

# An Expository Discussion on Singular Inequality Problems and Equilibrium Models in Economics<sup>1 2</sup>

D. Goeleven

Institut de Recherche en Mathématiques et Informatique Appliquées,  
Université de La Réunion  
Saint-Denis, Ile de La Réunion, France

## ABSTRACT

The aim of this paper is to provide a mathematical approach necessary for understanding and thorough analysis of some precise prices equilibrium models in economics.

## 1 Introduction

It is now well-known that the theory of complementarity systems has wide-ranging applications including problems of optimization and equilibrium for various economic models.

One of the model that appears generally in the applications consists in finding

$$x \in \mathbb{R}_+^N \quad (1.1)$$

such that

$$Mx + q \in \mathbb{R}_+^N \quad (1.2)$$

and

$$x^T(Mx + q) = 0, \quad (1.3)$$

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where  $q \in \mathbb{R}^N$  ( $N \in \mathbb{N}$ ) is a given vector and  $M \in \mathbb{R}^{N \times N}$  denotes a real matrix of order  $N$ .

It has been proved that Problem (1.1)-(1.3) constitutes a suitable setting in which several standard economic models can be framed. For example, if  $M$  denotes an excess supply matrix and  $x$  a vector price, then a solution to problem (1.1)-(1.3) corresponds to a free-disposal competitive equilibrium and the relation (1.3) means that the Walras law holds. Equilibrium problems in noncooperative  $N$ -persons games and Lindahl equilibrium models, invariant capital stock models have also been studied through the complementarity system (1.1)-(1.3) [4], [16]. These applications have originated the studies of complementarity systems for various classes of matrices. In particular, the problem consisting to find general conditions on the matrix  $M$  ensuring the solvability for each  $q \in \mathbb{R}^N$  of the system (1.1)-(1.3) has seen a strong development.

One says that  $M$  is a  $Q$ -matrix if problem (1.1)-(1.3) has at least one solution for each  $q \in \mathbb{R}^N$ . Note that it is easy to check that any  $Q$ -matrix satisfies the property

$$\ker M^T \cap \mathbb{R}_+^N = \{0\}. \quad (1.4)$$

Indeed, let  $z \in \ker M^T \cap \mathbb{R}_+^N$ . If for each  $q$  there exists  $u(q)$  satisfying (1.2) then we may write

$$0 \leq (Mu(q) + q)^T z$$

since  $z \geq 0$ . On the other hand  $z \in \ker M^T$  and thus  $(Mu(q))^T z = u(q)M^T z = 0$ , so that

$$0 \leq q^T z.$$

This last relation can be obtained for any  $q \in \mathbb{R}^N$  and thus necessarily  $z = 0$ . Several classes of matrices have been in this sense considered in the mathematical literature. For example, it is well-known that strictly copositive matrices and  $P$ -matrices are  $Q$ -matrices. Most of the fundamental results known in the theory of finite-dimensional linear complementarity problems are related to the theory of  $Q$ -matrices.

However, as we will see in this paper, the formulation of some concrete equilibrium models in economics lead to the study of complementarity problems involving singular matrices  $M$  that do not satisfy the property (1.4). On the other hand, the vector  $q$  involved in the models considered satisfies some properties that should be exploited together with those of the matrix  $M$  in order to obtain suitable results.

In this paper, we discuss such equilibrium models and we show that the panoply of mathematical tools recently developed in [6] by Goeleven, Stavroulakis, Salmon and Panagiotopoulos can be used in order to provide appropriate theoretical and numerical results.

The results in [6] have been developed so as to study a general class of inequality problems involving cocoercive matrices. A possibly singular matrix  $M$  is called cocoercive provided that there exists  $\sigma > 0$  such that

$$x^T M x \geq \sigma x^T M^T M x, \forall x \in \mathbb{R}^N.$$

Note that recent years have seen an arise of interest in cocoercive matrices [2], [3], [6], [13], [15].

In this expository paper, we will show that inequality problems involving cocoercive matrices are of particular interest in economics. More precisely, our aim is to provide a mathematical approach necessary for understanding and thorough analysis of some precise prices equilibrium models. We shall concentrate our mathematical discussion on the application so that variational inequalities theory can be seen as a whole.

We note that we have restricted this paper on some aspects of Variational Inequalities Theory. The theory has now been the subject of many works and can be extensively applied in a variety of Economics, Mechanics and Engineering fields. Various aspects of Variational Inequalities, Theory and Applications are discussed in the recent books [1], [5], [7] and [8].

## 2 Preliminaries

The complementarity model (1.1)-(1.3) is closely linked to some other formulations that are usually used as suitable intermediate step for the mathematical treatment of the former formulation.

In  $\mathbb{R}^N$  ( $N \in \mathbb{N} \setminus \{0\}$ ), one denotes by " $\leq$ " the ordering defined by the closed convex cone  $\mathbb{R}_+^N$ , that is

$$x \leq y \Leftrightarrow y - x \in \mathbb{R}_+^N$$

or equivalently

$$x \leq y \Leftrightarrow x_i \leq y_i, \forall i \in \{1, \dots, N\}.$$

The infimum vector  $\wedge\{x, y\}$  and the supremum vector  $\vee\{x, y\}$  are defined through the formulae

$$\wedge\{x, y\}_i = \min\{x_i, y_i\} \quad (i \in \{1, \dots, N\})$$

and

$$\vee\{x, y\}_i = \max\{x_i, y_i\} \quad (i \in \{1, \dots, N\}).$$

Using these notations, the problem (1.1)-(1.3) is equivalent to the nonlinear equation

$$\wedge\{x, Mx + q\} = 0. \quad (2.1)$$

Indeed, letting  $x, y$  be two vectors in  $\mathbb{R}^N$ , it is easy to check that

$$\wedge\{x, y\} = 0 \quad (2.2)$$

if and only if

$$x \geq 0, y \geq 0, x^T y = 0. \quad (2.3)$$

If (2.2) is satisfied then

$$x \geq \wedge\{x, y\} = 0,$$

$$y \geq \wedge\{x, y\} = 0$$

and moreover, if  $x_i > 0$  (resp.  $y_i > 0$ ) then necessarily  $y_i = 0$  (resp.  $x_i = 0$ ) since  $\wedge\{x, y\}_i = \min\{x_i, y_i\}$  and  $x_i, y_i \geq 0$ . It results that  $x^T y = 0$ . Reciprocally, (2.3) yields the relations

$$x_i > 0 \Rightarrow y_i = 0 \quad (i \in \{1, \dots, N\}),$$

$$y_i > 0 \Rightarrow x_i = 0 \quad (i \in \{1, \dots, N\}),$$

from which we deduce that

$$\wedge\{x, y\} = 0.$$

Another important equivalent formulation of Problem (1.1)-(1.3) consists in finding  $x \in \mathbb{R}_+^N$  such that

$$(Mx + q)^T(v - x) \geq 0, \forall v \in \mathbb{R}_+^N. \quad (2.4)$$

Indeed, let  $x$  be a solution of Problem (1.1)-(1.3). Then for  $v \in \mathbb{R}_+^N$ , (1.2) yields

$$(Mx + q)^T v \geq 0.$$

Using now (1.3) we see that

$$(Mx + q)^T(v - x) \geq 0.$$

This together with (1.1) entails that  $x$  solves Problem (2.4). Reciprocally, let  $x$  be a solution of Problem (2.4) and let  $h \in \mathbb{R}_+^N$  be given. Setting  $v = x + h$  in (2.4), we obtain  $(Mx + q)^T h \geq 0$ . This last relation holds for any  $h \in \mathbb{R}_+^N$ . It results that  $Mx + q \geq 0$  and  $(Mx + q)^T x \geq 0$ . Letting now  $v = 0$  in (2.4), we see that  $(Mx + q)^T x \leq 0$  and thus  $(Mx + q)^T x = 0$ . Both conditions (1.1)-(1.3) hold and the conclusion follows.

Problem (2.4) is called a "variational inequality problem" and it defines by itself a field of research in which various theoretical and numerical results have been developed.

The following result is a particular case of a general theorem stated in [6]. It will be the key of our further mathematical analysis.

**Theorem 2.1.** Let  $M$  be a nonzero cocoercive matrix with modulus  $\sigma > 0$ , i.e.

$$x^T M x \geq \sigma x^T M^T M x, \forall x \in \mathbb{R}^N. \quad (2.5)$$

If

$$q^T w > 0, \forall w \in \ker M \cap \mathbb{R}_+^N \setminus \{0\}, \quad (2.6)$$

then problem (2.4) has at least one solution. Moreover, if  $0 < \alpha < 2\sigma$  then the sequence  $\{x^n\}_{n \in \mathbb{N}}$  defined by the algorithm

$$x^0 \in \mathbb{R}^N,$$

$$x^{k+1} = \vee\{0, x^k - \alpha(Mx^k + q)\}$$

converges to a solution of Problem (2.4).

The inequality (2.5) can also be written as follows

$$x^T M x \geq \sigma \| M x \|^2, \forall x \in \mathbb{R}^N,$$

where  $\| \cdot \|$  denotes the euclidean norm in  $\mathbb{R}^N$ . It results that the matrix  $M$  is positive semidefinite and

$$\ker M = \{x \in \mathbb{R}^N : x^T M x = 0\}.$$

The class of cocoercive matrices has been the subject of several recent works [3], [6]. For example, any nonzero symmetric and positive semidefinite matrix is cocoercive with modulus  $\sigma = \| M \|^{\perp}$ , where

$$\| M \| = \max_{\lambda \in \text{sp}(M^T M)} \sqrt{\lambda},$$

with  $\text{sp}(M^T M)$  denoting the spectrum of the matrix  $M^T M$ .

If  $M$  and  $M^2$  are nonzero positive semidefinite then  $M$  is cocoercive with modulus  $\sigma = \frac{1}{2} \| M + M^T \|^{\perp}$ . General conditions ensuring that a normal positive semidefinite matrix is cocoercive are also given in [2] and [6].

If the matrix  $M$  is symmetric then Problem (2.4) is also equivalent to the optimal program

$$\min_{x \in \mathbb{R}_+^N} \frac{1}{2} x^T M x + q^T x. \quad (2.7)$$

**Remark 2.1.** i) If Problem (2.4) has a solution then

$$q^T w \geq 0, \forall w \in \ker M^T \cap \mathbb{R}_+^N.$$

Indeed, there exists  $x \in \mathbb{R}_+^N$  such that

$$(M x + q)^T (v - x) \geq 0, \forall v \in \mathbb{R}_+^N.$$

Let  $w \in \ker M^T \cap \mathbb{R}_+^N$  be fixed. Setting  $v = x + w$ , we get  $(M x + q)^T w \geq 0$ . Moreover  $(M x)^T w = x^T M^T w = 0$  and thus  $q^T w \geq 0$ .

ii) It results from remark i) that if  $M$  is a positive semidefinite matrix then for Problem (2.4) to have a solution, it is necessary that

$$q^T w \geq 0, \forall w \in \ker M \cap \mathbb{R}_+^N.$$

Indeed, if  $M$  is positive semidefinite then  $\ker M = \ker M^T$ .

### 3 A Mathematical Analysis of Prices Equilibrium Problems in Economics: A Symmetric Case.

Equilibrium prices of an economic model involving a good  $X$  and two countries can be described by the equilibrium relations

$$y_{1,S}(p_1) = y_{1,D}(p_1), \quad (3.1)$$

$$y_{2,S}(p_2) = y_{2,D}(p_2), \quad (3.2)$$

where  $y_{i,D}$ ,  $y_{i,S}$  and  $p_i$  denote respectively the quantity demanded, quantity supplied and price of good  $X$  in country  $i$  ( $i = 1, 2$ ).

The following discussion could be straightforwardly generalized to an economic model involving  $N$  countries ( $N \in \mathbb{N} \setminus \{0\}$ ). Here we prefer to choose  $N = 2$  in order to avoid an overloading of notations.

We suppose that the relationships that describe the variables  $y_{i,D}$  and  $y_{i,S}$  ( $i = 1, 2$ ) are linear. More precisely, we write

$$y_{i,D} = a_i - b_i p_i; \quad i = 1, 2 \quad (3.3)$$

$$y_{i,S} = c_i + d_i p_i; \quad i = 1, 2 \quad (3.4)$$

where

$$a_i > c_i, b_i > 0, d_i > 0; \quad i = 1, 2. \quad (3.5)$$

The equilibrium prices can be here simply computed by the formulae

$$\bar{p}_1 = \frac{a_1 - c_1}{b_1 + d_1} \quad (3.6)$$

and

$$\bar{p}_2 = \frac{a_2 - c_2}{b_2 + d_2} \quad (3.7)$$

Note that in many circumstances, the behavior of nonlinear models can be approximated by the behavior of a linear model like the one given in (3.1) and (3.2). Taylor's series are usually used to linearize economic models. We have indeed

$$y_{i,D}(p_i) \approx y_{i,D}(\bar{p}_i) + y'_{i,D}(\bar{p}_i)(p_i - \bar{p}_i); \quad i = 1, 2, \quad (3.8)$$

and

$$y_{i,S}(p_i) \approx y_{i,S}(\bar{p}_i) + y'_{i,S}(\bar{p}_i)(p_i - \bar{p}_i); \quad i = 1, 2, \quad (3.9)$$

provided that  $y_{i,D}$  and  $y_{i,S}$  are sufficiently smooth and the variables  $p_1$  and  $p_2$  are close to  $\bar{p}_1$  and  $\bar{p}_2$  respectively. However, the model in (3.1)-(3.2) cannot be used to describe the equilibrium prices of an economy allowing import-export of good  $X$  between the two mentioned countries. In this case, we write

$$y_{1,S}(p_1) + x_{21} = y_{1,D}(p_1) + x_{12} \quad (3.10)$$

and

$$y_{2,S}(p_2) + x_{12} = y_{2,D}(p_2) + x_{21} \quad (3.11)$$

where  $x_{ij}$  ( $i, j = 1, 2; i \neq j$ ) denotes the quantity of good  $X$  conveyed from country  $i$  to country  $j$ . Let us also denote by  $c_{ij}$  ( $i, j = 1, 2; i \neq j$ ) the cost of transporting from country  $i$  to country  $j$ . We can now complete the model (3.10)-(3.11) by writing some relationships between the import-export variables  $x_{12}, x_{21}$ , the prices  $p_1, p_2$  and the costs  $c_{12} > 0, c_{21} > 0$ .

The import-export variables are nonnegative and we may write

$$x_{12} \geq 0, \quad (3.12)$$

$$x_{21} \geq 0. \quad (3.13)$$

We express that the price of good  $X$  in country  $j$  must be lesser or equal than its price in country  $i$  majored by the cost of transporting from country  $i$  to country  $j$ , that are

$$p_1 + c_{12} \geq p_2 \quad (3.14)$$

$$p_2 + c_{21} \geq p_1. \quad (3.15)$$

Let us now express that import in country  $j$  from country  $i$  does not exist as soon as total expenses (price in country  $i$  + transport charges) exceed the home's price (in country  $j$ ), that are

$$p_1 + c_{12} > p_2 \Rightarrow x_{12} = 0, \quad (3.16)$$

$$p_2 + c_{21} > p_1 \Rightarrow x_{21} = 0. \quad (3.17)$$

Reciprocally, we express that the existence of import in country  $j$  from country  $i$  balance the expenses in both countries in the sense that

$$x_{12} > 0 \Rightarrow p_1 + c_{12} = p_2, \quad (3.18)$$

$$x_{21} > 0 \Rightarrow p_2 + c_{21} = p_1. \quad (3.19)$$

From (3.10) and (3.11), we deduce that

$$\begin{aligned} p_1 &= \frac{a_1 - c_1}{d_1 + b_1} + \frac{x_{12}}{d_1 + b_1} - \frac{x_{21}}{d_1 + b_1} \\ &= \bar{p}_1 + \frac{1}{d_1 + b_1}(x_{12} - x_{21}) \end{aligned} \quad (3.20)$$

and

$$\begin{aligned} p_2 &= \frac{a_2 - c_2}{d_2 + b_2} + \frac{x_{21}}{d_2 + b_2} - \frac{x_{12}}{d_2 + b_2} \\ &= \bar{p}_2 + \frac{1}{d_2 + b_2}(x_{21} - x_{12}). \end{aligned} \quad (3.21)$$

Thus

$$\begin{aligned} \begin{pmatrix} p_1 + c_{12} - p_2 \\ p_2 + c_{21} - p_1 \end{pmatrix} &= \begin{pmatrix} \frac{1}{d_1 + b_1} + \frac{1}{d_2 + b_2} \\ -1 \quad +1 \end{pmatrix} \begin{pmatrix} x_{12} \\ x_{21} \end{pmatrix} \\ &\quad + \begin{pmatrix} \bar{p}_1 + c_{12} - \bar{p}_2 \\ \bar{p}_2 + c_{21} - \bar{p}_1 \end{pmatrix}. \end{aligned}$$

Let us set

$$M = \begin{pmatrix} \frac{1}{d_1 + b_1} + \frac{1}{d_2 + b_2} \\ -1 \quad +1 \end{pmatrix} \quad (3.22)$$

and

$$q = \begin{pmatrix} \bar{p}_1 + c_{12} - \bar{p}_2 \\ \bar{p}_2 + c_{21} - \bar{p}_1 \end{pmatrix}. \quad (3.23)$$

Using the notations of Section 2, we may write the system (3.12)-(3.19) as

$$\wedge \left\{ \begin{pmatrix} p_1 + c_{12} - p_2 \\ p_2 + c_{21} - p_1 \end{pmatrix}, \begin{pmatrix} x_{12} \\ x_{21} \end{pmatrix} \right\} = 0. \quad (3.24)$$

This last model consists of a system of two nonlinear equations. Moreover the functions defining the relationships between the import-export variables, the prices and the costs are nonsmooth. It results that mathematical tools like Taylor's series cannot be used to linearize the model. The use of import-export variables introduce in consequence a substantial nonlinearity in the model. This nonlinearity must now be discussed by using appropriate mathematical tools.

Problem (3.24) consists to find  $x \in \mathbb{R}^2$  such that

$$\wedge \{Mx + q, x\} = 0 \quad (3.25)$$

with  $M$  and  $q$  as defined in (3.22) and (3.23) respectively. From Section 2, we know that problem (3.25) consists to find an import-export vector  $x \in \mathbb{R}^2$  satisfying the conditions

$$x \geq 0, \quad (3.26)$$

$$Mx + q \geq 0, \quad (3.27)$$

$$x^T(Mx + q) = 0. \quad (3.28)$$

We know also that this last problem is equivalent to find  $x \in \mathbb{R}_+^2$  such that

$$(Mx + q)^T(v - x) \geq 0, \forall v \in \mathbb{R}_+^2. \quad (3.29)$$

Here the matrix  $M$  in (3.24) is symmetric and positive semidefinite. Thus  $M$  is cocoercive with modulus  $\sigma = \|M\|^{-1}$ . We have also

$$\ker M = \left\{ \begin{pmatrix} \alpha \\ \alpha \end{pmatrix}; \alpha \in \mathbb{R} \right\}$$

and thus

$$\ker M \cap \mathbb{R}_+^2 \setminus \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} \alpha \\ \alpha \end{pmatrix}; \alpha > 0 \right\}.$$

Here

$$\|M\| = 2 \left( \frac{1}{d_1 + b_1} + \frac{1}{d_2 + b_2} \right)$$

and thus

$$\sigma = \frac{(d_1 + b_1)(d_2 + b_2)}{2(d_1 + b_1 + d_2 + b_2)}.$$



So, if  $w \in \ker M \cap \mathbb{R}_+^2 \setminus \{0\}$  then there exists  $\alpha > 0$  such that

$$w = \begin{pmatrix} \alpha \\ \alpha \end{pmatrix}$$

and

$$\begin{aligned} q^T w &= (\bar{p}_1 - \bar{p}_2 + c_{12})\alpha + (\bar{p}_2 - \bar{p}_1 + c_{21})\alpha \\ &= \alpha(c_{12} + c_{21}) > 0. \end{aligned}$$

The condition (2.6) in Theorem 2.1 is satisfied and the existence of at least one solution of Problem (3.29) follows.

Let us now check the uniqueness of the solution of problem (3.29). If we suppose by contradiction that there exists two distinct solutions  $x_1$  and  $x_2$  then

$$(Mx_1 + q)^T(v - x_1) \geq 0, \forall v \in \mathbb{R}_+^2$$

and

$$(Mx_2 + q)^T(v - x_2) \geq 0, \forall v \in \mathbb{R}_+^2.$$

In particular, we have

$$(Mx_1 + q)^T(x_1 - x_2) \leq 0,$$

$$(-Mx_2 - q)^T(x_1 - x_2) \leq 0,$$

so that

$$(M(x_1 - x_2))^T(x_1 - x_2) \leq 0.$$

It results that

$$(x_1 - x_2)^T M(x_1 - x_2) = 0$$

and thus

$$x_1 - x_2 \in \ker M.$$

Moreover, the symmetry of the matrix  $M$  entails that problem (3.29) is equivalent to problem (2.7) with  $N = 2$  and the data  $M$  and  $q$  as defined in (3.22) and (3.23) respectively. Thus

$$\frac{1}{2}x_1^T Mx_1 + q^T x_1 \leq \frac{1}{2}v^T Mv + q^T v, \forall v \in \mathbb{R}_+^2$$

and

$$\frac{1}{2}x_2^T Mx_2 + q^T x_2 \leq \frac{1}{2}v^T Mv + q^T v, \forall v \in \mathbb{R}_+^2.$$

It results that

$$\frac{1}{2}x_1^T Mx_1 + q^T x_1 = \frac{1}{2}x_2^T Mx_2 + q^T x_2.$$

On the other hand

$$\frac{1}{2}x_2^T Mx_2 = \frac{1}{2}(x_1 + w)^T M(x_1 + w)$$

for some  $w \in \ker M \setminus \{0\}$ . It results that

$$\frac{1}{2}x_2^T M x_2 = \frac{1}{2}x_1^T M x_1$$

and thus

$$q^T w = q^T (x_2 - x_1) = 0. \quad (3.30)$$

On the other hand, there exists  $\alpha \neq 0$  such that  $w = \begin{pmatrix} \alpha \\ \alpha \end{pmatrix}$  and thus

$$q^T w = \alpha(c_{12} + c_{21}) \neq 0,$$

which is a contradiction to (3.30).

The existence and uniqueness of the import-export vector determined by the non-linear model (3.24) entails the existence and uniqueness of the equilibrium prices through the formulae (3.20)-(3.21).

**Remark 3.1.** In the case of an economy involving  $N > 2$  countries, the solvability of the model can be proved by following the same arguments than above. If  $N > 2$  then the uniqueness of the import-export vector is not guaranteed. It is however possible to prove as above that the difference between two solutions belongs to the kernel of the matrix  $M$  involved in the model. This is in fact sufficient to yield the uniqueness of the corresponding equilibrium prices. For example, in the case of an economy involving three countries ( $N = 3$ ), the mathematical model (1.1)-(1.3) is involved with the following data

$$x = \begin{pmatrix} x_{12} \\ x_{21} \\ x_{13} \\ x_{31} \\ x_{23} \\ x_{32} \end{pmatrix},$$

$$M = \begin{pmatrix} \frac{1}{d_1+b_1} + \frac{1}{d_2+b_2} & + \frac{1}{d_1+b_1} & - \frac{1}{d_2+b_2} \\ + \frac{1}{d_1+b_1} & \frac{1}{d_1+b_1} + \frac{1}{d_3+b_3} & + \frac{1}{d_3+b_3} \\ - \frac{1}{d_2+b_2} & + \frac{1}{d_3+b_3} & \frac{1}{d_2+b_2} + \frac{1}{d_3+b_3} \end{pmatrix} \otimes \begin{pmatrix} +1 & -1 \\ -1 & +1 \end{pmatrix},$$

where  $\otimes$  denotes the Krönercker product, and

$$q = \begin{pmatrix} c_{12} + \bar{p}_1 - \bar{p}_2 \\ c_{21} + \bar{p}_2 - \bar{p}_1 \\ c_{13} + \bar{p}_1 - \bar{p}_3 \\ c_{31} + \bar{p}_3 - \bar{p}_1 \\ c_{23} + \bar{p}_2 - \bar{p}_3 \\ c_{32} + \bar{p}_3 - \bar{p}_2 \end{pmatrix}$$

where the variables and parameters are defined in a similar way that it has been done in the case  $N = 2$ . Here

$$\ker M = \{x \in \mathbb{R}^N : x_1 - x_2 = x_4 - x_3 = x_5 - x_6\}.$$

So, if  $x^1$  and  $x^2$  denote two import-export vectors solution of problem (1.1)-(1.3) then

$$(x_{12}^1 - x_{12}^2) - (x_{21}^1 - x_{21}^2) = (x_{31}^1 - x_{31}^2) - (x_{13}^1 - x_{13}^2) = (x_{23}^1 - x_{23}^2) - (x_{32}^1 - x_{32}^2).$$

The equilibrium prices are given by the formulae

$$p_1 = \frac{x_{12} + x_{13} - x_{21} - x_{31}}{d_1 + b_1} + \bar{p}_1$$

$$p_2 = \frac{x_{21} + x_{23} - x_{31} - x_{32}}{d_2 + b_2} + \bar{p}_2$$

$$p_3 = \frac{x_{31} + x_{32} - x_{13} - x_{23}}{d_3 + b_3} + \bar{p}_3.$$

We see that the equilibrium prices are uniquely determined. For example, setting  $p_{1,1} = p_1(x^1)$  and  $p_{1,2} = p_1(x^2)$ , we see that

$$\begin{aligned} p_{1,1} &= \frac{x_{12}^1 + x_{13}^1 - x_{21}^1 - x_{31}^1}{d_1 + b_1} + \bar{p}_1 \\ &= \frac{(x_{12}^2 - x_{21}^2) + (x_{13}^2 - x_{31}^2)}{d_1 + b_1} + \bar{p}_1 \\ &= p_{1,2}. \end{aligned}$$

So, the possible multiplicity of the import-export vector is not an obstacle to the uniqueness of the equilibrium prices.

We may now give a more precise characterization of the solution of our model.

**Case 1.** One supposes that

$$\bar{p}_2 \geq \bar{p}_1.$$

Then

$$\bar{p}_1 - \bar{p}_2 \leq 0 < c_{21}.$$

If

$$c_{12} \geq \bar{p}_2 - \bar{p}_1 \tag{3.31}$$

then

$$q = \begin{pmatrix} c_{12} - (\bar{p}_2 - \bar{p}_1) \\ c_{21} - (\bar{p}_1 - \bar{p}_2) \end{pmatrix} \geq 0,$$

and in this case

$$\bar{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

is the unique solution of the problem. Indeed,  $\bar{x} \geq 0$  and  $(M\bar{x} + q)^T(v - \bar{x}) = q^T v \geq 0, \forall v \in \mathbb{R}_+^2$ . It results that the inequality defined in problem (3.29) is satisfied. In

this case, import and export are not involved and the equilibrium prices are given by the formulae (3.6) and (3.7) (or also formulae (3.20) and (3.21) with  $x_{12} = x_{21} = 0$ ).

If

$$c_{12} < \bar{p}_2 - \bar{p}_1 \quad (\bar{p}_2 > \bar{p}_1) \quad (3.33)$$

then we show that the unique solution of the problem is given by

$$\bar{x} = \begin{pmatrix} \frac{\bar{p}_2 - \bar{p}_1 - c_{12}}{d_1 + b_1 + d_2 + b_2} \\ 0 \end{pmatrix}.$$

We have indeed

$$\begin{aligned} \bar{x} &\geq 0, \\ M\bar{x} + q &= \begin{pmatrix} +\left(\frac{1}{d_1 + b_1} + \frac{1}{d_2 + b_2}\right)x_{12} + \bar{p}_1 - \bar{p}_2 + c_{12} \\ -\left(\frac{1}{d_1 + b_1} + \frac{1}{d_2 + b_2}\right)x_{12} + \bar{p}_2 - \bar{p}_1 + c_{21} \end{pmatrix} \\ &= \begin{pmatrix} \bar{p}_2 - \bar{p}_1 - c_{12} + \bar{p}_1 - \bar{p}_2 + c_{12} \\ -\bar{p}_2 + \bar{p}_1 + c_{12} + \bar{p}_2 - \bar{p}_1 + c_{21} \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ c_{12} + c_{21} \end{pmatrix} \geq 0 \end{aligned}$$

and

$$\bar{x}^T(M\bar{x} + q) = x_{12} \times 0 + 0 \times (c_{12} + c_{21}) = 0.$$

If  $\bar{p}_2 > \bar{p}_1$  then there exists indeed a tendency to import in country 2 from country 1. Import occurs provided that the cost of transporting  $c_{12}$  satisfies the inequality  $c_{12} < \bar{p}_2 - \bar{p}_1$ . The equilibrium prices are given through formulae (3.20) and (3.21) by

$$\begin{aligned} p_1 &= \bar{p}_1 + \frac{x_{12}}{d_1 + b_1} \\ &= \left(1 - \frac{d_2 + b_2}{d_1 + b_1 + d_2 + b_2}\right)\bar{p}_1 + \frac{d_2 + b_2}{d_1 + b_1 + d_2 + b_2}\bar{p}_2 \\ &\quad - \frac{d_2 + b_2}{d_1 + b_1 + d_2 + b_2}c_{12} \end{aligned} \quad (3.34)$$

and

$$\begin{aligned} p_2 &= \bar{p}_2 - \frac{x_{12}}{d_2 + b_2} \\ &= \left(1 - \frac{d_1 + b_1}{d_1 + b_1 + d_2 + b_2}\right)\bar{p}_2 + \frac{d_1 + b_1}{d_1 + b_1 + d_2 + b_2}\bar{p}_1 \\ &\quad + \frac{d_1 + b_1}{d_1 + b_1 + d_2 + b_2}c_{12}. \end{aligned} \quad (3.35)$$

By using the fact that  $c_{12} < \bar{p}_2 - \bar{p}_1$  in (3.34) and (3.35), we see that  $p_1 > \bar{p}_1$  and  $\bar{p}_2 > p_2$ . Thus

$$\bar{p}_2 > \bar{p}_1, p_1 > \bar{p}_1, p_2 < \bar{p}_2. \quad (3.36)$$

Denoting  $\Delta^i p := p_i - \bar{p}_i$ , we see that if  $\bar{p}_2 > \bar{p}_1$  then  $\Delta^1 p > 0$  and  $\Delta^2 p < 0$ .

**Case 2.** One supposes that

$$\bar{p}_1 \geq \bar{p}_2.$$

Then

$$c_{12} > 0 \geq \bar{p}_2 - \bar{p}_1.$$

If

$$c_{21} \geq \bar{p}_1 - \bar{p}_2$$

then

$$\bar{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

is the unique solution of the problem since in this case

$$q \geq 0.$$

If

$$c_{21} < \bar{p}_1 - \bar{p}_2 \quad (\bar{p}_1 > \bar{p}_2) \quad (3.37)$$

then the unique solution of the problem is given by

$$\bar{x} = \begin{pmatrix} 0 \\ \frac{\bar{p}_1 - \bar{p}_2 - c_{21}}{d_1 + b_1 + d_2 + b_2} \end{pmatrix}.$$

Indeed,

$$\bar{x} \geq 0,$$

$$\begin{aligned} M\bar{x} + q &= \begin{pmatrix} -\left(\frac{1}{d_1 + b_1} + \frac{1}{d_2 + b_2}\right)x_{21} + \bar{p}_1 - \bar{p}_2 + c_{12} \\ +\left(\frac{1}{d_1 + b_1} + \frac{1}{d_2 + b_2}\right)x_{21} + \bar{p}_2 - \bar{p}_1 + c_{21} \end{pmatrix} \\ &= \begin{pmatrix} \bar{p}_2 - \bar{p}_1 + c_{21} + \bar{p}_1 - \bar{p}_2 + c_{12} \\ \bar{p}_1 - \bar{p}_2 - c_{21} + \bar{p}_2 - \bar{p}_1 + c_{21} \end{pmatrix} \\ &= \begin{pmatrix} c_{12} + c_{21} \\ 0 \end{pmatrix} \geq 0 \end{aligned}$$

and

$$\bar{x}^T(M\bar{x} + q) = 0 \times (c_{12} + c_{21}) + x_{21} \times 0 = 0.$$

The corresponding equilibrium prices are

$$p_1 = \left(1 - \frac{d_2 + b_2}{d_1 + b_1 + d_2 + b_2}\right)\bar{p}_1 + \frac{d_2 + b_2}{d_1 + b_1 + d_2 + b_2}\bar{p}_2$$

$$+ \frac{d_2 + b_2}{d_1 + b_1 + d_2 + b_2} c_{21} \quad (3.38)$$

and

$$p_2 = \left(1 - \frac{d_1 + b_1}{d_1 + b_1 + d_2 + b_2}\right) \bar{p}_2 + \frac{d_1 + b_1}{d_1 + b_1 + d_2 + b_2} \bar{p}_1 - \frac{d_1 + b_1}{d_1 + b_1 + d_2 + b_2} c_{21}. \quad (3.39)$$

We have  $\bar{p}_1 > p_1$  and  $p_2 > \bar{p}_2$  and thus  $\Delta^1 p < 0$  and  $\Delta^2 p > 0$ .

#### 4 A Mathematical Analysis of Prices Equilibrium Problems in Economics: A Nonsymmetric Case.

Let us now discuss the case of an economic model involving two goods  $X$  and  $Y$  whose offers and demands are relied in countries 1 and 2 by the relationships

$$y_{i,D}^X = a_i^X - b_i^X p_i^X + \beta_i^X p_i^Y; \quad i = 1, 2 \quad (4.1)$$

$$y_{i,S}^X = c_i^X + d_i^X p_i^X - \delta_i^X p_i^Y; \quad i = 1, 2 \quad (4.2)$$

$$y_{i,D}^Y = a_i^Y - b_i^Y p_i^Y + \beta_i^Y p_i^X; \quad i = 1, 2 \quad (4.3)$$

$$y_{i,S}^Y = c_i^Y + d_i^Y p_i^Y - \delta_i^Y p_i^X; \quad i = 1, 2 \quad (4.4)$$

where

$$a_i^X > c_i^X, b_i^X > 0, d_i^X > 0, \beta_i^X > 0, \delta_i^X \geq 0; \quad i = 1, 2,$$

$$a_i^Y > c_i^Y, b_i^Y > 0, d_i^Y > 0, \beta_i^Y > 0, \delta_i^Y \geq 0 \quad i = 1, 2.$$

The data and variables used in this section have the same meaning than in Section 3. The indexes  $X$  and  $Y$  are used to distinguish the data and variables relative to the good  $X$  from the ones relative to the good  $Y$ .

For an economy without import and export, the equilibrium relations

$$y_{i,S}^X(p_i^X, p_i^Y) = y_{i,D}^X(p_i^X, p_i^Y), \quad (4.5)$$

$$y_{i,S}^Y(p_i^X, p_i^Y) = y_{i,D}^Y(p_i^X, p_i^Y) \quad (4.6)$$

lead to the system

$$\begin{pmatrix} d_1^X + b_1^X & 0 & -(\delta_1^X + \beta_1^X) & 0 \\ 0 & d_2^X + b_2^X & 0 & -(\delta_2^X + \beta_2^X) \\ -(\delta_1^Y + \beta_1^Y) & 0 & d_1^Y + b_1^Y & 0 \\ 0 & -(\delta_2^Y + \beta_2^Y) & 0 & d_2^Y + b_2^Y \end{pmatrix} \begin{pmatrix} p_1^X \\ p_2^X \\ p_1^Y \\ p_2^Y \end{pmatrix} = \begin{pmatrix} a_1^X - c_1^X \\ a_2^X - c_2^X \\ a_1^Y - c_1^Y \\ a_2^Y - c_2^Y \end{pmatrix}.$$

It results that the equilibrium prices are given by

$$\bar{p}_1^X = \frac{(d_1^Y + b_1^Y)(a_1^X - c_1^X) + (\delta_1^X + \beta_1^X)(a_1^Y - c_1^Y)}{(d_1^X + b_1^X)(d_1^Y + b_1^Y) - (\delta_1^X + \beta_1^X)(\delta_1^Y + \beta_1^Y)},$$

$$\bar{p}_2^X = \frac{(d_2^Y + b_2^Y)(a_2^X - c_2^X) + (\delta_2^X + \beta_2^X)(a_2^Y - c_2^Y)}{(d_2^X + b_2^X)(d_2^Y + b_2^Y) - (\delta_2^X + \beta_2^X)(\delta_2^Y + \beta_2^Y)},$$

$$\bar{p}_1^Y = \frac{(\delta_1^Y + \beta_1^Y)(a_1^X - c_1^X) + (d_1^X + b_1^X)(a_1^Y - c_1^Y)}{(d_1^X + b_1^X)(d_1^Y + b_1^Y) - (\delta_1^X + \beta_1^X)(\delta_1^Y + \beta_1^Y)},$$

$$\bar{p}_2^Y = \frac{(\delta_2^Y + \beta_2^Y)(a_2^X - c_2^X) + (d_2^X + b_2^X)(a_2^Y - c_2^Y)}{(d_2^X + b_2^X)(d_2^Y + b_2^Y) - (\delta_2^X + \beta_2^X)(\delta_2^Y + \beta_2^Y)}.$$

It is now necessary to assume that

$$\Delta_i := (d_i^X + b_i^X)(d_i^Y + b_i^Y) - (\delta_i^X + \beta_i^X)(\delta_i^Y + \beta_i^Y) > 0$$

to have

$$\bar{p}_i^X \geq 0, \bar{p}_i^Y \geq 0 \quad (i = 1, 2).$$

If import-export of the goods is tolerated then the previous model must be completed by the relationships

$$x_{12}^X \geq 0, x_{21}^X \geq 0, x_{12}^Y \geq 0, x_{21}^Y \geq 0 \quad (4.7)$$

$$p_1^X + c_{12}^X - p_2^X \geq 0 \quad (4.8)$$

$$p_2^X + c_{21}^X - p_1^X \geq 0 \quad (4.9)$$

$$p_1^Y + c_{12}^Y - p_2^Y \geq 0 \quad (4.10)$$

$$p_2^Y + c_{21}^Y - p_1^Y \geq 0 \quad (4.11)$$

$$x_{12}^X > 0 \Rightarrow p_1^X + c_{12}^X - p_2^X = 0, \quad (4.12)$$

$$x_{21}^X > 0 \Rightarrow p_2^X + c_{21}^X - p_1^X = 0, \quad (4.13)$$

$$x_{12}^Y > 0 \Rightarrow p_1^Y + c_{12}^Y - p_2^Y = 0, \quad (4.14)$$

$$x_{21}^Y > 0 \Rightarrow p_2^Y + c_{21}^Y - p_1^Y = 0, \quad (4.15)$$

$$p_1^X + c_{12}^X - p_2^X > 0 \Rightarrow x_{12}^X = 0, \quad (4.16)$$

$$p_2^X + c_{21}^X - p_1^X > 0 \Rightarrow x_{21}^X = 0, \quad (4.17)$$

$$p_1^Y + c_{12}^Y - p_2^Y > 0 \Rightarrow x_{12}^Y = 0, \quad (4.18)$$

$$p_2^Y + c_{21}^Y - p_1^Y > 0 \Rightarrow x_{21}^Y = 0, \quad (4.19)$$

together with the equilibrium equations

$$y_{1,S}^X + x_{21}^X = y_{1,D}^X + x_{12}^X \quad (4.20)$$

$$y_{2,S}^X + x_{12}^X = y_{2,D}^X + x_{21}^X \quad (4.21)$$

$$y_{1,S}^Y + x_{21}^Y = y_{1,D}^Y + x_{12}^Y \quad (4.22)$$

$$y_{2,S}^Y + x_{12}^Y = y_{2,D}^Y + x_{21}^Y. \quad (4.23)$$

From (4.20)-(4.23), we deduce that

$$\begin{pmatrix} p_1^X \\ p_2^X \\ p_1^Y \\ p_2^Y \end{pmatrix} = \begin{pmatrix} \frac{d_1^Y + b_1^Y}{\Delta_1} & 0 & \frac{\delta_1^X + \beta_1^X}{\Delta_1} & 0 \\ 0 & \frac{d_2^Y + b_2^Y}{\Delta_2} & 0 & \frac{\delta_2^X + \beta_2^X}{\Delta_2} \\ \frac{\delta_1^Y + \beta_1^Y}{\Delta_1} & 0 & \frac{d_1^X + b_1^X}{\Delta_1} & 0 \\ 0 & \frac{\delta_2^Y + \beta_2^Y}{\Delta_2} & 0 & \frac{d_2^X + b_2^X}{\Delta_2} \end{pmatrix} \begin{pmatrix} a_1^X - c_1^X + x_{12}^X - x_{21}^X \\ a_2^X - c_2^X + x_{21}^X - x_{12}^X \\ a_1^Y - c_1^Y + x_{12}^Y - x_{21}^Y \\ a_2^Y - c_2^Y + x_{21}^Y - x_{12}^Y \end{pmatrix} \quad (4.24)$$

and as above, we formulate our problem as the variational inequality problem: Find  $x \in \mathbb{R}_+^4$  such that

$$(Mx + q)^T(v - x) \geq 0, \forall v \in \mathbb{R}_+^4, \quad (4.25)$$

where

$$\begin{aligned} M &= \begin{pmatrix} +a & -a & +b & -b \\ -a & +a & -b & +b \\ +c & -c & +d & -d \\ -c & +c & -d & +d \end{pmatrix} \\ &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} +1 & -1 \\ -1 & +1 \end{pmatrix} \end{aligned} \quad (4.26)$$

with

$$a := \frac{d_1^Y + b_1^Y}{\Delta_1} + \frac{d_2^Y + b_2^Y}{\Delta_2} \quad (4.27)$$

$$b := \frac{\delta_1^X + \beta_1^X}{\Delta_1} + \frac{\delta_2^Y + \beta_2^Y}{\Delta_2} \quad (4.28)$$

$$c := \frac{\delta_1^Y + \beta_1^Y}{\Delta_1} + \frac{\delta_2^X + \beta_2^X}{\Delta_2} \quad (4.29)$$

$$d := \frac{d_1^X + b_1^X}{\Delta_1} + \frac{d_2^X + b_2^X}{\Delta_2} \quad (4.30)$$

and

$$q = \begin{pmatrix} \bar{p}_1^X - \bar{p}_2^X + c_{12}^X \\ \bar{p}_2^X - \bar{p}_1^X + c_{21}^X \\ \bar{p}_1^Y - \bar{p}_2^Y + c_{12}^Y \\ \bar{p}_2^Y - \bar{p}_1^Y + c_{21}^Y \end{pmatrix}. \quad (4.31)$$

The matrix  $M$  is not necessarily positive semidefinite. However, if the matrix

$$\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is positive semidefinite then the matrix

$$M = \Lambda \otimes \begin{pmatrix} +1 & -1 \\ -1 & +1 \end{pmatrix}$$



is also positive semidefinite. We have

$$\begin{aligned}x^T \Lambda x &= ax_1^2 + (b+c)x_1x_2 + dx_2^2 \\ &\geq a|x_1|^2 - (b+c)|x_1||x_2| + d|x_2|^2.\end{aligned}$$

It results that if

$$b+c \leq 2\sqrt{ad}$$

then

$$x^T \Lambda x \geq (\sqrt{a}|x_1| - \sqrt{d}|x_2|)^2.$$

Thus the matrices  $\Lambda$  and  $M$  are positive semidefinite.

Let us now characterize the kernel of the matrix  $M$ . The system  $Mx = 0$  reads here

$$\begin{aligned}a(x_1 - x_2) &= b(x_4 - x_3) \\ c(x_1 - x_2) &= d(x_4 - x_3).\end{aligned}$$

It results that if

$$bc \neq ad$$

then

$$\ker M = \left\{ \begin{pmatrix} \alpha \\ \alpha \\ \beta \\ \beta \end{pmatrix}; \alpha, \beta \in \mathbb{R} \right\}. \quad (4.32)$$

Note now that if the condition

$$b+c < 2\sqrt{ad}, \quad (4.33)$$

ensuring that  $M$  is positive semidefinite holds, then  $4ad > (b+c)^2 \geq 4bc$  and thus  $ad > bc$ . Let us now give a condition ensuring the cocoercivity of the matrix  $M$ . We know that if  $M$  and  $M^2$  are positive semidefinite then  $M$  is cocoercive with modulus  $\sigma = \|M + M^T\|^{-1}$  [6]. We have

$$M^2 = \begin{pmatrix} 2a^2 + 2bc & 2ab + 2bd \\ 2ac + 2dc & 2bc + 2d^2 \end{pmatrix} \otimes \begin{pmatrix} +1 & -1 \\ -1 & +1 \end{pmatrix}.$$

It results that if  $(a+d)(b+c) \leq 2\sqrt{(a^2+bc)(d^2+bc)}$  then  $M^2$  is positive semidefinite. It is easy to check that both this last condition and (4.33) are satisfied provided that

$$b+c < 2 \min\left\{\sqrt{ad}, \frac{ad+bc}{a+d}\right\}. \quad (4.34)$$

So, we suppose that condition (4.34) is satisfied. Then any  $w \in \ker M \cap \mathbb{R}_+^4 \setminus \{0\}$  can be written as

$$w = \begin{pmatrix} \alpha \\ \alpha \\ \beta \\ \beta \end{pmatrix}$$

for some  $\alpha, \beta \geq 0$ ,  $\alpha + \beta \neq 0$ . A simple computation gives

$$q^T w = \alpha(c_{12}^X + c_{21}^X) + \beta(c_{12}^Y + c_{21}^Y) > 0.$$

The condition (2.6) in Theorem 2.1 is satisfied and the existence of at least one solution follows. If  $\bar{x}$  and  $\tilde{x}$  are two import-export vectors, solution of the model, then  $(\bar{x} - \tilde{x})^T M(\bar{x} - \tilde{x}) = 0$  and thus

$$\bar{x} - \tilde{x} \in \ker M$$

since  $M$  is cocoercive. Thus

$$\bar{x}_{12}^X - \tilde{x}_{12}^X = \bar{x}_{21}^X - \tilde{x}_{21}^X$$

and

$$\bar{x}_{12}^Y - \tilde{x}_{12}^Y = \bar{x}_{21}^Y - \tilde{x}_{21}^Y$$

or also

$$\bar{x}_{12}^X - \tilde{x}_{21}^X = \bar{x}_{12}^Y - \tilde{x}_{21}^Y \quad (4.35)$$

and

$$\bar{x}_{12}^Y - \tilde{x}_{21}^Y = \bar{x}_{12}^X - \tilde{x}_{21}^X. \quad (4.36)$$

The relationships (4.35) and (4.36) entail the uniqueness of the equilibrium prices which are determined through formula (4.24). So, multiplicity of the possible import-export variables does not affect the uniqueness of the equilibrium prices.

As in Section 3, we can use the model so as to deduce a precise information on the import-export variables and equilibrium prices.

If for example,

$$\begin{aligned} \bar{p}_2^X &> \bar{p}_1^X, \bar{p}_1^Y > \bar{p}_2^Y, \\ c_{12}^X &< \bar{p}_2^X - \bar{p}_1^X, c_{21}^Y < \bar{p}_1^Y - \bar{p}_2^Y \end{aligned}$$

then

$$c_{21}^X + \bar{p}_2^X - \bar{p}_1^X > 0, c_{12}^Y + \bar{p}_1^Y - \bar{p}_2^Y > 0$$

and a solution of the problem is given by

$$\bar{x}_{21}^X = 0, \bar{x}_{12}^Y = 0$$

and

$$\begin{pmatrix} +a & -b \\ -c & +d \end{pmatrix} \begin{pmatrix} \bar{x}_{12}^X \\ \bar{x}_{21}^Y \end{pmatrix} = \begin{pmatrix} \bar{p}_2^X - \bar{p}_1^X - c_{12}^X \\ \bar{p}_1^Y - \bar{p}_2^Y - c_{21}^Y \end{pmatrix}. \quad (4.37)$$

Note that here

$$\begin{pmatrix} +a & -b \\ -c & +d \end{pmatrix} = ad - bc > 0.$$

We get

$$\begin{pmatrix} \bar{x}_{12}^X \\ \bar{x}_{21}^Y \end{pmatrix} = \frac{1}{ad-bc} \begin{pmatrix} d(\bar{p}_2^X - \bar{p}_1^X) + b(\bar{p}_1^Y - \bar{p}_2^Y) - dc_{12}^X - bc_{21}^Y \\ c(\bar{p}_2^X - \bar{p}_1^X) + a(\bar{p}_1^Y - \bar{p}_2^Y) - cc_{12}^X - ac_{21}^Y \end{pmatrix}.$$

The corresponding equilibrium prices can be determined through formula (4.24). For example, we obtain

$$\begin{aligned} p_1^X &= \bar{p}_1^X \left[ 1 - \frac{(d_1^Y + b_1^Y)d - (\delta_1^X + \beta_1^X)c}{\Delta_1(ad-bc)} \right] + \bar{p}_2^X \left[ \frac{(d_1^Y + b_1^Y)d - (\delta_1^X + \beta_1^X)c}{\Delta_1(ad-bc)} \right] \\ &+ \bar{p}_1^Y \left[ \frac{(d_1^Y + b_1^Y)b - (\delta_1^X + \beta_1^X)a}{\Delta_1(ad-bc)} \right] + \bar{p}_2^Y \left[ \frac{-(d_1^Y + b_1^Y)b + (\delta_1^X + \beta_1^X)a}{\Delta_1(ad-bc)} \right] \\ &- c_{12}^X \left[ \frac{(d_1^Y + b_1^Y)d - (\delta_1^X + \beta_1^X)c}{\Delta_1(ad-bc)} \right] - c_{21}^Y \left[ \frac{(d_1^Y + b_1^Y)b - (\delta_1^X + \beta_1^X)a}{\Delta_1(ad-bc)} \right]. \end{aligned}$$

We see that a simple improvement of the model considered in Section 3 increases seriously the difficulty of the analytical treatment.

## 5 Numerical Applications

Analytical treatment of complementarity models being too heavy for problems involving a great number of data, the algorithm stated in Theorem 2.1 is of great interest.

**Example 5.1.** Let us consider the model given in Section 3 with the data:

$a_1 = 10$	$a_2 = 15$	$c_{12} = 4$
$b_1 = 1$	$b_2 = 0.5$	$c_{21} = 4$
$c_1 = 0$	$c_2 = 1$	
$d_1 = 0.6$	$d_2 = 0.8$	

We have

$$\bar{p}_1 \approx 6.25, \bar{p}_2 \approx 10.77$$

$$M \approx \begin{pmatrix} +1.32 & -1.32 \\ -1.32 & +1.32 \end{pmatrix}, \quad q \approx \begin{pmatrix} -0.52 \\ +8.52 \end{pmatrix}$$

and

$$\sigma \approx 0.36.$$

Using the formulae given in Section 3, we find

$$x_{12} \approx 0.39, x_{21} = 0.$$

Starting with  $x^0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ , the algorithm in Theorem 2.1 gives the iterates

$i$	$x_1^i$	$x_2^i$
0	0	0
1	0.26	0
2	0.35	0
3	0.38	0
4	0.39	0

The corresponding equilibrium prices are

$$p_1 \approx 6.49, p_2 \approx 10.47.$$

**Example 5.2.** Let us consider the model given in Section 4 with the data:

$a_1^X = 20$	$a_2^X = 15$	$a_1^Y = 15$	$a_2^Y = 20$	$c_{12}^X = 0.3$
$b_1^X = 1$	$b_2^X = 0.5$	$b_1^Y = 2$	$b_2^Y = 1$	$c_{21}^X = 2$
$c_1^X = 0$	$c_2^X = 1$	$c_1^Y = 5$	$c_2^Y = 3$	$c_{12}^Y = 1$
$d_1^X = 0.6$	$d_2^X = 0.8$	$d_1^Y = 0.6$	$d_2^Y = 0.3$	$c_{21}^Y = 2$
$\beta_1^X = 0.1$	$\beta_2^X = 0.1$	$\beta_1^Y = 0.3$	$\beta_2^Y = 0.1$	
$\delta_1^X = 0.3$	$\delta_2^X = 0.1$	$\delta_1^Y = 0.1$	$\delta_2^Y = 0.4$	

We have

$$\bar{p}_1^X = 14, \bar{p}_2^X \approx 13.58,$$

$$\bar{p}_1^Y = 6, \bar{p}_2^Y \approx 18.3,$$

$$M \approx \begin{pmatrix} +1.47 & -1.47 & +0.41 & -0.41 \\ -1.47 & +1.47 & -0.41 & +0.41 \\ +0.23 & -0.23 & +1.21 & -1.21 \\ -0.23 & +0.23 & -1.21 & +1.21 \end{pmatrix}, q \approx \begin{pmatrix} +0.72 \\ -0.12 \\ -12.3 \\ +15.3 \end{pmatrix}$$

and

$$\sigma \approx 0.07.$$

Some initial numerical tests suggest us to start the algorithm of Theorem 2.1 with

$$x^0 = \begin{pmatrix} 0 \\ 2 \\ 9 \\ 0 \end{pmatrix}.$$

The following iterates are obtained ( $\alpha := 0.1$ ).

$i$	$x_1^i$	$x_2^i$	$x_3^i$	$x_4^i$	$ \wedge\{Mx + q, 0\} $
1	0	2.0870	9.1870	0	3.4384
2	0	2.1689	9.3534	0	2.7819
3	0	2.2455	9.5015	0	2.2523
4	0	2.3170	9.6335	0	1.8245
5	0	2.3834	9.7511	0	1.4786
6	0	2.4448	9.8560	0	1.1986
7	0	2.5095	9.9497	0	0.9719
8	0	2.5537	10.0333	0	0.7882
9	0	2.6097	10.1080	0	0.6392
10	0	2.6457	10.1748	0	0.5184
11	0	2.6859	10.2345	0	0.4205
12	0	2.7227	10.2879	0	0.3410
13	0	2.7563	10.3357	0	0.2766
14	0	2.7869	10.3785	0	0.2243
15	0	2.8147	10.4168	0	0.1818
16	0	2.8400	10.4511	0	0.1474
17	0	2.8630	10.4818	0	0.1195
18	0	2.8839	10.5094	0	0.0969
19	0	2.9029	10.5341	0	0.0785
20	0	2.9201	10.5562	0	0.0636
21	0	2.9356	10.5761	0	0.0516
22	0	2.9497	10.5939	0	0.0418
23	0	2.9624	10.6099	0	0.0339
24	0	2.9740	10.6242	0	0.0274
25	0	2.9844	10.6371	0	0.0222
26	0	2.9938	10.6486	0	0.0180
27	0	3.0023	10.6590	0	0.0146
28	0	3.0100	10.6683	0	0.0118
29	0	3.0169	10.6767	0	0.0096

Thus

$$x_{12}^X = 0, x_{21}^X \approx 3.0169, x_{12}^Y \approx 10.6767, x_{21}^Y = 0$$

and the corresponding equilibrium prices are

$$p_1^X \approx 13.104, p_2^X \approx 14.754,$$

$$p_1^Y \approx 20.67, p_2^Y \approx 6.33.$$

## 6 The Existence Problem Revisited

In Section 4, we have used Theorem 2.1 in order to deduce the existence of a solution of the model. The condition

$$b + c < 2 \min\{\sqrt{ad}, \frac{ad + bc}{a + d}\} \quad (6.1)$$

has been assumed. Indeed, we have shown that condition (6.1) ensures that the matrix  $M$  is cocoercive. Moreover (6.1) implies that

$$ad > bc \quad (6.2)$$

so that  $\ker M$  has the form described in (4.32).

In this Section, we show that condition (6.2) is sufficient to guarantee the solvability of the model.

We set

$$X_1 := \begin{pmatrix} x_{12}^X \\ x_{21}^X \end{pmatrix}, X_2 := \begin{pmatrix} x_{12}^Y \\ x_{21}^Y \end{pmatrix}.$$

The model in Section 4 can be split into two subproblems given by

$$X_1 \geq 0, \quad (6.3)$$

$$\begin{pmatrix} +a & -a \\ -a & +a \end{pmatrix} X_1 + \begin{pmatrix} +b & -b \\ -b & +b \end{pmatrix} X_2 + \begin{pmatrix} \bar{p}_1^X - \bar{p}_2^X + c_{12}^X \\ \bar{p}_2^X - \bar{p}_1^X + c_{21}^X \end{pmatrix} \geq 0, \quad (6.4)$$

$$X_1^T \left[ \begin{pmatrix} +a & -a \\ -a & +a \end{pmatrix} X_1 + \begin{pmatrix} +b & -b \\ -b & +b \end{pmatrix} X_2 + \begin{pmatrix} \bar{p}_1^X - \bar{p}_2^X + c_{12}^X \\ \bar{p}_2^X - \bar{p}_1^X + c_{21}^X \end{pmatrix} \right] = 0, \quad (6.5)$$

and

$$X_2 \geq 0, \quad (6.6)$$

$$\begin{pmatrix} +c & -c \\ -c & +c \end{pmatrix} X_1 + \begin{pmatrix} +d & -d \\ -d & +d \end{pmatrix} X_2 + \begin{pmatrix} \bar{p}_1^Y - \bar{p}_2^Y + c_{12}^Y \\ \bar{p}_2^Y - \bar{p}_1^Y + c_{21}^Y \end{pmatrix} \geq 0, \quad (6.7)$$

$$X_2^T \left[ \begin{pmatrix} +c & -c \\ -c & +c \end{pmatrix} X_1 + \begin{pmatrix} +d & -d \\ -d & +d \end{pmatrix} X_2 + \begin{pmatrix} \bar{p}_1^Y - \bar{p}_2^Y + c_{12}^Y \\ \bar{p}_2^Y - \bar{p}_1^Y + c_{21}^Y \end{pmatrix} \right] = 0. \quad (6.8)$$

Let us first fix the vector  $X_2 \in \mathbb{R}^4$  and consider the problem which consists to find  $\bar{X}_1(X_2)$  satisfying the relations (6.3)-(6.5). The complementarity problem (6.3)-(6.5) is equivalent to the variational inequality problem which consists to find  $\bar{X}_1(X_2) \in \mathbb{R}_+^4$  such that

$$(M_1 \bar{X}_1 + q_1)^T (v - \bar{X}_1) \geq 0, \forall v \in \mathbb{R}_+^4,$$

with

$$M_1 = \begin{pmatrix} +a & -a \\ -a & +a \end{pmatrix}$$

and

$$q_1 = \begin{pmatrix} b(X_{2_1} - X_{2_2}) + \bar{p}_1^X - \bar{p}_2^X + c_{12}^X \\ b(X_{2_2} - X_{2_1}) + \bar{p}_2^X - \bar{p}_1^X + c_{21}^X \end{pmatrix}.$$

The matrix  $M_1$  is positive semidefinite, symmetric and

$$\ker M_1 = \left\{ \begin{pmatrix} \alpha \\ \alpha \end{pmatrix}; \alpha \in \mathbb{R} \right\}.$$

Thus if  $w \in \ker M_1 \cap \mathbb{R}_+^2 \setminus \{0\}$  then there exists  $\alpha > 0$  such that  $w = \begin{pmatrix} \alpha \\ \alpha \end{pmatrix}$ . We have

$$q_1^T w = \alpha(c_{12}^X + c_{21}^X) > 0$$

and we may apply Theorem 2.1 to conclude to the existence for any  $X_2 \in \mathbb{R}^2$  of at least one  $\bar{X}_1(X_2) \geq 0$  satisfying the complementarity problem (6.3)-(6.5). We prove the uniqueness of  $\bar{X}_1(X_2)$  by following the same arguments that the ones which have been used in Section 3. Let us now give a precise characterization of  $\bar{X}_1$  as a function of  $X_2$ .

If

$$\frac{\bar{p}_2^X - \bar{p}_1^X - c_{12}^X}{b} \leq X_{2_1} - X_{2_2} \leq \frac{\bar{p}_2^X - \bar{p}_1^X + c_{21}^X}{b}$$

then

$$\bar{X}_1(X_2) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Indeed, we have

$$q_1 = \begin{pmatrix} b(X_{2_1} - X_{2_2}) + \bar{p}_1^X - \bar{p}_2^X + c_{12}^X \\ b(X_{2_2} - X_{2_1}) + \bar{p}_2^X - \bar{p}_1^X + c_{21}^X \end{pmatrix} \geq 0.$$

If

$$X_{2_1} - X_{2_2} < \frac{\bar{p}_2^X - \bar{p}_1^X - c_{12}^X}{b}$$

then

$$\bar{X}_1(X_2) = \begin{pmatrix} \frac{b}{a}(X_{2_2} - X_{2_1}) + \frac{\bar{p}_2^X - \bar{p}_1^X - c_{12}^X}{a} \\ 0 \end{pmatrix}.$$

Indeed,

$$\bar{X}_1(X_2) \geq 0,$$

$$M_1 \bar{X}_1 + q_1 = \begin{pmatrix} 0 \\ c_{12}^X + c_{21}^X \end{pmatrix} \geq 0,$$

and

$$\bar{X}_1^T(M_1 \bar{X}_1 + q_1) = \bar{X}_1 \times 0 + 0 \times (c_{12}^X + c_{21}^X) = 0.$$

If

$$X_{2_1} - X_{2_2} > \frac{\bar{p}_2^X - \bar{p}_1^X + c_{21}^X}{b}$$

then

$$\bar{X}_1(X_2) = \begin{pmatrix} 0 \\ \frac{b}{a}(X_{2_1} - X_{2_2}) + \frac{\bar{p}_1^X - \bar{p}_2^X - c_{21}^X}{a} \end{pmatrix}.$$

We have,

$$\bar{X}_1(X_2) \geq 0,$$

$$M_1 \bar{X}_1 + q_1 = \begin{pmatrix} c_{21}^X + c_{12}^X \\ 0 \end{pmatrix} \geq 0,$$

and

$$\bar{X}_1^T(M_1 \bar{X}_1 + q_1) = 0 \times (c_{21}^X + c_{12}^X) + \bar{X}_{1_2} \times 0 = 0.$$

Let us set

$$F(X_2) := \begin{pmatrix} +c & -c \\ -c & +c \end{pmatrix} \bar{X}_1(X_2).$$

We know that if

$$\frac{\bar{p}_2^X - \bar{p}_1^X - c_{12}^X}{b} \leq X_{2_1} - X_{2_2} \leq \frac{\bar{p}_2^X - \bar{p}_1^X + c_{21}^X}{b}$$

then

$$F(X_2) = F_0(X_2) := \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

If

$$X_{2_1} - X_{2_2} < \frac{\bar{p}_2^X - \bar{p}_1^X - c_{12}^X}{b}$$

then

$$F(X_2) = F_-(X_2) := \begin{pmatrix} -\frac{bc}{a} & +\frac{bc}{a} \\ +\frac{bc}{a} & -\frac{bc}{a} \end{pmatrix} \begin{pmatrix} X_{2_1} \\ X_{2_2} \end{pmatrix} + \frac{c}{a} \begin{pmatrix} \bar{p}_2^X - \bar{p}_1^X - c_{12}^X \\ \bar{p}_1^X - \bar{p}_2^X + c_{12}^X \end{pmatrix}$$

and if

$$X_{2_1} - X_{2_2} > \frac{\bar{p}_2^X - \bar{p}_1^X + c_{21}^X}{b}$$

then

$$F(X_2) = F_+(X_2) := \begin{pmatrix} -\frac{bc}{a} & +\frac{bc}{a} \\ +\frac{bc}{a} & -\frac{bc}{a} \end{pmatrix} \begin{pmatrix} X_{2_1} \\ X_{2_2} \end{pmatrix} + \frac{c}{a} \begin{pmatrix} \bar{p}_2^X - \bar{p}_1^X + c_{21}^X \\ \bar{p}_1^X - \bar{p}_2^X - c_{21}^X \end{pmatrix}.$$



We set

$$\Lambda := \begin{pmatrix} -\frac{bc}{a} & +\frac{bc}{a} \\ +\frac{bc}{a} & -\frac{bc}{a} \end{pmatrix},$$

$$\gamma_1 = \frac{c}{a} \begin{pmatrix} \bar{p}_2^X - \bar{p}_1^X - c_{12}^X \\ \bar{p}_1^X - \bar{p}_2^X + c_{12}^X \end{pmatrix}$$

and

$$\gamma_2 = \frac{c}{a} \begin{pmatrix} \bar{p}_2^X - \bar{p}_1^X + c_{21}^X \\ \bar{p}_1^X - \bar{p}_2^X - c_{21}^X \end{pmatrix}.$$

The matrix  $\Lambda$  is negative semidefinite and thus

$$X_2^T F(X_2) \geq X_2^T \Lambda X_2 + \wedge \{0, \wedge \{X_2^T \gamma_1, X_2^T \gamma_2\}\}.$$

Setting

$$M_2 = \begin{pmatrix} +d & -d \\ -d & +d \end{pmatrix}$$

and

$$q_2 = \begin{pmatrix} \bar{p}_1^Y - \bar{p}_2^Y + c_{12}^Y \\ \bar{p}_2^Y - \bar{p}_1^Y + c_{21}^Y \end{pmatrix},$$

we consider now the inequality problem: Find  $\bar{X}_2 \in \mathbb{R}_+^2$  such that

$$(F(\bar{X}_2) + M_2 \bar{X}_2 + q_2)^T (v - \bar{X}_2) \geq 0, \forall v \in \mathbb{R}_+^2. \quad (6.9)$$

The variational inequality (6.9) is nonlinear and Theorem 2.1 cannot be applied. The following result ensures that if  $da > bc$  then problem (6.9) (and thus problem (4.25)) admits at least one solution.

**Theorem 6.1.** *If  $da > bc$  then problem (6.9) has at least one solution.*

*Proof:* Let  $D_n = \{x \in \mathbb{R}_+^2 : \|x\| \leq n\}$ . For each  $n \in \mathbb{N} \setminus \{0\}$ , the set  $D_n$  is nonempty, compact and convex. Let us now remark that the mapping  $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is continuous. Indeed, the mapping  $F_0$  is continuous on

$$D(F_0) = \{x \in \mathbb{R}^2 : \frac{\bar{p}_2^X - \bar{p}_1^X - c_{12}^X}{b} < x_1 - x_2 < \frac{\bar{p}_2^X - \bar{p}_1^X + c_{12}^X}{b}\},$$

the mapping  $F_- : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is continuous on

$$D(F_-) = \{x \in \mathbb{R}^2 : x_1 - x_2 < \frac{\bar{p}_2^X - \bar{p}_1^X - c_{12}^X}{b}\},$$

and the mapping  $F_+ : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is continuous on

$$D(F_+) = \{x \in \mathbb{R}^2 : x_1 - x_2 > \frac{\bar{p}_2^X - \bar{p}_1^X + c_{21}^X}{b}\}.$$

Moreover, if  $x_1 - x_2 = \frac{\beta_2^x - \beta_1^x - c_{12}^x}{b}$  then  $F_-(x) = F_0(x) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$  and if  $x_1 - x_2 = \frac{\beta_2^x - \beta_1^x + c_{21}^x}{b}$  then  $F_+(x) = F_0(x) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ . We may apply the Hartman-Stampacchia theorem [9] to conclude to the existence of  $x_n \in D_n$  such that

$$(F(x_n) + M_2 x_n + q_2)^T (v - x_n) \geq 0, \forall v \in D_n. \quad (6.10)$$

We claim that the sequence  $\{x_n\}_{n \in \mathbb{N}}$  is bounded. Suppose by contradiction that  $\|x_n\| \rightarrow +\infty$ , then along a subsequence (again denoted by  $\{x_n\}$ ), we may suppose that  $\|x_n\| \neq 0$ ,  $\|x_n\| \rightarrow +\infty$  and

$$w_n := \frac{x_n}{\|x_n\|} \rightarrow w \in \mathbb{R}_+^2 \setminus \{0\}.$$

Setting  $v = 0$  in (6.10), we obtain

$$x_n^T F(x_n) + x_n^T M_2 x_n + q_2^T x_n \leq 0 \quad (6.11)$$

and thus

$$x_n^T (\Lambda + M_2) x_n + \wedge \{0, \wedge \{x_n^T \gamma_1, x_n^T \gamma_2\}\} + q_2^T x_n \leq 0. \quad (6.12)$$

Dividing (6.12) by  $\|x_n\|^2$ , we obtain

$$w_n^T (\Lambda + M_2) w_n + \wedge \{0, \wedge \{w_n^T \gamma_1, w_n^T \gamma_2\}\} \|x_n\|^{-1} + (q_2^T w_n) \|x_n\|^{-1} \leq 0.$$

Taking the limit as  $n \rightarrow +\infty$ , we obtain

$$w^T (\Lambda + M_2) w \leq 0.$$

The matrix  $\Lambda + M_2$  is positive semidefinite, symmetric and

$$\ker M = \left\{ \begin{pmatrix} \alpha \\ \alpha \end{pmatrix}; \alpha \in \mathbb{R} \right\}.$$

Thus  $w = \begin{pmatrix} \alpha \\ \alpha \end{pmatrix}$  for some  $\alpha > 0$ . From (6.12), we deduce also that

$$\wedge \{0, \wedge \{x_n^T \gamma_1, x_n^T \gamma_2\}\} + q_2^T x_n \leq 0.$$

Thus

$$\wedge \{0, \wedge \{w_n^T \gamma_1, w_n^T \gamma_2\}\} + q_2^T w_n \leq 0.$$

Taking the limit as  $n \rightarrow +\infty$ , we obtain

$$\wedge \{0, \wedge \{w^T \gamma_1, w^T \gamma_2\}\} + q_2^T w \leq 0.$$

However

$$w^T \gamma_1 = 0, w^T \gamma_2 = 0, q_2^T w = \alpha(c_{12}^Y + c_{21}^Y)$$

and we get the contradiction

$$\alpha(c_{12}^Y + c_{21}^Y) \leq 0.$$

Thus the sequence  $\{x_n\}$  is bounded and along a subsequence, we may assume that  $x_n \rightarrow x$ . Let  $v \in \mathbb{R}_+^2$  be given. There exists  $n_0 \in \mathbb{N}$  such that

$$(F(x_n) + M_2 x_n + q_2)^T (v - x_n) \geq 0, \forall n \geq n_0.$$

Taking the limit as  $n \rightarrow +\infty$ , we get

$$(F(x) + M_2 x + q_2)^T (v - x) \geq 0.$$

The argument above being true for any  $v \in \mathbb{R}_+^2$ , we may conclude.  $\square$

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