM$ is said to be periodic with period $\omega > 0$ if $\mathfrak{G}^\omega(v) = v$, where $\{\mathfrak{G}^t\}$ denotes the geodesic flow in SM.

Let v be any vector in SM, and let $\sigma(t)$ be the unique geodesic with $\sigma'(0) = v$. Then it follows immediately from the definitions that v is a periodic vector with period $\omega > 0 \Leftrightarrow \sigma(t)$ is a closed geodesic with period ω.

Definition Let M be a complete Riemannian manifold. We say that the closed geodesics in M are dense if the periodic vectors in SM are dense in SM.

It is a problem of classical interest to determine Riemannian manifolds in which the closed geodesics are dense. Geometrically this says that every geodesic σ is a limit of a sequence of closed geodesics $\{\sigma_n\}$. Of course, if $\omega_n > 0$ is the smallest period of σ_n, then $\omega_n \to \infty$ as $n \to \infty$ if σ is not a closed geodesic. If this were not the case, then σ would be closed with period ω^* for any cluster point ω^* of the sequence $\{\omega_n\}$.

It is also a classical problem to determine the smallest periods of all closed geodesics in M. The collection of these periods, counted with multiplicities, is called the length spectrum of M. In cases where it is not easy to compute the length spectrum it is also useful to consider the growth rate of the function $N : (0, \infty) \to \mathbb{R}$ given by $N(t)$ = the number of smallest periods ω of closed geodesics in M such that $\omega \leq t$. If M is a compact Riemannian manifold with negative sectional curvature, then the growth rate of $N(t)$ is pretty well understood. In this article we shall be primarily interested in describing the length spectrum of a compact 2-step nilmanifold $\Gamma\backslash N$ and also the marked length spectrum of $\Gamma\backslash N$. See (6.8e).

Closed geodesics in $\Gamma\backslash H$ and translated geodesics in H

Proposition A Let H be a connected Lie group with a left invariant metric, and let Γ be a discrete subgroup of H. Let $\pi : H \to \Gamma\backslash H$ be the covering projection, where Γ acts by left multiplications on H. Let $\sigma(t)$ be a geodesic of $\Gamma\backslash H$ such that $\sigma(t+\omega) = \sigma(t)$ for all $t \in \mathbb{R}$ and some $\omega > 0$. Let $\gamma(t)$ be a geodesic of H such that $\pi(\gamma(t)) = \sigma(t)$ for all $t \in \mathbb{R}$. Then there exists an element φ of Γ such that $\varphi \cdot \gamma(t) = \gamma(t+\omega)$ for all $t \in \mathbb{R}$.

Conversely, let $\gamma(t)$ be a geodesic of H such that $\varphi \cdot \gamma(t) = \gamma(t+\omega)$ for all $t \in \mathbb{R}$, some $\omega > 0$ and some element φ of Γ. If $\sigma(t) = \pi(\gamma(t))$, then $\sigma(t+\omega) = \sigma(t)$ for all $t \in \mathbb{R}$.

Remark We say that a geodesic $\gamma(t)$ of H is translated by an element φ of H if $\varphi \cdot \gamma(t) = \gamma(t+\omega)$ for all $t \in \mathbb{R}$ and some $\omega > 0$. The number ω is called a period of the element φ. The result above says that finding closed geodesics and their periods in $\Gamma\backslash H$ is equivalent to finding geodesics of H that are translated by elements φ of Γ and finding the periods of these elements φ.

In general this is a useful way to study the closed geodesics of $\Gamma\backslash H$ since the geometry of H is easier to deal with than the geometry of a quotient $\Gamma\backslash H$. This is particularly true when H is simply connected and 2-step nilpotent since in this case the Lie group exponential $\exp : \mathfrak{N} \to N$ makes N diffeomorphic to the Euclidean space $\mathfrak{N}$.

Proof of Proposition A The proof of the second assertion is immediate, so we prove only the first. Let $\sigma(t)$ be a geodesic of $\Gamma\backslash H$ such that $\sigma(t+\omega) = \sigma(t)$ for all $t \to \mathbb{R}$ and some $\omega > 0$. Let $\gamma(t)$ be a geodesic of H such that $\pi(\gamma(t)) = \sigma(t)$ for all $t \in \mathbb{R}$. The deck group of the regular covering $\pi : H \to \Gamma\backslash H$ consists of left multiplications by the elements of Γ. Since $\pi(\gamma(\omega)) = \sigma(\omega) = \sigma(0) = \pi(\gamma(0))$ there exists an element φ of Γ such that $\varphi \cdot \gamma(0) = \gamma(\omega)$. If $\gamma_1(t) = \gamma(t+\omega)$ and $\gamma_2(t) = \varphi \cdot \gamma(t)$, then $\gamma_1(0) = \gamma_2(0) = \gamma(\omega)$ and $\pi(\gamma_i(t)) = \sigma(t)$ for $i = 1, 2$ and all $t \in \mathbb{R}$. It follows that $\gamma_1(t) = \gamma_2(t)$ for all t since $\sigma(t)$ has a unique lift to N that begins at the point $\gamma(\omega)$.

Translated geodesics in H and periodic vectors of $\{\bar{\mathfrak{G}}^t\}$ in $\mathfrak{H}$

Proposition B Let H be a connected Lie group with Lie algebra $\mathfrak{H}$, and let $<,>$ be a left invariant metric on H. Let $\gamma(t)$ be a geodesic of H such that $\varphi \cdot (t) = \gamma(t+\omega)$ for all $t \in \mathbb{R}$, some $\varphi \in H$ and some $\omega > 0$. Let $\xi = \gamma'(0)$ and let $\xi_o = G(\xi) \in \mathfrak{H}$, where $G : TH \to \mathfrak{H}$ is the Gauss map. Then $\bar{\mathfrak{G}}^\omega(\xi_o) = \xi_o$, where $\{\bar{\mathfrak{G}}^t\}$ is the geodesic flow in $\mathfrak{H}$.

Conversely, suppose that $\bar{\mathfrak{G}}^\omega(\xi_o) = \xi_o$ for some $\omega > 0$ and some $\xi_o \in \mathfrak{H}$. Then $\varphi \cdot \gamma(t) = \gamma(t+\omega)$ for all $t \in \mathbb{R}$, where γ is the geodesic with $\gamma'(0) = \xi_o$ and $\varphi = \gamma(\omega)$.

Proof We note that $\varphi \cdot \gamma(t) = \gamma(t+\omega)$ for all $t \in \mathbb{R} \Leftrightarrow dL_\varphi(\gamma'(0)) = \gamma'(\omega) \Leftrightarrow$

$dL_\varphi(\xi) = \mathfrak{G}^\omega(\xi)$, where $\{\mathfrak{G}^t\}$ denotes the geodesic flow in TH. By Proposition A of (5.2) we know that $G \circ \mathfrak{G}^t = \bar{\mathfrak{G}}^{-t} \circ G$ for all $t \in \mathbb{R}$. Hence $\bar{\mathfrak{G}}^{-\omega}\xi_0 = \bar{\mathfrak{G}}^{-\omega}G(\xi) = G(\mathfrak{G}^\omega\xi) = G(dL_\varphi(\xi)) = G(\xi) = \xi_o$. Applying $\mathfrak{G}^\omega$ to the equation $\bar{\mathfrak{G}}^{-\omega}\xi_o = \xi_o$ proves the first assertion of the proposition.

Now suppose that $\bar{\mathfrak{G}}^\omega(\xi_o) = \xi_o$ for some $\omega > 0$ and some $\xi_o \in \mathfrak{H}$. To prove that $\varphi\cdot\gamma(t) = \gamma(t+\omega)$ for all $t \in \mathbb{R}$, where γ is the geodesic with $\gamma'(0) = \xi_o$ and $\varphi = \gamma(\omega)$, it suffices by the discussion above to show that $dL_\varphi(\xi_o) = \mathfrak{G}^\omega(\xi_o)$. Both $dL_\varphi(\xi_o)$ and $\mathfrak{G}^\omega(\xi_o)$ lie in the same tangent space of H, and hence to prove that they are equal it suffices to prove that their images under the Gauss map G are equal. We compute $G(dL_\varphi(\xi_o)) = G(\xi_o) = \xi_o$ and $G(\mathfrak{G}^\omega(\xi_o)) = (\bar{\mathfrak{G}}^{-\omega} \circ G)(\xi_o) = \bar{\mathfrak{G}}^{-\omega}(\xi_o) = \xi_o$. The proof of the proposition is complete. ■

Density of periodic vectors in $T(\Gamma\backslash H)$ and $\mathfrak{H}$

A classical problem in Riemannian geometry is to look for Riemannian manifolds M such that the geodesic flow in the unit tangent bundle SM has a dense set of periodic vectors. Typically the manifolds M considered are compact although sometimes it is sufficient for M to be noncompact with finite volume. If M is compact, then SM is also compact, and in this compact situation there is an abundance of tools from differential geometry, dynamical systems and ergodic theory to study the geodesic flow on SM.

In the context of this article it is more natural to consider the tangent bundle TM with its symplectic structure. However, we note that $v \in TM$ is a periodic vector for the geodesic flow in $TM \Leftrightarrow$ the unit vector $v/|v|$ is a periodic vector for the geodesic flow in SM. Therefore, if we are interested in proving that SM contains a dense set of periodic vectors for the geodesic flow, then it is equivalent to prove that TM contains a dense set of periodic vectors for the geodesic flow.

We now consider the case that $M = \Gamma\backslash H$, where H is a connected Lie group with a left invariant metric and Γ is a discrete subgroup of H that acts on H by left multiplications.

Proposition C Let H be a connected Lie group with a left invariant metric, and let Γ be a discrete subgroup of H. If the periodic vectors for the geodesic flow in $T(\Gamma\backslash H)$ are dense in $T(\Gamma\backslash H)$, then the periodic vectors for the geodesic flow in $\mathfrak{H}$ are dense in $\mathfrak{H}$.

Lemma Let $\pi : H \to \Gamma\backslash H$ be the covering projection, and let $G : TH \to \mathfrak{H}$ be the Gauss map. Let ξ be an element of TH. Let $\pi_*(\xi)$ be a vector of period $\omega > 0$ for the geodesic flow in $T(\Gamma\backslash H)$. Then $G(\xi)$ is a vector of period ω for the geodesic flow in $\mathfrak{H}$.

Proof of the lemma Let $\gamma(t)$ be the geodesic of H such that $\gamma'(0) = \xi$. Since $\pi_*(\xi)$ is tangent to the closed geodesic $\pi \circ \gamma$ with period ω it follows from Proposition A that $\varphi \cdot \gamma(t) = \gamma(t+\omega)$ for some $\varphi \in \Gamma$ and all $t \in \mathbb{R}$. Proposition B now says that $G(\xi)$ has period ω for the geodesic flow in $\mathfrak{H}$. ■

Proof of Proposition C Let $X = \{\xi \in TH : \pi_*(\xi)$ is periodic for the geodesic flow in $T(\Gamma\backslash H)\}$. The set X is dense in TH since $\pi_*(X)$ is dense in $T(\Gamma\backslash H)$ by hypothesis. Hence $G(X)$ is dense in $\mathfrak{H}$ since G is continuous and surjective. The lemma says that

$G(X)$ consists of periodic vectors for the geodesic flow in $\mathfrak{H}$. ■

6. Geometry of 2-step nilpotent Lie groups

In this section we apply the results and methods described in the earlier sections to connected 2-step nilpotent Lie groups with a left invariant metric.

6.1 Definitions and basic examples

6.1a Definition A finite dimensional Lie algebra $\mathfrak{H}$ is 2-step nilpotent if $\mathfrak{H}$ is not abelian and $[\mathfrak{H}, [\mathfrak{H}, \mathfrak{H}]] = \{0\}$. A Lie group H is 2-step nilpotent if its Lie algebra $\mathfrak{H}$ is 2- step nilpotent.

Clearly, a 2-step nilpotent Lie algebra has a nontrivial center that contains $[\mathfrak{H}, \mathfrak{H}]$. It is natural to study 2-step nilpotent Lie groups and Lie algebras. They are as close as possible to being abelian, but the differences from Euclidean space are interesting and challenging. In addition, they arise frequently in important areas, for example as the horospheres of symmetric spaces of strictly negative curvature. See for example, [EH, pp. 447-448].

In this section we will use the letter $\mathfrak{N}$ to denote a 2-step nilpotent Lie algebra, and N will denote the corresponding simply connected 2-step nilpotent Lie group with Lie algebra $\mathfrak{N}$.

6.1b Exponential and logarithm functions

It is known that if N is a simply connected nilpotent Lie group with Lie algebra $\mathfrak{N}$, then $\exp : \mathfrak{N} \to N$ is a diffeomorphism, where exp denotes the Lie group exponential map. In this case we let $\log : N \to \mathfrak{N}$ denote the inverse of the exponential function.

If N is 2-step nilpotent in addition, then the multiplication law in N can be expressed as follows in terms of the exponential map.

$\exp(X) \cdot \exp(Y) = \exp(X + Y + \frac{1}{2}[X, Y])$ for all $X, Y \in \mathfrak{N}$.

From this formula one can quickly derive additional formulas. Let X, Y be any elements of $\mathfrak{N}$ and write $\varphi = \exp(X)$ and $\psi = \exp(Y)$. Then

a) $\varphi\psi\varphi^{-1} = \exp(Y + [X, Y])$.

b) $[\varphi, \psi] = \varphi\psi\varphi^{-1}\psi^{-1} = \exp([X, Y])$.

c) φ commutes with $\psi \Leftrightarrow X$ commutes with Y.

d) $\log(\varphi \cdot \psi) = \log\varphi + \log\psi + \frac{1}{2}[\log\varphi, \log\psi]$

6.1c Examples of 2-step nilpotent Lie algebras

We begin with some important examples of 2-step nilpotent Lie algebras and Lie groups. There are many of them, and one never has to check the Jacobi identity!

Example 1 Free 2-step nilpotent Lie algebras

Let $k \geq 2$ be an integer. A 2-step nilpotent Lie algebra $\mathfrak{N}$ is said to be free on k generators if it admits a generating set $\{X_1, X_2, ..., X_k\}$ with the following property : Let $\mathfrak{N}'$ be any 2-step nilpotent Lie algebra and let $\{X_1', X_2', ..., X_k'\}$ be any subset of $\mathfrak{N}'$ with k elements. Then there exists a unique Lie algebra homomorphism $T : \mathfrak{N} \to \mathfrak{N}'$ such that $T(X_i) = X_i'$ for $1 \leq i \leq k$.

A simply connected 2-step nilpotent Lie group N is said to be free on k generators if its Lie algebra $\mathfrak{N}$ is free on k generators.

It follows easily from the definition that a free 2-step nilpotent Lie algebra $\mathfrak{N}$ on k generators is unique up to isomorphism, and that every generating set for $\mathfrak{N}$ has at least k generators. Moreover, if $\mathfrak{N}'$ is a 2-step nilpotent Lie algebra generated by a subset with k elements, then $\mathfrak{N}'$ is a homomorphic image of $\mathfrak{N}$. It follows that if $\mathfrak{N}'$ is any connected 2-step nilpotent Lie group whose Lie algebra $\mathfrak{N}'$ is generated by k elements, then N' is a homomorphic image of a simply connected 2-step nilpotent Lie group N whose Lie algebra is free on k generators.

The universal mapping property of a free 2-step nilpotent Lie algebra $\mathfrak{N}$ on k generators is useful. In principle, if one understands a particular geometric property thoroughly for $\mathfrak{N}$, then one has a tool to study that property on the homomorphic images of $\mathfrak{N}$.

Model of a 2-step nilpotent Lie algebra $\mathfrak{N}$ that is free on k generators

For any integer $k \geq 2$ let $\mathfrak{N}_k = \mathbb{R}^k \oplus \Lambda^2(\mathbb{R}^k)$ as a vector space, where $\Lambda^2(\mathbb{R}^k)$ denotes the second exterior power of $\mathbb{R}^k$. Define a Lie bracket operation in $\mathfrak{N}_k$ by
$[(x,y),(x',y')] = (0, x \wedge x')$.

It is easy to see that $\mathfrak{N}_k$ is 2-step nilpotent and that $\Lambda^2(\mathbb{R}^k)$ is the center of $\mathfrak{N}_k$. Moreover, any basis of $\mathbb{R}^k$ is a generating set for $\mathfrak{N}_k$. To see that $\mathfrak{N}_k$ is free on k generators fix any basis $\{v_1, v_2, ..., v_k\}$ for $\mathbb{R}^k$. If $\{X_1', X_2', ..., X_k'\}$ is any subset with k elements in a 2-step nilpotent Lie algebra $\mathfrak{N}'$, then let $T : \mathbb{R}^k \to \mathfrak{N}'$ be the unique linear transformation such that $T(v_i) = X_i'$ for all i. Extend T to a linear map from $\mathfrak{N}_k$ to $\mathfrak{N}'$ by requiring that $T(v_i \wedge v_j) = [X_i', X_j']'$, where $[,]'$ denotes the Lie bracket in $\mathfrak{N}'$. We leave it to the reader to check that T is a well defined Lie algebra homomorphism such that $T(x \wedge y) = [T(x), T(y)]$ for all $x, y \in \mathbb{R}^k$.

