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Pseudodifferential operators in $L^p(\mathbb{R}^n)$ spaces[1]

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ABSTRACT

We survey general results on the boundedness of pseudodifferential operators in $L^p(\mathbb{R}^n)$. We mainly consider operators with nonregular symbols which are general versions of Hörmander's class $S^m_{\rho,\delta}$. We treat the theory in a rather classic and elementary manner.

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RESUMEN

Se presentan resultados generales en el acotamiento de operadores seudo diferenciales en $L^p(\mathbb{R}^n)$. Se considera principalmente operadores con símbolos no regulares, los cuales son versiones generales de la clase de Hörmander's $S^m_{\rho,\delta}$. Se trata la teoría en una forma clásica y elemental.

Key words and phrases: *pseudodifferential operator, L^p boundedness in $\mathbb{R}^n$, nonregular symbols.*

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1 Introduction

The theory of pseudodifferential operators was born in the early 1960's and, thereafter, it evolved with the theory of partial differential equations. Therefore, many topics in these two theories are closely related, like the hypoellipticity of operators, the sharp form of Gårding's inequality, the parametrix of operators, and so on. In the theory of pseudodifferential operators, one of the most interesting topics is to investigate the behavior of pseudodifferential operators of Hörmander's class, $S^m_{\rho,\delta}$, in $L^p(\mathbb{R}^n)$ and Sobolev spaces. The behavior of operators in $L^p(\mathbb{R}^n)$ spaces plays an essential role in the theory of linear and nonlinear partial differential equations. In the present paper, we consider operators with nonregular symbols which are generalizations of Hörmander's class $S^m_{\rho,\delta}$.

We treat the theory of pseudodifferential operators in a rather classic manner, dealing mainly with their behavior in $L^p(\mathbb{R}^n)$ spaces. We present very elementary results and methods for the proof of the boundedness of pseudodifferential operators in $L^p(\mathbb{R}^n)$. We note that the results presented here may not be the best possible ones.

We do not treat symbols of the form $p(x,\xi,y)$. In the case of smooth symbols there is no difference between the cases $p(x,\xi)$ and $p(x,\xi,y)$. However, if we consider nonsmooth symbols, the behavior of the operators $p(X,D_x)$ and $p(X,D_x,Y)$ in $L^p(\mathbb{R}^n)$ may be slightly different. For example, when the symbol is of the form $p(x,\xi,y)$, Hörmander's Theorem 3.1 in Section 3 is a little different (see [9]). Recently, many authors (see, for example [12], [7]) have treated operators with symbols $p(x,\xi,y)$ by using modulation spaces or Besov spaces.

In Section 2, we recall fundamental results on the algebra and the asymptotic expansion formulas of symbols of pseudodifferential operators. In Section 3, we treat $L^2(\mathbb{R}^n)$ boundedness. In Section 4, we list well-known fundamental results on the behavior of pseudodifferential operators in $L^p(\mathbb{R}^n)$ spaces. However, the purpose of this section is to present boundedness results for operators with symbols whose order is, in some sense, lower than the critical order in $L^p(\mathbb{R}^n)$ spaces. In Section 5, we give a boundedness theorem from $L^\infty(\mathbb{R}^n)$ to BMO. For the case $\rho = 1$ the main results of the present paper on the $L^p(\mathbb{R}^n)$ boundedness of pseudodifferential operators are given in Section 6. In the last two sections, Sections 7 and 8, we consider symbols which may be useful when considering classes of pseudo (or partial) differential operators with magnetic potentials. The results in Sections 7 and 8 can be found in [14].

2 Fundamental properties of pseudodifferential operators

We use the notation found in [10]. Moreover we use a lot of constants C which are not the same at each occasion. For a point $x \in \mathbb{R}^n$ we write $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$ and $\langle x \rangle = \sqrt{1 + x_1^2 + \cdots + x_n^2}$. For a multi-integer $\alpha = (\alpha_1, \ldots, \alpha_n)$, we denote

$$\partial_x^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} = \frac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}},$$

where $|\alpha| = \alpha_1 + \cdots + \alpha_n$, and we write $D_x = (-i)\partial_x$. Hence

$$D_x^\alpha = D_{x_1}^{\alpha_1} \cdots D_{x_n}^{\alpha_n} = \left(\frac{1}{i}\right)^{|\alpha|} \frac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}}.$$

For a function $f(x,\xi)$ on $\mathbb{R}_x^n \times \mathbb{R}_\xi^n$ and multi-integers α and β we write

$$f_{(\beta)}^{(\alpha)}(x,\xi) = \partial_\xi^\alpha D_x^\beta f(x,\xi).$$

We begin with the definition of symbols of Hörmander's class $S^m_{\rho,\delta}$.

Definition 2.1. The set of smooth functions $p(x,\xi)$ on $\mathbb{R}_x{}^n \times \mathbb{R}_\xi{}^n$ which satisfy

$$|p_{(\beta)}^{(\alpha)}(x,\xi)| \le C_{\alpha,\beta} \langle \xi \rangle^{m-\rho|\alpha|+\delta|\beta|}$$

for any α and β is denoted by $S^m_{\rho,\delta}$.

For a function $p(x,\xi)$ in $S^m_{\rho,\delta}$, we define the pseudodifferential operator $p(X,D_x)$ by

$$p(X,D_x)u(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} p(x,\xi)\hat{u}(\xi)\, d\xi,$$

where $\hat{u}(\xi)$ denotes the Fourier transform of the function $u(x)$, that is,

$$\hat{u}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x)\, dx.$$

Hereafter, we denote integrals $\int_{\mathbb{R}^n} u(x)\,dx$ taken over $\mathbb{R}^n$ simply by $\int u(x)\,dx$.

In the present paper, we mainly treat the case $\delta < 1$. A very interesting study by David and Journé [4] considers pseudodifferential operators in $L^p(\mathbb{R}^n)$ for the case $\delta = \rho = 1$.

Norms in $L^p(\mathbb{R}^n)$ are

$$\|u\|_p = \left[\int |u(x)|^p\,dx\right]^{1/p} \qquad (1 \le p < \infty),$$

and

$$\|u\|_\infty = \operatorname{esssup}\{|u(x)| : x \in \mathbb{R}^n\} \qquad (p = \infty).$$

In order to define a new class of symbols we need to define basic weight functions.

Definition 2.2. A real valued smooth function, $\lambda(x,\xi)$, which satisfies the two conditions:

(i) There exists a constant $0 \le \sigma \le 1$ such that $1 \le \lambda(x,\xi) \le C\langle x\rangle^\sigma\langle\xi\rangle$.

(ii) There exists a constant $0 \le \delta < 1$ such that, for any multi-indices α and β, we have

$$|\lambda^{(\alpha)}_{(\beta)}(x,\xi)| \le C_{\alpha,\beta}\lambda(x,\xi)^{1-|\alpha|+\delta|\beta|}$$

for some constant $C_{\alpha,\beta}$,

is called a basic weight function.

Of course, the function $\langle\xi\rangle$ is a typical basic weight function. The function $\lambda(x,\xi) = \sqrt{1+|x|^2+|\xi|^2}$ is also a typical example of a basic weight function (see [1]). This $\lambda(x,\xi)$ can be used, for example, when we consider various harmonic oscillator problems. In [1], Boggiatto and Rodino consider a weight function $\lambda(x,\xi)$ which satisfies

$$c\langle\xi\rangle^\epsilon \le \lambda(x,\xi).$$

Here, however, we assume that

$$1 \le \lambda(x,\xi)$$

for applications to quantized Hamiltonians problems with magnetic vector potentials.

Let $a(x) = \big(a_1(x), a_2(x), \dots, a_n(x)\big)$ be an $\mathbb{R}^n$ valued function on $\mathbb{R}^n$ such that $|\partial^\alpha a_j(x)|$ are bounded for any $\alpha \ne 0$ and $j = 1,\dots,n$. In Sections 7 and 8, we shall use a basic weight function of the form

$$\lambda(x,\xi) = \langle x - a(x)\rangle = \sqrt{1+|\xi-a(x)|^2}.$$

A simple calculation gives the following lemma.

Lemma 2.1 *Let $\lambda(x,\xi)$ be a basic weight function. Then we have*

$$\lambda(x,\xi+\eta) \le C\langle\eta\rangle\lambda(x,\xi).$$

Definition 2.3. Let $\lambda(x,\xi)$ be a basic weight function and let m,ρ and δ be real numbers such that $0 \le \delta \le \rho \le 1$ and $\delta < 1$. Then the symbol class $S^m_{\rho,\delta,\lambda}$ is defined by

$$S^m_{\rho,\delta,\lambda} = \{p(x,\xi) : |p^{(\alpha)}_{(\beta)}(x,\xi)| \le C_{\alpha,\beta}\lambda(x,\xi)^{m-\rho|\alpha|+\delta|\beta|} \quad \text{for any } \alpha \text{ and } \beta\}.$$

We denote $S^{\infty}_{\rho,\delta,\lambda} = \bigcup_{m\in\mathbb{R}} S^m_{\rho,\delta,\lambda}$. For a symbol $p(x,\xi) \in S^{\infty}_{\rho,\delta,\lambda}$ we define the pseudodifferential operator $P = p(X, D_x)$ as above and write $p(X, D_x) \in S^{\infty}_{\rho,\delta,\lambda}$. If the symbol $p(x,\xi)$ belongs to $S^m_{\rho,\delta,\lambda}$ we write $p(X,D_x) \in S^m_{\rho,\delta,\lambda}$. The class $S^{\infty}_{\rho,\delta,\lambda}$ forms an algebra in the following sense.

Theorem 2.1 *Let $\lambda(x,\xi)$ be a basic weight function and let $0 \le \delta \le \rho \le 1$ and $\delta < 1$.*

(a) *If $p_j(X, D_x) \in S^{m_j}_{\rho,\delta,\lambda}$ $(j = 1,2)$, then $p_1(X,D_x) + P_2(X,D_x) \in S^m_{\rho,\delta,\lambda}$, where $m = \max\{m_1, m_2\}$.*

(b) *If $p_j(X,D_x) \in S^{m_j}_{\rho,\delta,\lambda}$ $(j = 1,2)$, then there exists a symbol $p(x,\xi) \in S^{m_1+m_2}_{\rho,\delta,\lambda}$ such that*

$$p(X,D_x)u(x) = p_1(X,D_x)p_2(X,D_x)u(x) \qquad (u \in \mathcal{S}),$$

and $p(x,\xi)$ has the following asymptotic expansion:

$$p(x,\xi) \sim \sum_{k=0}^{\infty} p_k(x,\xi),$$

where

$$p_k(x,\xi) = \sum_{|\alpha|=k} \frac{1}{\alpha!} p_1^{(\alpha)}(x,\xi) p_{2(\alpha)}(x,\xi) \in S^{m_1+m_2-(\rho-\delta)k}_{\rho,\delta,\lambda}.$$

(c) *If $p(X,D_x) \in S^m_{\rho,\delta,\lambda}$, then there exists a symbol $p^*(x,\xi) \in S^m_{\rho,\delta,\lambda}$ such that*

$$(p(X,D_x)u,\ v) = (u,\ p^*(X,D_x)v) \quad (u,\ v \in \mathcal{S}),$$

and $p^*(x,\xi)$ has the following asymptotic expansion:

$$p^*(x,\xi) \sim \sum_{k=0}^{\infty} p_k(x,\xi),$$

where

$$p_k(x,\xi) = \sum_{|\alpha|=k} \frac{(-1)^k}{\alpha!} \overline{p^{(\alpha)}_{(\alpha)}(x,\xi)} \in S^{m_1+m_2-(\rho-\delta)k}_{\rho,\delta,\lambda}.$$

3 Fundamental boundedness results in $L^2(\mathbb{R})$

We denote the set of bounded linear operators on a Banach space E by $\mathcal{L}(E)$. In [8], Hörmander gives an interesting result about the $L^2(\mathbb{R}^n)$ and $L^p(\mathbb{R}^n)$ boundedness of pseudodifferential operators. We start with $L^2(\mathbb{R}^n)$ boundedness results.

Theorem 3.1 *Let $0 \le \rho \le 1$ and $0 \le \delta \le 1$. Then*

$$S^m_{\rho,\delta} \subset \mathcal{L}(L^2(\mathbb{R}^n)) \implies m \le m_0 = \min\left[0, \frac{n}{2}(\rho-\delta)\right].$$

Hörmander shows that the converse is true if $0 \le \delta < \rho \le 1$. Moreover by Calderón and Vaillancourt [2], we have

Theorem 3.2 (Calderón and Vaillancourt) *Let $0 \le \delta < 1$ and $0 \le \rho \le 1$. Then the converse of Theorem 3.1 is true, that is, the inclusion $S^{m_0}_{\rho,\delta} \subset \mathcal{L}(L^2(\mathbb{R}^n))$ holds.*

The Calderón–Vaillancourt Theorem is generalized to the case of nonregular symbols.

Theorem 3.3 (See, for example, [3]) *Let $0 \le \delta \le \rho \le 1$ and $\delta < 1$. We put $\kappa = [n/2] + 1$. If a symbol $p(x, \xi)$ satisfies*

$$|p^{(\alpha)}_{(\beta)}(x, \xi)| \le C_{\alpha,\beta}\langle\xi\rangle^{-\rho|\alpha|+\delta|\beta|}$$

for any $|\alpha| \le \kappa$ and $|\beta| \le \kappa$, then the operator $p(X, D_x)$ is bounded in $L^2(\mathbb{R}^n)$, that is, there is a constant C such that

$$\|p(X, D_x)u\|_2 \le C\|u\|_2.$$

In particular, in the case $\delta = \rho = 0$, if a bounded symbol $p(x, \xi)$ is such that $|p^{(\alpha)}_{(\beta)}(x, \xi)|$ is bounded for any $|\alpha| \le \kappa$ and $|\beta| \le \kappa$, then the operator $p(X, D_x)$ is bounded in $L^2(\mathbb{R}^n)$.

