CUBO *a Mathematical Journal*
Vol. 6, № 1, (115-121). March 2004.

On Brouwer's Fixed Point Theorem[1]

N. Tarkhanov
Universität Potsdam
Institut für Mathematik
Postfach 60 15 53
14415 Potsdam
Germany
tarkhanov@math.uni-potsdam.de

ABSTRACT

The paper presents an explicit formula for the number of fixed points of a C^∞ map of a segment $[a, b] \subset \mathbb{R}$. While the formula can be derived from the Lefschetz fixed point theorem for general CW-complexes, the new proof is instructive and highlights the contributions of degenerate fixed points.

1 Introduction

In 1910 Brouwer (cf. [Bro12]) proved that any continuous map $f : S \to S$ of an n-dimensional simplex has at least one fixed point $p \in S$, i.e., $f(p) = p$. This result extends to continuous maps of any compact convex body in a finite dimensional topological vector space and has numerous applications to proving existence theorems for diverse equations. Moreover, the fixed point theorem generalises to infinite dimensional topological vector spaces. When thinking over this result, Lefschetz [Lef26] showed a formula which expresses the number of fixed points of a map of an orientable topological manifold through the traces of the corresponding pull-back operators on the cohomology. Although a fixed point theorem à la Lefschetz has nowadays been known for general CW-complexes, cf. [Dol72], it gives no explicit description of the contribution of a non-interior fixed point. In [BS91] an explicit formula is obtained for the contribution of a simple boundary fixed point. A fixed point on the boundary

[1] AMS Subject Classification: primary: 32S50; secondary: 58G10.
Key words and phrases: de Rham complex, Lefschetz number, fixed points.

contributes to the Lefschetz number only in the case when it is attracting. The purpose of the present paper is to explicitly evaluate contributions for arbitrary isolated fixed points on the boundary. While the proof of [BS91] is an exposition of arguments of [AB67] in the context of Boutet de Monvel's algebra [BdM71], the present paper uses a constructive approach elaborated in [Tar95].

To make the proof completely transparent we restrict ourselves to the case $n = 1$. Let $M = [a, b]$ be a closed interval on the real axis, and f be a smooth map of $[a, b]$ to itself.

2 The Neumann problem

In order to derive an integral formula for the Lefschetz number of f we need a parametrix of the de Rham complex on $[a, b]$

$$0 \longrightarrow \mathcal{E}[a,b] \xrightarrow{d} \mathcal{E}^1[a,b] \longrightarrow 0 \qquad (2.1)$$

where $\mathcal{E}^1[a, b]$ is the space of differential forms of degree 1 with C^∞ coefficients on $[a, b]$.

The Neumann problem at extreme step 1 consists of finding, for a given $F \in \mathcal{E}^1[a, b]$, a differential form $u \in \mathcal{E}^1[a, b]$ such that

$$\begin{array}{rcll} dd^*u & = & F & \text{in} \quad (a,b), \\ n(u) & = & 0 & \text{on} \quad \partial(a,b), \end{array} \qquad (2.2)$$

where d^* is the formal adjoint for d. Note that the Neumann boundary condition $n(u) = 0$ appears from the consideration of d^* as the Hilbert space adjoint, cf. for instance § 4.2 in [Tar95].

Write

$$\begin{array}{rcl} F & = & F_1(x)\,dx, \\ u & = & u_1(x)\,dx, \end{array}$$

then the problem (2.2) becomes just the Dirichlet problem for the function $u_1(x)$, namely

$$\begin{array}{rcll} -u_1'' & = & F_1 & \text{in} \quad (a,b), \\ u_1(a) & = & u_1(b) & = \quad 0. \end{array} \qquad (2.3)$$

The general solution of $-u_1'' = F_1$ is

$$u_1(x) = \Big(Ax + B\Big) - \int_a^x (x-y)F_1(y)dy,$$

and the substitution of this expression to the boundary conditions gives

$$\begin{array}{rcl} B & = & -aA, \\ A & = & \displaystyle\int_a^b \frac{b-y}{b-a} F_1(y)\,dy. \end{array}$$

The solution

$$u_1(x) = (x-a)\int_a^b \frac{b-y}{b-a} F_1(y)\,dy - \int_a^x (x-y)\,F_1(y)\,dy$$

is $L^2[a,b]$-orthogonal to the space of solutions of the homogeneous problem corresponding to (2.3), for this latter is zero. It follows that the Neumann operator at step 1 is

$$NF(x) = \int_a^b F(y)\Big((x-a)\frac{b-y}{b-a} - (x-y)\,\Theta(x-y)\Big)dy, \tag{2.4}$$

the integral being over $y \in [a,b]$.

3 A parametrix of the de Rham complex

Given any $F \in \mathcal{E}^1[a,b]$, set

$$\begin{aligned} PF(x) &= d^* NF(x) \\ &= \int_a^b \Big(\Theta(x-y) - \frac{b-y}{b-a}\Big)F(y), \end{aligned} \tag{3.1}$$

i.e., $P = d^* N$.

Lemma 3.1 *As defined in (3.1), the operator P satisfies*

$$\begin{aligned} P\,du &= u - S^0 u \quad \textit{for all} \quad u \in \mathcal{E}[a,b], \\ dPF &= F - S^1 F \quad \textit{for all} \quad F \in \mathcal{E}^1[a,b], \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} S^0 u &= u - \frac{1}{b-a}\int_a^b u(y)dy, \\ S^1 F &= 0. \end{aligned}$$

Proof. It suffices to prove only the first equality of (3.2), for the second one is obvious. To this end, write

$$\begin{aligned} P\,du(x) &= \int_a^b \Big(\Theta(x-y) - \frac{b-y}{b-a}\Big)u'(y)dy \\ &= \Big(u(x) - u(a)\Big) - \frac{b-y}{b-a}u(y)\Big|_a^b + \int_a^b u(y)\,d\frac{b-y}{b-a} \\ &= u(x) - \frac{1}{b-a}\int_a^b u(y)dy, \end{aligned}$$

as desired. □

4 A formula for the Lefschetz number

By the Lefschetz number of the map $f : [a,b] \to [a,b]$ is meant

$$L(f) = \mathrm{tr}\,(Hf)_0 - \mathrm{tr}\,(Hf)_1$$

where

$$\begin{array}{rcl} (Hf)_0 & : & H^0(\mathcal{E}^\cdot[a,b]) \to H^0(\mathcal{E}^\cdot[a,b]), \\ (Hf)_1 & : & H^1(\mathcal{E}^\cdot[a,b]) \to H^1(\mathcal{E}^\cdot[a,b]) \end{array}$$

are endomorphisms of the cohomology of the de Rham complex on $[a,b]$, induced by the pull-back operator $f^\sharp$ on differential forms. Since the cohomology is finite dimensional at every step, the traces of these endomorphisms are well defined.

Note that in fact $L(f) = 1$ in our special case, for $H^0(\mathcal{E}^\cdot[a,b]) \cong \mathbb{C}$ and $H^1(\mathcal{E}^\cdot[a,b]) \cong 0$.

Lemma 4.1 *Suppose all fixed points of the map f on $[a,b]$ are isolated. Then*

$$L(f) = \mathrm{p.v.} \int_a^b d\Big(\Theta(f(y) - y) - \frac{b-y}{b-a}\Big). \qquad (4.1)$$

Proof. Applying the pull-back operator $f^\sharp$ to both sides of equalities (3.2) we get

$$\begin{array}{rcl} (f^\sharp P)\,d & = & f^\sharp - f^\sharp S^0, \\ d\,(f^\sharp P) & = & f^\sharp - f^\sharp S^1, \end{array}$$

for $f^\sharp$ and d commute.

Since $f^\sharp P$ maps $\mathcal{E}^1[a,b]$ to $\mathcal{E}[a,b]$ we deduce that $f^\sharp$ and $f^\sharp S$ are homotopic endomorphisms of the de Rham complex. Hence they induce the same endomorphisms of the cohomology, i.e., $Hf = H(f^\sharp S)$. It follows that $L(f) = L(f^\sharp S)$.

We now observe that $f^\sharp S$ is a trace class endomorphism of the de Rham complex. By the alternating sum formula (cf. for instance Theorem 19.1.15 of [Hör85]) we obtain

$$\begin{array}{rcl} L(f) & = & \mathrm{Tr}\, f^\sharp S^0 - \mathrm{Tr}\, f^\sharp S^1 \\ & = & \displaystyle\int_a^b \Delta^\sharp\Big((f\times 1)^\sharp K_{S^0} - (f\times 1)^\sharp K_{S^1}\Big) \end{array}$$

where Δ stands for the diagonal map $[a,b] \to [a,b] \times [a,b]$, and K_S is the Schwartz kernel of S.

By assumption, the set $\mathrm{Fix}(f,[a,b])$ is discrete. Since the integrand is of class $L^1[a,b]$, we get

$$L(f) = \lim_{\varepsilon\to 0} \int_{[a,b]\setminus U_\varepsilon} \Delta^\sharp\Big((f\times 1)^\sharp K_{S^0} - (f\times 1)^\sharp K_{S^1}\Big)$$

where U_ε is the set of all points $y \in [a,b]$ whose distance to $\mathrm{Fix}(f,[a,b])$ is less than ε.

We now make use of equalities (3.2) to evaluate the integrand in the latter integral. Namely, they imply that

$$\begin{aligned} d'_y K_P &= -K_{S^0}, \\ d_x K_P &= -K_{S^1} \end{aligned}$$

away from the diagonal of $[a,b] \times [a,b]$. It follows that

$$\begin{aligned} \Delta^\sharp\Big((f\times 1)^\sharp K_{S^0} - (f\times 1)^\sharp K_{S^1}\Big) &= \Delta^\sharp\Big(- d'_y (f\times 1)^\sharp K_P + d_x (f\times 1)^\sharp K_P\Big) \\ &= d\Big(\Delta^\sharp (f\times 1)^\sharp K_P\Big) \end{aligned}$$

holds on $[a,b] \setminus U_\varepsilon$, whence

$$\begin{aligned} L(f) &= \lim_{\varepsilon\to 0} \int_{[a,b]\setminus U_\varepsilon} d\Big(\Delta^\sharp (f\times 1)^\sharp K_P\Big) \\ &= \text{p.v.} \int_a^b d\Big(\Theta(f(y)-y) - \frac{b-y}{b-a}\Big). \end{aligned}$$

This proves the formula. □

5 Fixed point theorem

The second term in the integral (4.1) is easily evaluated, hence this formula transforms to

$$L(f) = 1 + \text{p.v.} \int_a^b d\Theta(f(y)-y).$$

Theorem 5.1 *Let f be a C^∞ map of the segment $[a,b]$ with isolated fixed points. Then*

$$L(f) = 1 + \Theta(f(y)-y)\Big|_{a+}^{b-} + \sum_{p\in \mathrm{Fix}(f,(a,b))} \mu(p)$$

where $\mu(p)$ is the local degree of $1-f$ at p.

Proof. Write $a < p_1 < \ldots < p_N < b$ for the fixed points of f that lie in the open interval (a,b). Since the function $\Theta(f(y)-y)$ is constant away from the set of fixed points of f we get

$$L(f) = 1 + \int_{a+\varepsilon}^{p_1-\varepsilon} d\Theta(f(y)-y) + \sum_{k=1}^{N} \int_{p_k+\varepsilon}^{p_{k+1}-\varepsilon} d\Theta(f(y)-y) + \int_{p_N+\varepsilon}^{b-\varepsilon} d\Theta(f(y)-y)$$

for all $\varepsilon > 0$ small enough. Hence it follows that

$$L(f) = 1 - \Theta(f(y)-y)\Big|_{b-\varepsilon}^{a+\varepsilon} - \sum_{k=1}^{N} \Theta(f(y)-y)\Big|_{p_k-\varepsilon}^{p_k+\varepsilon}.$$

A passage to the limit when $\varepsilon \to 0+$ gives the desired formula, for the local degree of $1 - f$ at p is opposite to that of $f - 1$.

□

If a is a simple fixed point of f then the sign of $(1 - f)(a+)$ already uniquely determines the local degree of any smooth extension of $1 - f$ to a neighbourhood of a. Namely the local degree of $1 - f$ at a just amounts to $\operatorname{sign}(1 - f)(a+)$, or $\operatorname{sign}(1 - f'(a))$. The same reasoning applies to the case where b is a simple fixed point of f. However these arguments no longer work if a or b is not simple, for f can be extended to a smooth function in a neighbourhood of a respectively b in diverse manners. For this reason we need another specification of fixed points of f on the boundary. Suppose $f(a) = a$. Then a is said to be an attracting fixed point of f if $(1 - f)(a+) > 0$, and repulsing if $(1 - f)(a+) < 0$. If $f(b) = b$ then the fixed point b is called attracting if $(1 - f)(b-) < 0$, and repulsing if $(1 - f)(b-) > 0$. For the attracting fixed points on the boundary we define the local degree of $1 - f$ to be 1, and for the repulsing fixed points we define the local degree of $1 - f$ to be -1. Then Theorem 5.1 can be reformulated in the following way.

Corollary 5.2 *Let f be a C^∞ map of the segment $[a, b]$ with isolated fixed points. Then*

$$L(f) = \sum_{p \in \operatorname{Fix}(f,(a,b)) \cup \operatorname{Fix}^{(a)}(f,\partial(a,b))} \mu(p),$$

$\operatorname{Fix}^{(a)}(f, \partial(a, b))$ *being the set of attracting fixed points of f on the boundary.*

For the smooth maps of $[a, b]$ the fixed point theorem of Brouwer [Bro12] is an obvious consequence of Corollary 5.2 because $L(f) = 1$.

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