Let $N_k = \mathbb{R}^k \times \Lambda^2(\mathbb{R}^k)$, and define multiplication in N by
$(x,y) * (x',y') = (x + x', y + y' + \frac{1}{2}(x \wedge x'))$
Clearly N is simply connected, and it is easy to see that $\Lambda^2(\mathbb{R}^k)$ is the center of N. We leave it to the reader to check that N_k is a group with unit element $(0,0)$ and that $\mathfrak{N}_k$ is the Lie algebra of N_k.

Example 2 Heisenberg Lie algebras

In $\mathbb{R}^{2k+1}, k \geq 1$, we construct a 2-step nilpotent Lie algebra with 1-dimensional center as follows. Let $\{x_1, y_1, x_2, y_2, ..., x_k, y_k, z\}$ be a basis of $\mathbb{R}^{2k+1}$ and define a 2-step nilpotent structure on $\mathbb{R}^{2k+1}$ by requiring that $[x_i, y_i] = -[y_i, x_i] = z$ for $1 \leq i \leq k$ and setting all other brackets of basis vectors equal to zero. It is easy to check that this defines a 2-step nilpotent Lie algebra with 1-dimensional center spanned by z. This Lie algebra is called the Heisenberg Lie algebra of dimension $2k+1$.

It is not difficult to show that every 2-step nilpotent Lie algebra with 1- dimensional center must be isomorphic to the Heisenberg Lie algebra of dimension $2k+1$ for some $k \geq 1$. It is also easy to check that the Heisenberg Lie algebra of dimension 3 is a free 2-step nilpotent Lie algebra on 2 generators.

Example 3 P-tuples of random skew symmetric matrices

Let $p \geq 2$ and $q \geq 2$ be any integers. Let $C^1, \dots, C^p$ be any skew symmetric $q \times q$ matrices. Then we may define a 2-step nilpotent Lie algebra structure on $\mathfrak{N} = \mathbb{R}^q \oplus \mathbb{R}^p$ as follows. Let $\{v_1, v_2, \dots, v_q\}$ and $\{z_1, z_2, \dots, z_p\}$ be any bases for $\mathbb{R}^q$ and $\mathbb{R}^p$ respectively. Define $[v_i, v_j] = \sum_{k=1}^{p} C_{ij}^k z_k$ and require $\mathbb{R}^p$ to lie in the center of $\mathfrak{N}$. Now extend the bracket operation to be bilinear on $\mathbb{R}^k$.

If $\mathfrak{N}$ is any 2-step nilpotent Lie algebra, then we may write $\mathfrak{N} = \mathcal{V} \oplus \mathfrak{Z}$, where $\mathcal{V}$ is any subspace complementary to the center $\mathfrak{Z}$. By the definition of a 2-step nilpotent Lie algebra it follows that $[\mathcal{V}, \mathcal{V}] \subseteq \mathfrak{Z}$. If we identify $\mathcal{V}$ with $\mathbb{R}^q$ and $\mathfrak{Z}$ with $\mathbb{R}^p$, then $\mathfrak{N}$ is one of the examples above for suitable skew symmetric matrices $C^1, \dots, C^p$.

Example 4 Random subspaces of $so(n, \mathbb{R})$

We define a 2-step nilpotent Lie algebra $\mathfrak{N}$ with a preferred inner product $<,>$. One can show that every 2-step nilpotent Lie algebra $\mathfrak{N}$ is isomorphic as a Lie algebra to one of these examples.

Fix an integer $n \geq 2$, and an inner product $<,>$ on $\mathbb{R}^n$. Let $so(n, \mathbb{R})$ denote the Lie algebra of linear transformations on $\mathbb{R}^n$ that are skew symmetric relative to $<,>$. Define an inner product $<,>^*$ on $so(n, \mathbb{R})$ by $< X, Y >^* = -\text{trace } XY$. It is routine to check that $<,>^*$ is positive definite on $so(n, \mathbb{R})$ and that $\text{ad } X : so(n, \mathbb{R}) \to so(n, \mathbb{R})$ is skew symmetric relative to $<,>^*$. Indeed, these two properties characterize $<,>^*$ uniquely up to a constant multiple.

Now let W be any subspace of $so(n, \mathbb{R})$ of dimension p, and let $\mathfrak{N} = \mathbb{R}^n \oplus W$. Equip $\mathfrak{N}$ with the inner product induced from $<,>$ and $<,>^*$ that makes $\mathbb{R}^n$ and W orthogonal. We now define a Lie bracket operation on $\mathfrak{N}$ by requiring that W lie in the center of $\mathfrak{N}$ and that $< [X, Y], Z >^* = < Z(X), Y >$ for all X, Y in $\mathbb{R}^n$ and all Z in W. It is easy to check that the center of $\mathfrak{N} = U \oplus W$, where $U = \{X \in \mathbb{R}^n : Z(X) = 0$ for all $Z \in W\}$.

Example 5 Special subspaces of $so(n, \mathbb{R})$

A generic subspace W of $so(n, \mathbb{R})$ produces a generic 2-step nilpotent Lie algebra $\mathfrak{N} = \mathbb{R}^n \oplus W$, one without much interest or importance. If we require more from W, then $\mathfrak{N}$ exhibits nicer properties. For example, if $W = so(n, \mathbb{R})$, then $\mathfrak{N} = \mathbb{R}^n \oplus so(n, \mathbb{R})$ is a free 2-step nilpotent Lie algebra on n generators. This is a consequence of the natural isomorphism between $so(n, \mathbb{R})$ and $\Lambda^2(\mathbb{R}^n)$.

Another important example arises when W is a subalgebra of $so(n, \mathbb{R})$. A good way to find subalgebras is to consider representations $\rho : H \to GL(V)$ of a compact Lie group H on a finite dimensional real vector space V. If V is equipped with an inner product $<,>$ that is invariant under the elements of $\rho(H)$, then $W = d\rho(\mathfrak{H})$ is a subalgebra of $so(V,<,>)$.

A subspace W of $so(n,\mathbb{R})$ is said to be a Lie triple system if $[W,[W,W]] \subseteq W$. Clearly any subalgebra W of $so(n,\mathbb{R})$ is a Lie triple system, and it is not hard to show that if W is a Lie triple system in $so(n,\mathbb{R})$, then $W + [W,W]$ is a subalgebra of $\mathsf{SO}(n,\mathbb{R})$. One can further describe the difference between subalgebras of $so(n,\mathbb{R})$ and Lie triple systems in $so(n,\mathbb{R})$ in geometric terms. Let the special orthogonal group $\mathsf{SO}(n,\mathbb{R})$ be given a bi- invariant Riemannian metric, which is unique up to constant multiples. Then the totally geodesic submanifolds of $\mathsf{SO}(n,\mathbb{R})$ that contain the identity matrix are precisely the sets $\exp(W)$, where W is a Lie triple system in $so(n,\mathbb{R})$, and $\exp : so(n,\mathbb{R}) \to \mathsf{SO}(n,\mathbb{R})$ is the matrix exponential map. The totally geodesic subgroups of $\mathsf{SO}(n,\mathbb{R})$ are precisely the sets $\exp(W)$, where W is a subalgebra of $so(n,\mathbb{R})$.

Example 6 Representations of Clifford algebras and spaces of Heisenberg type

Representations of certain algebras produce important examples of Lie triple systems in $so(n,\mathbb{R})$. If $C\ell(p)$ denotes the real Clifford algebra determined by $\mathbb{R}^p$ with a fixed inner product $<,>$, then a linear map $j : \mathbb{R}^p \to \mathrm{End}\,(V)$ defines a representation of the Clifford algebra if $j(Z)^2 = -|Z|^2\,\mathrm{Id}$ for all $Z \in C\ell(p)$. Under this condition there is a natural extension of j to an algebra homomorphism from $C\ell(p)$ to $\mathrm{End}\,(V)$. The elements of $C\ell(p)$ that are finite products of elements of unit length in $\mathbb{R}^P$ form a compact group called $\mathrm{Pin}(p)$. If one equips V with an inner product $<,>$ that is invariant under $j(\mathrm{Pin}(p))$, then $W = j(\mathbb{R}^p)$ is a subspace of $so(V,<,>)$. From the condition above that defines a representation of $C\ell(p)$ it is not difficult to show that W is a Lie triple system of $so(V,<,>)$. The totally geodesic subspace $\exp(W)$ of $so(V)$ is in this case a sphere of dimension p.

The representations of the Clifford algebras $C\ell(p)$ are important in several contexts. The corresponding 2-step nilpotent Lie algebras $\mathfrak{N} = V \oplus j(\mathbb{R}^p)$ are those of Heisenberg type. These are the nicest possible 2-step nilpotent Lie algebras with a center of fixed dimension $p \geq 2$. The corresponding simply connected, metric, 2-step nilpotent Lie groups N are the model spaces for all simply connected, metric, 2-step nilpotent Lie groups in much the same way that the Riemannian symmetric spaces are the model spaces for all Riemannian manifolds.

The systematic study of the spaces N of Heisenberg type was initiated by A. Kaplan in [K1,2]. They have also been used by E. Damek and F. Ricci in [DR] to produce counterexamples to a long standing conjecture of Lichnerowicz that every harmonic Riemannian manifold must be a locally symmetric Riemannian manifold. Damek and Ricci showed that each of the Lie algebras $\mathfrak{N}$ arising from a representation of $C\ell(p)$ has a natural solvable extension $\mathfrak{S} = \mathbb{R} \oplus \mathfrak{N}$ that is the Lie algebra of a simply connected solvable group S that is also a harmonic space with nonpositive sectional curvature. This construction produces a symmetric space only when $p = 1, 3$ or 7, but examples exist for every positive integer p.

For further information on the geometry of spaces of Heisenberg type see [BTV].

Example 7 Spaces that are Heisenberg like

In a space N of Heisenberg type we have the very strong condition that $j(Z)^2 = -|Z|^2 Id$ for every $Z \in \mathfrak{Z}$. In particular every eigenvalue of $j(Z)$ is $\pm i\,|Z|$. A space N is called Heisenberg like if the eigenvalues of $j(Z)$ depend only on $|Z|$. The Heisenberg like spaces have interesting geometric properties. See for example [$GM2$] for further discussion and other references in the literature.

6.2 Geometry of a simply connected 2-step nilpotent Lie group N

We illustrate how one may study the geometry of N by computations in the Lie algebra $\mathfrak{N}$ with the metric $<,>$ that determines the left invariant metric on N.

6.2a The maps $j(\mathfrak{Z})$

Following Kaplan [$K1,2$] we decompose $\mathfrak{N}$ into an orthogonal direct sum $\mathfrak{N} = \mathcal{V} \oplus \mathfrak{Z}$, where $\mathfrak{Z}$ denotes the center of $\mathfrak{N}$ and $\mathcal{V} = \mathfrak{Z}^{\perp}$. For each $Z \in \mathfrak{Z}$ we define a skew symmetric linear transformation $j(Z) : \mathcal{V} \to \mathcal{V}$ by $j(Z)X = (adX)^*(Z)$, where $(adX)^*$ denotes the transpose of ad $X : \mathfrak{N} \to \mathfrak{Z}$ determined by $<,>$.

Equivalently, $j(Z) : \mathcal{V} \to \mathcal{V}$ is the unique linear map such that $< j(Z)X, Y > = < [X,Y], Z >$ for all $X, Y \in \mathcal{V}$ and $Z \in \mathfrak{Z}$. It is evident that $j : \mathfrak{Z} \to so(\mathcal{V})$ is a linear map.

Much of the geometry of N can be described by the maps $j(Z)$. See [$K1,2$] and [E] for further details. We present one example, which illustrates some of the challenge in studying the geometry of Lie groups with a left invariant metric and 2-step nilpotent groups with a left invariant metric in particular.

6.2b Ricci tensor

If M is any Riemannian manifold, then the Ricci tensor is a symmetric, bilinear form on each tangent space of M defined by the formula $\mathrm{Ric}(v,w) = \mathrm{trace}(R_{vw})$, where $(R_{vw})(z) = R(z,v)w$ and R denotes the curvature tensor of M.

Let H be a connected Lie group with a left invariant metric $<,>$ arising from an inner product $<,>$ on the Lie algebra $\mathfrak{H}$. Then the Ricci tensor Ric , like the curvature tensor R, may be defined by its values on left invariant vector fields of H, or equivalently, by its values in $\mathfrak{H} = T_e H$. Hence we may regard the Ricci tensor as a symmetric, bilinear map Ric: $\mathfrak{H} \times \mathfrak{H} \to \mathbb{R}$ given by the formula above.

If $\mathfrak{H}$ is 2-step nilpotent, then we can be more specific. See Proposition 2.5 of [$E1$] for a proof of the next result.

Proposition Let N be a simply connected, 2-step nilpotent Lie group with a left invariant metric $<,>$, and let $\mathfrak{N}$ denote the Lie algebra of N. Write $\mathfrak{N} = \mathcal{V} \oplus \mathfrak{Z}$, where $\mathfrak{Z}$ denotes the center of $\mathfrak{N}$ and $\mathcal{V} = \mathfrak{Z}^{\perp}$ denotes the orthogonal complement of $\mathfrak{Z}$. Then

1) $\mathrm{Ric}(X, Z) = 0$ for all $X \in \mathcal{V}$ and all $Z \in \mathfrak{Z}$.

2) For $X, Y \in \mathcal{V}$ $\quad \mathrm{Ric}(X, Y) = < T(X), Y >$, where $T : \mathcal{V} \to \mathcal{V}$ is the symmetric linear transformation given by $T = \frac{1}{2}\sum_{i=1}^{p} j(Z_i)^2$, where $\{Z_1, ..., Z_p\}$ is any orthonormal basis of $\mathfrak{Z}$. In particular T is negative definite on $\mathcal{V}$.

3) For $Z, Z^* \in \mathfrak{Z}$ $\quad \mathrm{Ric}(Z, Z^*) = -\frac{1}{4}\mathrm{trace}\{j(Z) \circ j(Z^*)\}$. In particular $\mathrm{Ric}(Z, Z) \geq 0$ for all $Z \in \mathfrak{Z}$ with equality $\Leftrightarrow j(Z) = 0$.

Remark The statement above says that the Ricci tensor is negative definite on $\mathcal{V}$, and positive semidefinite on $\mathfrak{Z}$. We shall see in the next result that the Ricci tensor is positive definite on $\mathfrak{Z}$ as well if we remove the Euclidean de Rham factor from N. This means that the Ricci tensor of a left invariant metric on N can never by positive semidefinite or negative semidefinite. Comparison theorem techniques (*cf.* [*CE*]), which have played a central role in studying Riemannian manifolds with sectional or Ricci curvatures of a fixed sign, cannot be used to study the geometry of 2-step nilpotent Lie groups N with a left invariant metric. The geometry of such groups is an interesting mixture of phenomena that occur in spaces of positive, negative and zero sectional curvature.

6.2c Euclidean de Rham factor

The next result is contained in Proposition 2.7 of [*E*1]

Proposition Let N be a simply connected, 2-step nilpotent Lie group with a left invariant metric $<,>$, and let $\mathfrak{N}$ denote the Lie algebra of N. Let $\mathcal{E} = \{Z \in \mathfrak{Z} : j(Z) \equiv 0$ in $\mathcal{V}\}$ and let $\mathfrak{N}^*$ denote the orthogonal complement of $\mathcal{E}$ in $\mathfrak{N}$. Let $N^* = \exp(\mathfrak{N}^*)$ and $E = \exp(\mathcal{E})$, where $\exp : \mathfrak{N} \to N$ is the Lie group exponential map. Then N^* and E are totally geodesic submanifolds of N, N is isometric to the Riemannian product $N^* \times E$ and E is isometric to the Euclidean de Rham factor of N.

6.2d Isometry group of N

Let H be a simply connected Lie group with a left invariant metric $<,>$, and let $I(H)$ denote the isometry group of $\{H, <,>\}$. If φ is any isometry of H, then set $h = \varphi(e)$ and $\psi = L_{h^{-1}} \circ \varphi$. Then clearly $\psi \in K = I(H)_e = \{\varphi \in I(H) : \varphi(e) = e\}$ and $I(H) = H \cdot K$, where we identify H with the left translation subgroup $H' = \{L_h : h \in H\} \subseteq I(H)$.

If $H = N$, a simply connected Lie group (not necessarily 2-step), then E. Wilson proved in [Wi] that $K = \mathrm{Aut}(N) \cap I(N)$; that is, every isometry of N that fixes the identity of N is an automorphism of the Lie group N. It follows immediately in this case that N is a normal subgroup of $I(N)$.

We obtain

Proposition A [Wi] Let N be a simply connected nilpotent Lie group with a left invariant metric $<,>$, and let $K = \mathrm{Aut}(N) \cap I(N)$. Then $I(N) = K \cdot N = N \cdot K$ and $N = \{L_n : n \in N\}$ is a normal subgroup of N.

Remark J. Lauret in [La 1, 3, 5] has computed the group K in some important special cases, including those in which the Lie algebra $\mathfrak{N}$ is either of Heisenberg type or can be expressed as $\mathbb{R}^n \oplus \mathfrak{G}$, where $\mathfrak{G} \subseteq so(n, \mathbb{R})$ is the Lie algebra of a compact subgroup G of $SO(n, \mathbb{R})$ and $N = \mathbb{R}^n \oplus \mathfrak{G}$ has the bracket structure defined in example 4 of (6.1c).

Left invariant metrics with maximal symmetry

Proposition B Let N be a simply connected nilpotent Lie group , and let K_o be a maximal compact subgroup of $\mathrm{Aut}(N)$. Then

1) There exists a left invariant metric $<,>_o$ on N such that $K_o = \mathrm{Aut}(N) \cap I(N, <,>_o)$.

2) If $<,>$ is any left invariant metric on N, then $\dim I(N, <,>) \leq \dim I(N, <,>_o)$. If equality holds, then $K = \mathrm{Aut}(N) \cap I(N, <,>)$ and $\varphi K_o \varphi^{-1}$ have the same identity component for some $\varphi \in \mathrm{Aut}(N)$.

Proof The proposition in (4.1) shows that there exists a left invariant metric $<,>_o$ such that $K_o = \mathrm{Aut}(N) \cap I(N, <,>_o)$. Now let $<,>$ be any left invariant metric on N and set $K = \mathrm{Aut}(N) \cap I(N, <,>)$. By the discussion in (4.1)K is a compact subgroup of $\mathrm{Aut}(N)$. Let $K_o{}^*$ be a maximal compact subgroup of $\mathrm{Aut}(N)$ such that $K \subseteq K_o{}^*$. Then $K_o{}^* = \varphi K_o \varphi^{-1}$ for some $\varphi \in \mathrm{Aut}(N)$ ([I], [Mal 2]). By Proposition A we have $\dim I(N, <,>) = \dim K + \dim N \leq \dim K_o{}^* + \dim N = \dim K_o + \dim N = \dim I(N, <,>_o)$. Moreover, it is clear that if $\dim I(N, <,>) = \dim I(N, <,>_o)$, then K and $K_o{}^* = \varphi K_o \varphi$ must have the same identity component. ■

6.3 Nonsingular 2-step nilpotent Lie algebras

Let $\mathfrak{N}$ be a 2-step nilpotent Lie algebra. We say that $\mathfrak{N}$ is nonsingular if $\mathrm{ad}X : \mathfrak{N} \to \mathfrak{Z}$ is surjective for all $X \in \mathfrak{N} - \mathfrak{Z}$. It is an easy exercise to show :

Proposition A Let $\mathfrak{N}$ be a 2-step nilpotent Lie algebra. Then the following properties are equivalent :

1) $\mathfrak{N}$ is nonsingular

2) For every inner product $<,>$ on $\mathfrak{N}$ and every nonzero element Z of $\mathfrak{Z}$ the linear map $j(Z)$ is nonsingular.

3) For some inner product $<,>$ on $\mathfrak{N}$ and every nonzero element Z of $\mathfrak{Z}$ the linear map $j(Z)$ is nonsingular.

Coadjoint action description

We can also define the nonsingularity of $\mathfrak{N}$ in terms of the coadjoint action of $\mathfrak{N}$ on its dual space $\mathfrak{N}^*$. Recall from example 2 of (1.4) that for every $X \in \mathfrak{N}$ we obtain a transformation $\mathrm{ad}^* X : \mathfrak{N}^* \to \mathfrak{N}^*$ given by $\mathrm{ad}^* X(\omega)(Y) = -\omega([X, Y])$ for all $Y \in \mathfrak{N}$. For every $\omega \in \mathfrak{N}^*$ define $\mathfrak{N}_\omega = \{X \in \mathfrak{N} : \mathrm{ad}^* X(\omega) = 0\}$.

We list the following useful facts whose proofs are straightforward.

a) $\mathfrak{N}_\omega \supseteq \mathfrak{Z}$ for all $\omega \in \mathfrak{N}^*$.

b) $\mathfrak{N}_\omega = \mathfrak{N} \Leftrightarrow \omega \equiv 0$ on $[\mathfrak{N}, \mathfrak{N}]$

We now relate the coadjoint action to the behavior of the maps $j(\mathfrak{Z})$ defined by an inner product on $\mathfrak{N}$. Fix an inner product $<,>$ on $\mathfrak{N}$ and let $T : \mathfrak{N}^* \to \mathfrak{N}$ be the isomorphism such that $< T(\omega), \xi >= \omega(\xi)$ for all $\omega \in \mathfrak{N}^*$ and all $\xi \in \mathfrak{N}$. For $\omega \in \mathfrak{N}^*$ let Z_ω denote the $\mathfrak{Z}$-component of $T(\omega)$. From the definitions one then obtains

c) Let $X \in \mathcal{V}$ and $\omega \in \mathfrak{N}^*$ be given. Then $X \in \mathfrak{N}_\omega \Leftrightarrow X \in \ker j(Z_\omega)$.

As an immediate corollary of c) we obtain

d) Let $\omega \in \mathfrak{N}^*$. Then $\mathfrak{N}_\omega = \mathfrak{Z} \Leftrightarrow j(Z_\omega)$ is nonsingular on $\mathcal{V}$.

One also has the following result whose proof is left as an exercise.

Proposition B Let $\mathfrak{N}$ be a 2-step nilpotent Lie algebra. Then the following properties are equivalent :

1) $\mathrm{ad}X : \mathfrak{N} \to \mathfrak{Z}$ is surjective if $X \in \mathfrak{N} - \mathfrak{Z}$.

2) $\mathfrak{N}_\omega = \mathfrak{Z}$ if ω is not identically zero on $\mathfrak{Z}$.

6.4 Almost nonsingular 2-step nilpotent Lie algebras

Let $\mathfrak{N}$ be a 2-step nilpotent Lie algebra. We say that $\mathfrak{N}$ is almost nonsingular if $\mathfrak{N}_\omega = \mathfrak{Z}$ for some nonzero $\omega \in \mathfrak{N}^*$. The next result is analogous to the Proposition in (6.3), and as before, we leave the proof as an exercise.

Proposition Let $\mathfrak{N}$ be a 2-step nilpotent Lie algebra. Then the following properties are equivalent :

1) $\mathfrak{N}$ is almost nonsingular.

2) For every inner product $<,>$ on $\mathfrak{N}$ there exists a nonzero element Z of $\mathfrak{Z}$ such that $j(Z)$ is nonsingular on $\mathcal{V} = \mathfrak{Z}^\perp$.

3) For some inner product $<,>$ on $\mathfrak{N}$ there exists a nonzero element Z of $\mathfrak{Z}$ such that $j(Z)$ is nonsingular on $\mathcal{V} = \mathfrak{Z}^\perp$.

Corollary Let $\mathfrak{N}$ be an almost nonsingular 2-step nilpotent Lie algebra. Then

1) There exists a dense open subset O of $\mathfrak{Z}$ such that $j(Z)$ is nonsingular for all Z in O.

2) There exists a dense open subset O^* of $\mathfrak{N}^*$ such that $\mathfrak{N}_\omega = \mathfrak{Z}$ for all in O^*.

Proof 1) One can show that $\{Z \in \mathfrak{Z} : \det j(Z) = 0\}$ is the set of common zeros of a finite set of polynomial equations in any set of linear coordinate variables for $\mathfrak{N}$. It follows from d) in (6.3) that if $\mathfrak{N}$ is almost nonsingular, then $\{Z \in \mathfrak{Z} : j(Z)$ is invertible$\}$ is a dense open subset O of $\mathfrak{Z}$.

2) This follows from 1) and d) in (6.3).

6.5 Rank of a 2-step nilpotent Lie algebra

To define the rank of a 2-step nilpotent Lie algebra we use the notation and discussion of the coadjoint action in (6.3).

Definition Let $\mathfrak{N}$ be a 2-step nilpotent Lie algebra. We define

$\mathrm{rank}(\mathfrak{N}) = 1 + \min\{\dim(\mathfrak{N}_\omega/\mathfrak{Z}) : \omega \in \mathfrak{N}^*\}$

The discussion above shows that if $\mathfrak{N}$ is nonsingular or almost nonsingular, then $\mathrm{rank}(\mathfrak{N}) = 1$.

For an example of rank 2 consider an irreducible representation of the compact 3-dimensional Lie group $H = SU(2)$ on an odd dimensional real vector space V. (There is one of these V for every odd integer.) Identify V with $\mathbb{R}^n$ and choose an inner product $<,>$ invariant under $SU(2)$; that is, $H = SU(2)$ is a subgroup of $\mathrm{SO}(n,\mathbb{R})$. Then $\mathfrak{H}$ is a subalgebra of $so(n,\mathbb{R})$, and $\mathfrak{N} = \mathbb{R}^n \oplus \mathfrak{H}$ becomes a 2-step nilpotent Lie algebra with an inner product $<,>$ by the process described above in example 5 of (6.1c). One can show that $\mathfrak{H}$ is the center of $\mathfrak{N}$, and $j(Z)$ has a 1-dimensional kernel for each Z in $\mathfrak{Z}$ since the dimension of $V = \mathbb{R}^n$ is odd. Hence $\mathrm{rank}(\mathfrak{N}) = 2$ for all of these examples.

Not much attention has been paid to 2-step nilpotent Lie algebras of rank ≥ 2.

6.6 The Hamiltonian foliation and the symplectic leaves in $\mathfrak{N}$

Now let $\mathfrak{N}$ be a 2-step nilpotent Lie algebra with an inner product $<,>$, and let $\mathfrak{N}$ be equipped with the Poisson structure defined by $<,>$ in example 4 of (3.7). By the discussion of that example the Hamiltonian foliation $\mathcal{H}$ in $\mathfrak{N}$ is given by $\mathcal{H}(A) = \{(ad\xi)^t(A) : \xi \in \mathfrak{N}\}$, and the symplectic leaves in $\mathfrak{N}$ have the form $L(A) = \{Ad(n)^t(A) : n \in N\}$, where $(ad\xi)^t$ and $Ad(n)^t$ denote the metric transpose of ad $\xi : \mathfrak{N} \to \mathfrak{N}$ and $Ad(n) : \mathfrak{N} \to \mathfrak{N}$ respectively.

Proposition A Let $A = X + Z \in \mathfrak{N}$, where $X \in \mathcal{V} = \mathfrak{Z}^\perp$ and $Z \in \mathfrak{Z}$. Then

1) $\mathcal{H}(A) = j(Z)(\mathcal{V}) = \{j(Z)(Y) : Y \in \mathcal{V}\}$, where $j(Z) : \mathcal{V} \to \mathcal{V}$ is the skew symmetric map defined in (6.2a).

2) $L(A) = A + \mathcal{H}(A) = \{A + j(Z)(Y) : Y \in \mathcal{V}\}$.

Remark It follows from this proposition that if $A \in \mathcal{V}$, then $L(A) = \{A\}$. Moreover, if $j(Z)$ is nonzero whenever Z is nonzero, then $\mathcal{V} = \{A \in \mathfrak{N} : L(A) = \{A\}\}$. One may show that $j(Z)$ is nonzero whenever Z is nonzero $\Leftrightarrow$ $\mathfrak{N}$ cannot be written as a nontrivial Lie algebra direct sum $\mathfrak{N}_1 \oplus \mathfrak{A}$, where $\mathfrak{A}$ is abelian. In practice one can always reduce to this case that $\mathfrak{N}$ has no abelian factor $\mathfrak{A}$. Hence, in this case Proposition A gives a nice characterization of $\mathcal{V} = \mathfrak{Z}^\perp$ in terms of the Poisson structure $\{,\}$ on $\mathfrak{N}$ defined by $<,>$.

Proof of 1) This is an immediate consequence of the next result

Lemma 1) $\mathrm{ad}(Z)^t = 0$ for all $Z \in \mathfrak{Z}$.

2) $\mathrm{ad}(X)^t Y = 0$ for all $X, Y \in \mathcal{V}$.

3) $\mathrm{ad}(X)^t Z = j(Z)X$ for all $X \in \mathcal{V}$ and all $Z \in \mathfrak{Z}$.

Proof of the lemma

1) If $Z \in \mathfrak{Z}$ and $\xi, \xi' \in \mathfrak{N}$ are given, then $< (adZ)^t(\xi), \xi' > = < \xi, [Z, \xi'] > = 0$.

2) Let $X, Y \in \mathcal{V}$ and $\xi \in \mathfrak{N}$ be given. Then $< (adX)^t Y, \xi > = < Y, [X, \xi] > = 0$ since $[X, \xi] \in \mathfrak{Z}$.

3) Let $X \in \mathcal{V}$, $Z \in \mathfrak{Z}$ and $\xi \in \mathfrak{N}$ be given. Write $\xi = X' + Z'$, where $X' \in \mathcal{V}$ and $Z' \in \mathfrak{Z}$. Then $< (adX)^t Z, \xi > = < Z, [X, \xi] > = < Z, [X, X'] > = < j(Z)X, X' > = < j(Z)X, \xi >$ by the definition of $j(Z)$. ∎

Proof of 2) Let $n \in N$ be given. Then $n = \exp(\xi)$ for a unique element $\xi \in \mathfrak{N}$ since

$\exp : \mathfrak{N} \to N$ is a diffeomorphism by (6.1b). It follows that $Ad(n) = Ad(\exp(\xi)) = e^{ad(\xi)} = Id + ad\xi$ since $(ad\xi)^k = 0$ for all $k \geq 2$. Hence $L(A) = \{Ad(n)^t(A) : n \in N\} = A + \{(ad\xi)^t(A) : \xi \in \mathfrak{N}\} = A + \mathcal{H}(A)$. ∎

Casimir functions in the almost nonsingular case

Proposition B Let $\mathfrak{N}$ be an almost nonsingular 2-step nilpotent Lie algebra. Let $<,>$ be an inner product on $\mathfrak{N}$, and let $\{,\}$ be the Poisson structure on $\mathfrak{N}$ determined by $<,>$. Let $\bar{f} : \mathfrak{N} \to \mathbb{R}$ be a C^∞ Casimir function, and let $g : \mathfrak{Z} \to \mathbb{R}$ be the restriction of $\bar{f}$ to $\mathfrak{Z} \subseteq \mathfrak{N}$. Then $\bar{f} = g \circ \pi_3$, where $\pi_3 : \mathfrak{N} \to \mathfrak{Z}$ denotes orthogonal projection.

Remark This result is the converse (in the almost nonsingular 2-step nilpotent case) of a result proved for an arbitrary metric Lie algebra $\{\mathfrak{H}, <,>\}$in the discussion of example 1 of (3.9).

Proof of the proposition Let $A = X + Z \in \mathfrak{N}$ be given. By the corollary in (6.4) there exists a sequence $\{Z_n\} \subseteq \mathfrak{Z}$ such that $Z_n \to Z$ as $n \to \infty$ and $j(Z_n) : \mathcal{V} \to \mathcal{V}$ is nonsingular for every n. By Proposition A the symplectic leaf through Z_n is given by $L(Z_n) = Z_n + j(Z_n)\mathcal{V} = Z_n + \mathcal{V}$ for every n. If $A_n = X + Z_n$, then $A_n \to A$ as $n \to \infty$ and $A_n \in L(Z_n)$ for every n. Since the Casimir function $\bar{f} : \mathfrak{N} \to \mathbb{R}$ is constant on symplectic leaves it follows that $\bar{f}(A_n) = \bar{f}(Z_n)$ for every n. Hence $\bar{f}(A) = \lim_{n\to\infty} \bar{f}(A_n) = \lim_{n\to\infty} \bar{f}(Z_n) = \bar{f}(Z) = g(Z) = (g \circ \pi_3)(A)$. ∎

6.7 Lattices in simply connected 2-step nilpotent Lie groups

6.7a Definition A discrete subgroup Γ of a simply connected nilpotent Lie group N is said to be a lattice if the quotient manifold $\Gamma \backslash N$ is compact.

We require that Γ act by left translations on N so that Γ will be a group of isometries of N with respect to any left invariant inner product. At this point we require only that N be nilpotent, not necessarily 2-step nilpotent. The quotient space $\Gamma\backslash N$ will be called a compact nilmanifold.

It is always desirable to work with compact manifolds if possible, but in fact lattices in a simply connected nilpotent Lie group usually do not exist. We will be more precise on this point shortly in the case of 2-step nilpotent Lie groups.

6.7b The criterion of Mal'cev

Mal'cev in [Mal 1] found a necessary and sufficient condition for a simply connected nilpotent Lie group N to admit a lattice Γ.

Proposition Let N be a simply connected nilpotent Lie group with Lie algebra $\mathfrak{N}$. Then N admits a lattice $\Gamma \Leftrightarrow$ there exists a basis $\mathfrak{B} = \{\xi_1, \xi_2, ..., \xi_n\}$ for $\mathfrak{N}$ with rational structure constants; that is, $[\xi_i, \xi_j] = \sum_{k=1}^{n} C_{ij}^k \xi_k$, where the constants $\{C_{ij}^k\}$ are rational numbers.

We say that $\mathfrak{N}_{\mathbb{Q}} = \mathbb{Q} - \text{span}(\mathfrak{B})$ defines a <u>rational structure</u> on $\mathfrak{N}$; that is, $\mathfrak{N}_{\mathbb{Q}}$ is a Lie algebra over $\mathbb{Q}$ and $\dim_{\mathbb{Q}} \mathfrak{N}_{\mathbb{Q}} = \dim_{\mathbb{R}} \mathfrak{N}$.

Mal'cev also proved a correspondence between rational structures on $\mathfrak{N}$ and commensurabilty classes of lattices on N. Two lattices Γ and Γ^* on N are said to be <u>commensurable</u> if $\Gamma \cap \Gamma^*$ is a finite index subgroup of both Γ and Γ^*.

Proposition Let N be a simply connected nilpotent Lie group with Lie algebra $\mathfrak{N}$.

1) If Γ is a lattice in N, then $\mathfrak{N}_{\mathbb{Q}} = \mathbb{Q} - \text{span}(\log \Gamma)$ is a rational structure on $\mathfrak{N}$.

2) If Γ and Γ^* are lattices in N, then $\mathbb{Q} - \text{span}(\log \Gamma) = \mathbb{Q} - \text{span}(\log \Gamma^*) \Leftrightarrow \Gamma$ and Γ^* are commensurable.

3) If $\mathfrak{B}$ is any basis of $\mathfrak{N}$ with rational structure constants and $\mathfrak{L} = \mathbb{Z} - \text{span}(\mathfrak{B})$, then the subgroup in N generated by $\exp(\mathfrak{L})$ is a lattice Γ.

6.7c The space $\mathfrak{N}_2(p,q)$ of 2-step nilpotent Lie algebras

Consider the collection of all 2-step nilpotent Lie algebras $\mathfrak{N}$ of dimension $n \geq 3$ with a center of dimension $p \geq 1$. If $p = 1$, then $\mathfrak{N}$ is isomorphic to one of the Heisenberg Lie algebras as we observed above in Example 2 of (6.1c). The case $p = 2$ is also special and for a fixed integer $n \geq 4$ a generic 2-step nilpotent Lie algebra $\mathfrak{N}$ of dimension n with 2-dimensional center is unique up to isomorphism.

We consider the case where $p \geq 3$.

Definition $\mathfrak{N}_2(p,q) = \{$2-step nilpotent Lie algebras $\mathfrak{N}$ such that $\dim \mathfrak{N} = p + q$, $\mathfrak{N}$ has center $\mathfrak{Z}$ of dimension p and $[\mathfrak{N}, \mathfrak{N}] = \mathfrak{Z}\}$.

For convenience we define $D = \frac{1}{2}q(q-1)$. The condition $[\mathfrak{N}, \mathfrak{N}] = \mathfrak{Z}$ implies that $p \leq D$ (details omitted) and hence $q \geq 3$ since $p \geq 3$. It is therefore no loss of generality to consider the structure of the space $\mathfrak{N}_2(p,q)$, where $3 \leq p \leq D$ and $q \geq 3$. One now has the following structure result whose proof may be found in [*E*4]. Recall that $G(p, p+q)$ denotes the compact connected Grassmann manifold of p-dimensional subspaces of $\mathbb{R}^{p+q}$.

Proposition Let p, q be integers with $3 \leq q$ and $3 \leq p \leq D = \frac{1}{2}q(q-1)$. Then $\mathfrak{N}_2(p,q)$ is a smooth manifold of dimension $pq + pD$ that fibers over $G(p, p+q)$. Moreover,

1) The fibration $Z : \mathfrak{N}_2(p,q) \to G(p, p+q)$ is the map that sends an element $\mathfrak{N}$ of $\mathfrak{N}_2(p,q)$ to its center $\mathfrak{Z}$ in $G(p, p+q)$.

2) The fiber F is the set of elements $C = (C^1, C^2, \ldots, C^p) \in so(q, \mathbb{R})^p$ such that

a) $\cap_{i=1}^{p} \ker(C^i) = \{0\}$ and

b) $\{(C^1, C^2, \ldots, C^p)$ are linearly independent in $so(q, \mathbb{R})$.

The fiber F is a dense open subset of $so(q, \mathbb{R})^p$.

Remark Once we are given the fiber bundle structure of $\mathfrak{N}_2(p,q)$ the dimension of $\mathfrak{N}_2(p,q)$ follows immediately. Note that 2*b*) is only possible since $p \leq D = \frac{1}{2}q(q-1) = \dim so(q, \mathbb{R})$.

The group $G = GL(p+q, \mathbb{R})$ acts on $\mathfrak{N}_2(p,q)$ in a natural way, and the orbits of G are the isomorphism classes of elements in $\mathfrak{N}_2(p,q)$. Hence the space of isomorphism classes in $\mathfrak{N}_2(p,q)$ may be identified with the orbit space $\mathfrak{N}_2(p,q)/G$ and given the quotient topology.

Problem: What can one say about the topology of $\mathfrak{N}_2(p,q)/G$? In particular, what can one say about the fundamental group of $\mathfrak{N}_2(p,q)/G$?

6.7d The scarcity of rational Lie algebras in $\mathfrak{N}_2(p,q)$

Call a 2-step nilpotent Lie algebra $\mathfrak{N}$ rational if it admits a basis $\mathfrak{B}$ with rational structure constants. Since there are only countably many choices of rational structure constants $\{C_{ij}^k\}$ it follows that the rational elements of $\mathfrak{N}_2(p,q)$ consist of the union of countably many orbits of $G = GL(p+q,\mathbb{R})$. If the dimension of every G-orbit in $\mathfrak{N}_2(p,q)$ is smaller than the dimension of $\mathfrak{N}_2(p,q)$, then we conclude that the rational elements of $\mathfrak{N}_2(p,q)$ form a set of measure zero in $\mathfrak{N}_2(p,q)$.

For a given integer $p \geq 3$ there exists a positive integer $q_o = q_o(p)$ such that if $q \geq q_o$, then the dimension of $\mathfrak{N}_2(p,q)$ is greater than the dimension of any G-orbit in $\mathfrak{N}_2(p,q)$. We outline the proof and refer the reader to $[E4]$ for details.

One may show that the dimension of a G-orbit in $\mathfrak{N}_2(p,q)$ is at most p^2+pq+q^2-1. The dimension of $\mathfrak{N}_2(p,q)$ is $pq+pD$. The condition $p^2+pq+q^2-1 < pq+pD$ is equivalent to the condition $(^*)$ $q^2(1-\frac{1}{2}p)+q(\frac{1}{2}p)+(p^2-1) < 0$. For fixed $p \geq 3$ the condition $(^*)$ is satisfied for sufficiently large q since $1-\frac{1}{2}p \leq -\frac{1}{2}$. Whether one obtains the smallest value of q_o from the smallest value of q_o that satisfies $(^*)$ is unknown to me.

6.7e Spaces N that admit lattices

By the Mal'cev criterion we may identify the simply connected 2-step nilpotent Lie groups that admit lattices with the elements of $\mathfrak{N}_2(p,q)$ that admit rational structures. By the argument above, most elements of $\mathfrak{N}_2(p,q)$ do not admit rational structures, but there are definitely large classes of elements of $\mathfrak{N}_2(p,q)$ that do. These elements , which form a null set of $\mathfrak{N}_2(p,q)$ in general, are the really interesting ones. It is an interesting problem to find criteria guaranteeing that an element $\mathfrak{N}$ of $\mathfrak{N}_2(p,q)$ admit a rational structure. Once we have found one rational structure on an element $\mathfrak{N}$ of $\mathfrak{N}_2(p,q)$ we can then try to describe the space of all rational structures on $\mathfrak{N}$.

For the moment we remark only that $\mathfrak{N}$ admits a rational structure if it arises from a Lie triple system W in $so(n,\mathbb{R})$ as described in Example 5 of (6.1c). See $[E3]$ for details. As noted earlier the Lie triple system examples include all examples of Heisenberg type and all examples arising from finite dimensional real representations of compact Lie groups. See also $[CD]$.

Problem What can you say about the space of rational structures on a 2-step nilpotent Lie algebra that arises from a Lie triple system W in $so(n,\mathbb{R})$?

6.7f Riemannian submersion structure of compact 2-step nilmanifolds with a left invariant metric

Let N be a 2-step nilpotent Lie group that admits a lattice Γ. If we let Γ act by left translations on N, then Γ acts by isometries relative to any left invariant metric $<,>$ on N. The left invariant metric $<,>$ on N descends to a Riemannian metric on $\Gamma\backslash N$ and the projection $p : N \to \Gamma\backslash N$ becomes a Riemannian covering map. By abuse of language we refer to the metric on $\Gamma\backslash N$ as left invariant.

The isometry group $I(\Gamma\backslash N)$ does not act transitively on $\Gamma\backslash N$. In fact the identity component $I_o(\Gamma\backslash N)$ is a torus of dimension $p = \dim \mathfrak{Z}$, where $\mathfrak{Z}$ is the center of $\mathfrak{N}$. The orbits of $I_o(\Gamma\backslash N)$ are flat, totally geodesic p-tori that are the fibers of a Riemannian submersion onto a flat torus. More precisely we have the following result which is Proposition 5.5 of $[E1]$:

Proposition Let Γ be a lattice in a simply connected, 2-step nilpotent Lie group with a left invariant metric. Let $\mathfrak{N}$ be the Lie algebra of N, and write $\mathfrak{N} = \mathcal{V} \oplus \mathfrak{Z}$ as in (6.2a). Let $\pi_{\mathcal{V}} : \mathfrak{N} \to \mathcal{V}$ denote the orthogonal projection onto $\mathcal{V}$, and let $T_B = \mathcal{V}/\pi_{\mathcal{V}}(\log \Gamma)$. Then

1) T_B is a flat torus of dimension q, where $q = \dim \mathcal{V}$.
2) There exists a Riemannian submersion $\pi : \Gamma\backslash N \to T_B$ whose fibers are the orbits of $I_o(\Gamma\backslash N)$, all of which are isometric to a single flat torus T_F.

The result above says that $\Gamma\backslash N$ is a principal torus bundle over T_B whose fibers are totally geodesic and isometric to T_F.

6.8 Geodesic flow in a compact 2-step nilmanifold with a left invariant metric

6.8a Formula for the geodesic flow in TN and $\mathfrak{N}$

Let N be a simply connected 2-step nilpotent Lie group with a left invariant metric $\langle , \rangle$, and let $\mathfrak{N}$ be the Lie algebra of N. We identify TN with $N \times \mathfrak{N}$ as in (4.2b) under the diffeomorphism $(n, A) \to dL_n(A)$ for $n \in N$ and $A \in \mathfrak{N} = T_eN$. We write $\mathfrak{N} = \mathcal{V} \oplus \mathfrak{Z}$ as in (6.2a).

The geodesic flow $\{\mathfrak{G}^t\}$ in TN has a simple description. Note that the flow maps $\mathfrak{G}^t$ commute with the maps $\{dL_n : n \in N\}$. From Proposition 3.2 of $[E1]$ we obtain the following

Proposition Let $\xi = (n, A) \in TN$. Let $\gamma_\xi(t)$ denote the geodesic of N with initial velocity ξ, and write $A = X + Z$, where $X \in \mathcal{V}$ and $Z \in \mathfrak{Z}$. Then $\mathfrak{G}^t(n, A) = (\gamma_\xi(t), e^{tj(Z)}X + Z)$ for all $t \in \mathbb{R}$.

As an immediate consequence of the result above and Proposition A of (5.2) we obtain a description of the geodesic flow maps $\bar{\mathfrak{G}}^t$ in $\mathfrak{N}$.

Corollary 1 Let $\bar{\mathfrak{G}}^t$ be the flow maps of the geodesic vector field $\bar{\mathfrak{G}}$ in $\mathfrak{N}$. Then $\bar{\mathfrak{G}}^t(X + Z) = e^{-tj(Z)}X + Z$ for all $X \in \mathcal{V}, Z \in \mathfrak{Z}$ and $t \in \mathbb{R}$.

We use the result above to describe the zero locus of the geodesic vector field $\bar{\mathfrak{G}}$ in $\mathfrak{N}$.

Corollary 2 Let $\bar{\mathfrak{G}}$ be the geodesic vector field in $\mathfrak{N}$. Then $\bar{\mathfrak{G}}(\xi) = 0$ if $\xi \in \mathcal{V} \cap \mathfrak{Z}$. If $\mathfrak{N}$ is nonsingular, then $\mathcal{V} \cup \mathfrak{Z} = \{\xi \in \mathfrak{N} : \bar{\mathfrak{G}}(\xi) = 0\}$.

Proof This is an immediate consequence of Corollary 1 and Proposition A of (6.3) since $\bar{\mathfrak{G}}(\xi) = 0 \Leftrightarrow \bar{\mathfrak{G}}^t(\xi) = \xi$ for all $t \in \mathbb{R}$. ∎

6.8b First integrals for the geodesic flow

The canonical $\mathfrak{Z}$-valued first integral

Define a map $\varphi : TN \to \mathfrak{Z}$ by $\varphi(n, X + Z) = Z$ for $n \in N, X \in \mathcal{V}$ and $Z \in \mathfrak{Z}$.

It is clear from the previous proposition that φ is a $\mathfrak{Z}$-valued first integral for the geodesic flow $\{\mathfrak{G}^t\}$ in TN. This first integral has a simple relationship to the first integrals produced by the momentum map $J : TN \to \mathfrak{N}^*$.

First integrals in TN from the momentum map $J : TN \to \mathfrak{N}^*$

We recall some results of (5.3) in this special case where $\mathfrak{H} = \mathfrak{N}$, a 2-step nilpotent Lie algebra. By the discussion in (3.11e) the momentum map $J : TN \to \mathfrak{N}^*$ is equivalent to the map $\hat{J} : \mathfrak{N} \to C^\infty(TN)$ given by $\hat{J}(A) = \theta(\lambda(A))$, where θ is the canonical 1-form on $TN, \lambda(A) \in \mathfrak{X}(TN)$ is the vector field on TN whose flow transformations are $\{\lambda_{e^{tA}}\}$, and $\lambda_n = dL_n$ for all $n \in N$. From (5.3a) we have the formula

$\hat{J}(A)(n, B) =< B, Ad(n^{-1})A >$ for all $(n, B) \in N \times \mathfrak{N} = TN$

The function $\hat{J}(A)$ is N-invariant on $TN \Leftrightarrow Ad(n)A = A$ for all $n \in N \Leftrightarrow A \in \mathfrak{Z}$. If $A \in \mathfrak{Z}$, then we obtain immediately :

Proposition Let $\varphi : TN \to \mathfrak{Z}$ be the canonical $\mathfrak{Z}$-valued first integral for the geodesic flow $\{\mathfrak{G}^t\}$ in TN. Then $< \varphi(\xi), A >= \hat{J}(A)(\xi)$ for all $A \in \mathfrak{Z}$ and all $\xi \in TN$.

First integrals for the geodesic flow in $\mathfrak{N}$.

We observed in (5.3b) that the first integrals $\bar{f} : \mathfrak{N} \to \mathbb{R}$ for the geodesic flow $\bar{\mathfrak{G}}^t$ in $\mathfrak{N}$ are in one-one correspondence with the N - invariant first integrals $f : TN \to \mathbb{R}$ for the geodesic flow $\mathfrak{G}^t$ in TN. The correspondence is given by the relation $f(n, A) = \bar{f}(A)$ for all $(n, A) \in N \times \mathfrak{N} = TN$.

We discuss some polynomial first integrals for the geodesic flow $\{\bar{\mathfrak{G}}^t\}$ in $\mathfrak{N}$ that are particular to the 2-step nilpotent case. By the discussion in (5.3b) a C^∞ function $\bar{f} : \mathfrak{N} \to \mathbb{R}$ is a first integral for the geodesic flow $\{\bar{\mathfrak{G}}^t\}$ in $\mathfrak{N} \Leftrightarrow \{\bar{f}, \bar{E}\} = 0$, where $\bar{E} : \mathfrak{N} \to \mathbb{R}$ is the energy function given by $\bar{E}(A) = \frac{1}{2} < A, A >$ for all $A \in \mathfrak{N}$. We also observed in (5.3b) that if $\bar{f} : \mathfrak{N} \to \mathbb{R}$ is a polynomial function, then we may reduce to the case that $\bar{f}$ is homogeneous of a given degree.

Linear first integrals

Proposition Let $\mathfrak{N}$ be a 2-step nilpotent Lie algebra with an inner product $<,>$, and let $^\# : \mathfrak{N} \to \mathfrak{N}^*$ be the canonical isomorphism given by $A^\#(B) =< A, B >$ for $A, B \in \mathfrak{N}$. Let $\bar{f} = A^\# : \mathfrak{N} \to \mathbb{R}$. Then

1) $\{\bar{f}, \bar{E}\} = 0 \Leftrightarrow A \in \mathfrak{Z}$, the center of $\mathfrak{N}$.

2) If $\{\bar{f}, \bar{E}\} = 0$, then $\bar{f}$ is a Casimir function.

Proof of 1) By Proposition C of (5.3b) we know that $\{\bar{f}, \bar{E}\} = 0 \Leftrightarrow < \alpha, [A, \alpha] >= 0$ for all $\alpha \in \mathfrak{N}$. If $A \in \mathfrak{Z}$, then clearly $\{\bar{f}, \bar{E}\} = 0$. Conversely, suppose that $\{\bar{f}, \bar{E}\} = 0$ and write $A = X + Z$ for $X \in V$ and $Z \in \mathfrak{Z}$. Let $\alpha = X' + Z'$ be an arbitrary element of $\mathfrak{N}$, where $X' \in V$ and $Z' \in \mathfrak{Z}$. Then $0 =< \alpha, [A, \alpha] >=< X' + Z', [X + Z, X' + Z'] >=< Z', [X, X'] >$. If we set $Z' = [X, X']$, then it follows that $[X, X'] = 0$ for all $X' \in V$. Hence $X \in V \cap \mathfrak{Z} = \{0\}$, and we conclude that $A = Z \in \mathfrak{Z}$. This proves 1).

Proof of 2) This is an immediate consequence of 1) and the discussion in example 1 of (3.9) since $\bar{f}$ depends only on the center $\mathfrak{Z}$.

Quadratic polynomials in the almost nonsingular case

We consider next the homogeneous polynomials of degree 2. Any such polynomial function $\bar{f} : \mathfrak{N} \to \mathbb{R}$ can be expressed as $\bar{f}(\xi) = < S(\xi), \xi >$, where $S : \mathfrak{N} \to \mathfrak{N}$ is a linear map that is symmetric relative to $<,>$. By the discussion in (5.3b) the polynomial is a first integral for the geodesic flow $\bar{\mathfrak{G}}^t$ in $\mathfrak{N} \Leftrightarrow$

(*) $< S(A), \nabla_A A >= 0$ for all $A \in \mathfrak{N}$.

Let $A \in \mathfrak{N}$ and write $A = X + Z$, where $X \in \mathcal{V}$ and $Z \in \mathfrak{Z}$. From (2.2) of [E1] it follows that $\nabla_A A = -j(Z)X$.

In the case that $\mathfrak{N}$ is almost nonsingular we obtain a nice characterization of the symmetric linear maps $S : \mathfrak{N} \to \mathfrak{N}$ that satisfy (*) above.

Proposition Let $\mathfrak{N}$ be an almost nonsingular 2-step nilpotent Lie algebra. Let $S : \mathfrak{N} \to \mathfrak{N}$ be a symmetric linear transformation and define $\bar{f} : \mathfrak{N} \to \mathbb{R}$ by $\bar{f}(\xi) = < S(\xi), \xi >$ for all $\xi \in \mathfrak{N}$. Then $\bar{f}$ is a first integral for the geodesic flow $\bar{\mathfrak{G}}^t$ in $\mathfrak{N} \Leftrightarrow$

1) $S(\mathcal{V}) \subseteq \mathcal{V}$ and $S(\mathfrak{Z}) \subseteq \mathfrak{Z}$.

2) S commutes with $j(Z)$ for all $Z \in \mathfrak{Z}$.

Remarks

1) It is easy to see that 1) and 2) are equivalent to 1) and 2)$'$: $[S(A), B] = [A, S(B)]$ for all $A, B \in \mathfrak{N}$.

2) Let $S : \mathfrak{N} \to \mathfrak{N}$ be a symmetric linear map such that $S(\mathcal{V}) \subseteq \mathcal{V}$ and $S(\mathfrak{Z}) \subseteq \mathfrak{Z}$. Write $S = S_1 + S_2$, where $S_1 = S$ on $\mathcal{V}$ and $S_1 \equiv 0$ on $\mathfrak{Z}$; $S_2 \equiv 0$ on $\mathcal{V}$ and $S_2 = S$ on $\mathfrak{Z}$. If $\bar{f} : \mathfrak{N} \to \mathbb{R}$ is given by $\bar{f}(\xi) = < S(\xi), \xi >$, then $\bar{f} = \bar{f}_1 + \bar{f}_2$, where $\bar{f}_1 = < S_1(\xi), \xi >$ and $\bar{f}_2 = < S_2(\xi), \xi >$. By the discussion in example 1 of (3.9) $\bar{f}_2$ is a Casimir function, and hence $\bar{f}$ is a first integral for $\{\bar{\mathfrak{G}}^t\} \Leftrightarrow \bar{f}_1$ is a first integral for $\{\bar{\mathfrak{G}}^t\}$.

From the proposition above and the discussion of the previous paragraph we now conclude: If $\mathfrak{N}$ is an almost nonsingular 2-step nilpotent Lie algebra, then the polynomial first integrals for $\bar{\mathfrak{G}}^t$ that are homogeneous of degree 2 are in one-one correspondence, modulo Casimir functions, to symmetric linear maps $S : \mathcal{V} \to \mathcal{V}$ such that S commutes with $j(Z)$ for all $Z \in \mathfrak{Z}$.

Note that condition 2) in the proposition implies that $j(\mathfrak{Z})$ must leave invariant each of the eigenspaces of S restricted to $\mathcal{V}$.

Proof of the Proposition Let $S : \mathfrak{N} \to \mathfrak{N}$ be a symmetric linear transformation and define $\bar{f} : \mathfrak{N} \to \mathbb{R}$ by $\bar{f}(\xi) = < S(\xi), \xi >$ for all $\xi \in \mathfrak{N}$. By the discussion above we must characterize all such maps S with the property that $0 = < S(X+Z), j(Z)(X) > = < S(X), j(Z)(X) > + < S(Z), j(Z)(X) >$ for all $X \in \mathcal{V}$ and $Z \in \mathfrak{Z}$.

Assume first that this equation holds for some symmetric linear map S. The first term on the right hand side is linear in Z while the second term is quadratic in Z. Hence both terms must vanish identically and we have

a) $0 = < S(X), j(Z)(X) >$ for all $X \in \mathcal{V}$ and $Z \in \mathfrak{Z}$.

b) $0 = < S(Z), j(Z)(X) >$ for all $X \in \mathcal{V}$ and $Z \in \mathfrak{Z}$.

Define $S_{\mathcal{V}} : \mathfrak{N} \to \mathcal{V}$ by $S_{\mathcal{V}} = \pi_{\mathcal{V}} \circ S$, where $\pi_{\mathcal{V}}$ denotes orthogonal projection onto $\mathcal{V}$. The restriction of $S_{\mathcal{V}}$ to $\mathcal{V}$ is symmetric on $\mathcal{V}$ since S is symmetric on $\mathfrak{N}$. We may now replace S by $S_{\mathcal{V}}$ in the statements of 1) and 2) above since $j(Z)X \in \mathcal{V}$ for all $X \in \mathcal{V}$ and $Z \in \mathfrak{Z}$.

From a) we conclude that $< S_{\mathcal{V}} j(Z)X, X >= 0$ for all $X \in \mathcal{V}$ and $Z \in \mathfrak{Z}$ since

$S_{\mathcal{V}} : \mathcal{V} \to \mathcal{V}$ is symmetric. From a) we also conclude that $< j(Z)S_{\mathcal{V}}(X), X >= 0$ for all $X \in \mathcal{V}$ and $Z \in \mathfrak{Z}$ since $j(Z) : \mathcal{V} \to \mathcal{V}$ is skew symmetric. Hence we obtain

(*) $\quad 0 =< \{S_{\mathcal{V}}j(Z) - j(Z)S_{\mathcal{V}}\}(X), X >$ for all $X \in \mathcal{V}$ and $Z \in \mathfrak{Z}$

However, $S_{\mathcal{V}}j(Z) - j(Z)S_{\mathcal{V}}$ is symmetric on $\mathcal{V}$ for all $Z \in \mathfrak{Z}$ since $j(Z)$ is skew symmetric and $S_{\mathcal{V}}$ is symmetric. The equation (*) implies that every eigenvalue of $S_{\mathcal{V}}j(Z) - j(Z)S_{\mathcal{V}}$ is zero for all $Z \in \mathfrak{G}$. We conclude

(**) $\quad S_{\mathcal{V}}$ commutes with $j(Z)$ for all $Z \in \mathfrak{Z}$

So far we have not used the hypothesis that $\mathfrak{N}$ is almost nonsingular, but we now use this hypothesis to draw conclusions from b) above, where S is replaced by $S_{\mathcal{V}}$. If $\mathfrak{N}$ is almost nonsingular, then by the corollary in (6.4) there exists a dense open subset O of $\mathfrak{Z}$ such that $j(Z)$ is nonsingular for all Z in O. Condition b) above now implies that $S_{\mathcal{V}}(Z) = 0$ for all $Z \in O$. Since O is dense in $\mathfrak{Z}$ and S is continuous it follows that $S_{\mathcal{V}} \equiv 0$ on $\mathfrak{Z}$ or equivalently that $S(\mathfrak{Z}) \subseteq \mathfrak{Z}$. Since S is symmetric and $\mathcal{V} = \mathfrak{Z}^{\perp}$ it follows that $S(\mathcal{V}) \subseteq \mathcal{V}$. Hence $S = S_{\mathcal{V}}$ on $\mathcal{V}$ and (*) says that S commutes with $j(Z)$ for all Z.

We have verified that if $\bar{f}(\xi) =< S(\xi), \xi >$ is a first integral for the geodesic flow on $\mathfrak{N}$, then S must satisfy the two conditions of the proposition. Conversely, if these two conditions are satisfied, then it is immediately clear that b) holds. It remains only to verify that a) holds as well. Given $X \in \mathcal{V}$ and $Z \in \mathfrak{Z}$ we compute $< S(X), j(Z)X >=< X, Sj(Z)X >$ since S is symmetric. On the other hand $< S(X), j(Z)X >= - < j(Z)S(X), X >= - < X, Sj(Z)X >$ since $j(Z)$ is skew symmetric and S commutes with $j(Z)$. It follows that $< S(X), j(Z)X >= 0$ for all $X \in \mathcal{V}$ and $Z \in \mathfrak{Z}$, which proves that a) holds and completes the proof of the proposition. ■

Polynomial first integrals of Butler

L. Butler in [Bu 3] exhibited the following polynomial first integrals for the geodesic flow in $\mathfrak{N}$.

Proposition (L. Butler) For each integer i with $1 \le i \le \frac{1}{2}\dim \mathcal{V}$ define $\bar{f}_i : \mathfrak{N} \to \mathbb{R}$ by $\bar{f}_i(X + Z) =< X, j(Z)^{2i}X >$ for all $X \in \mathcal{V}$ and $Z \in \mathfrak{Z}$. Then each $\bar{f}_i$ is a first integral for the geodesic flow in $\mathfrak{N}$.

Note that each $\bar{f}_i$ is a polynomial function of degree $2i + 2$ in the variables of any linear coordinate system for $\mathfrak{N}$. To see why each $\bar{f}_i$ is a first integral we first observe that $j(Z)$ commutes with $e^{tj(Z)}$ for all t and $e^{tj(Z)}$ is an orthogonal linear transformation of $\mathcal{V}$ for all t since $j(Z)$ is skew symmetric. Recall from (6.8a) that the geodesic flow $\{\bar{\mathfrak{G}}^t\}$ on $\mathfrak{N}$ is given by $\bar{\mathfrak{G}}^t(X + Z) = e^{-tj(Z)}X + Z$. Hence $(\bar{f}_i \circ \bar{\mathfrak{G}}^t)(X + Z) =< e^{-tj(Z)}X, j(Z)^{2i}e^{-tj(Z)}X >=< e^{-tj(Z)}X, e^{-tj(Z)}j(Z)^{2i}X >=$ $< X, j(Z)^{2i}X >= \bar{f}_i(X + Z)$ for all $X \in \mathcal{V}, Z \in \mathfrak{Z}$ and $t \in \mathbb{R}$. It follows that each $\bar{f}_i$ is a first integral for $\bar{\mathfrak{G}}^t$ in $\mathfrak{N}$. ■

Remark The first integrals will in general be functionally independent for a generic 2-step nilpotent Lie algebra $\mathfrak{N}$. In the case of a 2-step nilpotent Lie algebra of Heisenberg type we obtain the simple expression $\bar{f}_i(X + Z) = (-1)^i |X|^2 |Z|^{2i}$ for all $X \in \mathcal{V}$ and $Z \in \mathfrak{Z}$.

Other first integrals

For simplicity we consider arbitrary continuous functions $\bar{f} : \mathfrak{N} \to \mathbb{R}$ that depend only on the $\mathcal{V}$-component of each vector in $\mathfrak{N} = \mathcal{V} \oplus \mathfrak{Z}$; that is, $\bar{f}(X + Z) = \bar{g}(X)$ for some continuous function $\bar{g} : \mathcal{V} \to \mathbb{R}$. (This is not unreasonable since the functions $\bar{f} : \mathfrak{N} \to \mathbb{R}$ that depend only on the $\mathfrak{Z}$-components of each vector are Casimir functions by the discussion in example 1 of (3.9)). Evidently such a function $\bar{f}$ is a first integral of the geodesic flow in $\mathfrak{N} \Leftrightarrow \bar{g}$ is constant on all curves $t \to e^{tj(Z)}X$ for all $X \in \mathcal{V}$ and all $Z \in \mathfrak{Z}$. Let $\mathfrak{H}$ denote the Lie subalgebra of $so(\mathcal{V})$ that is generated by $j(\mathfrak{Z}) = \{j(Z) : Z \in \mathfrak{Z}\}$, and let H denote the connected Lie subgroup of $SO(\mathcal{V})$ whose Lie algebra is $\mathfrak{H}$. If $\bar{g}$ is constant on all curves $t \to e^{tj(Z)}X$, then $\bar{g}$ must be constant on all orbits in $\mathcal{V}$ of $\bar{H}$, the closure in $SO(\mathcal{V})$ of H. To see this let $H' = \{\varphi \in SO(\mathcal{V}) : \bar{g}(\varphi X) = \bar{g}(X)$ for all $X \in \mathcal{V}\}$. Then H' is a closed subgroup of $O(\mathcal{V})$ whose Lie algebra contains $j(\mathfrak{Z})\}$.

Conversely, if $\bar{g} : \mathcal{V} \to \mathbb{R}$ is a continuous function that is constant on orbits of $\bar{H}$ in $\mathcal{V}$, then clearly $\bar{g}$ is constant on all curves $t \to e^{tj(Z)}X$ for $X \in \mathcal{V}$ and $Z \in \mathfrak{Z}$.

For first integrals $\bar{f} : \mathfrak{N} \to \mathbb{R}$ of this type we have reduced the problem, in a sense, to the problem of computing $\mathfrak{H}, \bar{H}$ and the orbits of $\bar{H}$ in $\mathcal{V}$ once we are given $j(\mathfrak{Z})$. For a generic 2-step nilpotent Lie algebra $\mathfrak{N} = \mathcal{V} \oplus \mathfrak{Z}$, the subalgebra $\mathfrak{H}$ of $so(\mathcal{V})$ generated by $j(\mathfrak{Z})$ is $so(\mathcal{V})$ itself. In this case $\bar{H} = SO(\mathcal{V})$, which acts transitively on all spheres of $\mathcal{V}$. Hence the functions $\bar{g} : \mathcal{V} \to \mathbb{R}$ that are constant on orbits of $\bar{H} = SO(\mathcal{V})$ are the functions of the form $\bar{g}(X) = f(|X|)$, where $f : (0, \infty) \to \mathbb{R}$ is any continuous function.

6.8c Density of closed geodesics in $\Gamma\backslash N$, a necessary condition

Let Γ be a lattice in N, and let $\pi : N \to \Gamma\backslash N$ be the covering projection. We saw in (5.4) that if $\xi \in TN$ is a vector such that $\pi_*(\xi)$ has period $\omega > 0$ relative to the geodesic flow $\{\mathfrak{G}^t\}$ in $T(\Gamma\backslash N)$, then $G(\xi)$ has period $\omega > 0$ relative to the geodesic flow $\{\bar{\mathfrak{G}}^t\}$ in $\mathfrak{N}$. We now take a closer look at what it means for a vector $A \in \mathfrak{N}$ to be periodic relative to $\{\bar{\mathfrak{G}}^t\}$ if $\mathfrak{N}$ is a 2-step nilpotent Lie algebra.

Regular vectors in 2-step nilpotent metric Lie algebras

Let $\mathfrak{N}$ be a 2-step nilpotent Lie algebra with an inner product $<,>$, and decompose $\mathfrak{N}$ into an orthogonal direct sum $\mathfrak{N} = \mathcal{V} \oplus \mathfrak{Z}$ as in (6.2*a*), where $\mathfrak{Z}$ denotes the center of $\mathfrak{N}$. For each element Z of $\mathfrak{Z}$ the transformation $j(Z)^2$ is symmetric and negative semidefinite since $j(Z)$ is skew symmetric. We may decompose $\mathcal{V}$ into an orthogonal direct sum $\mathcal{V} = W_o \oplus W_1 \oplus ... \oplus W_N$, where W_o is the kernel of $j(Z), j(Z)^2 = -\theta_i^2 Id$ on W_i, $1 \leq i \leq N$, and $\{-\theta_1^2, ..., -\theta_N^2\}$ are the distinct nonzero eigenvalues of $j(Z)^2$. Let N_o denote the maximum value of N as Z ranges over all nonzero vectors of $\mathfrak{Z}$, and let $\mathfrak{Z}_o = \{Z \in \mathfrak{Z} : j(Z)^2$ has N_o nonzero eigenvalues$\}$. One can show that $\mathfrak{Z}_o$ is a dense open subset of $\mathfrak{Z}$. See for example [GM 2, Proposition 1.19].

Definition of Regular vectors

Let A be any element of $\mathfrak{N}$ and write $A = X + Z$, where $X \in \mathcal{V}$ and $Z \in \mathfrak{Z}$. The vector A is said to be <u>regular</u> if Z belongs to $\mathfrak{Z}_o$ and X has a nonzero component in each subspace $W_i, 1 \le i \le N_o$, relative to the decomposition above of $\mathcal{V}$ into eigenspaces of $j(Z)^2$. Clearly the set of regular vectors of $\mathfrak{N}$ is a dense open subset of $\mathfrak{N}$ since $\mathfrak{Z}_o$ is a dense open subset of $\mathfrak{Z}$.