In the present paper, our starting point for the $L^2(\mathbb{R}^n)$ boundedness is Theorem 3.3. Then we have the following theorem.

Theorem 3.4 *Let $\lambda(x, \xi)$ be a basic weight function and assume that $0 \le \delta \le \rho \le 1$ and $\delta < 1$. If the symbol $p(x, \xi) \in S^{-\sigma}_{\rho,\delta,\lambda}$ for a positive σ, then the operator $p(X, D_x)$ is $L^2(\mathbb{R}^n)$ bounded, that is, there is a constant C such that*

$$\|p(X, D_x)u\|_2 \le C\|u\|_2$$

holds for any $u \in \mathcal{S}$.

Proof. If σ is greater than n, then, by Theorem 3.3, the operator $p(X, D_x)$ is $L^2(\mathbb{R}^n)$ bounded, because $\left|p^{(\alpha)}_{(\beta)}(x, \xi)\right|$ are bounded when $|\alpha| \le \kappa$ and $|\beta| \le \kappa$, where $\kappa = [n/2] + 1$. If the symbol $p(x, \xi)$ belongs to $S^{-\sigma}_{\rho,\delta,\lambda}$ for $\sigma > n/2$, then, by Theorem 2.1, we have

$$\begin{aligned}\|p(X, D_x)u\|_2{}^2 &= (p(X, D_x)u,\ p(X, D_x)u)\\ &= (u,\ p^*(X, D_x)p(X, D_x)u)\\ &= (u,\ \tilde{p}(X, D_x)u),\end{aligned}$$

where $p^*(x, \xi) \in S^{-\sigma}_{\rho,\delta,\lambda}$ and $\tilde{p}(x, \xi) \in S^{-2\sigma}_{\rho,\delta,\lambda}$. Since $2\sigma > n$, we have already seen that the operator $\tilde{p}(X, D_x)$ is $L^2(\mathbb{R}^n)$ bounded. So by Schwarz' inequality and the boundedness we have

$$\begin{aligned}\|p(X, D_x)u\|_2{}^2 &= (u,\ \tilde{p}(X, D_x)u)\\ &\le \|u\|_2\|\tilde{p}(X, D_x)u\|_2\\ &\le C^2\|u\|_2\end{aligned}$$

for any $u \in \mathcal{S}$. Hence we get the boundedness of the operator $p(X, D_x)$ with symbol $p(x,\xi) \in S^{-\sigma}_{\rho,\delta,\lambda}$, $(\sigma > n/2)$. Boundedness can be proved in a similar way when the symbol $p(x,\xi)$ belongs to $S^{-\sigma}_{\rho,\delta,\lambda}$ for $\sigma > n/4$. Repeating this procedure we can prove the theorem for any positive σ. ∎

Theorem 3.5 *Let $\lambda(x,\xi)$ be a basic weight function and $0 \le \delta < \rho \le 1$. If the symbol $p(x,\xi) \in S^0_{\rho,\delta,\lambda}$, then the operator $p(X,D)$ is $L^2(\mathbb{R}^n)$ bounded.*

Proof. Putting

$$|p|_0 = \sup\{|p(x,\xi)| : (x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n\}$$

and

$$q(x,\xi) = \sqrt{2|p|_0 - |p(x,\xi)|^2},$$

we can see that $q(x,\xi) \in S^0_{\rho,\delta,\lambda}$. We have

$$\begin{aligned} 0 \le \|q(X,D_x)u\|^2 &= (q(X,D_x)u,\ q(X,D_x)u) \\ &= (u,\ q^*(X,D_x)q(X,D_x)u). \end{aligned}$$

By the expansion formula, we can see that the symbol of the operator $q^*(X,D_x)$ can be written in the form

$$q^*(x,\xi) = \overline{q(x,\xi)} + q_1(x,\xi) := \bar{q}(x,\xi) + q_1(x,\xi)$$

where $q_1(x,\xi) \in S^{-(\rho-\delta)}_{\rho,\delta,\lambda}$ and we write $\bar{q}(x,\xi)$ for $\overline{q(x,\xi)}$. Hence we have

$$\begin{aligned} q^*(X,D_x)q(X,D_x) &= \bar{q}(X,D_x)q(X,D_x) + q_1(X,D_x)q(X,D_x) \\ &= \tilde{q}(X,D_x) + q_2(X,D_x), \end{aligned}$$

where $\tilde{q}(x,\xi) = |q(x,\xi)|^2$ and $q_2(x,\xi) \in S^{-(\rho-\delta)}_{\rho,\delta,\lambda}$. So we can write

$$0 \le (u,\ \tilde{q}(X,D_x)u + q_2(X,D_x)u).$$

Setting $\tilde{p}(x,\xi) = |p(x,\xi)|^2$, we have

$$\begin{aligned} \|p(X,D_x)u\|_2^2 = (p(X,D_x)u,\ p(X,D_x)u) &= (u,\ p^*(X,D_x)p(X,D_x)u) \\ &= (u,\ \tilde{p}(X,D_x)u) + (u,\ p_1(X,D_x)u), \end{aligned}$$

where $p_1(x,\xi) \in S^{-(\rho-\delta)}_{\rho,\delta,\lambda}$. Moreover, since

$$\tilde{q}(X,D_x) = 2|p|_0{}^2 - \tilde{p}(X,D_x),$$

we have

$$\begin{aligned} 0 &\le (u,\ 2|p|_0^2 u - \tilde{p}(X,D_x)u + q_2(X,D_x)u) \\ &\le 2|p|_0^2\|u\|_2^2 - \|p(X,D_x)u\|_2^2 + (u,\ p_1(X,D_x)u) + (u,\ q_2(X,D_x)u). \end{aligned}$$

Thus we have

$$\|p(X,D_x)u\|_2^2 \le 2|p|_0^2\|u\|_2^2 + (u,\ p_1(X,D_x)u) + (u,\ q_2(X,D_x)u).$$

Since $\rho-\delta>0$, by Theorem 3.4 we have

$$|(u,\ p_1(X,D_x)u)| \le \|u\|_2\|p_1(X,D_x)u\|_2 \le C\|u\|_2^2,$$
$$|(u,\ q_2(X,D_x)u)| \le \|u\|_2\|q_2(X,D_x)u\|_2 \le C\|u\|_2^2.$$

Combining these inequalities, we finally obtain

$$\|p(X,D_x)u\|_2^2 \le C\|u\|_2^2.$$

■

4 L^p boundedness of pseudodifferential operators with lower order symbols

In this section, we treat the case of the basic weight function $\lambda(x,\xi)=\langle\xi\rangle$. So the symbols $p(x,\xi)$ may belong to $S^{\infty}_{\rho,\delta}$ or to the generalized (nonregular) class of $S^{\infty}_{\rho,\delta}$. In particular, the case $\rho=1,\ \delta<1$ is important when we study the general boundedness in $L^p(\mathbb{R}^n)$ in relation to the class of Calderón–Zygmund operators (see [3]). We begin with results by Hörmander [8] and Fefferman [5].

For general $1\le p\le\infty$ we have the following theorem.

Theorem 4.1 (Hörmander [8]) *Let $0\le\delta\le\rho\le1$ and $\delta<1$. Then*

$$S^m_{\rho,\delta}\subset\mathcal{L}\big(L^p(\mathbb{R}^d)\big) \implies m\le -n(1-\rho)\left|\frac{1}{2}-\frac{1}{p}\right|.$$

Therefore we may consider that the order m_p, defined by

$$m_p=n(1-\rho)\left|\frac{1}{2}-\frac{1}{p}\right|, \tag{1}$$

is the critical decreasing order for the $L^p(\mathbb{R}^n)$ boundedness of pseudodifferential operators of Hörmander's class $S^{\infty}_{\rho,\delta}$.

It is known that, for $p=1$ and $p=\infty$, the converse to Hörmander's theorem does not hold. For $1<p<\infty$, C. Fefferman proved the connverse of Hörmander's theorem.

Theorem 4.2 (C. Fefferman [5]) *Let $1<p<\infty$, $0\le\delta\le\rho\le1$ and $\delta<1$ and set $m_p=n(1-\rho)\left|\frac{1}{2}-\frac{1}{p}\right|$. Then*

$$S^{-m_p}_{\rho,\delta}\subset\mathcal{L}\big(L^p(\mathbb{R}^n)\big).$$

On the boundedness of pseudodifferential operators, it is easy to treat operators with lower order symbols. Here "lower order" means that the decreasing order of the symbol at $|x| \to \infty$ is greater than m_p in some sense. We begin with a very elementary boundedness lemma.

Lemma 4.1 *Let the symbol $p(x,\xi)$ have support in $\{(x,\xi) : |\xi| \le R\}$ for some $R > 0$, and suppose that*

$$|p^{(\alpha)}(x,\xi)| \le C_\alpha \tag{2}$$

for $|\alpha| \le \kappa = [n/2] + 1$. Then the operator $p(X, D_x)$ is bounded in $L^p(\mathbb{R}^n)$ for $2 \le p \le \infty$. If the symbol is independent of the space variable x, that is, $p(x,\xi) = p(\xi)$, then the operator $p(X, D_x) = p(D_x)$ is bounded in $L^p(\mathbb{R}^n)$ for $1 \le p \le \infty$. Moreover, if inequality (2) holds for $|\alpha| \le n+1$, then the operator $p(X, D_x)$ is bounded in $L^p(\mathbb{R}^n)$ for $1 \le p \le \infty$.

Proof. For any $u \in \mathcal{S}$ we can write

$$p(X, D_x)u(x) = \int K(x, x-y)u(y)\,dy,$$

where

$$K(x,z) = \frac{1}{(2\pi)^n}\int e^{iz\cdot\xi}p(x,\xi)\,d\xi.$$

If the symbol $p(x,\xi)$ satisfies inequality (2) for α with $|\alpha| \le \kappa$, then by Plancherel's equality we have

$$\begin{aligned}\int |\langle z\rangle^\kappa K(x,z)|^2\,dz &= \sum_{|\alpha|\le\kappa} c_\alpha \int |z^\alpha K(x,z)|^2\,dz\\ &= \sum_{|\alpha|\le\kappa} c_\alpha \int |p^{(\alpha)}(x,\xi)|^2\,d\xi\\ &\le C^2.\end{aligned}$$

Therefore we have

$$\begin{aligned}\|p(X,D_x)u\|_2^2 &\le \int \left|\int |K(x,x-y)u(y)|\,dy\right|^2 dx\\ &\le \iint \langle x-y\rangle^{-2\kappa}|u(y)|^2\,dy \int \langle x-y\rangle^{2\kappa}|K(x,x-y)|^2\,dy\,dx\\ &= \iint \langle x-y\rangle^{-2\kappa}|u(y)|^2\,dy \int \langle z\rangle^{2\kappa}|K(x,z)|^2\,dz\,dx\\ &\le C^2 \iint \langle x-y\rangle^{-2\kappa}|u(y)|^2\,dy\,dx\\ &= C^2\|u\|_2^2.\end{aligned}$$

This means that the operator $p(X, D_x)$ is L^2 bounded. Moreover, we have

$$\begin{aligned} |p(X, D_x)u(x)| &\leq \int |K(x, x-y)||u(y)|\,dy \\ &\leq \int |K(x,z)|\,dz\,\|u\|_\infty \\ &\leq \left[\int \langle z\rangle^{-2\kappa}\,dz\right]^{1/2}\left[\int \langle z\rangle^{2\kappa}|K(x,z)|^2\,dz\right]^{1/2}\|u\|_\infty. \end{aligned}$$

Hence, the operator $p(X, D_x)$ is L^∞ bounded. So, by the Riesz–Thorin interpolation theorem, the operator $p(X, D_x)$ is $L^p(\mathbb{R}^n)$ bounded for $2 \leq p \leq \infty$.

If the symbol is independent of the space variable x, we have

$$p(D_x)u(x) = \int K(x-y)u(y)\,dy,$$

where

$$K(z) = \frac{1}{(2\pi)^n}\int e^{iz\cdot\xi}p(\xi)\,d\xi.$$

Hence, changing the order of the integration, the $L^1(\mathbb{R}^n)$ norm of $p(D_x)$ is

$$\begin{aligned} \int |p(D_x)u(x)|dx &\leq \iint |K(x-y)u(y)|\,dy\,dx \\ &\leq \left[\int |K(z)|\,dz\right]\left[\int |u(y)|\,dy\right]. \end{aligned}$$

As for the case of $L^\infty(\mathbb{R}^n)$ boundedness, we can prove that

$$\int |K(z)|\,dz \leq C$$

by Plancherel's formula. Hence we have

$$\|p(D_x)u\|_1 \leq C\|u\|_1.$$

Thus, by the Riesz-Thorin Theorem we have $L^p(\mathbb{R}^n)$ boundedness for $1 \leq p \leq \infty$.

If the symbol $p(x,\xi)$ satisfies inequality (2) for α with $|\alpha| \leq n+1$, then for any $|\alpha| \leq n+1$ we have

$$\begin{aligned} |z^\alpha K(x,z)| &= \frac{1}{(2\pi)^n}\left|\int e^{iz\cdot\xi}p^{(\alpha)}(x,\xi)\,d\xi\right| \\ &\leq \frac{1}{(2\pi)^n}\int \left|p^{(\alpha)}(x,\xi)\right|\,d\xi \leq C_\alpha. \end{aligned}$$

So, we have

$$|K(x,z)| \leq \frac{C}{\langle z\rangle^{n+1}}.$$

Therefore

$$\begin{aligned}\|p(X,D_x)u\|_1 &\le \int\int |K(x,x-y)u(y)|\,dy\,dx\\ &\le C\iint \langle x-y\rangle^{-n-1}|u(y)|\,dx\,dy\\ &\le C\|u\|_1.\end{aligned}$$

Thus, the operator $p(X,D_x)$ is bounded on $L^1(\mathbb{R}^n)$. So the operator $p(X,D_x)$ is $L^p(\mathbb{R}^n)$ bounded for $1\le p\le 2$. ■

Remark 4.1. As we have seen in the proof of the Lemma 4.1, when we estimate the $L^\infty(\mathbb{R}^n)$ norm we need to estimate the integral of the integral kernel $K(x,z)$ with respect to z. On the other hand when we estimate the $L^1(\mathbb{R}^n)$ norm, we have to estimate the kernel $K(x,z)$ itself, except for the case where it is independent of the variable x. This is why we often have to change the assumptions when we treat the $L^p(\mathbb{R}^n)$ boundedness for $1\le p\le 2$ or $2\le p\le\infty$.

Theorem 4.3 *Let $0\le\rho\le 1$ and $0<\sigma\le 1$, and suppose that the symbol $p(x,\xi)$ satisfies*

$$|p^{(\alpha)}(x,\xi)|\le C\omega(\langle\xi\rangle^{-\sigma})\langle\xi\rangle^{-(n/2)(1-\rho)-\rho|\alpha|}$$

for any $|\alpha|\le\kappa=[n/2]+1$, where $\omega(t)$ is a nonnegative and nondecreasing function on $[0,\infty)$ which satisfies

$$\int_0^1\frac{\omega(t)^2}{t}\,dt<\infty. \qquad (3)$$

Then the operator $p(X,D_x)$ is $L^2(\mathbb{R}^n)$ bounded, that is, there is a constant C such that

$$\|p(X,D_x)u\|_2\le C\|u\|_2 \quad \textit{for any } u\in\mathcal{S}.$$

Proof. By Lemma 4.1, we may assume that the support of the symbol $p(x,\xi)$ is contained in $\{(x,\xi):|\xi|\ge 4\}$. We take a nonnegative and smooth function $f(t)$ on $\mathbb{R}$ such that

$$\int_0^\infty\frac{f(t)^2}{t}\,dt=1,\qquad \operatorname{supp} f\subset\left[\frac{1}{2},1\right].$$

Then for any $\xi\ne 0$ we have

$$\int_0^\infty\frac{f(t|\xi|)^2}{t}\,dt=1.$$

Hence we can write

$$\begin{aligned}p(X,D_x)u(x)&=\frac{1}{(2\pi)^{n/2}}\int e^{ix\cdot\xi}p(x,\xi)\hat u(\xi)\,d\xi\\ &=\frac{1}{(2\pi)^{n/2}}\int_0^\infty\frac{1}{t}\,dt\int e^{ix\cdot\xi}p(x,\xi)f(t|\xi|)^2\hat u(\xi)\,d\xi\\ &=\frac{1}{(2\pi)^{n/2}}\int_0^\infty\frac{1}{t}\,dt\int e^{ix\cdot\xi}p(x,\xi)f(t|\xi|)\hat v_t(\xi)\,d\xi,\end{aligned}$$

where $v_t(x) = f(t|D_x|)u(x)$, that is, $\hat{v_t}(\xi) = f(t|\xi|)\hat{u}(\xi)$. Noting that the support of $p(x,\xi)$ is contained in $\{\xi : |\xi| \geq 4\}$, we can write

$$\begin{aligned} p(X,D_x)u(x) &= \frac{1}{(2\pi)^n}\int_0^{1/4}\frac{1}{t}\,dt\int\left(\int e^{i(x-y)\cdot\xi}p(x,\xi)f(t|\xi|)\,d\xi\right)v_t(y)\,dy \\ &= \int_0^{1/4}\frac{1}{t}\,dt\int K_t(x,z)v_t(x-tz)\,dz, \end{aligned}$$

where

$$K_t(x,z) = \frac{1}{(2\pi)^n}\int e^{iz\cdot\xi}p\left(x,\frac{1}{t}\xi\right)f(|\xi|)\,d\xi.$$

Now we split the integral in two parts:

$$\begin{aligned} p(X,D_x)u(x) &= \int_0^{1/4}\frac{1}{t}\,dt\int_{|z|\leq t^{\rho-1}}K_t(x,z)v_t(x-tz)\,dz \\ &\quad+\int_0^{1/4}\frac{1}{t}\,dt\int_{|z|\geq t^{\rho-1}}K_t(x,z)v_t(x-tz)\,dz \\ &= A(x)+B(x). \end{aligned}$$

Then by Schwarz' inequality we have

$$\begin{aligned} |A(x)|^2 \leq &\left[\int_0^{1/4}\frac{1}{t^{1-n(1-\rho)}}\,dt\int_{|z|\leq t^{\rho-1}}|v_t(x-tz)|^2\,dz\right] \\ &\times\left[\int_0^{1/4}\frac{1}{t^{1+n(1-\rho)}}\,dt\int_{|z|\leq t^{\rho-1}}|K_t(x,z)|^2\,dz\right]. \end{aligned}$$

From the Plancherel formula and the assumption on $p(x,\xi)$ we have

$$\begin{aligned} \int|K_t(x,z)|^2\,dz &= \frac{1}{(2\pi)^n}\int\left|p\left(x,\frac{1}{t}\xi\right)f(|\xi|)\right|^2 d\xi \\ &\leq C\int_{\{1/2\leq|\xi|\leq1\}}\omega\left(\frac{|\xi|^{-\sigma}}{t^{-\sigma}}\right)^2\left(\frac{|\xi|}{t}\right)^{-n(1-\rho)}d\xi \\ &\leq C\omega((2t)^\sigma)^2t^{n(1-\rho)}. \end{aligned}$$

Thus, we have

$$\int |A(x)|^2\,dx \le \int \left[\int_0^{1/4} \frac{1}{t^{1-n(1-\rho)}}\,dt \int_{|z|\le t^{\rho-1}} |v_t(x-tz)|^2\,dz\right]$$
$$\times \left[\int_0^{1/4} \frac{1}{t^{1+n(1-\rho)}}\,dt \int_{|z|\le t^{\rho-1}} |K_t(x,z)|^2\,dz\right]\,dx$$
$$\le C\int_0^{1/4} \frac{\omega\big((2t)^\sigma\big)^2}{t}\,dt$$
$$\times \int \left[\int_0^{1/4} \frac{1}{t^{1-n(1-\rho)}}\,dt \int_{|z|\le t^{\rho-1}} |v_t(x-tz)|^2\,dz\right]\,dx.$$

Therefore, from the assumption on $\omega(t)$ we have

$$\int |A(x)|^2\,dx \le C\int \left[\int_0^{1/4} \frac{1}{t^{1-n(1-\rho)}}\,dt \int_{|z|\le t^{\rho-1}} |v_t(x-tz)|^2\,dz\right]\,dx$$
$$\le C\int_0^{1/4} \frac{1}{t^{1-n(1-\rho)}}\,dt \int_{|z|\le t^{\rho-1}} \left[\int |v_t(x-tz)|^2\,dx\right]\,dz$$
$$= C\int_0^{1/4} \frac{1}{t}\,dt \int |v_t(x)|^2\,dx.$$

Then, since

$$\int_0^\infty \frac{1}{t}\,dt \int |v_t(x)|^2\,dx = \int_0^\infty \frac{1}{t}\,dt \int |\hat{v_t}(\xi)|^2 d\xi$$
$$= \int \left[\int_0^\infty \frac{f(t|\xi|)^2}{t}\,dt\right] |\hat{u}(\xi)|^2\,d\xi$$
$$= \|u\|_2^2,$$

we have

$$\int |A(x)|^2\,dx \le C\|u\|_2^2.$$

In order to estimate the L^2 norm of the term $B(x)$, for $|\alpha| = \kappa$ we need

$$z^\alpha K_t(x,z) = \frac{i^{|\alpha|}}{(2\pi)^n} \int e^{iz\cdot\xi} \partial_\xi^\alpha \left[p\left(x, \frac{1}{t}\xi\right) f(|\xi|)\right]\,d\xi$$
$$= \frac{i^\kappa}{(2\pi)^n} \sum_{\alpha'\le\alpha} \binom{\alpha'}{\alpha} \int e^{iz\cdot\xi} t^{-|\alpha'|} p^{(\alpha')}\left(x, \frac{1}{t}\xi\right) \partial_\xi^{\alpha-\alpha'} \{f(|\xi|)\}\,d\xi.$$

Therefore, we have

$$|z^\alpha K_t(x,z)| \le C \sum_{\alpha'\le\alpha} t^{-|\alpha'|} \left| \int e^{iz\cdot\xi} p^{(\alpha')}\left(x, \frac{1}{t}\xi\right) \partial_\xi^{\alpha-\alpha'} \{f(|\xi|)\}\,d\xi \right|$$

and

$$\int |z^\alpha K_t(x,z)|^2\,dz \le C \sum_{\alpha'\le\alpha} t^{-2|\alpha'|} \int \left| p^{(\alpha')}\left(x, \frac{1}{t}\xi\right) \partial_\xi^{\alpha-\alpha'}\{f(|\xi|)\}\right|^2 d\xi.$$

Since the support of the function $f(|\xi|)$ is contained in $1/2 \le |\xi| \le 1$, we have

$$\begin{aligned}\left|p^{(\alpha')}\left(x, \frac{1}{t}\xi\right)\right| &\le C\left|\frac{\xi}{t}\right|^{-n(1-\rho)/2-\rho|\alpha'|} \omega\left(\left|\frac{\xi}{t}\right|^{-\sigma}\right)\\ &\le Ct^{(n-2\kappa)(1-\rho)/2+|\alpha'|}\omega((2t)^\sigma)\end{aligned}$$

in the support of the integration. So we get

$$\begin{aligned}\int |z^\alpha K_t(x,z)|^2\,dz &\le Ct^{(n-2\kappa)(1-\rho)}\omega((2t)^\sigma)^2 \sum_{\alpha'\le\alpha}\int_{\{1/2\le|\xi|\le1\}} d\xi\\ &\le Ct^{(n-2\kappa)(1-\rho)}\omega((2t)^\sigma)^2.\end{aligned}$$

Thus we have

$$\int |z|^{2\kappa}|K_t(x,z)|^2\,dz \le Ct^{(n-2\kappa)(1-\rho)}\omega((2t)^\sigma)^2.$$

Writing $m = (n-2\kappa)(1-\rho)(\ge 0)$, we have

$$\begin{aligned}\int |B(x)|^2\,dx \le \int &\left[\int_0^{1/4} \frac{1}{t^{1-m}}\,dt \int_{|z|\ge t^{\rho-1}} |z|^{-2\kappa}|v_t(x-tz)|^2\,dz\right]\\ &\times\left[\int_0^{1/4}\frac{1}{t^{1+m}}\,dt\int_{|z|\ge t^{\rho-1}}|z|^{2\kappa}|K_t(x,z)|^2\,dz\right]dx.\end{aligned}$$

From the estimate of the L^2 norm of the kernel $|z|^\kappa|K_t(x,z)|$ we have

$$\begin{aligned}\int |B(x)|^2\,dx &\le C\int\left[\int_0^{1/4}\frac{1}{t^{1-m}}\,dt\int_{|z|\ge t^{\rho-1}}|z|^{-2\kappa}|v_t(x-tz)|^2\,dz\right]\\ &\qquad\times\left[\int_0^{1/4}\frac{\omega((2t)^\sigma)^2}{t}\,dt\right]dx\\ &\le C\int\left[\int_0^{1/4}\frac{1}{t^{1-m}}\,dt\int_{|z|\ge t^{\rho-1}}|z|^{-2\kappa}|v_t(x-tz)|^2\,dz\right]dx.\end{aligned}$$

So, changing the order of integration, we have

$$\begin{aligned}\int |B(x)|^2\,dx &\le C\int_0^{1/4}\frac{1}{t^{1-m}}\left[\int_{|z|\ge t^{\rho-1}}|z|^{-2\kappa}\left(\int|v_t(x-tz)|^2\,dx\right)dz\right]dt\\ &\le C\int_0^{1/4}\frac{1}{t}\left[\int|v_t(x)|^2\,dx\right]dt\\ &\le C\|u\|_2^2.\end{aligned}$$

Combining the L^2 norm of $A(x)$ we have

$$\|p(X,D_x)u\|_2 \le C\|u\|_2.$$

■

Remark 4.2. We note that the order $n(1-\rho)/2$ of the symbols in Theorem 4.3 is equal to $m_\infty = m_1$ as defined in (1). Moreover we note that in this theorem we do not need the continuity of the symbols $p(x,\xi)$ in the variable x.

When $\rho = 1$, we see that $m_p = 0$ for $1 \le p \le \infty$. Therefore, combining the Calderón–Vaillancourt Theorem with Theorem 4.3 we have the following corollary.

Corollary 4.1 *Let $0 \le \delta < 1$. If the symbol $p(x,\xi)$ satisfies*

$$|p^{(\alpha)}(x,\xi)| \le C\langle\xi\rangle^{-|\alpha|},$$
$$|p^{(\alpha)}(x,\xi) - p^{(\alpha)}(y,\xi)| \le C\omega(|x-y|\langle\xi\rangle^{\delta})\langle\xi\rangle^{-|\alpha|}$$

for $|\alpha| \le \kappa = [n/2]+1$, where $\omega(t)$ is the same as in Theorem 4.3, then the operator $p(X,D_x)$ is $L^2(\mathbb{R}^n)$ bounded.

Proof. We take a smooth function $\varphi(x)$ with support in $\{x : |x| \le 1\}$ and with integral

$$\int \varphi(x)\,dx = 1.$$

We define a new symbol $q(x,\xi)$ by

$$q(x,\xi) = \langle\xi\rangle^{n\delta'}\int \varphi(\langle\xi\rangle^{\delta'}(x-y))p(y,\xi)\,dy$$
$$= \int \varphi(z)p(x-\langle\xi\rangle^{-\delta'}z,\xi)\,dz$$

where $\delta < \delta' < 1$. Then it is not difficult to see that $q(x,\xi)$ satisfies

$$|q^{(\alpha)}_{(\beta)}(x,\xi)| \le C\langle\xi\rangle^{-|\alpha|+\delta'|\beta|}$$

for any $|\alpha| \le \kappa$ and any β. Hence by the Calderón–Vaillancourt Theorem, the operator $q(X,D_x)$ is $L^2(\mathbb{R}^n)$ bounded.

Moreover, from the assumption we can see that

$$r(x,\xi) = p(x,\xi) - q(x,\xi)$$

satisfies the conditions in Theorem 4.3. Hence the operator $r(X,D_x)$ is also $L^2(\mathbb{R}^n)$ bounded. Therefore the operator $p(X,D_x) = q(X,D_x)+r(X,D_x)$ is $L^2(\mathbb{R}^n)$ bounded.

■

Under a slightly stronger condition than the one in Theorem 4.3, we have the following $L^\infty(\mathbb{R}^n)$ boundedness.

Theorem 4.4 *Let $0 \le \rho \le 1$ and $0 < \sigma \le 1$, and suppose that the symbol $p(x,\xi)$ satisfies*

$$|p^{(\alpha)}(x,\xi)| \le C\omega(\langle\xi\rangle^{-\sigma})\langle\xi\rangle^{-(n/2)(1-\rho)-\rho|\alpha|}$$

for any $|\alpha| \le \kappa = [n/2]+1$, where $\omega(t)$ is a nonnegative and nondecreasing function on $[0,\infty)$ which satisfies

$$\int_0^1 \frac{\omega(t)}{t}\,dt < \infty. \tag{4}$$

Then the operator $p(X,D_x)$ is $L^\infty(\mathbb{R}^n)$ bounded, that is, there is a constant C such that

$$\|p(X,D_x)u\|_\infty \le C\|u\|_\infty \quad \text{for any } u \in \mathcal{S}.$$

Proof. As in the proof of Theorem 4.3 we may assume that the support of the symbol $p(x,\xi)$ is contained in $\{\xi : |\xi| \ge 4\}$. We take a nonnegative and smooth function $f(t)$ such that

$$\int_0^\infty \frac{f(t)}{t}\,dt = 1, \qquad \operatorname{supp} f \subset \left[\frac{1}{2},1\right].$$

Then as before, for any $\xi \ne 0$ we have

$$\int_0^\infty \frac{f(|\xi|t)}{t}\,dt = 1.$$

Then we can write

$$p(X,D_x)u(x) = \int_0^{1/4} \frac{1}{t}\,dt \int K_t(x,z)u(x-tz)\,dz,$$

where

$$K_t(x,z) = \frac{1}{(2\pi)^n}\int e^{iz\cdot\xi} p\left(x,\frac{1}{t}\xi\right) f(|\xi|)\,d\xi.$$

We divide the integral of $K_t(x,z)$ in two parts:

$$\int |K_t(x,z)|\,dz = \int_{|z|\le t^{\rho-1}} |K_t(x,z)|\,dz + \int_{|z|\ge t^{\rho-1}} |K_t(x,z)|\,dz = A(x) + B(x).$$

Then we have

$$\begin{aligned}
A(x) &\le \left[\int_{|z|\le t^{\rho-1}} dz\right]^{1/2} \left[\int_{|z|\le t^{\rho-1}} |K_t(x,z)|^2\,dz\right]^{1/2} \\
&\le Ct^{n(\rho-1)/2}\left[\int \left|p\left(x,\frac{1}{t}\xi\right)\right|^2 f(|\xi|)^2\,d\xi\right]^{1/2} \\
&\le Ct^{n(\rho-1)/2}\left[\int_{\{1/2\le|\xi|\le1\}} \left|\frac{1}{t}\xi\right|^{n(\rho-1)} \omega\left(\left|\frac{\xi}{t}\right|^{-\sigma}\right)^2 d\xi\right]^{1/2} \\
&\le C\omega((2t)^\sigma).
\end{aligned}$$

For $\kappa = [n/2] + 1$ we have

$$\begin{aligned} B(x) &\le \left[\int_{|z|\ge t^{\rho-1}} |z|^{-2\kappa}\, dz\right]^{1/2} \left[\int_{|z|\ge t^{\rho-1}} |z|^{2\kappa} |K_t(x,z)|^2\, dz\right]^{1/2} \\ &\le Ct^{(n-2\kappa)(\rho-1)/2} \left[\int \sum_{|\alpha|=\kappa} t^{-2\kappa} \left|p^{(\alpha)}\left(x, \frac{1}{t}\xi\right)\right|^2 f(|\xi|)^2\, d\xi\right]^{1/2} \\ &\le Ct^{(n-2\kappa)(\rho-1)/2} \left[\int_{\{1/2\le|\xi|\le 1\}} t^{-2\kappa} \left|\frac{1}{t}\xi\right|^{n(\rho-1)-2\kappa\rho} \omega\left(\left|\frac{\xi}{t}\right|^{-\sigma}\right)^2 d\xi\right]^{1/2} \\ &\le C\omega((2t)^\sigma). \end{aligned}$$

Hence

$$\int |K_t(x,z)|\, dz \le C\omega((2t)^\sigma).$$

Therefore we have

$$\begin{aligned} |p(X,D_x)u(x)| &\le \int_0^{1/4} \frac{1}{t}\, dt \int |K_t(x,z)|\, |u(x-tz)|\, dz \\ &\le C\int_0^\infty \frac{\omega((2t)^\sigma)}{t}\, dt \|u\|_\infty \\ &\le C\|u\|_\infty. \end{aligned}$$

■

Remark 4.3. The condition in Theorem 4.4 is a little stronger than the one in Theorem 4.3. In fact it is easy to see that inequality (4) implies inequality(3).

We have the following corollary to Theorem 4.4.

Corollary 4.2 *Under the condition in Theorem 4.4, the operator* $p(X, D_x)$ *is* $L^p(\mathbb{R}^n)$ *bounded for* $2 \le p \le \infty$, *and we have*

$$\|p(X,D_x)u\|_p \le C\|u\|_p \qquad (u \in \mathcal{S}),$$

where the constant C *is independent of* $2 \le p \le \infty$.

For the $L^1(\mathbb{R}^n)$ boundedness we have to put a stronger condition than in the case of the $L^2(\mathbb{R}^n)$ and $L^\infty(\mathbb{R}^n)$ boundedness.

Theorem 4.5 *Let* $0 \le \rho \le 1$ *and* $\sigma > 0$. *Assume that the symbol* $p(x,\xi)$ *satisfies the inequality*

$$|p^{(\alpha)}(x,\xi)| \le C\omega(\langle\xi\rangle^{-\sigma})\langle\xi\rangle^{-n(1-\rho)-\rho|\alpha|}$$

for any $|\alpha| \le n+1$, *where* $\omega(t)$ *satisfies the same condition as in Theorem 4.4. Then the operator* $p(X, D_x)$ *is* $L^1(\mathbb{R}^n)$ *bounded.*

Proof. As in the proof of Theorem 4.4 we may assume that the support of the symbol $p(x,\xi)$ is contained in $\{(x,\xi) : |\xi| \geq 4\}$. Then, taking a smooth function $f(t)$ as before, we write

$$p(X,D_x)u(x) = \int_0^{1/4} \frac{1}{t}\,dt \int K_t(x,z)u(x-tz)\,dz,$$

where

$$K_t(x,z) = \frac{1}{(2\pi)^n}\int e^{iz\cdot\xi} p\left(x, \frac{1}{t}\xi\right) f(|\xi|)\,d\xi,$$

and write

$$\begin{aligned}\int K_t(x,z)u(x-tz)\,dz &= \int_{|z|\leq t^{-(1-\rho)}} K_t(x,z)u(x-tz)\,dz \\ &\quad + \int_{|z|\geq t^{-(1-\rho)}} K_t(x,z)u(x-tz)\,dz \\ &= A_t(x) + B_t(x).\end{aligned}$$

Since

$$\begin{aligned}|K_t(x,z)| &\leq \int \left|p\left(x,\frac{1}{t}\xi\right)\right| f(|\xi|)\,d\xi \\ &\leq C\int_{\frac{1}{2}\leq|\xi|\leq 1} \left|\frac{1}{t}\xi\right|^{-n(1-\rho)} \omega\left(\left|\frac{1}{t}\xi\right|^{-\sigma}\right) d\xi \\ &\leq Ct^{n(1-\rho)}\omega((2t)^\sigma),\end{aligned}$$

we have

$$\begin{aligned}\int |A_t(x)|\,dx &\leq \int \left[\int_{|z|\leq t^{-(1-\rho)}} |K_t(x,z)u(x-tz)|\,dz\right] dx \\ &\leq Ct^{n(1-\rho)}\omega((2t)^\sigma) \int \left[\int_{|z|\leq t^{-(1-\rho)}} |u(x-tz)|\,dz\right] dx \\ &\leq Ct^{n(1-\rho)}\omega((2t)^\sigma) \int_{|z|\leq t^{-(1-\rho)}} \int |u(x-tz)|\,dx\,dz \\ &= C\omega((2t)^\sigma)\|u\|_1.\end{aligned}$$

In order to estimate the L^1 norm of $B(x)$, for $|\alpha| = n+1$ we have

$$\begin{aligned}|z^\alpha K_t(x,z)| &\leq t^{-n-1}\int \left|p^{(\alpha)}\left(x,\frac{1}{t}\xi\right)\right| f\left(\frac{1}{t}\xi\right) d\xi \\ &\leq Ct^{-n-1}\int_{1/2\leq|\xi|\leq 1} \left|\frac{1}{t}\xi\right|^{-n(1-\rho)-\rho(n+1)} \omega\left(\left|\frac{1}{t}\xi\right|^{-\sigma}\right) d\xi \\ &\leq Ct^{-(1-\rho)}\omega((2t)^\sigma).\end{aligned}$$

Hence we have

$$\int |B_t(x)|\,dx \le \int \left[\int_{|z|\ge t^{-(1-\rho)}} |z|^{-n+1}|z|^{n+1}|K_t(x,z)|\,|u(x-tz)|\,dz \right] dx$$
$$\le Ct^{-(1-\rho)}\omega((2t)^\sigma) \int \left[\int_{|z|\ge t^{-(1-\rho)}} |z|^{-n-1}|u(x-tz)|\,dz \right] dx$$
$$\le Ct^{-(1-\rho)}\omega((2t)^\sigma) \int_{|z|\ge t^{-(1-\rho)}} |z|^{-n-1} \left[\int |u(x-tz)|\,dx \right] dz$$
$$\le C\omega((2t)^\sigma)\|u\|_1.$$

Finally, we obtain

$$\int |p(X,D_x)u(x)|\,dx \le \int_0^{1/4} \frac{1}{t}\,dt \iint |K_t(x,z)u(x-tz)|\,dz\,dx$$
$$\le \int_0^{1/4} \frac{\omega((2t)^\sigma)}{t}\,dt\,\|u\|_1$$
$$\le C\|u\|_1.$$

■

Again, by the Riesz–Thorin interpolation theorem, we can get the $L^p(\mathbb{R}^n)$ boundedness for $1 \le p \le 2$.

Corollary 4.3 *If the symbol $p(x,\xi)$ satisfies the same condition as in Theorem 4.5, then the operator $p(X,D_x)$ is $L^p(\mathbb{R}^n)$ bounded for $1 \le p \le \infty$, and we have*

$$\|p(X,D_x)u\|_p \le C\|u\|_p \qquad (u \in \mathcal{S}),$$

where the constant C is independent of $1 \le p \le \infty$.

In the case $0 \le \rho < 1$, the decreasing order, $n(1-\rho)$, of the symbols in Theorem 4.5 does not coincide with the optimal decreasing order $m_1 = n(1-\rho)/2$. In this sense, the assumption of Theorem 4.5 is too strong. However, in the case $\rho = 1$, since $m_p = 0$, we can get the $L^1(\mathbb{R}^n)$ boundedness without using the regularity of the symbols in the space variable, x, for operators with symbols which have almost optimal decreasing order.

We give here the result for the case $\rho = 1$ as a corollary, which is only a special, but important, case of Corollary 4.3.

Corollary 4.4 *Let $\sigma > 0$. Assume that the symbol $p(x,\xi)$ satisfies*

$$\left|p^{(\alpha)}(x,\xi)\right| \le C\omega(\langle\xi\rangle^{-\sigma})\langle\xi\rangle^{-|\alpha|}$$

for $|\alpha| \le n+1$, where $\omega(t)$ is a nonnegative and nondecreasing function on $[0,\infty)$ which satisfies

$$\int_0^1 \frac{\omega(t)}{t}\,dt < \infty.$$

Then the operator $p(X, D_x)$ *is* $L^p(\mathbb{R}^n)$ *bounded for* $1 \le p \le \infty$ *and there exists a constant* C *which is independent of* $1 \le p \le \infty$, *such that*

$$\|p(X, D_x)u\|_p \le C\|u\|_p \quad (u \in \mathcal{S}).$$

If the symbols $p(x,\xi)$ are independent of the space variable, x, then we can get better results than the assertions in Theorem 4.5 and Corollaries 4.3 and 4.4. In fact we have the following theorem.

Theorem 4.6 *Let* $0 \le \rho \le 1$ *and* $\sigma > 0$ *and assume that the symbol* $p(\xi)$ *satisfies the inequality*

$$|p^{(\alpha)}(\xi)| \le C\omega(\langle\xi\rangle^{-\sigma})\langle\xi\rangle^{-n(1-\rho)/2-\rho|\alpha|}$$

for any $|\alpha| \le \kappa = [n/2]+1$, *where* $\omega(t)$ *satisfies the same condition as in Theorem 4.4. Then the operator* $p(D_x)$ *is* $L^1(\mathbb{R}^n)$ *bounded.*

Proof. As usual, by Lemma 4.1 we may assume that the support of the symbol $p(\xi)$ is contained in $\{\xi : |\xi| \ge 4\}$. Taking a smooth function $f(t)$ such that

$$\operatorname{supp} f \subset \left[\frac{1}{2}, 1\right] \quad \text{and} \quad \int_0^\infty \frac{f(t)}{t}\,dt = 1,$$

we have

$$p(D_x)u(x) = \frac{1}{(2\pi)^n}\int_0^{1/2}\frac{1}{t}\,dt\int K_t(z)u(x-tz)\,dz,$$

where

$$K_t(z) = \int e^{iz\cdot\xi}p\left(\frac{1}{t}\xi\right)f(|\xi|)\,d\xi.$$

Then, as in the proof of the $L^\infty(\mathbb{R}^n)$ boundedness, writing

$$\begin{aligned}\int |K_t(z)|dz &\le \int_{|z|\le t^{-(1-\rho)}} |K_t(z)|\,dz + \int_{|z|\ge t^{-(1-\rho)}} |K_t(z)|\,dz\\ &:= A_t + B_t,\end{aligned}$$

we have

$$\begin{aligned}A_t &\le \left[\int_{|z|\le t^{-(1-\rho)}} dz\right]^{1/2}\left[\int_{|z|\le t^{-(1-\rho)}} |K_t(z)|^2\,dz\right]^{1/2}\\ &\le Ct^{-n(1-\rho)/2}\left[\int \left|p\left(\frac{1}{t}\xi\right)f(|\xi|)\right|^2 d\xi\right]^{1/2}\\ &\le C\omega((2t)^\sigma)\end{aligned}$$

and

$$B_t \le \left[\int_{|z|\ge t^{-(1-\rho)}} |z|^{-2\kappa}\,dz\right]^{1/2}\left[\int_{|z|\ge t^{-(1-\rho)}} |z|^{2\kappa}|K_t(z)|^2\,dz\right]^{1/2}$$
$$\le Ct^{(1-\rho)(\kappa-n/2)}\sum_{|\alpha|=\kappa} t^{-\kappa}\left[\int\left|p^{(\alpha)}\left(\frac{1}{t}\xi\right)f(|\xi|)\right|^2 d\xi\right]^{1/2}$$
$$\le C\omega((2t)^\sigma).$$

Hence we have

$$\int |K_t(z)|\,dz \le C\omega((2t)^\sigma).$$

Therefore, changing the order of the integration, we obtain

$$\int |p(D_x)u(x)|\,dx \le \int_0^{1/2}\frac{1}{t}\,dt\iint |K_t(z)u(x-tz)|\,dzdx$$
$$\le \int_0^{1/2}\frac{1}{t}\,dt\int |K_t(z)|\,dz\|u\|_1$$
$$\le C\int_0^{1/2}\frac{\omega((2t)^\sigma)}{t}\,dt\|u\|_1$$
$$\le C\|u\|_1.$$

■

If we combine Theorems 4.4 and 4.6, the Riesz-Thorin interpolation theorem implies the following Corollary.

Corollary 4.5 *If the symbol $p(\xi)$ satisfies the same condition as in Theorem 4.6, then the operator $p(D_x)$ is $L^p(\mathbb{R}^n)$ bounded for $1 \le p \le \infty$, and we have*

$$\|p(D_x)u\|_p \le C\|u\|_p \qquad (u \in \mathcal{S}),$$

where the constant C is independent of $1 \le p \le \infty$.

5 Behavior in $L^\infty(\mathbb{R}^n)$ space

We first note that the results in this section are essentially found in Nagase [13].

Let Q be a cube in $\mathbb{R}^n$ with sides parallel to the coordinate axes, and $|Q|$ be its Lebesgue measure. For a function $u(x)$ defined on $\mathbb{R}^n$, we define its bounded mean oscillation (BMO) norm by

$$\|u\|_* := \|u\|_{BMO} = \sup_Q \frac{1}{|Q|}\int_Q |u(x) - u_Q|\,dx,$$

where u_Q denotes the mean value of $u(x)$ on Q, that is,

$$u_Q = \frac{1}{|Q|}\int_Q u(x)\,dx.$$

We let $BMO = \{u(x) : ||u||_* < \infty\}$ denote the space of BMO functions on $\mathbb{R}^n$. Then we easily see that $L^\infty(\mathbb{R}^n) \subset BMO$ and

$$||u||_* \leq 2||u||_\infty \qquad (\forall u \in L^\infty(\mathbb{R}^n)).$$

The proof of the following Theorem 5.1 is given in [13]. The result itself has been essentially derived by C. Fefferman [5] (see also, Li and Wang [11]). The proof of the theorem is a little long but we give it here. Theorem 5.1 will be used in the proof of our main Theorem 5.2 in this section.

Theorem 5.1 *Let $\rho > 0$ and $\delta < 1$ satisfy $0 \leq \delta \leq \rho \leq 1$. Assume that, for α and β with $|\alpha|, |\beta| \leq \kappa = [n/2] + 1$, the symbol $p(x,\xi)$ satisfies*

$$|p^{(\alpha)}_{(\beta)}(x,\xi)| \leq C_{\alpha,\beta}\langle\xi\rangle^{-n(1-\rho)/2-\rho|\alpha|+\delta|\beta|}.$$

Then the operator $p(X, D_x)$ is bounded from $L^\infty(\mathbb{R}^n)$ to BMO, and we have

$$||p(X, D_x)u||_* \leq C||u||_\infty \quad \forall u \in L^\infty(\mathbb{R}^n).$$

Proof. As before, we may assume that the support of $p(x,\xi)$ is contained in $\{(x,\xi) : |\xi| \geq 4\}$ and $p(x,\xi)$ satisfies

$$|p^{(\alpha)}_{(\beta)}(x,\xi)| \leq C|\xi|^{-n(1-\rho)/2-\rho|\alpha|+\delta|\beta|} \qquad (|\xi| \geq 4)$$

for $|\alpha|, |\beta| \leq \kappa$. Moreover, by the Calderón–Vaillancourt theorem we have

$$||p(X, D_x)|D_x|^{n(1-\rho)/2}u||_2 \leq C||u||_2$$

because the symbol $p(x,\xi)|\xi|^{n(1-\rho)/2}$ satisfies the conditions of Theorem 3.3. As before, we take a nonnegative function $f(t) \in C_0^\infty(\mathbb{R})$ such that

$$\operatorname{supp} f \subset \left[\frac{1}{2}, 1\right], \qquad \int_0^\infty \frac{f(t)}{t}\,dt = 1.$$

Thus, we have

$$\int_0^\infty \frac{f(t|\xi|)}{t}\,dt = 1 \qquad (|\xi| \neq 0).$$

We consider a cube $Q = \{x : |x_j - a_j^0| \leq d/2\}$ with $d \leq 1$, and take a function $\psi(\xi)$ such that

$$\operatorname{supp}\psi \subset \{|\xi| \leq 2\}, \qquad 0 \leq \psi(\xi) \leq 1,$$

and $\psi(\xi) = 1$ for $\xi \le 1$. We set $\psi_d(\xi) = \psi(d\xi)$. We split the symbol $p(x,\xi)$ as

$$\begin{aligned} p(x,\xi) &= p(x,\xi)\psi_d(\xi) + p(x,\xi)\big(1 - \psi_d(\xi)\big) \\ &:= p_0(x,\xi) + p_1(x,\xi). \end{aligned}$$

We begin with the estimate for the operator $p_0(X, D_x)$. As before, we can write

$$p_0(X, D_x)u(x) = \frac{1}{(2\pi)^{n/2}} \int e^{ix\cdot\xi} p_0(x,\xi)\hat{u}(\xi)\, d\xi,$$

and

$$\begin{aligned} D_{x_j}\{p_0(X,D_x)u(x)\} &= \frac{1}{(2\pi)^{n/2}} \int e^{ix\cdot\xi}\{p_{0(e_j)}(x,\xi) + \xi_j p_0(x,\xi)\}\hat{u}(\xi)\, d\xi \\ &= \frac{1}{(2\pi)^{n/2}} \int e^{ix\cdot\xi}\{p_{(e_j)}(x,\xi) + \xi_j p(x,\xi)\}\psi_d(\xi)\hat{u}(\xi)\, d\xi \\ &= \frac{1}{(2\pi)^{n/2}} \int e^{ix\cdot\xi} p^j(x,\xi)\hat{u}(\xi)\, d\xi, \end{aligned}$$

where

$$\begin{aligned} p^j(x,\xi) &= p_{0(e_j)}(x,\xi) + \xi_j p_0(x,\xi) \\ &= \{p_{(e_j)}(x,\xi) + \xi_j p(x,\xi)\}\psi_d(\xi). \end{aligned}$$

Hence, using $f(t)$ we can write

$$\begin{aligned} D_{x_j}\{p_0(X,D_x)u(x)\} &= \frac{1}{(2\pi)^n}\int_0^\infty \frac{1}{t}dt \iint e^{i(x-y)\cdot\xi} p^j(x,\xi) f(t|\xi|)u(y)\, dy d\xi \\ &= \frac{1}{(2\pi)^n}\int_0^\infty \frac{1}{t}dt \iint e^{iz\cdot\xi} p^j\left(x, \frac{1}{t}\xi\right) f(|\xi|)u(x-tz)\, d\xi dz \\ &= \frac{1}{(2\pi)^n}\int_0^\infty \frac{1}{t}dt \int K_{0,j}(t,x,z)u(x-tz)\, dz, \end{aligned}$$

where

$$\begin{aligned} K_{0,j}(t,x,z) &= \int e^{iz\cdot\xi} p^j\left(x,\frac{1}{t}\xi\right) f(|\xi|)\, d\xi \\ &= \int e^{iz\cdot\xi}\left[p_{(e_j)}\left(x,\frac{1}{t}\xi\right) + \frac{1}{t}\xi_j p\left(x,\frac{1}{t}\xi\right)\right]\psi_d\left(\frac{1}{t}\xi\right) f(|\xi|)\, d\xi. \end{aligned}$$

On the support of the integrand of the kernel $K_{0,j}(t,x,z)$, we have

$$\frac{1}{t}|\xi| \ge 2, \qquad \frac{1}{2} \le |\xi| \le 2, \qquad \frac{1}{t}|\xi| \le \frac{2}{d}.$$

Therefore, on this support we get $d/4 \le t$. Thus we can write

$$D_{x_j}\{p_0(X,D_x)u(x)\} = \frac{1}{(2\pi)^n}\int_{d/4}^{1/2} \frac{1}{t}dt \int K_{0,j}(t,x,z)u(x-tz)\, dz.$$

Then we write

$$\int |K_{0,j}(t,x,z)|\,dz = \int_{\{|z|\le t^{-(1-\rho)}\}} |K_{0,j}(t,x,z)|\,dz + \int_{\{|z|\ge t^{-(1-\rho)}\}} |K_{0,j}(t,x,z)|\,dz$$
$$:= I + II.$$

By Schwarz' inequality and Plancherel's formula we have

$$I \le \left[\int_{\{|z|\le t^{-(1-\rho)}\}} dz\right]^{1/2} \left[\int_{\{|z|\le t^{-(1-\rho)}\}} |K_{0,j}(t,x,z)|^2\,dz\right]^{1/2}$$
$$\le Ct^{-n(1-\rho)/2}\left[\int \left|p^j\left(x,\frac{1}{t}\,\xi\right)\right|^2 f(|\xi|)^2\,d\xi\right]^{1/2}.$$

By the definition of $p^j(x,\xi)$, we can see that

$$\left|p^j\left(x,\frac{1}{t}\,\xi\right)\right| \le C\left(\frac{1}{t}|\xi|\right)^{1-n(1-\rho)/2} \qquad \left(\frac{1}{2}\le|\xi|\le 1\right).$$

Therefore, we have

$$I \le Ct^{-1}\left[\int_{\{1/2\le|\xi|\le 1\}} d\xi\right]^{1/2} = \frac{C}{t}.$$

It is not difficult to see that the symbol $p^j(x,\xi)$ satisfies the estimate

$$\left|p^{j\,(\alpha)}(x,\xi)\right| \le C|\xi|^{-m_\infty+1-\rho|\alpha|} \qquad \left(\frac{1}{2}\le|\xi|\le 1\right)$$

for $|\alpha|\le\kappa$, where the constant C is independent of $0<d\le 1$. Hence we have

$$\left|\partial_\xi^\alpha\left[p^j\left(x,\frac{1}{t}\,\xi\right)\right]\right| \le Ct^{-|\alpha|}\left|\frac{1}{t}\,\xi\right|^{-m_\infty+1-\rho|\alpha|}$$
$$\le Ct^{m_\infty-1-(1-\rho)|\alpha|} \qquad \left(\frac{1}{2}\le|\xi|\le 1\right)$$

for $|\alpha|\le\kappa$. Therefore, for any α with $|\alpha|\le\kappa$ we have

$$\left|\partial_\xi^\alpha\left[p^j\left(x,\frac{1}{t}\,\xi\right)\right\}\right| \le Ct^{m_\infty-1-(1-\rho)\kappa} \qquad \left(\frac{1}{2}\le|\xi|\le 1\right).$$

Thus, again by Schwarz' inequality and Plancherel's formula, we have

$$II \le \left[\int_{|z|\ge t^{-(1-\rho)}} |z|^{-2\kappa}\,dz\right]^{1/2}\left[\int_{|z|\ge t^{-(1-\rho)}} |z|^{2\kappa}|K_{0,j}(t,x,z)|^2\,dz\right]^{1/2}$$

$$\le Ct^{-(n/2-\kappa)(1-\rho)}\sum_{|\alpha|=\kappa}\left[\int\left|\partial_\xi^\alpha\left[p^j\left(x,\frac{1}{t}\xi\right)\right]\right|^2 d\xi\right]^{1/2}$$

$$\le Ct^{-(n/2-\kappa)(1-\rho)}t^{m_\infty-1-(1-\rho)\kappa}$$

$$\le C\frac{1}{t}.$$

Hence we have

$$|D_{x_j}[p_0(X,D_x)u(x)]| \le C\int_{d/4}^{1/2}\frac{1}{t^2}\,dt\,\|u\|_\infty$$

$$\le \frac{C}{d}\|u\|_\infty.$$

Using this estimate we can see that

$$|p_0(X,D_x)u(x)-p_0(X,D_x)u(y)| \le \frac{C|x-y|}{d}\|u\|_\infty.$$

Therefore, writing

$$(p_0(X,D_x)u)_Q = \frac{1}{|Q|}\int_Q p_0(X,D_x)u(y)\,dy,$$

we obtain

$$\frac{1}{|Q|}\int_Q |p_0(X,D_x)u(x)-(p_0(X,D_x)u)_Q|\,dx$$

$$\le \frac{1}{|Q|^2}\int_Q\int_Q |p_0(X,D_x)u(x)-p_0(X,D_x)u(y)|\,dy\,dx$$

$$\le C\frac{1}{|Q|^2}\int_Q\int_Q\frac{|x-y|}{d}\,dy\,dx\|u\|_\infty$$

$$\le C\|u\|_\infty. \tag{5}$$

Now we have to estimate the term $p_1(X,D_x)u(x)$. We take a function $\chi(x)\in C_0^\infty(\mathbb{R}^n)$ such that

$$\begin{cases}\chi(x)=1 & \text{for } x=(x_1,x_2,\dots,x_n) \text{ with } |x_j|\le 2 \quad (j=1,2,\dots,n),\\ \chi(x)=0 & \text{for } x=(x_1,x_2,\dots,x_n) \text{ with } |x_j|\ge 4 \quad (j=1,2,\dots,n).\end{cases}$$

Denoting the center of the cube Q by x^0, we set

$$\chi_d(x)=\chi(d^{-\rho}(x-x^0))$$

and write

$$p_1(X,D_x)u(x) = p_1(X,D_x)\{\chi_d u\}(x) + p_1(X,D_x)\{(1-\chi_d)u\}(x)$$
$$:= Iu(x) + IIu(x).$$

Then, by Schwarz' inequality we have

$$(Iu)_Q = \left|\frac{1}{|Q|}\int_Q Iu(x)\,dx\right| \le \left[\frac{1}{|Q|}\int_Q |p_1(X,D_x)\{\chi_d u\}(x)|^2\,dx\right]^{1/2}.$$

Now we write the symbol $p_1(x,\xi)$ as

$$p_1(x,\xi) = p(x,\xi)|\xi|^{n(1-\rho)/2}(1-\psi_d(\xi))|\xi|^{-n(1-\rho)/2}$$

and note that $p(x,\xi)|\xi|^{n(1-\rho)/2}$ satisfies the conditions of the Calderón-Vaillancourt theorem. So, using the $L^2(\mathbb{R}^n)$ boundedness of the operator $p(X,D_x)|D_x|^{n(1-\rho)/2}$, we have

$$\|p_1(X,D_x)\chi_d u\|_2 \le C\|(1-\psi_d(D_x))|D_x|^{-n(1-\rho)/2}\{\chi_d u\}\|_2$$
$$\le Cd^{n(1-\rho)/2}\|\{\chi_d u\}\|_2$$
$$\le Cd^{n(1-\rho)/2}\|\chi_d\|_2\,\|u\|_\infty.$$

By the definition of $\chi_d(x)$ we have

$$\|\chi_d\|_2 = d^{n\rho/2}\|\chi\|_2.$$

Thus we have

$$\|p_1(X,D_x)\chi_d u\|_2 \le Cd^{n/2}\|u\|_\infty,$$

and, therefore,

$$(Iu)_Q \le C\|u\|_\infty.$$

In order to estimate the term $IIu(x)$ we write

$$IIu(x) = \frac{1}{(2\pi)^n}\int_0^d \frac{1}{t}\,dt \int K_1(t,x,z)\{(1-\chi_d)u\}(x-tz)\,dz,$$

where

$$K_1(t,x,z) = \int e^{iz\cdot\xi}p\left(x,\frac{1}{t}\xi\right)\left[1-\psi_d\left(\frac{1}{t}\xi\right)\right]f(|\xi|)\,d\xi.$$

Here, in the support of the integrand of the integral kernel $K_1(t,x,z)$, for $x = (x_1,x_2,\dots,x_n) \in Q$ we have

$$d^{-\rho}|x_j - tz_j - x_j^0| \ge 2, \qquad |x_j - x_j^0| \le \frac{d}{2}.$$

Thus, we have

$$|z| \ge t^{-1}d^\rho.$$

Hence, for $x \in Q$ we have

$$\begin{aligned}
\int |K_1(t,x,z)|\,dz &= \int_{|z|\geq t^{-1}d^\rho} |K_1(t,x,z)|\,dz \\
&\leq \left[\int_{|z|\geq t^{-1}d^\rho} |z|^{-2\kappa}\,dz\right]^{1/2} \left[\int_{|z|\geq t^{-1}d^\rho} |z|^{2\kappa}|K_1(t,x,z)|^2\,dz\right]^{1/2} \\
&\leq Ct^{\kappa-n/2}d^{-\rho(\kappa-n/2)} \sum_{|\alpha|=\kappa} \left[\int z^{2\alpha}|K_1(t,x,z)|^2\,dz\right]^{1/2} \\
&\leq Ct^{\kappa-n/2}d^{-\rho(\kappa-n/2)} \sum_{|\alpha|=\kappa} \left\{\int \left|\partial_\xi^\alpha\left[p(x,\frac{1}{t}\xi)(1-\psi_d(\frac{1}{t}\xi))f(|\xi|)\right]\right|^2 d\xi\right\}^{1/2} \\
&\leq Ct^{\kappa-n/2}d^{-\rho(\kappa-n/2)}t^{-(1-\rho)(\kappa-n/2)} \\
&\leq Ct^{\rho(\kappa-n/2)}d^{-\rho(\kappa-n/2)}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
|IIu(x)| &\leq C\|u\|_\infty \int_0^d \frac{1}{t}\,dt \int |K_1(t,x,z)|\,dz \\
&\leq C\|u\|_\infty \int_0^d \frac{1}{t^{1-\rho(\kappa-n/2)}}\,dt d^{-\rho(\kappa-n/2)} \leq C\|u\|_\infty.
\end{aligned}$$

Here we used the condition $\rho > 0$. Thus combining the estimate for $Iu(x)$ we have

$$\frac{1}{|Q|}\int_Q |p_1(X,D_x)u(x) - (p_1(X,D_x)u)_Q|\,dx \leq C\|u\|_\infty. \tag{6}$$

Therefore, from (5) and (6) we have

$$\frac{1}{|Q|}\int_Q |p(X,D_x)u(x) - (p(X,D_x)u)_Q|\,dx \leq C\|u\|_\infty,$$

and, since the cube Q is arbitrary, we finally obtain

$$\|p(X,D_x)u\|_* \leq C\|u\|_\infty.$$

■

As a corollary to Theorem 5.1 we have the following boundedness result.

Corollary 5.1 ([5]) *Let $2 \leq p < \infty$, $0 \leq \delta \leq \rho \leq 1$, $\delta < 1$, and $0 < \rho$. If the symbol $p(x,\xi)$ satisfies*

$$\left|p_{(\beta)}^{(\alpha)}(x,\xi)\right| \leq C\langle\xi\rangle^{-m_p-\rho|\alpha|+\delta|\beta|}$$

for $|\alpha| \leq \kappa = [n/2]+1$ and $|\beta| \leq \kappa$, then the operator $p(X,D_x)$ is $L^p(\mathbb{R}^n)$ bounded and we have

$$\|p(X,D_x)u\|_p \leq C_p\|u\|_p.$$

The proof of Corollary 5.1 can be done by using the Fefferman–Stein interpolation theorem [6]).

Using the symbol approximation (or regularization) and Theorem 5.1 we can prove a boundedness theorem for operators with symbols which have a weak regularity in the space variable, x, and the critical decreasing order. For the symbol approximation the following two lemmas play an essential role.

Lemma 5.1 (See [13]) *Let $0 < \delta < 1$ and $\psi(x)$ be in $\mathcal{S}$. Then $\psi(\langle\xi\rangle^{\delta}x)$ belongs to $S^0_{1,\delta}$ and satisfies*

$$\partial_\xi^\alpha\{\psi(\langle\xi\rangle^\delta x)\} = \sum_{|\alpha'|\le|\alpha|} \psi_{\alpha,\alpha'}(\xi)\{\langle\xi\rangle^\delta x\}^{\alpha'}\psi^{(\alpha')}(\langle\xi\rangle^\delta x)$$

for any α, where $\psi^{(\alpha')}(z) = \partial_z^{\alpha'}\psi(z)$ and $\psi_{\alpha,\alpha'}(\xi) \in S^{-|\alpha|}_{1,0}$.

Lemma 5.2 (See [10]) *Let $0 < \tau < 1$ and $\psi(x)$ be in $\mathcal{S}$. Then, for any β, $\partial_\xi^\beta\psi(\langle\xi\rangle^{-\tau}(\zeta-\xi))$ satisfies*

$$\partial_\xi^\beta\{\psi(\langle\xi\rangle^{-\tau}(\zeta-\xi))\} = \sum_{|\gamma|\le|\beta|,\gamma_1\le\gamma} \psi_{\beta,\gamma,\gamma_1}(\xi)\{\langle\xi\rangle^{-\tau}(\zeta-\xi)\}^{\gamma_1}\psi^{(\gamma)}(\langle\xi\rangle^{-\tau}(\zeta-\xi)),$$

where $\psi_{\beta,\gamma,\gamma_1}(\xi) \in S^{-(|\beta|-(1-\tau)|\gamma-\gamma_1|)}_{1,0}$.

We shall use Lemma 5.2 in the next Section 6. As one of the main results of this section we have the following theorem. By Theorem 4.4 and the Fefferman–Stein interpolation theorem we can prove an $L^p(\mathbb{R}^n)$ boundedness theorem for operators with symbols with critical decreasing order m_p for $2 \le p < \infty$.

Theorem 5.2 *Let $0 \le \delta < \rho \le 1$ and $\delta < 1$, and suppose that the symbol $p(x,\xi)$ satisfies*

$$\left|p^{(\alpha)}(x,\xi)\right| \le C\langle\xi\rangle^{-n(1-\rho)/2-\rho|\alpha|}$$

and

$$\left|p^{(\alpha)}(x,\xi) - p^{(\alpha)}(y,\xi)\right| \le C\omega(|x-y|\langle\xi\rangle^\delta)\langle\xi\rangle^{-n(1-\rho)/2-\rho|\alpha|}$$

for any $|\alpha| \le \kappa = [n/2]+1$, where $\omega(t)$ is a nonnegative and nondecreasing function on $[0,\infty)$ which satisfies

$$\int_0^1 \frac{\omega(t)}{t}\,dt < \infty.$$

Then the operator $p(X,D_x)$ is bounded from $L^\infty(\mathbb{R}^n)$ to BMO and we have

$$\|p(X,D_x)u\|_* \le C\|u\|_\infty.$$

Proof. As usual, we may assume that the support of the symbol is contained in $\{(x,\xi) : |\xi| \geq 4\}$. Take a smooth function $\varphi(x)$ on $\mathbb{R}^n$ such that the support of $\varphi(x)$ is compact and $\int \varphi(x)\,dx = 1$. We define a new symbol $\bar{p}(x,\xi)$ by

$$\begin{aligned}\bar{p}(x,\xi) &= \int \varphi(y)p(x - \langle\xi\rangle^{-\delta'}y, \xi)\,dy \\ &= \langle\xi\rangle^{\delta' n}\int \varphi(\langle\xi\rangle^{\delta'}(x-y))p(y,\xi)\,dy,\end{aligned}$$

where δ' is a constant such that $\delta < \delta' < \rho$, and set

$$q(x,\xi) = p(x,\xi) - \bar{p}(x,\xi).$$

Then, by Lemma 5.1 we can see that the symbol $\bar{p}(x,\xi)$ satisfies

$$\left|\bar{p}^{(\alpha)}_{(\beta)}(x,\xi)\right| \leq C_\beta\langle\xi\rangle^{-n(1-\rho)/2-\rho|\alpha|+\delta'|\beta|}$$

for any $|\alpha| \leq \kappa$ and β. Therefore, by Theorem 5.1, the operator $\bar{p}(X,D_x)$ is bounded from $L^\infty(\mathbb{R}^n)$ to BMO, and we have

$$\|\bar{p}(X,D_x)u\|_* \leq C\|u\|_\infty.$$

On the other hand, we can see that the symbol $q(x,\xi)$ satisfies

$$\left|q^{(\alpha)}(x,\xi)\right| \leq C\omega(\langle\xi\rangle^{-\delta'+\delta})\langle\xi\rangle^{-n(1-\rho)/2-\rho|\alpha|}$$

for any $|\alpha| \leq \kappa$. Since $\delta' - \delta > 0$, the symbol $q(x,\xi)$ satisfies the conditions of Theorem 4.4 and we have

$$\|q(X,D_x)u\|_\infty \leq C\|u\|_\infty.$$

Finally, we obtain

$$\begin{aligned}\|p(X,D_x)u\|_* &\leq \|\bar{p}(X,D_x)u\|_* + \|q(X,D_x)u\|_* \\ &\leq C\|u\|_\infty + 2\|q(X,D_x)u\|_\infty \\ &\leq C\|u\|_\infty.\end{aligned}$$

∎

When $\rho = 1$, by using Theorem 5.2 we can show a slightly more general result than Corollary 5.1.

Corollary 5.2 *Let $\delta < 1$. Assume that the symbol $p(x,\xi)$ satisfies*

$$\left|p^{(\alpha)}(x,\xi)\right| \leq C\langle\xi\rangle^{-|\alpha|},$$
$$\left|p^{(\alpha)}(x,\xi) - p^{(\alpha)}(y,\xi)\right| \leq C\omega(\langle\xi\rangle^{\delta}|x-y|)\langle\xi\rangle^{-|\alpha|},$$

for any $|\alpha| \le \kappa$, where $\omega(t)$ is a nonnegative and nondecreasing function on $[0,\infty)$ such that

$$\int_0^1 \frac{\omega(t)}{t}\, dt < \infty.$$

Then the operator $p(X, D_x)$ is $L^p(\mathbb{R}^n)$ bounded for $2 \le p < \infty$.

Proof. We have already seen in Corollary 4.1 that the operator is $L^2(\mathbb{R}^n)$ bounded when $p(x,\xi)$ satisfies the conditions of Theorem 5.2. Also we have seen that the operator is bounded from $L^\infty(\mathbb{R}^n)$ to BMO by Theorem 5.2. Therefore the boundedness follows from the Fefferman-Stein interpolation theorem. ■

6 $L^p(\mathbb{R}^n)$ estimates for $1 < p < \infty$

In this section we consider only the case $\rho = 1$. When $2 \le p < \infty$, we have already seen in Corollary 5.1 that we can get the boundedness result even in the case $\rho < 1$. However for, the case $1 < p < 2$ and $\rho < 1$, we need a slightly different argument to get the $L^p(\mathbb{R}^n)$ boundedness.

Theorem 6.1 *Let $\delta < 1$ and assume that the symbol $p(x,\xi)$ satisfies*

$$\left|p^{(\alpha)}(x,\xi)\right| \le C\langle\xi\rangle^{-|\alpha|},$$
$$\left|p^{(\alpha)}(x,\xi) - p^{(\alpha)}(y,\xi)\right| \le C\omega(\langle\xi\rangle^{\delta}|x-y|)\langle\xi\rangle^{-|\alpha|},$$

for $|\alpha| \le n+2$, where $\omega(t)$ is a nonnegative and nondecreasing function on $[0,\infty)$ such that

$$\int_0^1 \frac{\omega(t)}{t} dt < \infty.$$

Then the operator $p(X, D_x)$ is $L^p(\mathbb{R}^n)$ bounded for $1 < p < \infty$.

Proof. We have already seen that the operator $p(X, D_x)$ is $L^p(\mathbb{R}^n)$ bounded for $2 \le p < \infty$. So we need only consider the case $1 < p < 2$. Let $\varphi(x)$ be a $C^\infty(\mathbb{R}^n)$ even function with support in $\{x : |x| \le 1\}$ and $\int \varphi(x)\, dx = 1$. We define a new symbol $\tilde{p}(x,\xi)$ by

$$\begin{aligned}
\tilde{p}(x,\xi) &= \langle\xi\rangle^{n\tau} \int \varphi(\langle\xi\rangle^{\tau}(x-y))p(y,\xi)\, dy \\
&= \langle\xi\rangle^{n\tau} \int \varphi(\langle\xi\rangle^{\tau} y)p(x-y,\xi)\, dy \\
&= \int \varphi(z)p(x - \langle\xi\rangle^{-\tau} y,\xi)\, dy,
\end{aligned}$$

where $\delta < \tau < 1$. Then, as before, we can see that the symbol $\bar{p}(x,\xi)$ satisfies

$$\left|\bar{p}^{(\alpha)}_{(\beta)}(x,\xi)\right| \leq C\langle\xi\rangle^{-|\alpha|+\tau|\beta|}$$

for any β and $|\alpha| \leq n+2$. Moreover we can see that the symbol $q(x,\xi) = p(x,\xi) - \bar{p}(x,\xi)$ satisfies

$$|q(x,\xi)| \leq C\omega(\langle\xi\rangle^{-(\tau-\delta)})\langle\xi\rangle^{-|\alpha|}$$

for $|\alpha| \leq n+2$. Hence, by Theorem 4.5 we can see that the operator $q(X, D_x)$ is $L^p(\mathbb{R}^n)$ bounded for $1 \leq p \leq \infty$. So we have to prove the boundedness for the operator $\bar{p}(X, D_x)$.

We define another new symbol $\bar{\bar{p}}(x,\xi)$ by

$$\begin{aligned}\bar{\bar{p}}(x,\xi) &= \langle\xi\rangle^{-n\rho}\int \varphi(\langle\xi\rangle^{-\rho}(\xi-\zeta))\bar{p}(x,\zeta)\,d\zeta \\ &= \langle\xi\rangle^{-n\rho}\int \varphi(\langle\xi\rangle^{-\rho}\zeta)\bar{p}(x,\xi-\zeta)\,d\zeta \\ &= \int \varphi(\zeta)\bar{p}(x,\xi-\langle\xi\rangle^{\rho}\zeta)\,d\zeta,\end{aligned}$$

where $\tau < \rho < 1$. Then, by Lemma 5.2 we can see that the symbol $\bar{\bar{p}}(x,\xi)$ belongs to the symbol class $S^0_{\rho,\tau}$ and satisfies

$$\left|\bar{\bar{p}}^{(\alpha)}_{(\beta)}(x,\xi)\right| \leq C_{\alpha,\beta}\langle\xi\rangle^{-|\alpha|+\tau|\beta|} \tag{7}$$

for any β and $|\alpha| \leq n+2$. Moreover we can see that the symbol $r(x,\xi) = \bar{p}(x,\xi) - \bar{\bar{p}}(x,\xi)$ satisfies

$$\left|r^{(\alpha)}(x,\xi)\right| \leq C\langle\xi\rangle^{-|\alpha|-(\rho-\tau)}$$

for $|\alpha| \leq n+1$. Hence, again, we can see that the operator $r(X, D_x)$ is $L^p(\mathbb{R}^n)$ bounded for $1 \leq p \leq \infty$. Writing the operator $p(X, D_x)$ as

$$p(X,D_x) = \bar{\bar{p}}(X,D_x) + r(x,\xi) + q(X,D_x),$$

we need only show the boundedness of the operator $\bar{\bar{p}}(X, D_x)$. Since $\bar{\bar{p}}(X, D_x)$ belongs to $S^0_{\rho,\tau}$, we can use the algebra of the symbol class $S^{\infty}_{\rho,\tau}$. For u and v in $\mathcal{S}$ we have

$$(\bar{\bar{p}}(X,D_x)u, v) = (u, \bar{\bar{p}}^*(X,D_x)v),$$

where the symbol $\bar{\bar{p}}^*(x,\xi)$ belongs to $S^{\infty}_{\rho,\tau}$ and has the asymptotic expansion

$$\bar{\bar{p}}^*(x,\xi) \sim \sum_{\alpha} \frac{(-i)^{|\alpha|}}{\alpha!}\overline{\bar{\bar{p}}^{(\alpha)}_{(\alpha)}(x,\xi)}.$$

Hence, using (7) and $\rho > \tau$ we have

$$\left|\bar{\bar{p}}^{*(\alpha)}{}_{(\beta)}(x,\xi)\right| \leq C\langle\xi\rangle^{-|\alpha|+\tau|\beta|}$$

for $|\alpha| \le n+1$ and any β. This implies

$$\|\bar{\tilde{p}}^*(X, D_x)v\|_{p'} \le C\|v\|_{p'},$$

where $1/p + 1/p' = 1$. So, by a duality argument we have

$$\|p(X, D_x)u\|_p \le C\|u\|_p.$$

■

7 Pseudodifferential operators with magnetic potentials

As we stated in Section 3, if $0 \le \delta \le \rho \le 1$ and $\delta < 1$, then the pseudodifferential operator $p(X, D_x)$ with symbol $S^0_{\rho,\delta,\lambda}$ is $L^2(\mathbb{R}^n)$ bounded where λ is any basic function.

Let $a(x) = (a_1(x), \dots, a_n(x))$ be an $\mathbb{R}^n$ valued function on $\mathbb{R}^n$, where $a_j(x)$, $j = 1, 2, \dots, n$, are real valued smooth functions whose derivatives, $|\partial^\alpha a_j(x)|$, are bounded for any $\alpha \ne 0$. We consider the basic function

$$\lambda(x,\xi) = \sqrt{1 + |\xi - a(x)|^2}. \tag{8}$$

Thus, $\lambda(x,\xi)$ satisfies the following inequalities:

(a) $\quad 1 \le \lambda(x,\xi) \le C\langle x\rangle\langle\xi\rangle$,

(b) $\quad |\lambda^{(\alpha)}_{(\beta)}(x,\xi)| \le C_{\alpha,\beta}\lambda(x,\xi)^{1-|\alpha|}$.

In this section, we restrict attention to the basic weight function (8) and consider the symbol class $S^m_{\rho,\delta,\lambda}$ where $0 \le \delta < \rho = 1$. Then the problem is to show the $L^p(\mathbb{R}^n)$ boundedness of the operator $p(X, D_x) \in S^0_{1,\delta,\lambda}$ for general $1 < p < \infty$. However, this problem is still open (see Section 8). We present here a slightly weaker boundedness result, which corresponds to the case of lower order operators.

We first prove the following lemma.

Lemma 7.1 *Let $p(x,\xi)$ be in $S^{-\sigma}_{0,\delta,\lambda}$ for some positive $\sigma > n$. Then the operator $p(X, D_x)$ is $L^p(\mathbb{R}^n)$ bounded for $1 \le p \le \infty$.*

Proof. From the definition of the operator $p(x, D_x)$, we have

$$\begin{aligned} p(X, D_x)u(x) &= \frac{1}{(2\pi)^n}\iint e^{i(x-y)\cdot\xi}p(x,\xi)u(y)\,dy\,d\xi \\ &= \int K(x, x-y)u(y)\,dy \qquad (u \in \mathcal{S}), \end{aligned}$$

where

$$K(x,z) = \frac{1}{(2\pi)^n}\int e^{iz\cdot\xi}p(x,\xi)\,d\xi.$$

Since $\sigma > n$, for any multi-index α we have

$$\begin{aligned}|z^\alpha K(x,z)| &= \frac{1}{(2\pi)^n}\left|\int e^{iz\cdot\xi}p^{(\alpha)}(x,\xi)\,d\xi\right| \\ &\le \frac{1}{(2\pi)^n}\int \left|p^{(\alpha)}(x,\xi)\right| d\xi \\ &\le C\int \langle \xi - a(x)\rangle^{-\sigma}\,d\xi \\ &= C\int \langle\xi\rangle^{-\sigma} d\xi = C_\alpha.\end{aligned}$$

Therefore, we obtain

$$|K(x,z)| \le C\langle z\rangle^{-n-1}$$

and

$$\|p(X,D_x)u\|_p \le C\|u\|_p \quad (u \in \mathcal{S}).$$

■

For $2 \le p \le \infty$, we can get a slightly stronger result than Lemma 7.1.

Lemma 7.2 *Let the symbol $p(x,\xi)$ satisfy*

$$|p^{(\alpha)}(x,\xi)| \le C_\alpha \lambda(x,\xi)^{-\sigma-|\alpha|}$$

for any α with $|\alpha| \le \kappa = [n/2]+1$, where σ is a positive constant. Then the operator $p(X,D_x)$ is $L^p(\mathbb{R}^n)$ bounded and we have

$$\|p(X,D_x)u\|_p \le C\|u\|_p \qquad (u \in \mathcal{S}),$$

where the constant C is independent of $2 \le p \le \infty$.

Proof. From the definition of the operator $p(X,D_x)$ we have

$$\begin{aligned}p(X,D_x)u(x) &= \frac{1}{(2\pi)^n}\iint e^{i(x-y)\cdot\xi}p(x,\xi)u(y)\,dy\,d\xi \\ &= \int K(x,x-y)u(y)\,dy,\end{aligned}$$

where the integral kernel $K(x,z)$ is

$$K(x,z) = \frac{1}{(2\pi)^n}\int e^{iz\cdot\xi}p(x,\xi)\,d\xi.$$

For $L^\infty(\mathbb{R}^n)$ boundedness we have

$$\begin{aligned}\int |K(x,x-y)u(y)|\,dy &\le \int |K(x,x-y)|\,dy\,\|u\|_\infty \\ &\le \int \langle z\rangle^{-\kappa}\langle z\rangle^{\kappa}|K(x,z)|\,dz\,\|u\|_\infty \\ &\le C\|u\|_\infty\left[\int \langle z\rangle^{2\kappa}|K(x,z)|^2\,dz\right]^{1/2}.\end{aligned}$$

From the assumption on the symbol $p(x,\xi)$ we have

$$\begin{aligned}\int \langle z\rangle^{2\kappa}|K(x,z)|^2\,dz &= \sum_{|\alpha|\le\kappa} c_\alpha \int |z^\alpha K(x,z)|^2\,dz\\ &= \sum_{|\alpha|\le\kappa} c_\alpha \int |p^{(\alpha)}(x,\xi)|^2 d\xi\\ &\le C\int \langle \xi - a(x)\rangle^{-2\sigma} d\xi\\ &= C\int \langle\xi\rangle^{-2\sigma} d\xi = C_\infty, \end{aligned}\tag{9}$$

where the last constant, C_∞, is independent of the variable x. Hence we have

$$|p(X,D_x)u(x)| \le C_\infty \|u\|_\infty.$$

Therefore, we can get the $L^\infty(\mathbb{R}^n)$ boundedness with norm bounds not greater than C_∞.

For the $L^2(\mathbb{R}^n)$ boundedness, using estimate (9), we have

$$\begin{aligned}\int |p(X,D_x)u(x)|^2\,dx &= \int \left|\int K(x,x-y)u(y)\,dy\right|^2 dx\\ &\le \int \left[\int \langle x-y\rangle^{-2\kappa}|u(y)|^2\,dy\right]\\ &\quad\times \left[\int \langle x-y\rangle^{2\kappa}|K(x,x-y)|^2\,dy\right] dx\\ &\le C_\infty \iint \langle x-y\rangle^{-2\kappa}|u(y)|^2\,dy\,dx\\ &= C'_\infty\|u\|_2.\end{aligned}$$

Thus we obtain the $L^2(\mathbb{R}^n)$ estimate. Hence the lemma follows from the Riesz–Thorin interpolation (see [17]). ∎

Remark 7.1. In Lemmas 7.1 and 7.2, we do not need the assumption that the derivatives $\partial_\alpha a_j(x)$ are bounded for any j and $\alpha \neq 0$. In the proofs we use only the fact that the functions $a_j(x)$ are real valued and measurable for $j = 1,2,\dots,n$.

Theorem 7.1 *Let $a(x)$ be as in Lemma 7.2 and $\lambda(x,\xi)$ as in Definition 2. Choose a nonnegative and nondecreasing function $\omega(t)$ on $[0,\infty)$ such that*

$$\int_0^1 \frac{\omega(t)}{t}\,dt < \infty,$$

and assume that the symbol $p(x,\xi)$ satisfies

$$|p^{(\alpha)}(x,\xi)| \le C_\alpha \lambda(x,\xi)^{-|\alpha|}\omega(\lambda(x,\xi)^{-1})$$

for any α with $|\alpha| \le n+1$. Then the pseudodifferential operator $p(X,D_x)$ is $L^p(\mathbb{R}^n)$ bounded for $1 \le p \le \infty$.

Proof. By Lemma 7.2, we may assume that the support of the symbol $p(x,\xi)$ is contained in $\{(x,\xi) : |\xi - a(x)| \geq 2\}$. Now we take a smooth nonnegative function $f(t)$ such that the support of $f(t)$ is contained in the interval $[1/2, 1]$ and

$$\int_0^\infty \frac{f(t)}{t}\,dt = 1.$$

Since the support of the symbol $p(x,\xi)$ is contained in $\{(x,\xi) : |\xi - a(x)| \geq 2\}$, we have

$$\begin{aligned}
p(X, D_x)u(x) &= \frac{1}{(2\pi)^n}\int_0^1 \frac{1}{t}\,dt \iint e^{i(x-y)\cdot\xi} p(x,\xi) f(t|\xi - a(x)|) u(y)\,dy\,d\xi \\
&= \frac{1}{(2\pi)^n}\int_0^1 \frac{1}{t^{n+1}}\,dt \iint e^{i\frac{(x-y)}{t}\cdot\xi} e^{i(x-y)\cdot a(x)} p\left(x, \frac{\xi}{t} + a(x)\right) f(|\xi|) u(y)\,d\xi\,dy \\
&= \frac{1}{(2\pi)^n}\int_0^1 \frac{1}{t}\,dt \int e^{itz\cdot a(x)} K_t(x,z) u(x - tz)\,dz,
\end{aligned}$$

where

$$K_t(x,z) = \int e^{iz\cdot\xi} p\left(x, \frac{\xi}{t} + a(x)\right) f(|\xi|)\,d\xi.$$

If we put $\tilde{p}(x,\xi) = p(x, \xi + a(x))$, then it is easy to see that

$$|\tilde{p}^{(\alpha)}(x,\xi)| \leq C_\alpha \langle\xi\rangle^{-|\alpha|}\omega(\langle\xi\rangle^{-1})$$

for $|\alpha| \leq n+1$. Since the equality

$$z^\alpha K_t(x,z) = i^{|\alpha|} \sum_{\alpha' \leq \alpha} \frac{1}{t^{|\alpha'|}} \binom{\alpha}{\alpha'} \int e^{iz\cdot\xi} \tilde{p}^{(\alpha')}\left(x, \frac{\xi}{t}\right) \partial_\xi^{\alpha-\alpha'} f(|\xi|)\,d\xi$$

holds for $|\alpha| \leq n+1$, we have

$$\begin{aligned}
|z^\alpha K_t(x,z)| &\leq \sum_{\alpha' \leq \alpha} \frac{1}{t^{|\alpha'|}} \binom{\alpha}{\alpha'} \int \left| \tilde{p}^{(\alpha)}\left(x, \frac{\xi}{t}\right) \partial_\xi^{\alpha-\alpha'} f(|\xi|) \right| d\xi \\
&\leq C \sum_{\alpha' \leq \alpha} \frac{1}{t^{|\alpha'|}} \binom{\alpha}{\alpha'} \int_{\frac{1}{2}\leq|\xi|\leq 1} \left|\frac{\xi}{t}\right|^{|\alpha'|} \omega\left(\left|\frac{\xi}{t}\right|^{-1}\right) d\xi \\
&\leq C\omega(t)
\end{aligned}$$

for $|\alpha| \leq n+1$. Therefore we have

$$|K_t(x,z)| \leq C\langle\xi\rangle^{-n-1}\omega(t). \tag{10}$$

By inequality (10) and the equality

$$p(X, D_x)u(x) = \frac{1}{(2\pi)^n}\int_0^1 \frac{1}{t}\,dt \int e^{itz\cdot a(x)} K_t(x,z) u(x - tz)\,dz$$

we can see that the operator $p(X, D_x)$ is L^1 and L^∞ bounded. That is, the inequalities

$$\|p(X, D_x)u\|_1 \le C\|u\|_1, \qquad \|p(X, D_x)u\|_\infty \le C\|u\|_\infty$$

hold. So by the Riesz–Thorin interpolation theorem we have the L^p boundedness for $1 \le p \le \infty$. ■ When $2 \le p$, we can show a slightly more general result than Theorem 7.1, by using Plancherel's formula.

Theorem 7.2 *Let $a(x)$ and $\lambda(x,\xi)$ be the same as in Theorem 7.1. Choose a non-negative and nondecreasing function $\omega(t)$ on $[0,\infty)$ such that*

$$\int_0^1 \frac{\omega(t)}{t}\, dt < \infty.$$

Assume that the symbol $p(x,\xi)$ satisfies

$$|p^{(\alpha)}(x,\xi)| \le C_\alpha \lambda(x,\xi)^{-|\alpha|}\omega(\lambda(x,\xi)^{-1})$$

for any α with $|\alpha| \le \kappa = \left[\frac{n}{2}\right] + 1$. Then the pseudodifferential operator $p(X, D_x)$ is $L^p(\mathbb{R}^n)$ bounded for $2 \le p \le \infty$.

Proof. We first show the L^∞ boundedness. We write the operator $p(X, D_x)$, as in the proof of Theorem 7.1, in the form

$$p(X, D_x)u(x) = \frac{1}{(2\pi)^n}\int_0^1 \frac{dt}{t}\int e^{itz\cdot a(x)} K_t(x,z)u(x-tz)\, dz, \tag{11}$$

where

$$K_t(x,z) = \int e^{iz\cdot\xi} p\left(x, \frac{\xi}{t} + a(x)\right) f(|\xi|)\, d\xi. \tag{12}$$

Then,

$$\begin{aligned}\int |K_t(x,z)|\, dz &= \int \langle z\rangle^{-\kappa}\langle z\rangle^{\kappa}|K_t(x,z)|\, dz \\ &\le \left[\int \langle z\rangle^{-2\kappa}\, dz\right]^{1/2}\left[\int \langle z\rangle^{2\kappa}|K_t(x,z)|^2\, dz\right]^{1/2} \\ &\le C\sum_{|\alpha|\le\kappa}\left[\int |z^\alpha K_t(x,z)|^2\, dz\right]^{1/2}\end{aligned}$$

and Plancherel's equality gives

$$\begin{aligned}\int |z^\alpha K_t(x,z)|^2\, dz &= (2\pi)^n \int \left|\partial_\xi^\alpha \left[\bar{p}\left(x, \frac{\xi}{t}\right) f(|\xi|)\right]\right|^2 d\xi \\ &\le C_\alpha \omega(t).\end{aligned}$$

Hence, we obtain

$$|p(X, D_x)u(x)| \leq C\|u\|_\infty.$$

In order to show the L^2 boundedness of the operator $p(X, D_x)$, using representation (11)-(12), we have

$$\|p(X, D_x)u(x)\|_2 \leq \frac{1}{(2\pi)^n}\int_0^1 \frac{dt}{t}\left\|\int e^{itza(\cdot)}K_t(\cdot, z)u(\cdot - tz)\,dz\right\|_2.$$

It follows that

$$\begin{aligned}\left\|\int e^{itz\cdot a(\cdot)}K_t(\cdot, z)u(\cdot - tz)\,dz\right\|_2^2 &= \int\left|\int e^{itz\cdot a(x)}K_t(x, z)u(x - tz)\,dz\right|^2 dx \\ &\leq \int\left|\int |K_t(x, z)u(x - tz)|\,dz\right|^2 dx.\end{aligned}$$

Hence, by Schwarz' inequality we have

$$\left|\int |K_t(x, z)u(x - tz)|\,dz\right|^2 \leq \int \langle z\rangle^{2\kappa}|K_t(x, z)|^2\,dz \int \langle z\rangle^{-2\kappa}|u(x - tz)|^2\,dz.$$

As above, we can see that

$$\begin{aligned}\int \langle z\rangle^{2\kappa}|K_t(x, z)|^2\,dz &\leq \sum_{|\alpha|\leq\kappa}\int |z^\alpha K_t(x, z)|^2\,dz \\ &= \sum_{|\alpha|\leq\kappa}\int \left|\partial_\xi\left[p\left(x, \frac{\xi}{t} + a(x)\right)f(|\xi|)\right]\right|^2 d\xi \\ &\leq C\omega(t)^2.\end{aligned}$$

Therefore we obtain

$$\begin{aligned}\left\|\int e^{itz\cdot a(\cdot)}K_t(\cdot, z)u(\cdot - tz)\,dz\right\|_2^2 &\leq C\int\int \langle z\rangle^{-2\kappa}|u(x - tz)|^2\,dz\,dx \\ &\leq C\omega(t)^2\|u\|_2^2.\end{aligned}$$

Thus, from the assumption on $\omega(t)$ we have the L^2 estimate

$$\|p(X, D_x)u\|_2 \leq C\|u\|_2.$$

Again, by the Riesz–Thorin interpolation theorem, we have the L^p boundedness for $2 \leq p \leq \infty$. ■

8 Conjectures

As was seen in the previous sections, we can expect that the following $L^p(\mathbb{R}^n)$ boundedness theorem holds.

Conjecture 1 *If the vector function* $a(x) = (a_1(x), \dots, a_n(x))$, *which defines the basic weight function* (8), *satisfies*

$$|\partial^\alpha a_j(x)| \le C_\alpha \tag{13}$$

for any $\alpha \ne 0$, *then, for* $1 < p < \infty$, *the operator* $p(X, D_x)$ *in* $S^0_{1,\delta,\lambda}$ *is* $L^p(\mathbb{R}^n)$ *bounded, that is, the inclusion*

$$S^0_{1,\delta,\lambda} \subset \mathcal{L}(L^p(\mathbb{R}^n))$$

holds.

As we stated in Section 3, it is known that if the vector function $a(x)$ satisfies the estimates (13), the operators in $S^0_{1,\delta,\lambda}$, with $\delta < 1$, are $L^2(\mathbb{R}^n)$ bounded. So if we can show weak type $(1,1)$ estimates or boundedness from $L^\infty(\mathbb{R}^n)$ to BMO, then we can get Conjecture 1, that is, $L^p(\mathbb{R}^n)$ boundedness for $1 < p < \infty$, by using interpolation theorems (see, for example, [16], [6]). Therefore, the fundamental conjecture is

Conjecture 2 *If the vector function* $a(x) = (a_1(x), \dots, a_n(x))$, *which defines the basic weight function* (8), *satisfies*

$$|\partial^\alpha a_j(x)| \le C_\alpha$$

for any $\alpha \ne 0$, *then the operator* $p(X, D_x)$ *in* $S^0_{1,\delta,\lambda}$ *is bounded from* $L^\infty(\mathbb{R}^n)$ *to* BMO, *that is, there is a constant* C *such that*

$$\|p(X, D_x)u\|_{BMO} \le C\|u\|_\infty.$$

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References

[1] Boggiatto, P. and Rodino, L., *Quantization and pseudo-differential operators*, Cubo, **5**(1) (2003) 237–272.

[2] Calderón, A. P. and Vaillancourt, R., *A class of bounded pseudo-differential operators*, Proc. Nat. Acad. Sci. U.S.A., **69** (1972) 1185–1187.

[3] Coifman, R. R. and Meyer, Y., *Au-delà des opérateurs pseudo-différentiels*, Astérisque, **57**, 1978.

[4] David, G. and Journé, J. L., *A boundedness criterion for generalized Calderón–Zygmund operators*, Ann. of Math., **120** (1984) 371–397.

[5] Fefferman, C., *L^p-bounds for pseudo-differential operators*, Israel J. Math., **14** (1972) 413–417.

[6] Fefferman, C. and Stein, E., *H^p-spaces of several variables*, Acta Math. **129** (1972) 137–193.

[7] Gröchenig, K. and Heil, C., *Modulation spaces and pseudodifferential operators*, Integral Equations Operator Theory **34** (1999) 439–457.

[8] Hörmander, L., *Pseudo-differential operators and hypoelliptic equations*, In Proc. Symposium on Singular Integrals, Amer. