CUBO *a Mathematical Journal*
Vol. 6, Nº 1, (139-157). March 2004.

G-bundles over a projective manifold

Indranil Biswas
School of Mathematics
Tata Institute of Fundamental Research
Homi Bhabha Road
Mumbai 400005, India
indranil@math.tifr.res.in

ABSTRACT

Vector bundles constitute an extensively studied topic in algebraic geometry. Principal bundles are emerging as the natural generalization of vector bundles. This is an exposition on the basic aspects of principal bundles.

1 Semistable principal bundles

Let G be a connected algebraic group over $\mathbb{C}$, the field of complex numbers. (See [Bo], [Sp] for algebraic groups.) A left G–action on a complex variety Z is an algebraic map

$$\psi : G \times Z \longrightarrow Z$$

satisfying the following two conditions:

1. $\psi(g, \psi(h, z)) = \psi(gh, z)$ for all $g, h \in G$ and $z \in Z$; and
2. $\psi(e, z) = z$ for all $z \in Z$, where e is the identity element of G.

If the action ψ is clear from the context, then $\psi(g, z)$ will also be denoted by gz. Similarly, a right action of G on Z is defined by a map

$$\phi : Z \times G \longrightarrow Z$$

with $\phi(z, e) = z$ and $\phi(\phi(z, g), h) = \phi(z, gh)$.

Let M be a connected smooth projective variety over $\mathbb{C}$. A *Principal G-bundle* over M is a smooth complex variety E equipped with an action of G on the right and an algebraic morphism

$$p : E \longrightarrow M \tag{1.1}$$

satisfying the following three conditions

1. the map p is smooth and surjective;
2. the map p is a morphism of G-spaces, with the action of G on M being the trivial one, or in other words, G acts along the fibers of the projection p;
3. the map to the fiber product over M

$$E \times G \longrightarrow E \times_M E$$

defined by $(z\,,g) \longmapsto (z\,,zg)$ is an isomorphism.

Note that we do *not* assume E to be locally trivial in Zariski topology. The third condition ensures that the action of G on E free and it acts transitively on each fiber of the map p. A G-bundle E over M is called *trivial* if it is isomorphic to $M \times G$ as a G-space (the action of G on $M \times G$ is defined by the right action of G on itself).

Let Z be a complex variety equipped with a left action of G. For a principal G-bundle E over M, define a "twisted diagonal" action of G on $E \times Z$ as follows. The action of any $g \in G$ sends a point $(x\,,z) \in E \times Z$ to $(xg\,,g^{-1}z) \in E \times Z$. (See [Gi, p. 114, Définition 1.3.1].) The quotient

$$E(Z) := \frac{E \times Z}{G} \tag{1.2}$$

is a fiber bundle over M with each fiber isomorphic to Z. This $E(Z)$ is called the *associated bundle*, associated to E for Z.

Let $\mathfrak{g}$ denote the Lie algebra of G. The group G acts on $\mathfrak{g}$ as conjugation. For a principal G-bundle E, the associated vector bundle $E(\mathfrak{g})$ (defined as in (1.2)) is called the *adjoint vector bundle*, and it is denoted by $\mathrm{ad}(E)$.

Let $\rho : G \longrightarrow H$ be a homomorphism of algebraic groups. Using ρ, the group G acts on the left of H. More precisely, the action of $g \in G$ sends any $h \in H$ to $\rho(g)h \in H$.

For a principal G-bundle E over M, the group H acts on the right of the associated fiber bundle $E(H)$ defined as in (1.2). The action of any $h \in H$ sends a point $(z, g) \in E(H)$, where $z \in E$ and $g \in H$, to the point $(z, gh) \in E(H)$. This action of H on $E(H)$ is free and it is transitive on the fibers of the natural projection of $E(H)$ to M. Consequently, $E(H)$ is a principal H-bundle over M. This construction of a principal bundle is known as the *extension of structure group*. Note that there is a morphism

$$E \longrightarrow E(H) \tag{1.3}$$

that sends any $z \in E$ to $(z, e) \in (E \times H)/G$, where $e \in H$ is the identity element.

Let H_1 be a closed algebraic subgroup of G. For a principal G–bundle E, consider the quotient space $E/H_1 \cong E(G/H_1)$. Let

$$q : E \longrightarrow E/H_1$$

be the quotient map. Any section

$$\sigma : M \longrightarrow E(G/H_1) \tag{1.4}$$

of the fiber bundle has the property that the inverse image $q^{-1}(\sigma(M))$ is a principal H_1 bundle over M. In other words, $q^{-1}(\sigma(M)) \subset E$ is closed under the action of H_1 (for the action of G on E), and the restriction of the projection p (in (1.1)) to $q^{-1}(\sigma(M))$ makes it into a principal H_1–bundle over M.

Conversely, if $E_{H_1} \subset E$ is a subvariety which is closed under the action of H_1 (for the action of G on E), and H_1 acts transitively on the fibers of the projection of E_{H_1} to M, then E_{H_1} is a principal H_1 bundle. Furthermore, sending any $x \in M$ to the H_1 orbit $E_{H_1} \cap p^{-1}(x) \subset p^{-1}(x)$ in the right G–space $p^{-1}(x)$ we get a section of the fiber bundle $E(G/H_1)$ over M. This construction of a principal H_1–bundle as a subvariety of E is known as *reduction of the structure group* of E. So giving a reduction of the structure group of the G–bundle E to the subgroup H_1 is equivalent to giving a section of the fiber bundle $E(G/H_1)$ over M.

Let G be a complex connected reductive algebraic group. This means that any finite dimensional complex G–module V decomposes as a direct sum of irreducible G–modules. We recall that a G–module V_1 is called irreducible if there are no proper positive dimensional complex linear subspace of V_1 left invariant by the action of G. So $\mathrm{GL}(n,\mathbb{C})$, $\mathrm{SL}(n,\mathbb{C})$, $\mathrm{Sp}(n,\mathbb{C})$, $\mathrm{SO}(n,\mathbb{C})$ are some of the reductive groups. On the other hand, the group of upper triangular $n \times n$ matrices is not reductive. The standard action of it on $\mathbb{C}^n$ cannot be expressed as a direct sum of irreducible modules.

Let $Z_0 \subset G$ denote the connected component of the center of G containing the identity element. We note that G is reductive if and only if Z_0 is isomorphic to a product of copies of $\mathbb{C}^*$ and the quotient group G/Z_0 is semisimple. A complex group H is semisimple if and only if the Killing form on the Lie algebra of H is nondegenerate.