Resonant vectors of $\mathfrak{Z}$

A nonzero vector Z of $\mathfrak{Z}$ is said to be resonant if the 1-parameter group $\{e^{tj(Z)}\}$ in $SO(\mathcal{V})$ passes through the identity of $SO(\mathcal{V})$ at some positive time ω. It is not difficult to show that Z is resonant $\Leftrightarrow$ the ratio of any two nonzero eigenvalues of $j(Z)$ is a rational number. Note that the ratio of any two nonzero eigenvalues of $j(Z)$ must be a real number since the eigenvalues of a skew symmetric linear map are purely imaginary.

If Z is a resonant vector of $\mathfrak{Z}$, then clearly cZ is also a resonant vector of $\mathfrak{Z}$.

Regular periodic vectors of $\{\mathfrak{G}^t\}$ in $\mathfrak{N}$

Proposition Let $\mathfrak{N}$ be a 2-step nilpotent Lie algebra with an inner product $<,>$. Let $A = X + Z$ be an element of $\mathfrak{N}$ that is regular and periodic relative to the geodesic flow $\{\mathfrak{G}^t\}$ in $\mathfrak{Z}$. Then Z is a resonant vector in $\mathfrak{Z}$.

Proof Let $\mathcal{V} = W_o \oplus W_1 \oplus ... \oplus W_{N_o}$ be the orthogonal direct sum decomposition of $\mathcal{V}$ into eigenspaces of $j(Z)^2$ that was described above. By hypothesis X has a nonzero component X_i in each eigenspace W_i for $1 \le i \le N_o$. Since $j(Z)^2 = -\theta_i^2 Id$ on $W_i, 1 \le i \le N_o$, it is routine to show

$(^*) e^{tj(Z)} = cos(t\theta_i)Id + (1/\theta_i)sin(t\theta_i)j(Z)$ on each W_i.

By hypothesis $\mathfrak{G}^\omega(A) = A$ for some positive number ω, and by the corollary in (6.8a) this means that $e^{\omega j(Z)}X = X$. Since $j(Z)$ leaves invariant each subspace W_i it follows that $e^{\omega j(Z)}$ does also. Hence $e^{\omega j(Z)}X_i = X_i$ for $1 \le i \le N_o$. Each component X_i is nonzero since $A = X + Z$ is regular, and it follows from $(^*)$ that $\omega\theta_i = 2\pi N_i$, where N_i is an integer for $1 \le i \le N_o$. It follows that θ_i/θ_j is rational for $1 \le i, j \le N_o$. The numbers $\{\theta_i/\theta_j, 1 \le i, j \le N_o\}$ are the ratios of the nonzero eigenvalues of $j(Z)$, and hence by the discussion above we conclude that Z is resonant. ■

The next result was first proved in [Mas, Theorem 4] using the argument of the proposition above. Although nonsingularity is part of the hypothesis in that result, it is not needed in the proof.

Corollary Let N be a simply connected, 2-step nilpotent Lie group with Lie algebra $\mathfrak{N}$ and a left invariant metric $<,>$. Let Γ be a lattice in N. If the periodic vectors for the geodesic flow in $T(\Gamma\backslash N)$ are dense in $T(\Gamma\backslash N)$, then the resonant vectors of $\mathfrak{Z}$ are dense in $\mathfrak{Z}$.

Proof If the periodic vectors for the geodesic flow $\{\mathfrak{G}^t\}$ in $T(\Gamma\backslash N)$ are dense in $T(\Gamma\backslash N)$, then by Proposition C of (5.4) the periodic vectors for the geodesic flow $\{\mathfrak{G}^t\}$ in $\mathfrak{N}$ are dense in $\mathfrak{N}$. The set of regular vectors of $\mathfrak{N}$ is dense and open in $\mathfrak{N}$, and hence there is a dense set of regular vectors in $\mathfrak{N}$ that are periodic for $\{\mathfrak{G}^t\}$. Each regular periodic vector is resonant by the proposition above, and the proof is now complete. ■

Examples with a dense set of resonant vectors

In view of the corollary above it is natural to look for examples of 2-step nilpotent Lie algebras with inner products $<,>$ such that the resonant vectors in $\mathfrak{z}$ form a dense subset of $\mathfrak{z}$. We will restrict our remarks to examples of the form $\mathfrak{N} = \mathbb{R}^n \oplus W$, where W is a subspace of $so(n,\mathbb{R})$, and $\mathfrak{N}$ has the inner product and 2-step nilpotent structure defined in example 4 of (6.1c). We omit the proofs of the assertions below.

Recall from example 4 of (6.1c) that the center of $\mathfrak{N} = \mathbb{R}^n \oplus W$ is given by $\mathfrak{z} = U \oplus W$, where $U = \{X \in \mathbb{R}^n : Z(X) = 0 \text{ for all } Z \in W\}$.

Example 1 ([DeC]) Let $\dim W = k$, where $k \geq \operatorname{rank} so(n,\mathbb{R})$. Then there exists a dense open subset O of $G(k, so(n,\mathbb{R}))$, the Grassmann manifold of k-dimensional subspaces of $so(n,\mathbb{R})$, such that if $W \in O$, then the resonant vectors of $\mathfrak{z}$ are dense in $\mathfrak{z}$.

The rank of a Lie algebra $\mathfrak{h}$ is the dimensional of a maximal abelian subspace of $\mathfrak{h}$. Rank $so(n,\mathbb{R}) = j$ where $n = 2j$ or $2j + 1$.

Example 2 Let W be the subalgebra of a compact connected subgroup G of the special orthogonal group $\mathrm{SO}(n,\mathbb{R})$, and suppose that G has no fixed points in $\mathbb{R}^n$. Then the resonant vectors are dense in $\mathfrak{z} = W$ for $\mathfrak{N} = \mathbb{R}^n \oplus W$.

If $W = su(2,\mathbb{R})$ and $\mathbb{R}^n$ is irreducible relative to W, then $\mathfrak{z} = W$ and every nonzero vector of $\mathfrak{z}$ is resonant. If $\dim W \geq 2$, then $W = su(2,\mathbb{R})$ is the only Lie algebra of a compact subgroup of $\mathrm{SO}(n,\mathbb{R})$ in which all vectors of $\mathfrak{z}$ are resonant.

Example 3 Let $\mathfrak{N}$ be a 2-step nilpotent Lie algebra of Heisenberg type. Then every nonzero vector of $\mathfrak{z}$ is resonant. In particular $j(Z)^2 = -|Z|^2\,Id$, so the ratio of any two nonzero eigenvalues of $j(Z)$ is 1 or -1.

Recall from example 6 of (6.1c) that every Lie algebra $\mathfrak{N}$ of Heisenberg type may be written as $\mathfrak{N} = \mathbb{R}^n \oplus W$, where $W = j(\mathbb{R}^p)$ and $j : C\ell(p) \to \operatorname{End}(\mathbb{R}^n)$ is a representation of the Clifford algebra.

Remark Recall that a 2-step nilpotent Lie algebra $\mathfrak{N}$ with an inner product $<,>$ is said to be Heisenberg like if the eigenvalues of $j(Z)$ depend only on $|Z|$. It follows that either all nonzero vectors of $\mathfrak{z}$ are resonant or none of them are since cZ is resonant if Z is resonant for any nonzero number c.

Lie algebras that are Heisenberg like are plentiful (cf. [GM 2]).

6.8d Density of closed geodesics in $\Gamma\backslash N$, sufficient conditions

We list sufficent conditions in the order of historical appearance.

Results of Eberlein, Mast, Lee-Park and DeMeyer

Proposition A ([E1, Proposition 5.6]) Let N be a simply connected 2-step nilpotent Lie group with a left invariant metric such that the Lie algebra $\mathfrak{N}$ is of Heisenberg type. Then the closed geodesics are dense in $\Gamma\backslash N$ for any lattice subgroup Γ of N.

We note that the necessary condition of a dense set of resonant vectors in $\mathfrak{z}$ is satisfied by example 3 above if $\mathfrak{N}$ has Heisenberg type.

Proposition B ([Mas, Theorem 3]) Let N be a simply connected 2-step nilpotent Lie group with a left invariant metric such that the Lie algebra $\mathfrak{N}$ is nonsingular and every nonzero vector in $\mathfrak{z}$ is resonant. Then the closed geodesics are dense in $\Gamma\backslash N$

for any lattice subgroup Γ of N.

Proposition C ([LP, Theorem 3.3]) Let N be a simply connected 2-step nilpotent Lie group with a left invariant metric such that the Lie algebra $\mathfrak{N}$ is almost nonsingular and $\mathfrak{Z}$ contains a dense set of resonant vectors. Then the closed geodesics are dense in $\Gamma\backslash N$ for any lattice subgroup Γ of N.

L. DeMeyer was the first to show that the condition of almost nonsingularity is not a necessary condition for the density of closed geodesics.

Proposition D ([DeM, Main Theorem]) Let $\mathbb{R}^n$ be an irreducible $\mathfrak{su}(2,\mathbb{R})$ module, where n is any odd integer ≥ 5. Let $\mathfrak{N} = \mathbb{R}^n \oplus \mathfrak{su}(2,\mathbb{R})$ be given the inner product $<,>$ and the 2-step nilpotent Lie algebra structure defined in example 4 of (6.1c). If $\{N, <,>\}$ is the corresponding simply connected, 2-step nilpotent Lie group with left invariant metric, then the closed geodesics are dense in $\Gamma\backslash N$ for any lattice subgroup Γ of N.

The fact that n is an odd integer means that the skew symmetric map $j(Z) : \mathbb{R}^n \to \mathbb{R}^n$ has a nontrivial kernel for every vector $Z \in \mathfrak{su}(2,\mathbb{R})$. For every odd integer $n \geq 3, \mathbb{R}^n$ has the structure of an irreducible $\mathfrak{su}(2,\mathbb{R})$ module (cf. [DeM]). If $n = 3$ we may take $\mathbb{R}^3 = \mathfrak{su}(2,\mathbb{R})$ and the Lie algebra structure of $\mathfrak{su}(2,\mathbb{R})$ makes $\mathbb{R}^3$ an $\mathfrak{su}(2,\mathbb{R})$ module. In this important special case the technique of proof of [DeM] does not work , and it is still unknown if Proposition D holds here.

For $\mathfrak{N} = \mathbb{R}^n \oplus \mathfrak{su}(2,\mathbb{R})$ every vector in $\mathfrak{Z} = \mathfrak{su}(2,\mathbb{R})$ is resonant. If n is an even integer, then it is easy to see that $\mathfrak{N} = \mathbb{R}^n \oplus \mathfrak{su}(2,\mathbb{R})$ is a nonsingular 2-step nilpotent Lie algebra. Hence in this case one may apply the result of [Mas] or [LP].

Problem The Lie algebras $\mathfrak{N} = \mathbb{R}^n \oplus \mathfrak{su}(2,\mathbb{R})$ of [DeM] all have rank 2 in the sense of (6.5). Can one find criteria for a metric 2-step nilpotent Lie algebra $\{\mathfrak{N}, <,>\}$ of rank $k \geq 2$ such that if Γ is any lattice in the corresponding simply connected, 2-step nilpotent Lie group N with left invariant metric $<,>$, then the closed geodesics in $\Gamma\backslash N$ are dense ?

6.8e Length spectrum and maximal length spectrum Isospectrality

A classical problem of geometry and analysis is to try to recover an operator on a space X, up to a suitable notion of equivalence, from the spectrum of its eigenvalues. For example, a self adjoint linear operator on a finite dimensional real inner product space X is completely determined by its eigenvalues up to conjugation by an element of the orthogonal group of X. If M is a Riemannian manifold and X is a suitable space of functions or differential forms equipped with a natural inner product, then the Laplace operator Δ on X is self adjoint. In this context the problem has been historically to find conditions under which M is determined up to isometry by the eigenvalues of Δ. There is an enormous literature on this subject, and we make no attempt to summarize it here. If M is a compact nilmanifold $\Gamma\backslash N$, then C. Gordon and E. Wilson showed that the space of compact nilmanifolds with the same Laplace spectrum as M on C^∞ functions, modulo isometric equivalence, is not a single point but is a nice finite dimensional double coset space determined by the almost inner automorphisms of N. For a precise statement of their result see Theorem 5.5 of [GW]. We shall return to the notion of almost inner automorphism shortly.

A related problem in Riemannian geometry is to find conditions under which a compact Riemannian manifold M is determined up to isometry by the lengths of its closed geodesics, with the multiplicities of these lengths being counted suitably. In this case the flow transformations $\{\mathfrak{G}^t\}$ of the geodesic flow play the role of the Laplacian Δ, the periodic vectors play the role of the eigenfunctions of Δ and the lengths of the closed geodesics play the role of the spectrum of Δ. The set of lengths of the closed geodesics of M is called the length spectrum (without multiplicities) of M.

There are relations between the Laplace spectrum and the length spectrum (cf. [CdV], [DG]), but in general these cannot be written down in closed form except in special cases. For compact surfaces with Gaussian curvature $K \equiv -1$, or more generally for compact locally symmetric spaces of rank 1 and negative sectional curvature, the Selberg trace formula gives an explicit relation between the length spectrum and the Laplace spectrum ([DeG], [Ga], [McK]). In the case of compact 2-step nilmanifolds $\Gamma\backslash N$, where N has 1-dimensional center, an explicit relation is given in [Pe].

Free homotopy classes of closed curves

Let M be any complete Riemannian manifold, and let $\mathcal{C}(M)$ denote the set of free homotopy classes of closed curves in M. A free homotopy class C is called trivial if it contains the constant curves. For every free homotopy class $C \in \mathcal{C}(M)$ let $\ell(C)$ denote the lengths of all closed geodesics that belong to C. If M has nonpositive sectional curvature, then $\ell(C)$ is a single positive number for each nontrivial $C \in \mathcal{C}(M)$. However, in general the set $\ell(C)$ may contain more than one real number, perhaps even infinitely many. We shall see that for a compact 2-step nilmanifold $\Gamma\backslash N$ each set $\ell(C)$ contains a largest element and a smallest element, and these elements in general are different unless the Lie algebra $\mathfrak{N}$ is nonsingular and the closed curves in C do not belong to the center of $\pi_1(\Gamma\backslash N)$. Moreover, the largest element has a simple explicit formula in terms of $\log \Gamma \subseteq \mathfrak{N}$. See Propositions A and D below for precise statements.

By elementary covering space theory there is a bijection between $\mathcal{C}(M)$ and the conjugacy classes in the fundamental group $\pi_1(M)$. Hence if M_1 and M_2 are complete Riemannian manifolds with isomorphic fundamental groups, then any isomorphism $T : \pi_1(M_1) \to (M_2)$ induces a bijection $T_* : \mathcal{C}(M_1) \to \mathcal{C}(M_2)$.

Length and marked length spectrum

Let M be a compact Riemannian manifold. For each positive number ω let $n(\omega)$ denote the number of free homotopy classes of closed curves in M that contain a closed geodesic of length ω. The number $n(\omega)$, the multiplicity of ω, is always finite by the compactness of M.

Let $\mathfrak{L}(M) = \{(\omega, n(\omega)) : n(\omega) \neq 0\}$. The set $\mathfrak{L}(M)$ is called the length spectrum of M (with multiplicities).

Now let M_1 and M_2 be compact Riemannian manifolds with isomorphic fundamental groups. We say that M_1 and M_2 have the same marked length spectrum if there exists an isomorphism $T : \pi_1(M_1) \to \pi_1(M_2)$ such that $\ell(T_*(C)) = \ell(C)$ for all $C \in \mathcal{C}(M_1)$, where $T_* : \mathcal{C}(M_1) \to \mathcal{C}(M_2)$ is the bijection induced by T. Clearly any two compact Riemannian manifolds with the same marked length spectrum have the same length spectrum.

Length spectrum in compact 2-step nilmanifolds

Let N be a simply connected, 2-step nilpotent Lie group with a left invariant metric $<,>$, and let Γ be a lattice in N. By the discussion in (5.4) the lengths of the closed geodesics in $\Gamma\backslash N$ are precisely the positive numbers ω such that $\varphi \cdot \gamma(t) = \gamma(t+\omega)$ for some $\varphi \in \Gamma$ and all $t \in \mathbb{R}$. The next two results, which are Proposition 4.5 and its corollary in [E1], give useful information about these periods ω of the elements φ of N.

Proposition A Let N be a simply connected, 2-step nilpotent Lie group with Lie algebra $\mathfrak{N}$, and let $<,>$ be a left invariant metric on N. Let φ be any element of N and write $\varphi = \exp(V^* + Z^*)$, where $V^* \in \mathcal{V}$ and $Z^* \in \mathfrak{Z}$. Let Z^{**} be the component of Z^* orthogonal to $[V^*, \mathfrak{N}]$ and let $\omega^* = \{|V^*|^2 + |Z^{**}|^2\}^{1/2}$. Let $\gamma(t)$ be a unit speed geodesic of N such that $\varphi \cdot \gamma(t) = \gamma(t+\omega)$ for all $t \in \mathbb{R}$. Then

1) $|V^*| \le \omega \le \omega^*$.
2) $\omega = \omega^* \Leftrightarrow$ the following conditions hold
   a) $\gamma(t) = n \cdot \exp(\frac{t}{\omega^*}(V^* + Z^{**}))$ for all $t \in \mathbb{R}$, where $n = \gamma(0)$.
   b) $Z^{**} = Z^* + [V^*, \xi]$, where $\xi = \log n$.
3) $\omega = |V^*| \Leftrightarrow Z^* = 0 \Leftrightarrow \gamma(t) = n.\exp(\frac{tV^*}{|V^*|})$ for all $t \in \mathbb{R}$.

Corollary B Let N be a simply connected, 2-step nilpotent Lie group with Lie algebra $\mathfrak{N}$, and let $<,>$ be a left invariant metric on N. Let φ be any element of N that does not lie in the center of N and write $\varphi = \exp(V^* + Z^*)$, where $0 \ne V^* \in \mathcal{V}$ and $Z^* \in \mathfrak{Z}$. Assume that Z^* lies in $[V^*, \mathfrak{N}]$. Then

1) φ has a unique period $\omega = |V^*|$.
2) Let $\gamma(t)$ be a unit speed geodesic in N with $\gamma(0) = n = \exp(\xi)$, where $\xi \in \mathfrak{N}$. Then $\varphi \cdot \gamma(t) = \gamma(t+\omega)$ for all $t \in \mathbb{R} \Leftrightarrow [\xi, V^*] = Z^*$ and $\gamma(t) = n \cdot \exp(\frac{tV^*}{\omega})$ for all $t \in \mathbb{R}$.

Remarks 1) By Proposition A of (6.3) the corollary applies to any noncentral element φ of N if $\mathfrak{N}$ is nonsingular.

2) The proposition above and its corollary give information about the largest and smallest periods of an element φ of N. There is also an explicit but more complicated formula for all of the periods of φ. See Theorem 2.4 of [GM 2].

Periods of central elements of N

If φ is an element in the center of N, then the geodesics translated by φ and the periods of φ have markedly different behavior than those of the noncentral elements of N.

The next two results are Propositions 4.9 and 4.11 of [E1].

Proposition C Let N be a simply connected, 2-step nilpotent Lie group with Lie algebra $\mathfrak{N}$, and let $<,>$ be a left invariant metric on N. Let φ be any element of N that lies in the center of N. Let $n \in N$ be arbitrary and let γ be any unit speed geodesic of N such that $\gamma(0) = n$ and $\gamma(\omega) = \varphi \cdot n$ for some positive number ω. Then $\varphi \cdot \gamma(t) = \gamma(t + \omega)$ for all $t \in \mathbb{R}$.

Proposition D Let N be a simply connected, 2-step nilpotent Lie group with Lie algebra $\mathfrak{N}$, and let $<,>$ be a left invariant metric on N. Let $\varphi = \exp(Z)$ be a central element of N such that $j(Z) \neq 0$ for some $Z \in \mathfrak{Z}$. Then there exists a positive integer N such that φ^n has at least two distinct periods for all integers $n \geq N$.

Remark If N has no Euclidean de Rham factor, then $j(Z) \neq 0$ for every nonzero element Z of $\mathfrak{Z}$ by (6.2c).

Maximal length spectrum and marked maximal length spectrum

By Proposition A above each nonidentity element φ of N has a unique maximal period $\omega^* = \omega^*(\varphi)$, whose formula is given in the statement of the proposition. Now let Γ be any lattice in N, and let φ be a nonidentity element of Γ. The correspondence between free homotopy classes of closed curves in $\Gamma\backslash N$ and conjugacy classes in $\Gamma = \pi_1(\Gamma\backslash N)$ shows that $\omega^*(\varphi)$ is the largest length of a closed geodesic in the free homotopy class of closed curves determined by the conjugacy class of φ. For each free homotopy class C of closed curves in $\Gamma\backslash N$ we let $\ell^*(C)$ denote the largest length of a closed geodesic in C.

For each positive number ω^* let $n^*(\omega^*)$ denote the number of free homotopy classes of closed curves in $\Gamma\backslash N$ that contain a longest closed geodesic of length ω^*. Define $\mathfrak{L}^*(\Gamma\backslash N) = \{(\omega^*, n^*(\omega^*) : n^*(\omega^*) \neq 0)$. The set $\mathfrak{L}^*(\Gamma\backslash N)$ is called the maximal length spectrum of $\Gamma\backslash N$ (with multiplicities).

If Γ_1 and Γ_2 are lattices in simply connected, 2-step nilpotent Lie groups N_1 and N_2 with left invariant metrics, then we say that $\Gamma_1\backslash N_1$ and $\Gamma_2\backslash N_2$ have the same marked maximal length spectrum if there exists an isomorphism $T : \Gamma_1 \to \Gamma_2$ such that $\ell^*(T_*(C)) = \ell^*(C)$ for all $C \in \mathcal{C}(\Gamma_1\backslash N_1)$. Here T_* denotes the bijection between $\mathcal{C}(\Gamma_1\backslash N_1)$ and $\mathcal{C}(\Gamma_2\backslash N_2)$, where we recall that Γ_i is isomorphic to $\pi_1(\Gamma_i\backslash N_i)$ for $i = 1, 2$.

Almost inner automorphisms of Gordon-Wilson

C. Gordon and E. Wilson in [GW] introduced the important concept of an almost inner automorphism of a simply connected nilpotent Lie group N. We say that an automorphism $\varphi : N \to N$ is almost inner if for each element n of N there exists an element $a = a(n)$ of N such that $\varphi(n) = ana^{-1}$. The usual inner automorphisms occur when a is a constant function on N.

Gordon and Wilson showed that the set $AI(N)$ of almost inner automorphisms of N forms a Lie subgroup of $\mathrm{Aut}(N)$ that in general is strictly larger than the group of inner automorphisms. The Lie algebra of $AI(N)$ consists of the almost inner derivations of $\mathfrak{N}$, where $\mathfrak{N}$ denotes the Lie algebra of N. A derivation $D : \mathfrak{N} \to \mathfrak{N}$ is almost inner if for every element $x \in \mathfrak{N}$ there exists an element $\xi(x) \in \mathfrak{N}$ such that $D(x) = [\xi(x), x]$, or equivalently, that $D(x) \in \mathrm{ad}\, x(\mathfrak{N})$ for all $x \in \mathfrak{N}$. The element $\xi(x)$ can clearly be replaced by $\xi(x) + Z(x)$, where $Z(x)$ is any element in the center of $\mathfrak{N}$. If $\xi : \mathfrak{N} \to \mathfrak{N}$ can be chosen so that ξ is continuous on $\mathfrak{Z}^\perp - \{0\}$, then D is said to be of continuous type.

If N is 2-step nilpotent and φ is an almost inner automorphism of N, then the differential $d\varphi \in \mathrm{Aut}(\mathfrak{N})$ can be written $d\varphi = Id + D$, where D is an almost inner derivation of $\mathfrak{N}$. One says that φ is of continuous type if D is of continuous type.

If Γ is a lattice in N, then an automorphism $\varphi : N \to N$ is called Γ-almost inner if for every $\gamma \in \Gamma$ there exists an element $a = a(\gamma)$ such that $\varphi(\gamma) = a\gamma a^{-1}$ for all $\gamma \in \Gamma$. The set $AI_\Gamma(N)$ of Γ-almost inner automorphisms is also a Lie subgroup of $\mathrm{Aut}(N)$ that in general is larger than the group of inner automorphisms of N.

Theorem E ([GW], [Gord 2]) Let N be a simply connected, nilpotent Lie group with a left invariant metric. Let Γ be a lattice in N and let φ be a Γ-almost inner automorphism of N. Then $\Gamma\backslash N$ and $\varphi(\Gamma)\backslash N$ have the same Laplace spectrum on functions and p-forms, $1 \leq p \leq \dim N$.

The manifolds $\Gamma\backslash N$ and $\varphi(\Gamma)\backslash N$ are Laplace isospectral on functions and forms if φ is a Γ-almost inner automorphism of N, but they are not isometric unless φ is an inner automorphism. See Theorem 5.5 of [GW] for a more complete statement.

Rigidity of marked maximal length spectrum

Let N be a simply connected, nilpotent Lie group with a left invariant metric, and let Γ be a lattice in N. If φ is a Γ- almost inner automorphism of N, then it is easy to see that the periods of γ are the same as the periods of $\varphi(\gamma)$ for every $\gamma \in \Gamma$. It follows that $\Gamma\backslash N$ and $\varphi(\Gamma)\backslash N$ have the same marked length spectrum and marked maximal length spectrum. Let N and N^* be simply connected Lie groups with left invariant metrics and suppose that there exists a group isomorphism $\varphi : N \to N^*$ that is also an isometry. Then it follows that $\Gamma\backslash N$ and $\varphi(\Gamma)\backslash N^*$ also have the same marked length spectrum and marked maximal length spectrum. Conversely, if we restrict ourselves to 2-step compact nilmanifolds $\Gamma\backslash N$ and $\Gamma^*\backslash N^*$, then these are the only two ways that the marked maximal length spectrum can be the same. More precisely, we have the following result, which is Theorem 5.20 of [E1].

Theorem F Let Γ, Γ^* be lattices in simply connected, 2-step nilpotent Lie groups with left invariant metrics. Assume that $\Gamma\backslash N$ and $\Gamma^*\backslash N^*$ have the same marked maximal length spectrum, and let $\psi : \Gamma \to \Gamma^*$ be an isomorphism that induces this marking. Then $\psi = (\psi_1 \circ \psi_2)|_\Gamma$, where ψ_2 is a Γ-almost inner automorphism of N and ψ_1 is an automorphism of N onto N^* that is also an isometry. In particular $\Gamma\backslash N$ and $\Gamma^*\backslash N^*$ have the same marked length spectrum and the same spectrum of the Laplacian on functions and differential forms.

Remark The result above says that in some sense the marked maximal length spectrum determines not only the marked length spectrum but also the Laplace spectra on functions and differential forms. Proposition A says that the maximal length spectrum of $\Gamma\backslash N$ can be computed from $\log \Gamma$ by a simple explicit formula. It is interesting to ask if the Laplace spectra of $\Gamma\backslash N$ on functions and forms can also somehow be computed explicitly from $\log \Gamma$.

6.8f The problem of geodesic conjugacy

Let M_1 and M_2 be Riemannian manifolds with geodesic flows $\{\mathfrak{G}_1^t\}$ and $\{\mathfrak{G}_2^t\}$ on the unit tangent bundles SM_1 and SM_2. One says that M_1 and M_2 are C^k geodesically conjugate, where $k \geq 0$ is an integer, if there exists a C^k homeomorphism $f : SM_1 \to SM_2$ such that $f \circ \mathfrak{G}_1^t = \mathfrak{G}_2^t \circ f$ for all $t \in \mathbb{R}$. If $k \geq 1$, then this condition is equivalent to the condition $f_*(\mathfrak{G}_1) = \mathfrak{G}_2$, where $\mathfrak{G}_1$ and $\mathfrak{G}_2$ denote the geodesic vector fields on SM_1 and SM_2.

Clearly M_1 and M_2 must have the same dimension n if they are geodesically conjugate. If $n \geq 3$, then it is not hard to see that the C^k geodesic conjugacy $f : SM_1 \to SM_2$ induces an isomorphism T between the fundamental groups of M_1 and M_2 and $T_* : C(M_1) \to C(M_2)$ preserves the marked length spectrum.

It is an interesting question to ask if the geodesic flow determines the isometry type of a Riemannian manifold. More precisely, if M_1 and M_2 are Riemannian manifolds that are C^k geodesically conjugate, then must M_1 and M_2 be isometric ? The answer to this question is no in general, but is yes in some important special cases that include the following :

1) M_1 and M_2 are compact surfaces with nonpositive Gaussian curvature and $k = 0$ ([Cr], [O]).

2) M_1 and M_2 are compact manifolds of arbitrary dimension and nonpositive sectional curvature, rank $M_1 \geq 2$ and $k = 0$ ([CEK]).

3) M_1 is a compact, locally symmetric space with negative sectional curvature, M_2 is a compact manifold with negative sectional curvature and $k \geq 1$ ([BCG 1, 2]).

Geodesic conjugacy for compact 2-step nilmanifolds

We shall consider the geodesic conjugacy problem in the case that both M_1 and M_2 are compact 2-step nilmanifolds of dimension $n \geq 3$. As we noted above if there exists a C^k geodesic conjugacy $f : SM_1 \to SM_2$, then M_1 and M_2 have the same marked length spectrum and in particular the same marked maximal length spectrum. In view of the rigidity result for marked maximal length spectra,Theorem F of (6.8e), we may reduce to the case that there exists a simply connected, 2-step nilpotent Lie group N with a left invariant metric, a lattice Γ in N and a Γ-almost inner automorphism φ of N such that $M_1 = \Gamma \backslash N$ and $M_2 = \varphi(\Gamma) \backslash N$.

To show that M_1 and M_2 are isometric is equivalent to showing that the Γ-almost inner automorphism φ above is actually an inner automorphism. By Theorem 2.12 of [GoM 2] the automorphism φ must be an almost inner automorphism of continuous type, as defined in the discussion of (6.8e). Almost inner automorphisms of continuous type that are not inner automorphisms are rare for a 2-step nilpotent Lie algebra. In fact, they don't exist for a generic 2-step nilpotent Lie algebra. See [GoM 2] for a more detailed discussion.

If we impose the additional condition that the geodesic conjugacy $f : SM_1 \to SM_2$ be the restriction of a symplectic diffeomorphism $f : TM_1 \to TM_2$, then M_1 and M_2 are isometric. See [GMS] for a proof.

6.9 Totally geodesic submanifolds and subgroups

Gauss map for submanifolds of a connected Lie group

Let H be a connected Lie group with Lie algebra $\mathfrak{H}$, and let $G : TH = H \times \mathfrak{H} \to \mathfrak{H}$ be the Gauss map. For an integer k with $1 \leq k \leq n = \dim H$ let $G(k, \mathfrak{H})$ denote the Grassmann manifold of k-dimensional subspaces of $\mathfrak{H}$. If S is a k-dimensional submanifold of H, then we may define a Gauss map $G : S \to G(k, \mathfrak{H})$ by $G(x) = G(T_xS) = \{G(v) : v \in T_xS \subseteq TH\}$. We abuse notation slightly by using the letter G and the name Gauss map to denote also this extension of the original Gauss map. Note that $G : T_hH = \{h\} \times H$ is an isomorphism for each $h \in H$ so that $G(T_xS)$ is a k-dimensional subspace of $\mathfrak{H}$ for every $x \in S$.

If $H = \mathbb{R}^n$, regarded as a simply connected abelian Lie group, then this Gauss map is just the standard Gauss map used in classical differential geometry to study submanifolds S of Euclidean space. In this more general context the Gauss map $G : S \to G(k, \mathfrak{H})$ should also be useful for studying the geometry of S for any choice of left invariant metric $<,>$ on H. In fact, comparing the geometry of S for different choices of $<,>$ is an interesting problem for a given connected Lie group H. This problem is nonexistent in the case that $H = \mathbb{R}^n$ since all left invariant metrics $<,>$ in this case are flat and isometric to each other.

The image of geodesics under the Gauss map

In this article we consider only the case that S is a totally geodesic submanifold of H and $H = N$, a simply connected, 2-step nilpotent Lie group with a left invariant metric $<,>$. First we consider the case that S is 1-dimensional ; that is, S is a geodesic $\gamma : (a, b) \to N$. Write $\gamma'(a) = (n, A) = dL_n(A)$ for $A \in \mathfrak{N}$, and write $A = X + Z$, where $X \in \mathcal{V}$ and $Z \in \mathfrak{Z}$ as in (6.2a). In this case $T_tS = \gamma'(t)$ for $t \in (a, b)$ and by the proposition in (6.8a) $G(S)$ is the curve in $\mathfrak{N}$ given by $G(\gamma'(t)) = e^{tj(Z)}X$. Note that either $G : S \to G(k, \mathfrak{H})$ is nonsingular or G is a constant map and $j(Z)X = 0$. In the latter case $\gamma(t) = n \cdot \exp(tA)$, the left translate of a 1-parameter subgroup of N, by the geodesic equations of N as found, for example, in Proposition 3.5 of [E1]. Equivalently, if S is a geodesic of N, then either $G(S)$ is a nonsingular curve in $\mathfrak{N}$ or $G(S)$ is a point and S is the left translate of a totally geodesic 1-dimensional subgroup of N.

The image of totally geodesic submanifolds under the Gauss map

If $\mathfrak{N}$ is a nonsingular 2-step nilpotent Lie algebra, then the Gauss map exhibits similar behavior for totally geodesic submanifolds of N with dimension $k \geq 2$ ([E2]). We do not know if nonsingularity is a necessary hypothesis or merely a convenience for the proofs in [E2]. We note that if S is a k-dimensional totally geodesic submanifold of a Euclidean space $H = \mathbb{R}^n$, then the Gauss map $G : S \to G(k, \mathbb{R}^n)$ is a constant map.

The next two results are Theorem 4.2 and Corollary 5.6 of [E2].

Proposition A Let N be a simply connected, 2-step nilpotent Lie group of dimension $n \geq 3$ whose Lie algebra $\mathfrak{N}$ is nonsingular. Let $<,>$ be a left invariant metric on N and let S be a totally geodesic submanifold of N of dimension $k \geq 2$. Let

$G : S \to G(k, \mathfrak{N})$ denote the Gauss map. Then either G is nonsingular at every point of S or G is a constant map and S is an open subset of $L_n(\bar{N})$, where n is any point of S and $\bar{N}$ is a totally geodesic subgroup of N.

Proposition B Let N be a simply connected, 2-step nilpotent Lie group of dimension $n \geq 3$ whose Lie algebra $\mathfrak{N}$ is nonsingular. Let $<,>$ be a left invariant metric on N and let S be a totally geodesic submanifold of N of dimension $k > \dim \mathfrak{Z}$, where $\mathfrak{Z}$ is the center of $\mathfrak{N}$. Then S is an open subset of $L_n(\bar{N})$, where n is any point of S and $\bar{N}$ is a totally geodesic subgroup of N.

Remark Proposition B says that if the dimension of a totally geodesic submanifold S of N is sufficiently large in the nonsingular case, then S must be an open subset of a left translate of a totally geodesic subgroup of N. It would be interesting to understand more about the Gauss images $G(S)$ when $\dim S \leq \dim \mathfrak{Z}$. Can one find a complete description as satisfactory as in the 1-dimensional case that S is a geodesic ?

7. Solvable extensions by $\mathbb{R}$ and homogeneous spaces of negative curvature

7.1 The criterion of Heintze

E. Heintze ([Hei]) and D.V. Alekseevski ([A]) were the first to study systematically the geometry of a simply connected homogeneous space of strictly negative sectional curvature. We discuss the method of Heintze, which is closest to our own discussion. Heintze's approach was generalized by R. Azencott and E. Wilson in [AW 1, 2] to the study of simply connected homogeneous spaces with nonpositive sectional curvature.

It was known that every simply connected homogeneous space M of nonpositive sectional curvature has a simply transitive solvable group S of isometries. Hence M can be regarded as a solvable Lie group S with an appropriate left invariant metric. To see this fix a point x of M and let $x : S \to M$ be the diffeomorphism given by $x(s) = s(x)$ for all $s \in S$. If S is given the metric $<,>_x$ that makes $x : S \to M$ an isometry, then it is easy to see that $<,>_x$ is a left invariant metric on S.

Heintze in [Hei] derived necessary and sufficent conditions for a solvable Lie algebra S to admit an inner product $<,>$ such that the corresponding simply connected Lie group S with left invariant metric $<,>$ has strictly negative sectional curvature.

Theorem ([Hei]) Let S be a solvable Lie algebra, and let S denote the simply connected solvable Lie group with Lie algebra S. Then the following are equivalent :

1) S admits an inner product $<,>$ such that S with the left invariant metric $<,>$ has negative sectional curvature.

2) S has the following properties :

a) $\mathfrak{N} = [\mathsf{S}, \mathsf{S}]$ has codimension 1 in S.

b) $\mathfrak{S}$ admits a derivation D such that the restriction of D to $\mathfrak{N}$ has eigenvalues with positive real parts.

Remark If $\mathfrak{S}$ is a solvable Lie algebra, then its commutator subalgebra $\mathfrak{N} = [\mathfrak{S}, \mathfrak{S}]$ is always nilpotent.

7.2 Examples

Riemannian symmetric spaces of negative sectional curvature

Let M be a simply connected Riemannian symmetric space with nonpositive sectional curvature that has no Euclidean de Rham factor. Then $G = I_o(M)$ is a semisimple Lie group and admits an Iwasawa decomposition $G = KAN$, where K is a maximal compact subgroup of G, A is an abelian subgroup and N is a nilpotent subgroup (cf. [Hel, Chapter VI]). If $S = AN$, then S is a solvable subgroup that acts simply transitively on M. If $\mathfrak{S}$ and $\mathfrak{N}$ are the Lie algebras of S and N, then $\mathfrak{N} = [\mathfrak{S}, \mathfrak{S}]$.

The group A is 1-dimensional $\Leftrightarrow$ the sectional curvature of M is strictly negative. In this case $\mathfrak{N} = [\mathfrak{S}, \mathfrak{S}]$ is a 2-step nilpotent Lie algebra of Heisenberg type, modulo multiplying the metric on M by a positive constant. If A_o is a unit vector in $\mathfrak{S}$ orthogonal to $\mathfrak{N}$ and if $D = \text{ad}\, A_o$, then D is a derivation such that $D = 2c\, Id$ on $\mathfrak{Z}$ and $D = c\, Id$ on $\mathcal{V} = \mathfrak{Z}^{\perp}$ for some nonzero constant c. Replacing A_o by $-A_o$ if necessary we may assume that c is positive. If the original metric on M is multiplied by a suitable positive constant, then we may assume that $c = 1$. See [Hei] for details.

3-step Carnot solvmanifolds

We now work backwards with the structure result of Heintze and the remarks above to guide us. Let $\mathfrak{N}$ be a 2-step nilpotent Lie algebra with an inner product $<,>$, and define a derivation D on $\mathfrak{N}$ by $D = Id$ on $\mathcal{V} = \mathfrak{Z}^{\perp}$ and $D = 2\, Id$ on $\mathfrak{N}$. Let $\mathfrak{S} = \mathbb{R} \oplus \mathfrak{N}$ and define a bracket operation $[,]$ on $\mathfrak{S}$ such that ad $(t, 0) = tD$ for all $t \in \mathbb{R}$ and $[,]$ agrees on $\mathfrak{N}$ with the bracket of $\mathfrak{N}$. It is easy to check that $\mathfrak{S}$ is solvable and $[\mathfrak{S}, \mathfrak{S}] = \mathfrak{N}$. Now let $\mathfrak{S}$ be given the inner product $<,>^*$ such that $\mathbb{R}$ and $\mathfrak{N}$ are orthogonal, $<,>^* = <,>$ on $\mathfrak{N}$ and $(1, 0)$ has length 1. Let S be the simply connected Lie group with Lie algebra $\mathfrak{S}$ and let $<,>^*$ also denote the corresponding left invariant metric on S. We call $\{S, <,>^*\}$ a 3-step Carnot solvmanifold.

Question 1 Under what conditions on $\{\mathfrak{N}, <,>\}$ does the 3-step Carnot solvmanifold $\{S, <,>^*\}$ have negative sectional curvature ?

Question 2 Under what conditions on $\{\mathfrak{N}, <,>\}$ is the 3-step Carnot solvmanifold $\{S, <,>^*\}$ a Riemannian symmetric space with negative sectional curvature ?

A partial answer to the first question is given by the next result, which is Proposition 3.10 of [EH].

Proposition A Let $\{S, <,>^*\}$ be a 3-step Carnot solvmanifold. For each $Z \in \mathfrak{Z}$, the center of $\mathfrak{N}$, let $j(Z) : \mathcal{V} \to \mathcal{V}$ be the skew symmetric linear map of (6.2a). Let $\|j\| = \max \{|j(Z)X| : X \in \mathcal{V}, Z \in \mathfrak{Z}$ and $|X| = |Z| = 1\}$. Let K denote the sectional curvature of $\{S, <,>^*\}$. Then

1) If $K \leq -1$ or $K \geq -4$, then $\|j\| \leq 2$. Moreover, $|j(Z)X| = 2|Z||X|$ for

nonzero vectors $Z \in \mathfrak{z}$ and $X \in V \Leftrightarrow j(Z)^2 X = -4|Z|^2 X$.

2) If $\dim \mathfrak{z} = 1$ and $||j|| \leq 2$, then $-4 \leq K \leq -1$.

3) Let $\dim \mathfrak{z}$ be arbitrary.

a) If $||j|| \leq 1$, then $K \leq -1$.

b) If $||j|| \leq \sqrt{2}$, then $K \geq -4$

A partial answer to the second question is given by the next result, which is contained in Proposition 3 of [Hei]. The condition (*) below is called the J^2 condition in [CDKR] and the next result is also a different formulation of Theorem 1.1 of [CDKR]

Proposition B Let $\{S, <,>^*\}$ be a 3-step Carnot solvmanifold with strictly negative sectional curvature such that $j(Z)^2 = -4|Z|^2 Id$ for all $Z \in \mathfrak{z}$. Then $\{S, <,>^*\}$ is a symmetric space $\Leftrightarrow$ for every nonzero vector X of V and every orthonormal basis $\{Z_1, ..., Z_p\}$ of $\mathfrak{z}$ we have

(*) $j(Z_r)j(Z_s)X \in \text{span}\{j(Z_1)X, ..., j(Z_p)X\}$ whenever $r \neq s, 1 \leq r, s \leq p$.

Damek-Ricci spaces

In [DR] E. Damek and F. Ricci proved the following important result

Theorem C Let $\{S, <,>^*\}$ be a 3-step Carnot solvmanifold such that $j(Z)^2 = -4|Z|^2 Id$ for all $Z \in \mathfrak{z}$. Then

1) The sectional curvature K of $\{S, <,>^*\}$ is nonpositive.

2) $\{S, <,>^*\}$ is a harmonic Riemannian manifold.

Remark 1 The result above gives an infinite family of counterexamples to a conjecture of Lichnerowicz [Li] that every harmonic Riemannian manifold must be a locally symmetric space. See [BTV, pp.11-12] for various equivalent definitions of a harmonic Riemannian manifold. The condition of being harmonic implies that an infinite sequence of curvature conditions hold (the Ledger conditions) and the first of these is the condition of being Einstein. The n^{th} of these conditions is called n-stein, for reasons that are left as an exercise for the reader.

Remark 2 The mysterious factor of 4 that occurs in the statements of Proposition B and Theorem C is a normalization that is necessary for the metric on $\{S, <,>^*\}$ to be Einstein. We could have avoided this renormalization of the definition of Heisenberg type by requiring in the definition of 3-step Carnot solvmanifold that $D = \frac{1}{2}Id$ on $V = \mathfrak{z}^{\perp}$ and $D = Id$ on $\mathfrak{z}$.

Recall that 2-step nilpotent Lie algebras $\{\mathfrak{N}, <,>\}$ of Heisenberg type arise from representations of the Clifford algebra $C\ell(p), p \geq 1$, and the center $\mathfrak{z}$ of $\mathfrak{N}$ in this case has dimension p. The 3-step Carnot solvmanifold $\{S, <,>^*\}$ corresponding to a representation of $C\ell(p)$ is a symmetric space only when $p = 1, 3$ or 7. In all other cases the harmonic manifold $\{S, <,>^*\}$ is nonsymmetric.

Remark 3 If $\{S, <,>^*\}$ is a nonsymmetric Damek-Ricci example, then the sectional curvature of $\{S, <,>^*\}$ is always zero on some 2-plane. Hence a Damek-Ricci example $\{S, <,>^*\}$ is symmetric $\Leftrightarrow$ the sectional curvature is strictly negative. See [Do] and [Lan] for a proof.

Examples of Leukert

For an integer $n \geq 2$ let $\mathfrak{G}$ be a semisimple subalgebra of $so(n, \mathbb{R})$ such that $\mathbb{R}^n$ is irreducible as a $\mathfrak{G}$-module. Let $\mathfrak{N} = \mathbb{R}^n \oplus \mathfrak{G}$ be the 2-step nilpotent Lie algebra with inner product $<,>$ that was defined in example 4 of (6.1c). Let $\{S, <,>^*\}$ be the 3-step Carnot solvmanifold constructed from $\{N, <,>\}$ as above. It is an interesting question to ask when the curvature of $\{\mathfrak{N}, <,>\}$ is negative or nonpositive. Leukert obtained a partial answer in [Le] in the case that $\mathfrak{G}$ is a simple Lie algebra. However, Leukert used a different normalization of the inner product on $so(n, \mathbb{R})$ in the construction of example 4 of (6.1c); namely, one defines $< Z, Z^* >= -(1/4n)$ trace ZZ^* for all $Z, Z^* \in so(n, \mathbb{R})$. It should not be surprising that this normalization of the inner product on $so(n, \mathbb{R})$ is also chosen so that the 3-step Carnot solvmanifold $\{S, <,>^*\}$ will be an Einstein manifold. See Proposition 3.21A of [EH] for further details.

In the next result we use the notation of Proposition A.

Theorem D ([Le]) Let $\{S, <,>^*\}$ be a 3-step Carnot solvmanifold with Lie algebra $\mathfrak{S}$ such that $[\mathfrak{S}, \mathfrak{S}] = \mathfrak{N} = \mathbb{R}^n \oplus \mathfrak{G}$, where $\mathfrak{G}$ is a simple Lie subalgebra of $so(n, \mathbb{R})$ and $\mathbb{R}^n$ is an irreducible $\mathfrak{G}$ - module. Then

1) If S has sectional curvature $K \leq 0$, then then $\|j\|^2 \leq 8$.

2) There are at most 9 examples $\mathfrak{N} = \mathbb{R}^n \oplus \mathfrak{G}$ where $\|j\|^2 \leq 8$. In the classification of complex simple Lie algebras these correspond to the following cases :

a) $\mathfrak{A}_1$ highest weights $\omega_1, 2\omega_1, 3\omega_1, 4\omega_1$

b) $\mathfrak{A}_2$ highest weights ω_1, ω_2

c) $\mathfrak{C}_2$ highest weight ω_1 (standard representation)

d) $\mathfrak{G}_2$ highest weights $\omega_1, 2\omega_1$

We recall that that in this classification the Lie algebras $\mathfrak{A}_1, \mathfrak{A}_2$, and $\mathfrak{C}_2$ correspond to $su(2)$, $su(3)$ and $sp(4)$ respectively. The Lie algebra $\mathfrak{G}_2$ is one of the exceptional Lie algebras.

In addition to the result above Leukert also showed that the sectional curvatures of the first three examples of a) are strictly negative. The sectional curvatures of the fourth example of a) are nonpositive, but there exist 2-planes with zero sectional curvature. It has not yet been determined if the examples of b), c) and d) actually have nonpositive sectional curvature. However, Leukert's computer calculations of the sectional curvature on random 2-planes suggest that the sectional curvature is also nonpositive in cases b), c) and d).

Examples of Iwasawa type

Let $\mathfrak{N}$ be a 2-step nilpotent Lie algebra with an inner product $<,>$, and let D be a derivation on $\mathfrak{N}$ that is a positive definite symmetric linear operator. The example that occurs in the definition of a 3-step Carnot solvmanifold is clearly the most primitive example of this type. We now mimic that definition. Let $\mathfrak{S} = \mathbb{R} \oplus \mathfrak{N}$ and define a bracket operation $[,]$ on $\mathfrak{S}$ such that ad $(t, 0) = tD$ for all $t \in \mathbb{R}$ and $[,]$ agrees on $\mathfrak{N}$ with the bracket of $\mathfrak{N}$. Again, it is easy to check that $\mathfrak{S}$ is solvable and $[\mathfrak{S}, \mathfrak{S}] = \mathfrak{N}$. Now let $\mathfrak{S}$ be given the inner product $<,>^*$ such that $\mathbb{R}$ and $\mathfrak{N}$ are orthogonal, $<,>^* = <,>$ on $\mathfrak{N}$ and $(1, 0)$ has length 1. Let S be the simply connected Lie group with Lie algebra $\mathfrak{S}$ and let $<,>^*$ also denote the corresponding left invariant metric on S. One calls $\{S,<,>^*\}$ a 3-step solvmanifold of Iwasawa type with algebraic rank 1.

More generally, a simply connected solvable Lie group S with a left invariant metric $<,>^*$ is said to be of Iwasawa type if

1) The subspace $\mathfrak{A} = \mathfrak{N}^{\perp}$ is abelian, where $\mathfrak{N} = [\mathfrak{S}, \mathfrak{S}]$.
2) For all nonzero A in $\mathfrak{A}$, ad A is symmetric and nonzero.
3) For some nonzero A in $\mathfrak{A}$, ad A is positive definite on $\mathfrak{N}$.

The algebraic rank of $\{S,<,>^*\}$ in this case is defined to be the dimension of $\mathfrak{A}$. Note that one does not require that $\mathfrak{N} = [\mathfrak{S}, \mathfrak{S}]$ be 2-step nilpotent.

Among the solvmanifolds $\{S,<,>^*\}$ of Iwasawa type it is natural to look for conditions under which $\{S,<,>^*\}$ must be a Damek-Ricci example.

Theorem E Let $\{S,<,>^*\}$ be a 3-step solvmanifold of Iwasawa type with algebraic rank 1. Then $\{S,<,>^*\}$ is a Damek-Ricci example under any of the following conditions :

1) The first two Ledger conditions (Einstein and 2-stein) hold.
2) $\{S,<,>^*\}$ is a harmonic Riemannian manifold.
3) $\{S,<,>^*\}$ has nonpositive sectional curvature.

Assertion 1) is proved in [BPR] and also in [Dr] in the special case that $\{S,<,>^*\}$ is a 3-step Carnot solvmanifold. Assertion 2) is an immediate consequence of 1) and 3) follows from 1) and a result of J. Heber [Heb] that a solvmanifold of Iwasawa type with nonpositive sectional curvature must have rank 1.

Theorem F ([Dr]) Let $\{S,<,>^*\}$ be a harmonic solvmanifold of Iwasawa type such that $\mathfrak{N} = [\mathfrak{S}, \mathfrak{S}]$ is a nonsingular 2-step nilpotent Lie algebra. Then $\{S,<,>^*\}$ is a Damek-Ricci example.

8. Other topics in the left invariant geometry of Lie groups

There are many interesting topics in the geometry of Lie groups with a left invariant metric that we have not discussed in this article. Some of these that are closely related to topics discussed above include Gelfand pairs, commutative spaces, d'Atri spaces and weakly symmetric spaces. For definitions, results and references to the vast literature on these topics see [La 1, 2, 3]. For results on the Laplace spectrum of compact nilmanifolds see [GG], [Gorn 1-4] and the references cited in these papers.

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