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Groups in action: from Euclidean to polynomial crystallographic groups

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Introduction

Although mathematics is often presented as consisting of a disjoint collection of subsets, such as analysis, geometry, algebra, combinatorics, topology, ..., many interesting results can only be achieved via a combination of techniques and results from different fields.

In this survey paper, we present a research topic in which one tries to build a bridge between a group theoretical topic on the one hand and a more geometrical or topological component on the other hand. In fact, the main idea is not really new, since group theory came into being as an abstraction of the notion of "symmetry".

The main goal of the research can be described rather vaguely as follows:

Find a geometrical context to describe the class of polycyclic-by-finite groups.

The class of polycyclic-by-finite groups will be described more detailed below, but for the moment one can think about all groups which are built up using only the infinite cyclic group of integers $\mathbb{Z}$ (with the addition as operation) and finite groups.

To be able to follow the ideas developed in this paper, the reader should be familiar with the basic concepts of group theory, such as "normal subgroup" and "quotient subgroup". It is an advantage (but not really neccesarry) to be acquainted with the concept of a topological space and compact subsets of a topological space.

1 Group actions

By *a geometrical context* as mentioned in the "main goal" above, we will in fact mean an interesting type of group actions.

Definition 1.1 *Let G be a group. We say that the group G acts (on the left) on a set X if a map*

$$\mu : G \times X \to X$$

is given, which satisfies the following two conditions:

1. $\forall g, h \in G,\ \forall x \in X :\ \mu(g, \mu(h, x)) = \mu(gh, x)$
2. $\forall x \in X : \mu(1, x) = x$

where 1 stands for the neutral element of the group G. The map μ is often omitted from the notations and one writes $g \cdot x$ or ${}^g x$ in stead of $\mu(g, x)$. We will use the later notation, and so the conditions of a group action are written as follows:

$${}^g({}^h x) = {}^{gh}x \text{ and } {}^1 x = x.$$

Given a group action of a group G on a set X, any element $g \in G$ determines a map

$$\lambda_g : X \to X :\ x \mapsto {}^g x.$$

Any such map λ_g determines a permutation (i.e. one-to-one and onto map) of X. In fact, if we denote the group of all permutations of$X$ by $\mathcal{S}(X)$, any group action determines a morphism

$$\lambda : G \to \mathcal{S}(X) : g \mapsto \lambda_g.$$

Conversely, any morphism $\lambda : G \to \mathcal{S}(X)$, determines a group action of G on X by letting ${}^g x = \lambda(g)(x)$.

One is often interested in actions for which $\lambda(G)$ lies in an interesting subgroup of $\mathcal{S}(X)$. For example, if X is a topological space, one can consider the group $\mathcal{H}(X)$ of homeomorphisms of X, if X is a smooth manifold, one can consider the group $C^\infty(X)$ of smooth diffemorphisms of X, if X is a metric space, one looks for actions into the group $\mathrm{Isom}(X)$ of isometries of X.

In this paper X will always be the set $\mathbb{R}^n$ for some n. This is a well known set which is a topological space, a smooth manifold and can be equipped with several metrics to turn it into a metric space. The most common of these metrics is of course the usual Euclidean metric.

Examples

1. Let G be any group and let H be a subgroup of G (e.g. one can take $H = G$). Then there are two well known actions of H on G:

   (a) The left multiplication or left translation action where

   $$\forall h \in H,\ \forall g \in G :\ {}^h g = h \cdot g$$

   (where $h \cdot g$ is just multiplication in G)

(b) The conjugation action, defined by

$$\forall h \in H,\ \forall g \in G:\ {}^{h}g = h \cdot g \cdot h^{-1}.$$

2. Let us specialise the above action to the situation where G is the group $\mathbb{R}^n$ (the group operation is of course addition) and H is the subgroup $\mathbb{Z}^n$. It is obvious that the conjugation action is trivial (as is always the case if G is abelian, or more general if H is central), while for the left translation action we have that whenever $z \in \mathbb{Z}^n$ and $z \neq (0, 0, \ldots, 0)$ and for any $r \in \mathbb{R}^n$ that ${}^{z}r = z + r \neq r$.

3. Let D_∞ be the infinite dihedral group. This group is generated by two elements a and b, where a is an element of infinite order and b is an element of order 2, so $b^2 = 1$. Moreover, the relation $ba = a^{-1}b$ holds. It follows that any element of D_∞ can be written uniquely as a product of the form $a^k b^\epsilon$, where $k \in \mathbb{Z}$ and $\epsilon \in \{0, 1\}$. The group D_∞ acts on the real line $\mathbb{R}$ as follows:

$$\forall k \in \mathbb{Z}:\ \forall \epsilon \in \{0,1\},\ \forall r \in \mathbb{R}:\ {}^{a^k b^\epsilon}r = (-1)^\epsilon r + k.$$

4. Let $\varphi : H \to \mathrm{Aut}(G)$ be a morphism of groups. This is a special type of actions ($\mathrm{Aut}(G)$ is a special subset of $\mathcal{S}(G)$). Having such an action, we can define a new group, denoted by $G \rtimes H$ (or $G \rtimes_\varphi H$), whose elements are tuples $(g, h) \in G \times H$ and where the product is given by $(g_1, h_1)(g_2, h_2) = (g_1 \varphi(h_1)(g_2), h_1 h_2)$. This new product is called a semidirect product of G and H. For example, given any group G, we can always form the semidirect product $G \rtimes \mathrm{Aut}(G)$.

The left translation action of $\mathbb{Z}^n$ on $\mathbb{R}^n$ and the action of D_∞ on $\mathbb{R}$ are typical examples of what we mean by interesting types of actions. To be able to really describe what is *interesting*, we need a few more notions.

Definition 1.2 *Let G be a group acting on a set X. The stabilisor of an element $x \in X$ is given as*

$$G_x = \{g \in G \mid {}^{g}x = x\}.$$

One can check that the stabilisor of any point is a subgroup of G. In the example of the action of D_∞ on $\mathbb{R}$ we have that the stabilisor of the point $r = \frac{1}{2}$ is the subgroup $\{1, ab\}$, while the stabilisor of the point $r = \pi$ is the trivial subgroup $\{1\}$. If the action of a group G on X is trivial (${}^{g}x = x$, $\forall g \in G$, $\forall x \in X$) then all stabilisors are equal to the whole group G. Of course, this is really not the interesting case. On the contrary, it is obvious that we will be interested in group actions with small stabilisors and ideally with only trivial stabilisors. These kind of actions are called free.

Definition 1.3 *An action of a group G on a set X is said to be free if*

$$\forall g \in G:\ \forall x \in X:\ {}^{g}x = x \Rightarrow g = 1.$$

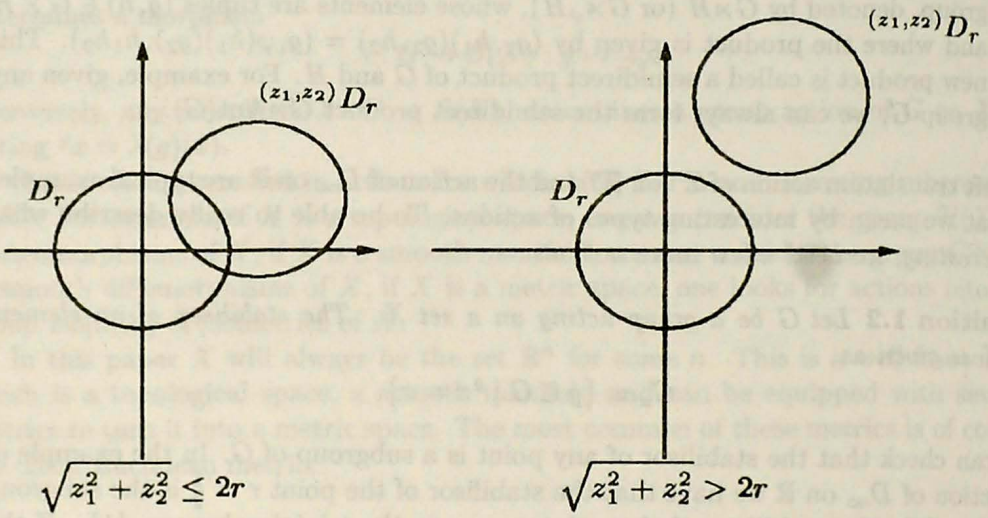
In fact, it will not be enough that under the action of a group G the points are moved away from there original position by all but finitely many elements of G, but they have to be moved away *far enough*. This idea is catched in the following notion:

Definition 1.4 *Let X be a locally compact topological space and $\lambda : G \to \mathcal{H}(X)$ denote an action of G on X. Then, G is said to act properly discontinuously if for any compact subset $C \subseteq X$, we have that*

$$\{g \in G \mid {}^gC \cap C \neq \emptyset\} \text{ is finite.}$$

When specialising to the situation where $X = \mathbb{R}^n$, we can say that an action is properly discontinuous if and only if for any $r \in \mathbb{R}$, we have that the set $\{g \in G \mid {}^gD_r \cap D_r \neq \emptyset\}$, where $D_r = \{x \in \mathbb{R}^n \mid \|x\| \leq r\}$ is the disk of radius r, is finite.

The action of $\mathbb{Z}^2$ on $\mathbb{R}^2$ by means of left translations is properly discontinuous. Indeed, fix an $r \in \mathbb{R}$. It is obvious that an element $(z_1, z_2) \in \mathbb{Z}^2$ moves any point over a distance $\sqrt{z_1^2 + z_2^2}$. This implies that whenever $\sqrt{z_1^2 + z_2^2} > 2r$ the image ${}^{(z_1,z_2)}D_r$ will have nothing in common with the original disk D_r. As there are only finitely many elements in $\mathbb{Z}^2$ satisfying $\sqrt{z_1^2 + z_2^2} \leq 2r$, the set $\{(z_1, z_2) \in \mathbb{Z}^2 \mid {}^{(z_1,z_2)}D_r \cap D_r \neq \emptyset\}$ is finite.

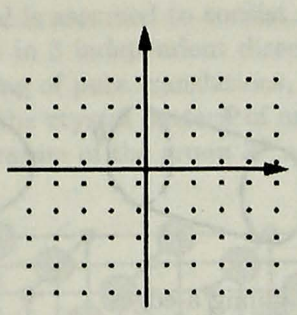


The action of $\mathbb{Z}^2$ on $\mathbb{R}^2$ is properly discontinuous

The last concept to be introduced in this section on group actions is that of orbits and orbit spaces.

Definition 1.5 *Let G act on a set X as usual. The orbit of a point x is the set $G \cdot x = \{{}^gx \mid\mid g \in G\}$.*

We illustrate this again by means of the left translation action of $\mathbb{Z}^2$ on $\mathbb{R}^2$.

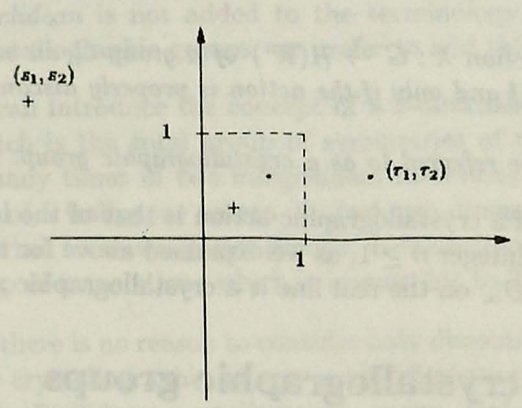


Orbit of a point under the action of $\mathbb{Z}^2$ on $\mathbb{R}^2$

A crucial object in this discussion is that of the orbit space of a group action.

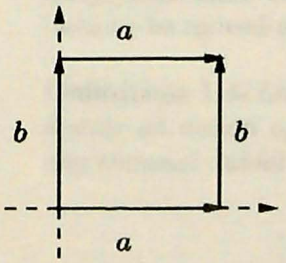
Definition 1.6 *Let G be a group acting on a set X. The set of all possible orbits is called the orbit space and is denoted by ${}_G\backslash X$*

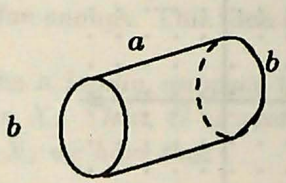
Thus ${}_G\backslash X = \{G \cdot x \mid x \in X\}$. Let us retrun to the example of $\mathbb{Z}^2$ acting on the plane. We claim that the orbit space can be identified with a torus. First of all note that for any point $(r_1, r_2) \in \mathbb{R}^2$, there exists a $(z_1, z_2) \in \mathbb{Z}^2$ such that ${}^{(z_1,z_2)}(r_1, r_2)$ belongs to the unit square $I \times I$, where $I = [0, 1]$.



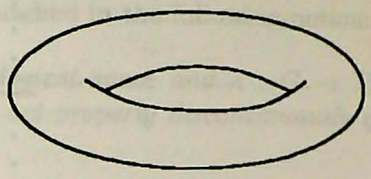
${}^{(-1,0)}(r_1, r_2) \in I \times I$ and ${}^{(2,-1)}(s_1, s_2) \in I \times I$.

Any orbit can be represented by a point in the unit square $I \times I$. However, some orbits are represented by 2 (or even 4) points. Indeed for any $r \in I$, we have that $(r, 0)$ and $(r, 1)$ lie in the same orbit, and the same holds for $(0, r)$ and $(1, r)$. So if we want to represent any orbit by exactly 1 point we have to indentify the two horizontal edges of the square and also the two vertical edges. This is suggested in the picture below. We leave it to the reader's imagination to see that one eventually obtains a torus.





After gluing a-edges



After gluing b-edges

Without going into detail, we want to stress the fact that the identification of $\mathbb{Z}^2\backslash\mathbb{R}^2$ with a two-dimensional torus is not merely an identification of sets, but is really an identification of topological spaces. Indeed, the representatives of two different orbits are close to each other on the torus if and only if the orbits themselves are close (in the sense that there are two points, one from each orbit, which are close to each other). To make the above observations rigourous, we should talk about the quotient topology and this would lead us to far.

The fact that we obtain a compact orbit space will also be considered as relevant. In our context of properly discontinuous actions on some space $\mathbb{R}^n$, the orbit space will be compact if and only if there exists a real number $r \in \mathbb{R}$, such that every orbit has at least one point inside the closed disc D_r.

Now, we have introduced all the necessary concepts to describe what we mean by an interesting geometrical context. This will in fact be the context of crystallographic groups and actions.

Definition 1.7 *An action $\lambda : G \to \mathcal{H}(\mathbb{R}^n)$ of a group G on some space $\mathbb{R}^n$ is said to be crystallographic if and only if the action is properly discontinuous and the orbit space $_G\backslash X$ is compact.*
The image $\lambda(G)$ will be referred to as a crystallographic group.

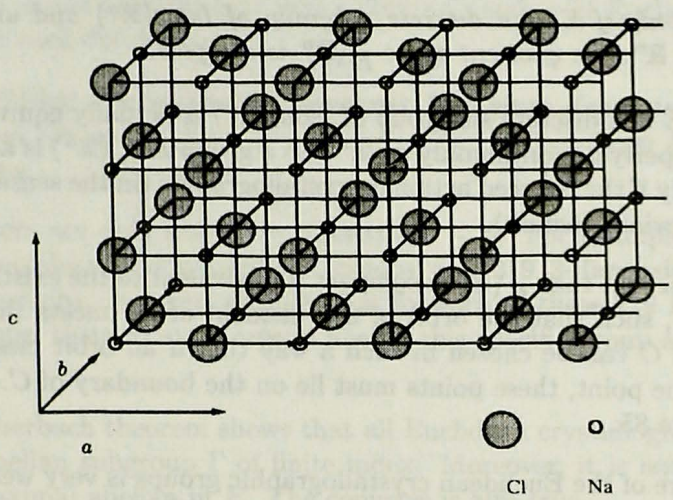
The typical example of a crystallographic action is that of the left translation action of $\mathbb{Z}^n$ on $\mathbb{R}^n$ for some integer $n \geq 1$, as we explained above for the case $n = 2$.

Also the action of D_∞ on the real line is a crystallographic action.

2 Euclidean crystallographic groups

The origin of the concept of a crystallographic group goes back to the study of genuine crystals. A crystal consists of a collection of atoms (or molecules) which are arranged in a very symmetric way. In fact, a crystal can be seen as a small pattern of atoms which is repeated over and over in three directions. To study these crystals mathematically, we will imaging that a crystal is infinite in size, so that this repetition of the pattern continues infinitely in all three directions. As crystals (how small they can seem) always consist of enormously many atoms, this idealisation is justified. To study a crystal, it is of great importance to study its group of symmetries. A symmetry of a crystal consists of a rigid motion of Euclidean 3-space mapping the crystal

exactly onto itself. As a crystal is assumed to consist of a pattern of atoms which is repeated infinitely many times in 3 independent directions, such a symmetry group always has a subgroup consisting of pure translations, which is isomorphic to $\mathbb{Z}^3$. To illustrate this, let us consider the crystal pattern of ordinary household salt (NaCl). In the picture below, the generators of the group $\mathbb{Z}^3$ are indicated by a, b and c.



Of course, this crystal also allows many more symmetries, like rotations and reflections. The total group of isometries is called a Euclidean crystallographic group. Often, the word *Euclidean* is not added to the terminology, but as we will treat different kinds of crystallographic groups, we prefer to add this to our terminology.

Analogously, we can introduce the concept of a 2-dimensional Euclidean crystallographic group, which is the total group of symmetries of a flat pattern which is repeated infinitely many times in two independent directions of a plane. As an example, we can think of a wallpaper patern. In fact two-dimensional crystallographic groups are often referred to as wallpaper groups. It is obvious that such a wallpaper group has a subgroup of translations which is isomorphic to $\mathbb{Z}^2$.

Mathematically, there is no reason to consider only dimensions 2 and 3, and therefore we can consider crystallographic groups in any dimension. We will first give the general definition of a Euclidean crystallographic group. Only after a little while, we will be able to explain that this definition is really a generalization of the notion of a group of isometries of a crystal.

The Euclidean crystallographic groups will be subgroups of the group of rigid motions of $\mathbb{R}^n$. This group will be denoted by $\mathrm{Isom}(\mathbb{R}^n)$ and is in fact a semidirect product $\mathbb{R}^n \rtimes O(n)$, where $O(n)$ stands for the group of orthogonal $n \times n$–matrices. Note that $\mathrm{Isom}(\mathbb{R}^n)$ is a subgroup of the group $\mathrm{Aff}(\mathbb{R}^n) = \mathbb{R}^n \rtimes \mathrm{GL}(n, \mathbb{R})$ of invertible affine mappings. Any element of $\mathrm{Aff}(\mathbb{R}^n)$ can be written as a pair (a, A) consisting of the translational part a and the linear part A. Note that there is an action of $\mathrm{Aff}(\mathbb{R}^n)$

and thus of any subgroup of $\mathrm{Aff}(\mathbb{R}^n)$ (and a fortiori of any subgroup of $\mathrm{Isom}(\mathbb{R}^n)$) which is given by

$$\forall (a,A) \in \mathrm{Aff}(\mathbb{R}^n),\ \forall x \in \mathbb{R}^n :\ {}^{(a,A)}x = a + Ax.$$

Definition 2.1 *A subgroup $E \subseteq \mathrm{Isom}(\mathbb{R}^n)$ is said to be a Euclidean crystallographic group, if and only if E is a discrete subgroup of $Isom(\mathbb{R}^n)$ and under the induced action of E on $\mathbb{R}^n$, the quotient space ${}_E\backslash\mathbb{R}^n$ is compact.*

The fact that E is a discrete subgroup of $\mathrm{Isom}(\mathbb{R}^n)$ is actually equivalent to the fact that E acts properly discontinuouly on $\mathbb{R}^n$. So a group $E \subseteq (\mathbb{R}^n)$ is a crystallographic group if and only if the induced action is crystallographic (in the sense of the definition given in the previous section).

Note that the fact that ${}_E\backslash\mathbb{R}^n$ is compact is equivalent to the existing of a bounded subset C of $\mathbb{R}^n$, such that the orbit of any element of $\mathbb{R}^n$ meets that subset C. In fact, the subset C can be chosen in such a way that if an orbit meets the subset C in more than one point, these points must lie on the boundary of C. We refer to the example on page 85.

The structure of the Euclidean crystallographic groups is very well understood by three famous theorems which are due to Bieberbach ([7],[8],[9], see also [12] and [30]).

Theorem 2.2 (First Bieberbach Theorem) *Let $E \subseteq \mathrm{Isom}(\mathbb{R}^n)$ be a Euclidean crystallographic group and let $\Gamma = E \cap \mathbb{R}^n$ be its subgroup of pure translations. Then $\Gamma \cong \mathbb{Z}^n$ and Γ spans the whole vectorspace $\mathbb{R}^n$. Moreover E/Γ is finite.*

Let us interpret this theorem in terms of the orbits of the action of E on $\mathbb{R}^n$. As E/Γ is finite, we can find a finite number of elements $e_1, e_2, \ldots, e_k \in E$ such that any other element $e \in E$ can be written as a product of the form $e = \gamma e_i$, for some $i \in \{1, 2, \ldots k\}$ and some $\gamma \in \Gamma$.

Let C still denote the set introduced above. Then any element of $\mathbb{R}^n$ can be written as

$$x = {}^{e}c = {}^{\gamma e_i}c = {}^{\gamma}({}^{e_i}c).$$

Therefore, we can consider $D = {}^{e_1}C \cup {}^{e_2}C \cup \cdots \cup {}^{e_k}C$, which is still a bounded subset of $\mathbb{R}^n$. With this notation, we obtain that any element x of $\mathbb{R}^n$ can be written as $x = {}^{\gamma}d$, for some $d \in D$. Otherwise stated, this says that $\mathbb{R}^n$ can be built up from a bounded piece which is being translated infinitely many times in n linear independent directions.

This already shows that the mathematical concept of a cystallographic group can be seens as a generalization of the 3-dimensional crystallographic groups we started from.

The second Bieberbach theorem shows that algebraic equivalence of Euclidean crystallographic groups is the same as geometrical equivalence.

Theorem 2.3 (Second Bieberbach Theorem) *Let $E, E' \subseteq \mathrm{Isom}(\mathbb{R}^n)$ be two Euclidean crystallographic groups and suppose that $\varphi : E \to E'$ is an isomorphism. Then there exists an element $\alpha \in \mathrm{Aff}(\mathbb{R}^n)$ such that*

$$\forall e \in E : \ \varphi(e) = \alpha e \alpha^{-1}.$$

The last theorem shows that there is only a limited number of Euclidean crystallographic groups in each dimension.

Theorem 2.4 (Third Bieberbach Theorem) *In a given dimension n, there are, up to isomorphism (or up to affine conjugation), only finitely many Euclidean crystallographic groups.*

The exact numbers are only known in dimensions ≤ 6. For example, there are 17 2-dimensional Euclidean crystallographic groups and 219 3-dimensional Euclidean crystallographic groups. It is remarkable, that for each of these 219 Euclidean crystallographic groups, there exists a crystal having this specific group as its symmetry group.

The first Bieberbach theorem shows that all Euclidean crystallographic groups E contain a free abelian subgroup Γ of finite index. Moreover, it is not hard to check that this Γ is maximal abelian in E. The converse is also true

Theorem 2.5 *[31] Let E be a group containing a finitely generated free abelian subgroup Γ of finite index which is maximal abelian in E. Then there exists an injective homomorphism $\varphi : E \to \mathrm{Isom}(\mathbb{R}^n)$ such that $\varphi(E)$ is a Euclidean crystallographic group.*

This shows that class of Euclidean crystallographic groups has a nice algebraic description. Groups satisfying this algebraic description are sometimes called abstract Euclidean crystallographic groups.

We now turn to the slightly more general setting of Euclidean crystallographic actions.

Definition 2.6 *A Euclidean crystallographic action of a group E on $\mathbb{R}^n$ consists of a morphism $\varphi : E \to \mathrm{Isom}(\mathbb{R}^n)$ letting E act crystallographically on $\mathbb{R}^n$.*

The fact that E acts crystallographically on $\mathbb{R}^n$ is equivalent to the fact that the kernel $\mathrm{Ker}(\varphi)$ of φ is finite and that $\varphi(E)$ is a Euclidean crystallographic group.

As we will often work with groups containing a normal finite index subgroup satisfying a ceratin group theoretic property, let us introduce the notion of groups which are *virtually P*:

Definition 2.7 *Let $\mathcal{P}$ denote a property of groups (e.g. $\mathcal{P}$ means abelian), then we say that a group G is virtually $\mathcal{P}$ if G contains a normal subgroup of finite index which is $\mathcal{P}$.*

Now, given any finitely generated virtually abelian group E, there exists a unique maximal finite normal subgroup H of E. One can show that E/H then contains a free abelian subgroup of finite index which is maximal abelian, thus E/H is an abstract Euclidean crystallographic group and we can find a monomorphism $\psi : E/H \to \mathrm{Isom}(\mathbb{R}^n)$, with $\psi(E/H)$ Euclidean crystallographic. If we then consider the composition $\varphi : E \xrightarrow{p} E/H \xrightarrow{\psi} \mathrm{Isom}(\mathbb{R}^n)$ we obtain that E admits a Euclidean crystallographic action. The converse also holds, any group E admitting a Euclidean crystallographic action has an abelian subgroup of finite index. We can conclude with the following:

> **Conclusion:** The class of groups admitting a Euclidean crystallographic action is exactly the class of finitely generated virtually abelian groups. Moreover, given a group E, then there is, up to affine conjugation, only one possible Euclidean crystallographic action of E

The uniqueness statement follows from the second Bieberbach theorem and from the fact that one can show that for any Euclidean crystallographic action $\varphi : E \to \mathrm{Isom}(\mathbb{R}^n)$ the kernel of φ is exactly the unique maximal finite normal subgroup H of E.

The conclusion above shows that there is a nice geometric context describing the class of virtually abelian groups. In the rest of the paper we will try to extend the concept of Euclidean crystallographic groups to obtain geometric characterizations of other classes of groups.

3 Almost crystallographic groups

To define Euclidean crystallographic groups, we needed the group $\mathrm{Isom}(\mathbb{R}^n) = \mathbb{R}^n \rtimes O(n)$ of isometries of $\mathbb{R}^n$. The group $O(n)$ is a compact group of automorphisms of the abelian group $\mathbb{R}^n$. As a first generalization, we will consider other groups, together with a compact group of automorphisms. As we do not want to go to far away from the original point of view, we will consider a class of groups which is very close to being abelian and which are called nilpotent groups.

To be able to define nilpotent groups, we need the concept of commutators and commutator subgroups.

Definition 3.1 *Let G be a group.*
If $g, h \in G$, then the commutator of g and h equals $[g,h] = g^{-1}h^{-1}gh$.
If $H, K \subseteq G$, then $[H,K]$ is the group generated by all commutators $[h,k]$, where $h \in H$ and $k \in K$.

Note that a group G is abelian if and only if $[G,G]$ is the trivial group.

This inspires us to define the lower central series of a group:

Definition 3.2 *Let G be a group, then the lower central series of G is defined recursively by $\gamma_1(G) = G$, and for all non-negative integers i: $\gamma_{i+1}(G) = [G, \gamma_i(G)]$.*

The lower central series is a decreasing series

$$G = \gamma_1(G) \supseteq \gamma_2(G) \supseteq \gamma_3(G) \supseteq \gamma_4(G) \supseteq \cdots$$

Definition 3.3 *A group G is said to be nilpotent if there exists an integer c for which $\gamma_{c+1}(G) = 1$. The smallest c for which this happens is called the nilpotency class of the group.*

The abelian groups are precisely the nilpotent groups of class 1.

The nilpotent groups we need are the simply connected, connected nilpotent Lie groups. These are groups N which are homeomorphic to a space $\mathbb{R}^n$ (and n is also the dimension of the Lie group N) and which can be realized as a group of unitriangular matrices (upper triangular matrices with 1's on the diagonal). The easiest non-abelian example of such a group is the Heisenberg group

$$H = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid\mid x, y, z \in \mathbb{R} \right\}.$$

of all unitriangular 3×3–matrices. The group of continuous automorphisms of N, which we denote by $\mathrm{Aut}(N)$ (although it is not the full group of autmorphisms of N as an abstract group) has a natural topology (the compact-open topology). In case N is the abelian group $\mathbb{R}^n$, $\mathrm{Aut}(\mathbb{R}^n) = \mathrm{GL}(n, \mathbb{R})$. Just like the group $\mathrm{Aff}(\mathbb{R}^n)$ acts on $\mathbb{R}^n$, we also have that the group $N \rtimes \mathrm{Aut}(N)$, which we will denote by $\mathrm{Aff}(N)$, acts on N in the following way:

$$\forall m, n \in N : \ \forall \alpha \in \mathrm{Aut}(N) : \ {}^{(m,\alpha)}n = m \cdot \alpha(n).$$

Let C be any compact subgroup of $\mathrm{Aut}(N)$. Thus C plays the role of $O(n)$. In fact, given C, there exists a metric on N for which $N \rtimes C \subseteq \mathrm{Isom}(N)$. So any subgroup of $N \rtimes C$ acts as a group of isometries on N. We can now define the notion of an almost-crystallographic group in a completely analogous way to the original notion of a Euclidean crystallographic group.

Definition 3.4 *A subgroup $E \subseteq N \rtimes C$ is said to be an almost crystallographic group, if and only if the action of E is a discrete subgroup of $N \rtimes C$ and under the induced action of E on N, the quotient space ${}_E\backslash N$ is compact.*

Equivalently, we can say that an almost crystallographic group is a subgroup $E \subseteq N \rtimes C$ acting crystallographically on N. If N is of dimension n, we say that all corresponding almost crystallographic groups are n-dimensional.

Each of the three Bieberbach Theorems have been generalized to the almost crystallographic case (see [13] for more details)

Theorem 3.5 *[3] Let $E \subseteq N \rtimes C$ be an almost crystallographic group and let $\Gamma = E \cap N$ be its subgroup of "pure translations".*
Then Γ is a uniform lattice of N and E/Γ is finite.

Γ being a uniform lattice of N means that Γ is a discrete subgroup of N, such that $\Gamma\backslash N$ is compact. In other words, Γ is acting crystallographically on N, thus Γ itself is already an almost crystallographic group. It follows that Γ is finitely generated and torsion free ([25], [28]).

Moreover, from the proof of this first Bieberbach-like theorem, it follows that Γ is a maximal nilpotent subgroup of E. The group Γ can be described completely in algebraic terms, it is the unique normal and nilpotent subgroup which is also maximal nilpotent in E. (In most cases E has other maximal nilpotent groups, but these are not normal in E). It follows that if E and E' are isomorphic almost crystallographic groups (possibly built using different nilpotent Lie groups), then also their translation subgroups Γ and Γ' are isomorphic. It is a result due to Malcev ([25]), that when there exists an isomorphism between uniform lattices of two connected and simply connected nilpotent Lie groups, then there also exists an isomorphism between the nilpotent Lie groups themselves. Therefore, in the investigation of isomorphisms between almost crystallographic groups, we can restrict our attention to almost crystallographic groups built using the same nilpotent Lie group.

Theorem 3.6 *([24]) Let N be a connected and simply connected nilpotent Lie group, and let C and C' be two compact subgroups of* $\mathrm{Aut}(N)$. *Assume that $E \subseteq N \rtimes C$ and $E' \subseteq N \rtimes C'$ are two almost crystallographic groups and that $\varphi : E \to E'$ is an isomorphism, then there exists an element $\alpha \in \mathrm{Aff}(N)$, such that*

$$\forall e \in E : \quad \varphi(e) = \alpha e \alpha^{-1}.$$

The generalization of the third Bieberbach theorem is not that straightforward. Indeed, it is no longer true that there are only finitely many almost crystallographic groups in each dimension.

Example: For any integer $k > 0$ let

$$\Gamma_k = \left\{ \begin{pmatrix} 1 & x & \frac{z}{k} \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{Z} \right\}.$$

Then each Γ_k is a uniform lattice in the three dimensional Heisenberg group H, and thus each Γ_k is a three-dimensional almost crystallographic group. The reader can check that

$$\Gamma_k / [\Gamma_k, \Gamma_k] = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_k.$$

which are pairwise non-isomorphic. It follows that all groups Γ_k are pairwise non-isomorphic, and hence there exist infinitely many almost crystallographic groups (built on the same nilpotent Lie group) in dimension 3.

Hence, if we want to generalise the third Bieberbach theorem, we have to be careful. We can look at the third Bieberbach theorem in a different way. In the Euclidean space, fixing the dimension, say n, is the same as fixing the translational subgroup (up to isomorphim) $\mathbb{Z}^n$. We can therefore formulate the third Bieberbach theorem as follows: given a group $\mathbb{Z}^n$, there are only finitely many crystallographic

groups, having this group as its subgroup of pure translations. It is this form of the third Bieberbach theorem, which has been generalized to the almost crystallographic case.

Theorem 3.7 *[21] Let Γ be a torsion free, finitely generated nilpotent group, then there are only finitely many almost crystallographic groups, having this group N as its subgroup of pure translations.*

The algebraic characterization also has a straightforward generalization

Theorem 3.8 *([24]) Let E be a group containing a finitely generated torsion free nilpotent subgroup Γ of finite index which is maximal nilpotent in E. Then there exists a simply connected, connected nilpotent Lie group N, a compact subgroup $C \subseteq \mathrm{Aut}(N)$ and an injective homomorphism $\varphi : E \to N \rtimes C$ such that $\varphi(E)$ is an almost crystallographic group.*

Just as in the case of Euclidean crystallographic groups, we can now turn to the situation of almost crystallographic actions.

Definition 3.9 *An almost crystallographic action of a group E consists of a morphism $\varphi : E \to N \rtimes C$, where N is a connected and simply connected nilpotent Lie group and C is a compact subgroup of $\mathrm{Aut}(N)$, letting E act crystallographically on N.*

Following the same reasoning as in the case of Euclidean crystallographic actions, we come to the following

> **Conclusion:** The class of groups admitting an almost crystallographic action is exactly the class of finitely generated virtually nilpotent groups. Moreover, given such a group E, then there is, up to conjugation in $\mathrm{Aff}(N)$, only one possible almost crystallographic action of E

The uniqueness statement follows from the second Bieberbach Theorem and form the fact that one can show that any for Euclidean crystallographic action $\varphi : E \to \mathrm{Isom}(\mathbb{R}^n)$ the kernel of φ is exactly the unique maximal finite normal subgroup H of E.

It follows that we have found a notion of crystallographic actions that catch exactly the class of finitely generated virtually nilpotent groups.

4 Polycyclic-by-finite groups

Now that we have found a geometric context determining all finitely generated nilpotent-by-finite groups, we move on to the next class of groups, which we try to describe geometrically.

From an algebraic point of view, the next class to be investigated is the class of polycyclic-by-finite groups. (General references for this section are [27] and [29]).

Definition 4.1 *Let Γ be a group, then Γ is polycyclic if and only if Γ admits a descending series*

$$\Gamma_1 = \Gamma \supseteq \Gamma_2 \supseteq \Gamma_3 \supseteq \cdots \supseteq \Gamma_r \subseteq \Gamma_{r+1} = 1 \tag{1}$$

such that $\Gamma_{i+1} \lhd \Gamma_i$ and any quotient Γ_i/Γ_{i+1} is cyclic.

A group is said to be polycyclic-by-finite if it is virtually polycyclic (has a polycyclic normal subgroup of finite index). The Hirsch length of a polycyclic group is defined as the number of infinite cyclic quotients Γ_i/Γ_{i+1} in the series (1) above. We will denote the Hirsch length of Γ by $h(\Gamma)$. It can be shown that $h(\Gamma)$ is independent of the series (1) in the definition above. Moreover, we can extend the definition of Hirsch length to polycyclic-by-finite groups, by defining the Hirsch length of a polycyclic-by-finite group, to be equal to the Hirsch length of a finite index polycyclic subgroup. Again, this is independent of the chosen subgroup.

Example: Any finitely generated abelian group is polycyclic.

So, the class of polycyclic-by-finite groups contains the class of finitely generated virtually abelian groups.

Proposition 4.2 *Any finitely generated nilpotent group G is polycyclic.*

Sketch of the proof: As G is nilpotent, we know that $\gamma_{c+1}(G) = 1$ for c sufficiently large. Consider the lower central series:

$$G = \gamma_1(G) \supseteq \gamma_2(G) \supseteq \gamma_3(G) \supseteq \cdots \supseteq \gamma_c(G) \supseteq \gamma_{c+1}(G) = 1.$$

For any $i \in \{1, 2, \ldots, c\}$, we have that $[\gamma_i G, \gamma_i(G)] \subseteq \gamma_{i+1}(G)$, and therefore $\gamma_i(G)/\gamma_{i+1}(G)$ is an abelian group. One can show that all the groups $\gamma_i(G)$, and so also their quotients $\gamma_i(G)/\gamma_{i+1}(G)$, are finitely generated. By refining the lower central series, we find a series of subgroups of G, satisfying the fact that the quotient of two successive terms in this series is a cyclic group. □

This proposition shows that the class of polycyclic-by-finite groups contains all virtually nilpotent groups (i.e. all groups admitting an almost crystallographic action). We are therefore considering a generalization of the previous classes of groups. On the other hand, the definition shows that polycyclic-by-finite groups are built up from cyclic groups and finite groups (both belonging to the class of virtually abelian groups) and therefore the class of polycyclic-by-finite groups is still rather *close* to the classes we considered thusfar.

The class of polycyclic-by-finite groups behaves better under algebraic constructions than the two previous classes. This is the content of the following theorem.

Theorem 4.3 *Let Γ be a group with a normal subgroup N. Then Γ is polycyclic-by-finite if and only if N and Γ/N are polycyclic-by-finite.*
Moreover, if Γ is polycyclic-by-finite, then $h(\Gamma) = h(N) + h(\Gamma/N)$.

The analoguous theorem for virtually nilpotent or virtually abelian groups does not hold.

Another group theoretical result on polycyclic-by-finite groups is the following theorem

Theorem 4.4 *Let Γ be an infinite polycyclic-by-finite group, then there exists an integer $k > 0$, such that Γ has a normal subgroup N which is isomorphic to $\mathbb{Z}^k$.*

This theorem will allow us to study polycyclic-by-finite groups by induction on the Hirsch length. Indeed, if Γ is of Hirsch length $h(\Gamma)$, then the group $N \cong \mathbb{Z}^k$ in the theorem above is of Hirsch length k, and therefore, the quotient is of Hirsch length $h(\Gamma) - k$ (use theorem 4.3). In this way, we can, up to a certain level, reduce the study of Γ to the study of the free abelian group $\mathbb{Z}^k$ (which we know very well) and a polycyclic-by-finite group of smaller Hirsch length.

Finally there is a theorem which says that if we have given a polycyclic-by-finite group Γ then there is at most one dimension n such that Γ acts crystallographically on $\mathbb{R}^n$.

Theorem 4.5 *Let $\varphi : \Gamma \to \mathcal{H}(\mathbb{R}^n)$ be a crystallographic action of a polycyclic-by-finite group, then $n = h(\Gamma)$.*
Moreover, the kernel of φ equals the unique maximal finite normal subgroup of Γ.

5 Affine crystallographic groups

In going from the Euclidean crystallographic groups to the almost crystallographic groups we changed the space (to be honest, we only changed the metric since a connected and simply connected nilpotent Lie group is homeomorphic to $\mathbb{R}^n$) and considered isometries of this new space. We can also stick to the space $\mathbb{R}^n$ and allow more motions.

Definition 5.1 *A subgroup $E \subseteq \mathrm{Aff}(\mathbb{R}^n)$ is said to be an affine crystallographic group, if and only if E acts crystallographically on $\mathbb{R}^n$.*
More generally, a group E is said to act affine crystallographically if there exists a morphism $\varphi : E \to \mathrm{Aff}(\mathbb{R}^n)$ letting E act crystallographically on $\mathbb{R}^n$.

We remark that it is no longer true that a discrete subgroup of $\mathrm{Aff}(\mathbb{R}^n)$ automatically acts properly discontinuously on $\mathbb{R}^n$.

As $\mathrm{Isom}(\mathbb{R}^n) \subseteq \mathrm{Aff}(\mathbb{R}^n)$, all Euclidean crystallographic groups are also affine crystallographic groups.

For the moment, there exist no nice generalizations of all of the three Bieberbach theorems to the case of affine crystallographic groups.

A generalization of the first Bieberbach theorem should tell us something about the algebraic structure of the affine crystallographic groups. In fact there are historically two important questions which are each others converse.

Milnor's question (1977): ([26]) Is it true that any torsion free polycyclic-by-finite group can be realized as an affine crystallographic group?

Auslander's question (1964): ([4]) Is it true that any affine crystallographic group is polycyclic-by-finite?

The fact that Milnor's question was formulated only for torsion free groups is of minor importance (Milnor was only interested in groups acting freely on $\mathbb{R}^n$, and this implies that the group has no torsion).

Auslander did not really formulate the above as a question, but rather as a theorem. Unfortunately, there is an irreparable gap in the alledged proof of this theorem.

If we would be able to give postive answers to both Milnor's and Auslander's question (or slight generalizations of these questions), we would obtain that the class of affine crystallographic groups is exactly the class of polycylic-by-finite groups and we would have reached our goal. And this would be a very nice first Bieberbach theorem in the case of affine crystallographic groups.

Unfortunately, we cannot not longer hope for a positive answer to Milnor's question, because of the following

Theorem 5.2 *([5]) There exists a finitely generated torsion free nilpotent (and hence polycyclic) group, which cannot be realised as an affine crystallographic group.*

This theorem shows that the class of groups acting affine crystallographically does not contain all polycyclic-by-finite groups, not even all virtually nilpotent groups. From this point of view, the group $\mathrm{Aff}(\mathbb{R}^n)$ is still not large enough. In fact, not too much is known about groups admitting an affine crystallographic action ([11], [10], [18], [17], [14], [22], ...).

On the other hand there is more hope that Auslander's question might have a positive answer. A postive answer is known up to dimension 6.

Theorem 5.3 *([2], [1]) Let n be a integer, with $n \leq 6$ and let $E \subseteq \mathrm{Aff}(\mathbb{R}^n)$ be an affine crystallographic group, then E is a polycyclic-by-finite group.*

This result gives us hope that the group $\mathrm{Aff}(\mathbb{R}^n)$ is not already too large, so that it allows crystallographic actions from non polycyclic-by-finite groups. However, Auslander's question seems to be very hard and it is doubtful that it will be solved completely in the near future.

On the contrary to the first Bieberbach theorem, there does exist a generalization of the second Bieberbach theorem. To be able to formulate this second Bieberbach theorem, we have to introduce a new group of motions of $\mathbb{R}^n$, called the group of polynomial automorphisms of $\mathbb{R}^n$.

Definition 5.4 *A map $\mu : \mathbb{R}^n \to \mathbb{R}^n$ is called a polynomial automorphism of $\mathbb{R}^n$ if μ is bijective and both μ and μ^{-1} are expressed by polynomial expressions in the usual coordinates of $\mathbb{R}^n$.*
The group of all polynomial automorphisms of $\mathbb{R}^n$ is denoted by $\mathrm{P}(\mathbb{R}^n)$.

As any invertible affine map is polynomial of degree 1, $\mathrm{Aff}(\mathbb{R}^n) \subseteq \mathrm{P}(R^n)$. For $n = 1$, we have that $\mathrm{Aff}(\mathbb{R}) = \mathrm{P}(\mathbb{R})$. For example although the map $\mu : x \mapsto x^3$ is polynomial and bijective, its iverse is not polynomial and therefore $\mu \notin \mathrm{P}(\mathbb{R})$.

For $n \geq 2$ the group $\mathrm{P}(\mathbb{R}^n)$ is much bigger than $\mathrm{Aff}(\mathbb{R}^n)$. To illustrate this, look at the map

$$\mu : \mathbb{R}^2 \to \mathbb{R}^2 : (x, y) \mapsto (x + p(y), y)$$

where $p(y)$ is any polynomial in the variable y. Then μ is bijective, polynomial and its inverse maps (x, y) to $(x - p(y), y)$ and therefore μ^{-1} is also polynomial. As this holds for any polynomial $p(y)$, this shows that $\mathrm{P}(\mathbb{R}^2)$ is *infinite dimensional* (to describe the elements you need an infinite number of parameters), while $\mathrm{Aff}(\mathbb{R}^2)$ is clearly finite dimensional.

We are now ready to formulate the generalization of the second Bieberbach theorem to the situation of affine crystallographic groups.

Theorem 5.5 *Let $E, E' \subseteq \mathrm{Aff}(\mathbb{R}^n)$ be two polycyclic-by-finite and affine crystallographic groups and suppose that $\varphi : E \to E'$ is an isomorphism. Then there exists an element $\alpha \in \mathrm{P}(\mathbb{R}^n)$ such that*

$$\forall e \in E : \quad \varphi(e) = \alpha e \alpha^{-1}.$$

Finally, let us note that it is not clear anymore how a third Bieberbach theorem in the setting of affine crystallographic groups should look like. (Although there are some results in [23]).

6 Polynomial crystallographic groups

In the previous section, we learned about a new group of transformations of $\mathbb{R}^n$, namely the group $\mathrm{P}(\mathbb{R}^n)$ of polynomial automorphisms of $\mathbb{R}^n$. On the other hand, we also saw that the class of groups acting affine cystallographically does not contain all polycyclic-by-finite groups.

It is therefore natural to look at the class of groups acting via maps inside $\mathrm{P}(\mathbb{R}^n)$.

Definition 6.1 *A subgroup $E \subseteq \mathrm{P}(\mathbb{R}^n)$ is said to be a polynomial crystallographic group, if and only if E acts crystallographically on $\mathbb{R}^n$.*
More generally, a group E is said to act polynomial crystallographically if there exists a morphism $\varphi : E \to \mathrm{P}(\mathbb{R}^n)$ letting E act crystallographically on $\mathbb{R}^n$.

It is obvious that any group acting affine crystallographically is a fortiori also acting polynomially crystallographically. The first real result which was obtained with respect to polynomial crystallographic groups, was the fact this new class of groups also contains the groups acting almost crystallographically.

Theorem 6.2 *[20] Let E be a group which is a finitely generated group and is virtually nilpotent of class c. Then E admits a polynomial crystallographic action $\varphi : E \to \mathrm{P}(\mathbb{R}^n)$ such that $\forall e \in E$, the degree of the map $\varphi(e)$ is bounded above by c.*

This theorem implies that this notion of crystallographic actions can been seen as a generalization of the three other crystallographic actions seen thusfar.

The first goal is to achieve the fact that all polycyclic-by-finite groups act polynomially crystallographic. It is natural to try induction on the Hirsch length of the polycyclic-by-finite group.

First of all note that a polycyclic-by-finite group is of Hirsch length 0 if and only if this group is finite. Moreover, the trivial action of a finite group on a point (which is by convention the space $\mathbb{R}^0$) is a polynomial crystallographic action. This means that the basis of our inductive approach is settled.

Now, assume that Γ is an infinite polycyclic-by-finite group. It follows from Theorem 4.4 that there exists a normal subgroup $N \cong \mathbb{Z}^k$ $(k > 0)$ of Γ. By the induction hypothesis we may assume that Γ/N admits a polynomial crystallographic action

$$\bar{\rho} : \Gamma/N \to \mathrm{P}(\mathbb{R}^K)$$

where $K = h(\Gamma/N)$ (thus $h(\Gamma) = k + K$). On the other hand, we also know that there is an action of N on $\mathbb{R}^k$, by pure translations, which is crystallographic. It is natural to wonder if these both actions cannot be combined into one action of Γ. By Theorem 4.5 we need to consider the space $\mathbb{R}^{k+K}$. We will consider this space as being the direct sum $\mathbb{R}^{k+K} = \mathbb{R}^k \oplus \mathbb{R}^K$, and therefore any element of $\mathbb{R}^{k+K}$ can be written as a pair (x, y), where $x \in \mathbb{R}^k$ and $y \in \mathbb{R}^K$.

Definition 6.3 $\mathrm{P}_{K,k}(\mathbb{R}^{k+K})$ *is the subset of* $\mathrm{P}(\mathbb{R}^{k+K})$ *consisting of those polynomial maps*

$$p : \mathbb{R}^{k+K} \to \mathbb{R}^{k+K} : (x, y) \mapsto (Ax + \lambda(y), h(y)),$$

where $x \in \mathbb{R}^k$, $y \in \mathbb{R}^K$, $A \in \mathrm{GL}(k, \mathbb{R})$, $h \in \mathrm{P}(\mathbb{R}^K)$ *and* $\lambda : \mathbb{R}^K \to \mathbb{R}^k$ *is a polynomial map.*

We leave it to the reader to verify that $\mathrm{P}_{K,k}(\mathbb{R}^{k+K})$ is a subgroup of $\mathrm{P}(\mathbb{R}^{k+K})$.

Using this group $\mathrm{P}(\mathbb{R}^{k+K})$ we were able to prove that any polycyclic-by-finite group admits a polynomial crystallographic action.

Theorem 6.4 *[19] Let* Γ *be an infinite polycyclic-by-finite group with a normal subgroup* $N \cong \mathbb{Z}^k$ *amd such that there is a polynomial crystallographic action of bounded degree*

$$\bar{\rho} : \Gamma/N \to \mathrm{P}(\mathbb{R}^K).$$

Then there exists also a polynomial crystallographic action of bounded degree of the form

$$\rho : \Gamma \to \mathrm{P}_{K,k}(\mathbb{R}^{k+K})$$

such that the action of a given $\gamma \in \Gamma$ *is given by*

$$\rho(\gamma)(x, y) = (A_\gamma x + \lambda_\gamma(y), \bar{\rho}(\bar{\gamma})(y)).$$

where $\bar{\gamma}$ *denotes the natural projection of* $\gamma \in \Gamma$ *in* Γ/N *and where* $A_\gamma \in \mathrm{GL}(k, \mathbb{R})$ *and* $\lambda_\gamma : \mathbb{R}^K \to \mathbb{R}^k$ *is a polynomial map. Moreover, this action can be taken so that the group* N *acts as pure translations on the first* k *factors of* $\mathbb{R}^{k+K}$.

This theorem can now be applied in an inductive manner and we reach the following main existence result.

Theorem 6.5 *([19], see also [15]) Any polycyclic-by-finite group Γ acts polynomial crystallographically on $\mathbb{R}^{h(\Gamma)}$. Moreover, one can suppose that this action is of bounded degree.*

This theorem really shows that the groups $\mathrm{P}(\mathbb{R}^n)$ are certainly big enough to catch all polycyclic-by-finite groups via the corresponding notion of crystallographic actions. We really want these groups to be the only ones appearing. This is, we really want an analogue of the conjecture of Auslander to hold in the polynomial case.

Conjecture 6.6 Generalized Auslander conjecture
If $\rho : \Gamma \to \mathrm{P}(\mathbb{R}^n)$ is a crystallographic action, then Γ is polycyclic-by-finite.

We have not really a clue on how to attack this problem (it is even harder than the affine case, which is far from being solved).

Maybe we have to restrict our attention to actions of bounded degree, although we believe that this really makes no difference.

Conjecture 6.7 *Let $\rho : \Gamma \to \mathrm{P}(\mathbb{R}^{k+K})$ be a polynomial crystallographic action, then $\rho(\Gamma)$ is of bounded degree.*

Some positive evidence for both conjectures can be found in [16].

Note that a positive answer to the generalized Auslander conjecture would imply that we have discovered a geometrical context to describe the class of polycyclic-by-finite groups. Moreover, it turns out that there is also a nice generalization of the second Bieberbach theorem

Theorem 6.8 *[6] Let $E, E' \subseteq \mathrm{P}(\mathbb{R}^n)$ be two polynomial crystallographic groups of bounded degree and suppose that $\varphi : E \to E'$ is an isomorphism.*
Then there exists an element $\alpha \in \mathrm{P}(\mathbb{R}^n)$ such that

$$\forall e \in E : \quad \varphi(e) = \alpha e \alpha^{-1}.$$

Note that this is the first time that the conjugation occurs inside the given group of homeomorphisms, which makes it even more nice. We can conclude with :

> For any polycyclic-by-finite group Γ there exists, up to conjugation exactly one, representation $\rho : \Gamma \to \mathrm{P}(\mathbb{R}^n)$ letting Γ act crystallographically and which is of bounded degree.

Together with a (hoped for) positive answer to the generalized Auslander's conjecture this is actually a sharp geometrical description of the notion of polycyclic-by-finite group.

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