Let M be a connected complex projective manifold of dimension d. Fix an ample line bundle $\mathcal{O}_M(1)$ on M. The degree of any torsionfree coherent sheaf W on M will be defined as

$$\mathrm{degree}(W) := \int_M c_1(W) \wedge (c_1(\mathcal{O}_M(1)))^{d-1} \in \mathbb{Z}.$$

Note that $c_1(W) \wedge (c_1(\mathcal{O}_M(1)))^{d-1} \in H^{2d}(M,\mathbb{Q})$. The integral $\int_M$ is the cap product with the oriented top homology class of M.

Let W' be a coherent subsheaf defined over a nonempty Zariski open subset $U \subset M$ such that the complex codimension of the complement $M \setminus U$ is at least two. Let

$\iota : U \longrightarrow M$ denote the inclusion map. Note that the direct image $\iota_* W'$ is a coherent sheaf on M. The *degree* of W' is defined to be the degree of the direct image $\iota_* W'$.

A torsionfree coherent sheaf W over M is called *semistable* if for any nonzero subsheaf $W' \subset W$ the partial inequality

$$\frac{\text{degree}(W')}{\text{rank}(W')} \leq \frac{\text{degree}(W)}{\text{rank}(W)}$$

is valid. If the strict inequality "<" is valid for subsheaves satisfying the extra condition that W' is a proper subsheaf with W/W' torsionfree, then W is called *stable* (see [Ko]).

The quotient degree(W)/rank(W) is called the *slope* of W and is usually denoted by $\mu(W)$. The sheaf W is called *polystable* if it is a direct sum of stable sheaves of same slope. In particular, any polystable sheaf is semistable.

A. Ramanathan extended the notion of (semi)stability to principal bundles [Ra1], [Ra2], [Ra3]. We will recall the definition.

A *parabolic subgroup* of the reductive group is a connected Zariski closed proper subgroup $P \subset G$ with G/P compact. A parabolic subgroup of G is called *maximal* if it is not properly contained in another parabolic subgroup of G. A subgroup P of $\mathrm{GL}(n, \mathbb{C})$ is a maximal parabolic subgroup if and only if there is a linear subspace $V \subset \mathbb{C}^n$ with $\dim V \in [1, n-1]$ such that for the standard action of $\mathrm{GL}(n, \mathbb{C})$ on $\mathbb{C}^n$ the subgroup that leaves V invariant coincides with P. So the quotient space $\mathrm{GL}(n, \mathbb{C})/P$, where P is a maximal parabolic subgroup, is isomorphic to a Grassmannian.

Let G be a connected complex reductive algebraic group. The center of G will be denoted by $Z(G)$.

Take a principal G-bundle E over X. Let

$$E_P \subset E|_U$$

be a reduction of structure group of E to a maximal parabolic subgroup $P \subset G$ over a nonempty Zariski open subset $U \subset M$ with the codimension of the complement $M \setminus U$ being at least two. We recall that the reduction of structure group is defined by a section $\sigma : U \longrightarrow E|_U/P$ as in (1.4). Let T_{rel} denote the relative tangent bundle over $E|_U/P$ for the projection of $E|_U/P$ to M. So T_{rel} is the subbundle of the holomorphic tangent bundle of $E|_U/P$ defined by the kernel of the differential of the projection of $E|_U/P$ to M. The pull back $\sigma^* T_{\text{rel}}$ is a vector bundle over U.

The G bundle E is called *semistable* (respectively, *stable*) if in *every* such situation describe above, the degree of the pull back $\sigma^* T_{\text{rel}}$ is nonnegative (respectively, strictly positive).

For a connected complex algebraic group H, let $R_u(H)$ denote the unipotent radical of H. So $R_u(H)$ is the maximal connected normal solvable subgroup of H. The quotient $H/R_u(H)$ is reductive. See [Bo] for the details. The quotient $H/R_u(H)$ is called the *Levi factor*, and it is denoted by $L(H)$.

If P is a parabolic subgroup of reductive group G, then the Levi factor $L(P)$ can be identified with a subgroup of P as follows. Fix a maximal torus T of G contained in P. Let $L(P)' \subset P$ be the maximal T invariant reductive subgroup of P. The natural projection of P to the Levi factor $L(P)$ sends $L(P)'$ isomorphically to $L(P)$. This way the Levi factor of P is realized as a subgroup of P.

The G-bundle E is called *polystable* if either E is stable or there is a Levi factor $L(P) \subset P \subset G$ of a parabolic subgroup P and a reduction, over M, of structure group $E_{L(P)} \subset E$ to the subgroup $L(P) \subset G$ such that

1. the $L(P)$-bundle $E_{L(P)}$ is stable;

2. for any character $\chi : L(P) \longrightarrow \mathbb{C}^*$ with the property that the restriction of χ to the center $Z(G)$ of G is trivial, the degree of the associated line bundle $E_{L(P)}(\mathbb{C})$ is zero.

Note that the line bundle $E_{L(P)}(\mathbb{C})$ is the associated bundle (as in (1.2)) for the action of $L(P)$ on $\mathbb{C}$ defined by the character χ.

A $\mathrm{GL}(n,\mathbb{C})$-bundle E_{GL} is stable or semistable or polystable if and only if the vector bundle of rank n associated to E_{GL} by the standard representation of $\mathrm{GL}(n,\mathbb{C})$ is stable or semistable or polystable respectively.

Ramanathan constructed the moduli space of semistable G-bundles over a Riemann surface as an irreducible normal projective variety (see [Ra2] and [Ra3]). See [FMW] for an extensive study of the moduli space of semistable principal bundles over an elliptic curve.

Given any holomorphic vector bundle V on M, there is a unique filtration

$$0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_{l-1} \subset V_l = V$$

of coherent subsheaves such that each quotient V_i/V_{i-1}, $i \in [1,l]$, is a torsionfree semistable sheaf and $\mu(V_i/V_{i-1}) > \mu(V_{i+1}/V_i)$ for all $i \in [1, l-1]$. This filtration is known as the *Harder–Narasimhan filtration* or the *canonical filtration* (see [Ko]).

This property of vector bundles, namely the existence of a unique canonical filtration, extends to principal bundles. It will be explained in the rest of this section.

Let $\mathfrak{p}$ be the Lie algebra of a parabolic subgroup P of G. Let $\mathfrak{u}(P)$ denote the nilpotent radical of $\mathfrak{p}$. So $\mathfrak{u}(P)$ is the Lie algebra of $R_u(P)$. Set

$$\mathfrak{u}_2 := [\mathfrak{u}(P), \mathfrak{u}(P)]$$

to be the commutator. Since $R_u(P)$ is a normal subgroup of P, the adjoint action of P on its Lie algebra $\mathfrak{p}$ leaves the subalgebra $\mathfrak{u}(P)$ invariant. The induced action of P on $\mathfrak{u}(P)$ leaves the subalgebra $\mathfrak{u}_2$ invariant. So we have an induced action of P on the quotient $\mathfrak{u}(P)/\mathfrak{u}_2$.

In [AAB] the following analog of canonical filtration for a G-bundle is proved.

Theorem 1.1 *Let E be a principal G–bundle over M. Then either E is semistable or there is a nonempty Zariski open subset $U \subset M$, with $\mathrm{codim}(M \setminus U) \geq 2$, and a unique reduction of structure group of E over U to a parabolic subgroup, say P, of G such that, denoting the reduction by E_P, the following two conditions hold:*

1. *the principal $L(P)$–bundle obtained by extending the structure group of the P–bundle E_P over U using the quotient map of P to $L(P)$ is a semistable $L(P)$–bundle;*

2. *for any submodule V in the P–module $\mathfrak{u}(P)/\mathfrak{u}_2$, the associated vector bundle $E_P(V)$ over U is of positive degree.*

If we set $G = \mathrm{GL}(n, \mathbb{C})$, then it is straight–forward to check that the above theorem is equivalent to the existence, and uniqueness, of the canonical reduction of the rank n vector bundle associated to E for the standard action of $\mathrm{GL}(n, \mathbb{C})$ on $\mathbb{C}^n$.

The second condition in the above theorem can be replaced by the following equivalent condition:
For any nontrivial character χ of P which can be expressed as a nonnegative integral combination of simple roots, the line bundle associated to E_P for χ is of strictly positive degree. (See [AAB, p. 712, Theorem 6].)

The following proposition gives another equivalent formulation of the canonical reduction (see [AAB, p. 706, Porposition 4]):

Proposition 1.2 *Let E be a principal G–bundle over M which is not semistable and P. Let $E_P \subset E$ be a reduction of structure group of E to a parabolic subgroup P of G over a Zariski open subset $U \subset M$ with $\mathrm{codim}(M \setminus U) \geq 2$. Assume that the following three conditions are valid:*

1. *the principal $L(P)$–bundle over U, where $L(P)$ is the Levi factor of P, obtained by extending the structure group of E_P using the projection of P to $L(P)$ is a semistable $L(P)$–bundle;*

2. *if $Q \subset G$ is a parabolic subgroup properly containing P and $E_P \subset E_Q \subset E$ a reduction of structure group of E to Q over some Zariski open subset $U_1 \subset U$ with codim$(M \setminus U_1) \geq 2$, then the inequality*
$$\mathrm{degree}(\mathrm{ad}(E_Q)/\mathrm{ad}(E_P)) < 0$$
is valid;

3. *$\mu_{\mathrm{min}}(\mathrm{ad}(E_P)) = 0$, where μ_{min} of a vector bundle is the slope of the final quotient of the Harder–Narasimhan filtration of the vector bundle.*

Then the reduction E_P coincides with the canonical reduction of E in Theorem 1.1. Conversely, the canonical reduction in Theorem 1.1 satisfies all the above three conditions.

In the next section we will consider holomorphic connections on a principal bundle.

2 Holomorphic connections on a principal bundle

Let E be a principal G-bundle over M, where G is any complex algebraic group. For any analytic open subset $U \subset M$, consider the inverse image $p^{-1}(U)$ (the map p is as in (1.1)) which is a complex manifold equipped with an action of G. So G acts on the space of all holomorphic vector fields on $p^{-1}(U)$. Let $\mathcal{A}(U)$ denote the space of all holomorphic vector fields on $p^{-1}(U)$ that are invariant under that action of G. If θ is a holomorphic vector field on $p^{-1}(U)$ left invariant by the action of G and $y \in p^{-1}(U)$, then $\theta(y)$ determines $\theta|_{p^{-1}(p(y))}$. Indeed, this is an immediate consequence of the fact that G acts transitively on the fibers of p.

Note that if θ is a holomorphic vector field on $p^{-1}(U)$ left invariant by the action of G and f a holomorphic function on U, then the holomorphic vector field $(f \circ p) \cdot \theta$ on $p^{-1}(U)$ is also left invariant by the action of G.

Therefore, associating the vector space $\mathcal{A}(U)$ to any analytic open subset U of M we get a coherent analytic sheaf on M. By GAGA (see [Se]) this is a coherent algebraic sheaf. It is easy to see that this coherent sheaf is locally free of rank $\dim G + \dim M$. Indeed, this follows from the fact that $\dim E = \dim G + \dim M$.

Let $\mathrm{At}(E)$ denote the vector bundle over M defined by this coherent sheaf that associates $\mathcal{A}(U)$ to any U. This vector bundle $\mathrm{At}(E)$ was first constructed in [At2], and it is now known as the *Atiyah bundle* for E.

Let $T_{\mathrm{rel}} \subset TE$ be the relative tangent bundle for the projection p. So T_{rel} coincides with the subbundle of the holomorphic tangent bundle TE defined by the kernel of the differential

$$dp : TE \longrightarrow p^{*}TM$$

of the projection p. In other words, we have an exact sequence of holomorphic vector bundles

$$0 \longrightarrow T_{\mathrm{rel}} \longrightarrow TE \xrightarrow{dp} p^{*}TM \longrightarrow 0 \tag{2.1}$$

over E. Note that is exact sequence is compatible with the action of G. In other words, the action of G on TE preserves the subbundle T_{rel}. The induced action of G on the quotient $p^{*}TM$ is clearly the trivial action.

The coherent sheaf on M that associates to any open subset $U \subset M$ the space of all G-invariant holomorphic sections of $T_{\mathrm{rel}}|_{p^{-1}(U)}$ coincides with the coherent sheaf on M corresponding to the adjoint vector bundle $\mathrm{ad}(E)$. This follows from the fact that the space of all right invariant holomorphic vector fields on the algebraic group G is identified with the Lie algebra $\mathfrak{g}$. The identification is obtained by evaluating any vector field at the identity element.

Therefore, taking G-invariant sections for the exact sequence of vector bundles (2.1) we obtain an exact sequence of vector bundles

$$0 \longrightarrow \mathrm{ad}(E) \xrightarrow{\iota} \mathrm{At}(E) \xrightarrow{dp} TM \longrightarrow 0 \tag{2.2}$$

over M. This is known as the *Atiyah exact sequence*.

For $G = \mathrm{GL}(n, \mathbb{C})$, the Atiyah bundle of a $\mathrm{GL}(n, \mathbb{C})$-bundle E_{GL} can alternatively be described as follows. Let $E = E_{\mathrm{GL}}(\mathbb{C}^n)$ be the rank n vector bundle over M associated to E_{GL} for the standard action of $\mathrm{GL}(n, \mathbb{C})$ on $\mathbb{C}^n$. Let $\mathrm{Diff}^1_M(E, E)$ be the holomorphic vector bundle over M defined by the first order differential operators from E to E. Let

$$\sigma : \mathrm{Diff}^1_M(E, E) \longrightarrow TM \otimes \mathrm{End}(E)$$

be the symbol map. Note that $\mathrm{End}(E)$ has a trivial line subbundle defined by the identity automorphism of E. So we have an exact sequence of holomorphic vector bundles

$$0 \longrightarrow \mathrm{End}(E) \longrightarrow \sigma^{-1}(TM \otimes \mathrm{Id}) \xrightarrow{\sigma} TM \longrightarrow 0$$

over M. The Atiyah bundle

$$\mathrm{At}(E_{\mathrm{GL}}) \cong \sigma^{-1}(TX \otimes \mathrm{Id}) \subset \mathrm{Diff}^1_M(E, E)\,, \tag{2.3}$$

and the above exact sequence coincides with the Atiyah exact sequence obtained in (2.2) (note that adjoint bundle $\mathrm{ad}(E_{\mathrm{GL}})$ is naturally identified with $\mathrm{End}(E)$).

A *holomorphic connection* on a principal G-bundle E is a splitting of the Atiyah exact sequence (2.2), that is, a homomorphism of holomorphic vector bundles

$$D : TM \longrightarrow \mathrm{At}(E)$$

such that $(dp) \circ D$ is the identity automorphism of TM, where dp as in (2.2) [At2].

For the trivial G-bundle $E_0 = M \times G$ over M, we have $\mathrm{At}(E_0) \cong TM \bigoplus M \times \mathfrak{g}$, where $M \times \mathfrak{g}$ is the trivial vector bundle over M with $\mathfrak{g}$ as the fiber. Hence the trivial G-bundle E_0 has a natural connection, which is called the ***trivial connection***.

We recall that the exact sequence (2.2) was obtained from the exact sequence of vector bundles (2.1) over E by taking G-invariant sections. Therefore, giving a holomorphic connection on E is equivalent to giving a G-equivariant holomorphic splitting of the exact sequence (2.1).

For another such splitting D' of (2.2), the difference of the two splittings, namely $D - D'$, is a holomorphic homomorphism from TM to $\mathrm{ad}(E)$. Indeed, this is an immediate consequence of the fact that both define splittings. Conversely, for any holomorphic section s of $\Omega^1_M \bigotimes \mathrm{ad}(E)$, clearly

$$D + s : TM \longrightarrow \mathrm{At}(E)$$

is a splitting of (2.2) if D is so. In other words, the space of all holomorphic connections on E is an affine space for the vector space $H^0(M, \Omega^1_M \bigotimes \mathrm{ad}(E))$.

For any open subset U' of E, the space of all holomorphic vector fields over U is equipped with a Lie algebra structure defined by the Lie bracket operation. This Lie bracket operation preserves G-invariant vector fields. In other word, if $U' = p^{-1}(U)$, where $U \subset M$ is some open subset, and s and t are two G-invariant holomorphic vector fields on U', then the Lie bracket $[s\,, t]$ is also a G-invariant holomorphic vector fields on U'. Consequently, the space of all holomorphic sections of the Atiyah bundle

$\mathrm{At}(E)|_U$ over U is equipped with a Lie algebra structure. On the other hand, the space of all holomorphic vector fields over U is also equipped with a Lie algebra structure defined by the Lie bracket operation. It is easy to see that the homomorphism dp in (2.2) is compatible with the Lie algebra structure of the sections of $\mathrm{At}(E)|_U$ and TU, that is, $dp([s,t]) = [dp(s), dp(t)]$, where s and t are any holomorphic sections of $\mathrm{At}(E)|_U$. Indeed, this is an immediate consequence of the fact that the differential dp commutes with the Lie bracket operations on E and M respectively.

Since dp in (2.2) is compatible with the Lie algebra structures, the Lie algebra structure on the space of sections of $\mathrm{At}(E)$ induces a Lie algebra structure on the kernel of dp in (2.2) namely on the sections of the adjoint bundle $\mathrm{ad}(E)$. On the other hand, the fibers of $\mathrm{ad}(E)$ has a natural Lie algebra structure. Indeed, $\mathrm{ad}(E)$ is the vector bundle associated to E for the adjoint action of G on its Lie algebra $\mathfrak{g}$. Since the adjoint action of G on $\mathfrak{g}$ preserves the Lie algebra structure, the fibers of $\mathrm{ad}(E)$ get Lie algebra structure. This Lie algebra structure on the fibers of $\mathrm{ad}(E)$ induce a Lie algebra structure on the space of sections of $\mathrm{ad}(E)$. It is easy to see that this Lie algebra structure on the sections on $\mathrm{ad}(E)$ coincides with the one induced by Lie algebra structure on the space of sections of $\mathrm{At}(E)$. If we identify $\mathfrak{g}$ with the right invariant holomorphic vector fields on G, then the Lie algebra operation on $\mathfrak{g}$ coincides with the Lie bracket operation on the right invariant holomorphic vector fields on G.

Let $D : TM \longrightarrow \mathrm{At}(E)$ be a holomorphic splitting of the Atiyah exact sequence (2.2) defining a holomorphic connection on the G-bundle E. The image $D(TM)$ need not be closed under the Lie algebra structure of the space of sections of $\mathrm{At}(E)$. In other words, for two section s and t of $D(TM)$ over an open subset $U \subset M$, the Lie bracket $[s,t]$ need not be a section of $D(TM)$. Since the projection dp in (2.2) is compatible with the Lie algebra structures, we conclude that $[s,t]$ is a holomorphic section of $\mathrm{ad}(E)$ over U.

Using the above remark, we have a holomorphic homomorphism of vector bundles from the exterior power $\bigwedge^2 TM$ to $\mathrm{ad}(E)$. To explain this, for any point $x \in M$ and holomorphic tangent vectors $v, w \in T_xM$, let $\hat{v}$ and $\hat{v}$ be holomorphic vector fields defined around x with $\hat{v}(x) = v$ and $\hat{w}(x) = w$. Consider the evaluation $[D(\hat{v}), D(\hat{w})](x)$ at x of the Lie bracket $[D(\hat{v}), D(\hat{w})]$, where D as before is a holomorphic connection on E. It is straight-forward to check that $[D(\hat{v}), D(\hat{w})](x)$ does not depend on the choices of the vector fields $\hat{v}$ and $\hat{v}$ (it only depends on u, v and, of course, D). Consequently, we have

$$\mathcal{K}(D) \in H^0(M, \Omega^2_M \otimes \mathrm{ad}(E)) = H^0(M, \mathrm{Hom}(\wedge^2 TM, \mathrm{ad}(E))) \tag{2.4}$$

defined by

$$\langle \mathcal{K}(D), v \wedge w \rangle = [D(\hat{v}), D(\hat{w})](x) \in \mathrm{ad}(E)_x$$

for any $x \in M$ and $v, w \in T_xM$. This section $\mathcal{K}(D)$ is called the *curvature* of the holomorphic connection D. So the curvature measures the failure of the splitting D to be Lie algebra structure preserving.

In particular, if $\mathcal{K}(D) = 0$, then the splitting D is compatible with the Lie algebra structure of the sections of TM and $\mathrm{At}(E)$. If $\mathcal{K}(D) = 0$, then D is called a *flat connection*. Note that if M is a Riemann surface, that is, $\dim_{\mathbb{C}} M = 1$, then any holomorphic connection is automatically flat, as $\Omega^2_M = 0$.

Let

$$\rho : G \longrightarrow H$$

be a homomorphism of algebraic groups and E a principal G–bundle over M. Consider the map

$$\hat{\rho} : E \longrightarrow E(H)$$

defined in (1.3). If ν is a G invariant vector field on $p^{-1}(U) \subset E$, where p is the projection of E to M and U an open subset of M, then $d\hat{\rho}(\nu)$ is a H invariant vector field on $q^{-1}(U) \subset E(H)$, where q is the natural projection of the principal H–bundle $E(H)$ to M, and $d\hat{\rho}$ is the differential of the above map $\hat{\rho}$. Therefore, we have a homomorphism

$$\tilde{\rho} : \mathrm{At}(E) \longrightarrow \mathrm{At}(E(H))$$

of Atiyah bundles. If $D : TM \longrightarrow \mathrm{At}(E)$ is a holomorphic connection on E, then the composition homomorphism $\tilde{\rho} \circ D$ is a holomorphic connection on the principal H–bundle $E(H)$. In other words, a connection on principal bundle induces connections on principal bundles obtained by extension of structure group.

Giving a holomorphic connection D on a G–bundle E is equivalent to giving a $\mathfrak{g}$–valued holomorphic one–form

$$\omega \in H^0(E, \Omega^1_E \otimes \mathfrak{g}) \tag{2.5}$$

on E satisfying the following two conditions:

1. the form ω is G–equivariant for the adjoint action of G on its Lie algebra $\mathfrak{g}$ and the obvious action of G on E;
2. the restriction of the form ω to any fiber of the projection p to M coincides with the holomorphic Maurer–Cartan form.

Let

$$\psi : E \times G \longrightarrow E$$

be the action of G on E. For any point $z \in E$ and any $v \in \mathfrak{g}$, let $\hat{v} := d\psi(z, e)(0, v) \in T_z E$ be the image of v by the differential of ψ, where $e \in E$ is the identity element in G. The holomorphic Maurer–Cartan form on E, which is a holomorphic relative one–form on E with values in $\mathfrak{g}$ (that is, a holomorphic section of $T^*_{\mathrm{rel}} \otimes \mathfrak{g}$), sends $\hat{v}$ to v.

Given such a form ω on E, the kernel of ω defines a homomorphism from TM to $\mathrm{At}(E)$ splitting the Atiyah exact sequence. Conversely, given a splitting of the Atiyah exact sequence, there is a unique one–form ω on E satisfying the above two conditions and having the property that its kernel is the image of TM for the splitting homomorphism. Therefore, giving a holomorphic connection D on E is equivalent to giving a $\mathfrak{g}$-valued holomorphic one-form on E satisfying the above two conditions.

If E is a principal G-bundle over M admitting a holomorphic connection, then all the rational characteristic classes of E vanish [At2]. In particular, there are no topological obstructions for E to admit a flat holomorphic connection. A question due to Atiyah asks for the converse:

Question 2.1 *Let E be a principal G-bundle over a compact complex manifold admitting a holomorphic connection. Does E admit a flat holomorphic connection?*

In many special cases the answer to Question 2.1 is positive, but the general answer is not known.

3 Principal bundles on a Riemann surface

Let X be a connected smooth projective curve over $\mathbb{C}$, or equivalently, a compact connected Riemann surface. Let E be a holomorphic vector bundle over X. The vector bundle E is called *decomposable* if there are holomorphic subbundles W_1 and W_2 of E of positive ranks such that there is a holomorphic isomorphism

$$E \cong W_1 \oplus W_2 .$$

The vector bundle E is called *indecomposable* if it is not decomposable. Every vector bundle can be expressed as a direct sum of indecomposable vector bundle, and furthermore, the decomposition is unique up to a permutation of the direct summands [At1].

In (2.3) we saw that the Atiyah bundle for a vector bundle E coincides with the subbundle of $\mathrm{Diff}^1_X(E,E)$ whose image by the symbol map is $TX \otimes \mathrm{Id}$. Recall that a holomorphic connection is by definition a splitting of the Atiyah exact sequence (2.2).

Consequently, a holomorphic connection on a vector bundle E over X is a first order holomorphic differential operator

$$D \in H^0(X, \mathrm{Diff}^1_X(E, \Omega^1_X \otimes E))$$

whose symbol is the identity automorphism of E. It is easy to see that this condition on symbol is equivalent to the condition that D satisfies the Leibniz identity which says that

$$D(fs) = fD(s) + \partial(f) \otimes s$$

where f is a locally defined holomorphic function on X and s is a locally defined holomorphic section of E.

A holomorphic connection on E is same as a holomorphic connection on the corresponding principal $\mathrm{GL}(n,\mathbb{C})$-bundle, where $n = \mathrm{rank}(E)$. It was noted earlier that any holomorphic connection on a Riemann surface is automatically flat.

Let D be a holomorphic connection on E. So we have degree$(E) = 0$. Assume that $E \cong W_1 \bigoplus W_2$, where W_1 and W_2 are holomorphic subbundles of E. Let q denote the projection of E to W_1 defined using the decomposition. The composition

$$W_1 \hookrightarrow E \xrightarrow{D} \Omega^1_X \otimes E \xrightarrow{\mathrm{Id}\otimes q} \Omega^1_X \otimes W_1$$

defines a holomorphic connection on W_1. Consequently, degree$(W_1) = 0$.

Therefore, if a holomorphic vector bundle E over X admits a holomorphic connection, then every indecomposable component of E is of degree zero. The converse is also true. In other words, a holomorphic vector bundle E over X admits a holomorphic connection if and only if every indecomposable component of E is of degree zero [At2], [We].

Let G be a connected reductive algebraic group over $\mathbb{C}$. We will describe a condition on G-bundles that ensures the existence of a holomorphic connection.

For convenience, in this section by a parabolic subgroup of G we will mean what we defined in Section 1 to be parabolic subgroup and G itself. In other words, now a parabolic subgroup of G need not be a proper subgroup. The Levi factor for G is G itself, as the unipotent radical of G is the trivial group.

Henceforth, we will consider the Levi factor $L(P)$ of a parabolic subgroup P as a subgroup of P. (It was explained in Section 1 that we can do so.)

For a parabolic subgroup P of G, let

$$L_0(P) := \frac{L(P)}{[L(P)\,,L(P)]}$$

be the quotient by the commutator. So, $L_0(P)$ is the maximal abelian quotient of $L(P)$. Note that $L_0(P)$ is isomorphic to a product of copies of $\mathbb{C}^*$. Since there are no nontrivial (multiplicative) characters on $[L(P)\,,L(P)]$, the characters of $L(P)$ are in bijective correspondence with the characters of $L_0(P)$. Consequently, for a $L(P)$-bundle $E_{L(P)}$ on X, the corresponding $L_0(P)$-bundle $E_{L(P)}(L_0(P))$, obtained by extension of structure group, is topologically trivial if and only if for every character

$$\chi : L(P) \longrightarrow \mathbb{C}^*$$

degree$(E_{L(P)}(\mathbb{C})) = 0$, where $E_{L(P)}(\mathbb{C})$ is the line bundle over X associated to $E_{L(P)}$ for the action of P on $\mathbb{C}$ that factors through χ.

Let E be a principal G-bundle over X. Let

$$E_{L(P)} \hookrightarrow E$$

be a reduction of structure group of E to the Levi factor $L(P)$ of a parabolic subgroup P. Take a character

$$\chi : L(P) \longrightarrow \mathbb{C}^*$$

of $L(P)$. Let

$$\xi := \frac{E_{L(P)} \times \mathbb{C}}{P} = E_{L(P)}(\mathbb{C})$$

be the holomorphic line bundle over X, where the action of P on $\mathbb{C}$ is χ composed with the multiplication action of $\mathbb{C}^*$ on $\mathbb{C}$.

Lemma 3.1 *If the G-bundle E admits a holomorphic connection then* degree(ξ) = 0.

Proof. Let $\mathfrak{l}(P)$ denote the Lie algebra of $L(P)$. Consider the two $L(P)$ modules, namely $\mathfrak{l}(P)$ and $\mathfrak{g}$, equipped with the adjoint action of $L(P)$. Since $L(P)$ is reductive, the inclusion map

$$\iota : \mathfrak{l}(P) \hookrightarrow \mathfrak{g}$$

of $L(P)$-modules admits a splitting. Take a splitting

$$\phi : \mathfrak{g} \longrightarrow \mathfrak{l}(P). \tag{3.1}$$

So, ϕ is $L(P)$ equivariant and $\phi \circ \iota = \mathrm{Id}_{\mathfrak{l}(P)}$.

Let τ be the inclusion map of $E_{L(P)}$ in E. Let

$$\omega \in H^0(E, \Omega^1_E \otimes \mathfrak{g})$$

be a connection form, as in (2.5), on E. Then the form

$$\phi \circ \tau^*\omega \in H^0(E_{L(P)}, \Omega^1_{L(P)} \otimes \mathfrak{l}(P))$$

where ϕ as in (3.1), defines a holomorphic connection on the $L(P)$-bundle $E_{L(P)}$. Indeed, since ϕ is $L(P)$ equivariant, the form $\phi \circ \iota^*\omega$ evidently satisfies both the conditions needed to define a holomorphic connection.

The connection $\omega' := \phi \circ \tau^*\omega$ on $E_{L(P)}$ induces a holomorphic connection on the principal $\mathbb{C}^*$-bundle $E_{L(P)}(\mathbb{C}^*)$ obtained by extending the structure group using the homomorphism χ. Since $E_{L(P)}(\mathbb{C}^*)$ admits a holomorphic connection and ξ is the line bundle associated to $E_{L(P)}(\mathbb{C}^*)$ by the standard action, we conclude that ξ admits a holomorphic connection. Consequently, degree(ξ) = 0. This completes the proof of the lemma. ∎

It was proved in [AB] that the converse of Lemma 3.1 is also valid. In other words, the following theorem is valid ([AB, p. 342, Theorem 4.1]).

Theorem 3.2 *Let G be a connected reductive algebraic group over $\mathbb{C}$ and E a principal G-bundle over a compact connected Riemann surface X. The principal bundle E admits a holomorphic connection if and only if for every reduction*

$$E_{L(P)} \subset E,$$

where $L(P)$ is the Levi factor of some parabolic subgroup P, and for every character χ of $L(P)$, the degree of the associated line bundle $(E_{L(P)} \times \mathbb{C})/L(P)$ is zero.

If we set $G = \mathrm{GL}(n, \mathbb{C})$, then the criterion in Theorem 3.2 coincides with the earlier stated criterion, that is, the rank n vector bundle over X associated to E by the standard representation admits a holomorphic connection if and only if all the indecomposable components are of degree zero.

A corresponding criterion for the existence of a holomorphic connection in positive characteristics is established in [BS].

Let P be a parabolic subgroup of the reductive group G. We can ask the following question: what is a necessary and condition for the existence of a holomorphic connection on a given principal P-bundle over X?

Let E_P be a principal P-bundle over X. Let $E_P(L(P))$ be the principal $L(P)$-bundle obtained by extending the structure group using the projection of P to $P/R_u(P) = L(P)$. Now, a holomorphic connection on E_P induces a holomorphic connection on $E_P(L(P))$. Note that since $L(P)$ is reductive, we can use the criterion in Theorem 3.2 to decide if $E_P(L(P))$ admits a holomorphic connection.

Question 3.3 *Assume that $E_P(L(P))$ admits a holomorphic connection. Does this imply that the P-bundle E_P admits a holomorphic connection?*

4 Hermitian–Einstein connection on a principal bundle

In this section G will be a complex reductive algebraic group. We will recall the definition of a C^∞ connection (as opposed to the holomorphic connection defined in Section 2) on a G-bundle.

Let E be a principal G-bundle over M. The real cotangent bundle of the total space of E will be denoted by $T^*_{\mathbb{R}}E$. The real tangent bundle of E will be denoted by $T_{\mathbb{R}}E$. Since E is a complex manifold, the real tangent bundle $T_{\mathbb{R}}E$ is equipped with an almost complex structure.

A *C^∞ connection on E* is a C^∞ section

$$\omega' \in C^\infty(E; T^*_{\mathbb{R}}E \otimes_{\mathbb{R}} \mathfrak{g})$$

satisfying the following two conditions:

1. the $\mathfrak{g}$-valued one-form ω' on E is G-equivariant for the adjoint action of G on its Lie algebra $\mathfrak{g}$ and the obvious action of G on E;
2. the restriction of the form ω' to any fiber of the projection p to M coincides with the real Maurer–Cartan form.

A C^∞ analog of the Atiyah bundle can be defined (exactly as before). A C^∞ connection is a C^∞ splitting of the C^∞ Atiyah exact sequence. These constructions are straight-forward analog of those for the holomorphic case.

We note that a holomorphic connection on E gives a C^∞ connection on E. Indeed, the real part of a form ω as in (2.5) defining a holomorphic connection on E is a C^∞ connection on E.

A C^∞ connection ω' on E is called a *complex connection* if the homomorphism

$$F_{\omega'} : T_{\mathbb{R}}E \longrightarrow \mathfrak{g}$$

induced by ω' is compatible with the almost complex structures. For any point $z \in E$ and any tangent vector $v \in (T_{\mathbb{R}}E)_z$

$$F_{\omega'}(z)(v) := \langle \omega'(z), v \rangle \in \mathfrak{g},$$

where $\langle - , - \rangle$ is the contraction of $(T_{\mathbb{R}}E)_z$ with its dual $(T^*_{\mathbb{R}}E)_z$. The above compatibility condition means that $F_{\omega'}(z)$ commutes with the almost complex structures of $(T_{\mathbb{R}}E)_z$ and $\mathfrak{g}$ (since $\mathfrak{g}$ is a complex vector space, the underlying real vector space has an almost complex structure which is defined by multiplication with $\sqrt{-1}$).

A C^∞ connection on E obtained (as above) from a holomorphic connection on E is clearly a complex connection. However, not all complex connections arise from holomorphic connections.

For a holomorphic vector bundle E over M, a C^∞ connection ∇ on E is a complex connection if and only if the $(0,1)$-part of ∇ coincides with the Dolbeault operator for E defining the holomorphic structure of the vector bundle. (The connection operator ∇ decomposes as $\nabla^{1,0} + \nabla^{0,1}$ using the decomposition of complex one-forms on M as sum of forms of Hodge types $(0,1)$ and $(0,1)$.)

Let $K(G) \subset G$ be a maximal compact subgroup of G. Let

$$E_{K(G)} \subset E \tag{4.1}$$

be a C^∞ reduction of structure group of E. So, $E_{K(G)}$ is fiber bundle over M and $K(G)$ acts transitively on the fibers of $E_{K(G)}$. Note that a C^∞ reduction of structure group of E to $K(G)$ is given by a C^∞ section of the fiber bundle $E/K(G)$ over M.

A C^∞ reduction of structure group of E to the maximal compact subgroup $K(G)$ is called a *unitary* structure on E.

If $G = \mathrm{GL}(n, \mathbb{C})$, then $K(G) = U(n)$ (or a conjugate of it). It is easy to see that giving a C^∞ reduction of structure group of a principal $\mathrm{GL}(n,\mathbb{C})$-bundle E_{GL} to $U(n)$ is equivalent to giving a C^∞ Hermitian structure on the associated vector bundle of rank n (associated to E_{GL} for the standard action of $\mathrm{GL}(n,\mathbb{C})$ on $\mathbb{C}^n$).

Given a reduction $E_{K(G)}$ as in (4.1) and a C^∞ connection ∇ on $E_{K(G)}$, there is an induced connection on E. Note that E is an extension of structure group of the $K(G)$-bundle $E_{K(G)}$ to G (for the inclusion map of $K(G)$ in G). In Section 2 it was shown that a connection induces a connection on any extension of structure group.

Proposition 4.1 *Given a unitary structure $E_{K(G)} \subset E$ on a principal G-bundle E, there is a unique C^∞ connection ∇ on $E_{K(G)}$ with the property that the C^∞ connection on E induced by ∇ is a complex connection.*

A proof of the above proposition is given in [KN, p. 178, Theorem 10.1] (see also the remark in [KN, p. 185]).

For $G = \mathrm{GL}(n, \mathbb{C})$, the above proposition says that a holomorphic Hermitian vector bundle over a complex manifold has a unique connection which preserves the Hermitian structure as well as the $(0,1)$–part of the connection coincides with the Dolbeault operator defining the holomorphic structure of the vector bundle (see [Ko] for a proof for the case of vector bundles). This connection on the holomorphic Hermitian vector bundle E is called the *Chern connection* on E.

So, given a unitary structure on a G–bundle E, Proposition 4.1 gives a complex connection on E. A connection on E is called *unitary* if it arises this way for some unitary structure on E.

Let ∇ be a unitary connection on a G–bundle E over M. The curvature $\mathcal{K}(\nabla)$ is a C^∞ form of type $(1,1)$ with values in the adjoint bundle $\mathrm{ad}(E)$. In other words,

$$\mathcal{K}(\nabla) \in C^\infty(M; \Omega^{1,1}_M \otimes \mathrm{ad}(E)) .$$

Fix a Kähler form $\omega \in C^\infty(M; \Omega^{1,1}_M)$ on M such that the cohomology class in $H^2(M, \mathbb{R})$ represented by the closed form ω is a multiple of the Chern class $c_1(\mathcal{O}_M(1))$, where $\mathcal{O}_M(1)$ is the ample line bundle on M that has been fixed (the degree was defined using it).

Let $\Lambda : \Omega^{p,q}_M \longrightarrow \Omega^{p-1,q-1}_M$ be the adjoint of the multiplication by the Kähler form ω (see [Ko] for the details). So, we have

$$\Lambda(\mathcal{K}(\nabla)) \in C^\infty(M; \mathrm{ad}(E)) , \tag{4.2}$$

that is, $\Lambda(\mathcal{K}(\nabla))$ is a smooth section of the adjoint bundle.

Let $\mathfrak{z} \subset \mathfrak{g}$ be the center of the Lie algebra $\mathfrak{g}$. Note that the adjoint action of $g \in G$ on any $v \in \mathfrak{z}$ is the trivial action, that is, $\mathfrak{z}$ is fixed pointwise by the adjoint action of G on $\mathfrak{g}$. So any $v \in \mathfrak{z}$ gives a smooth section $\hat{v}$ of the adjoint bundle $\mathrm{ad}(E)$. To explain this, recall that

$$\mathrm{ad}(E) = \frac{E \times \mathfrak{g}}{G} .$$

So $v \in \mathfrak{z}$ defines a section $\hat{v}$ of $\mathrm{ad}(E)$ by sending any $x \in M$ to $(y, v) \in E \times \mathfrak{g}$, where y is any point in $p^{-1}(x)$ (the map p is the projection in (1.1)). That the element in the fiber $\mathrm{ad}(E)_x$ represented by (y, v) does not depend on the choice of $y \in p^{-1}(x)$ is an immediate consequence of the fact that G acts trivially on v and it acts transitively on $p^{-1}(x)$.

A unitary connection ∇ on a G–bundle E over M is called a *Hermitian–Einstein connection* if there an element $v \in \mathfrak{z}$ such that

$$\Lambda(\mathcal{K}(\nabla)) = \hat{v} \in C^\infty(M; \mathrm{ad}(E)) ,$$

where $\Lambda(\mathcal{K}(\nabla))$ is defined in (4.2).

In [RS] and [ABi] the following theorem is proved.

Theorem 4.2 *If E admits a Hermitian–Einstein connection, then E is polystable. Conversely, if E is polystable, then it has a unique Hermitian–Einstein connection.*

For vector bundles, this theorem was proved in [Do], [UY].

Let

$$\rho : \pi_1(M, x_0) \longrightarrow K(G)$$

be a homomorphism from the fundamental group to the maximal compact subgroup. Consider $\widetilde{M} \times G$, where $\widetilde{M}$ is the universal cover of M. The group $\pi_1(M, x_0)$ acts on $\widetilde{M}$ as deck transformations, and it acts on G via ρ, that is, the action of $\gamma \in \pi_1(M, x_0)$ sends $g \in G$ to $\rho(\gamma)g \in G$. The quotient

$$E_\rho := \frac{\widetilde{M} \times G}{\pi_1(M, x_0)}$$

for the diagonal action of $\pi_1(M, x_0)$ is a principal G–bundle over M. The projection of E_ρ to M is obtained from the natural projection of $\widetilde{M}$ to M.

Since we have $\rho(\pi_1(M, x_0)) \subset K(G)$, the trivial connection on the trivial G–bundle $\widetilde{M} \times G$ over $\widetilde{M}$ descends to a unitary flat connection on E_ρ. (Trivial connection was defined in Section 2.) All G–bundles over M with a flat unitary connection arise this way.

If a G–bundle E has a flat connection, then all the rational characteristic classes of E of positive degree vanish [At2]. Therefore, Theorem 4.2 has the following corollary:

Corollary 4.3 *A G–bundle E over M is constructed from a homomorphism from $\pi_1(M)$ to $K(G)$ (that is, it admits a flat unitary connection) if and only if E is polystable and all the rational characteristic classes of E of degrees one and two vanish.*

References

[At1] M. F. Atiyah: On the Krull–Schmidt theorem with application to sheaves. *Bull. Soc. Math. Fr.* **84** (1956), 307–317.

[At2] M. F. Atiyah: Complex analytic connections in fibre bundles. *Trans. Amer. Math. Soc.* **85** (1957), 181–207.

[AAB] B. Anchouche, H. Azad and I. Biswas: Harder–Narasimhan reduction for principal bundles over a compact Kähler manifold. *Math. Ann.* **323** (2002), 693–712.

[ABi] B. Anchouche and I. Biswas: Einstein–Hermitian connections on polystable principal bundles over a compact Kähler manifold. *Amer. Jour. Math.* **123** (2001), 207–228.

[AB] H. Azad and I. Biswas: On holomorphic principal bundles over a compact Riemann surface admitting a flat connection. *Math. Ann.* **322** (2002), 333–346.

[BS] I. Biswas and S. Subramanian: Connection and principal bundles over curves in positive characteristics. *Pacific Jour. Math.* **208** (2003), 11–22.

[Bo] A. Borel: *Linear algebraic groups.* Graduate Texts in Mathematics, 126. Springer-Verlag, New-York, 1991.

[Do] S.K. Donaldson: Infinite determinants, stable bundles and curvature. *Duke Math. Jour.* **54** (1987), 231–247.

[FMW] R. Friedman, J.W. Morgan and W. Witten: Principal G-bundles over elliptic curves. *Math. Res. Lett.* **5** (1998), 97–118.

[Gi] J. Giraud: *Cohomologie non abélienne.* Die Grundlehren der mathematischen Wissenschaften, Band 179, Springer–Verlag, Berlin–New York, 1971.

[Ko] S. Kobayashi: *Differential Geometry of Complex Vector Bundles.* Publications of the Math. Society of Japan 15. Iwanami Shoten Publishers and Princeton University Press, 1987.

[KN] S. Kobayashi and K. Nomizu: *Foundations of differential geometry. Vol. II.* Interscience Publishers, a division of John Wiley & Sons, New York-London, 1963.

[Ra1] A. Ramanathan: Stable principal bundles on a compact Riemann surface. *Math. Ann.* **213** (1975), 129–152.

[Ra2] A. Ramanathan: Moduli for principal bundles over algebraic curves: I. *Proc. Ind. Acad. Sci. (Math. Sci.)* **106** (1996), 301–328.

[Ra3] A. Ramanathan: Moduli for principal bundles over algebraic curves: II. *Proc. Ind. Acad. Sci. (Math. Sci.)* **106** (1996), 421–449.

[RS] A. Ramanathan and S. Subramanian: Einstein–Hermitian connections on principal bundles and stability. *Jour. Reine Angew. Math.* **390** (1988), 21–31.

[Se] J.-P. Serre: Géométrie algébrique et géométrie analytique. *Ann. Inst. Fourier* **6** (1955), 1–42.

[Sp] T.A. Springer: *Linear algebraic groups.* Second edition. Progress in Mathematics 9. Birkhäuser Boston, Inc., Boston, MA, 1998.

[UY] K. Uhlenbeck and S.-T. Yau: On the existence of Hermitian–Yang–Mills connections on stable vector bundles. *Commun. Pure Appl. Math.* **39** (1986), 257–293.

[We] A. Weil: Généralisation des fonctions abéliennes. *Jour. Math. Pures Appl.* **17** (1938), 47–87.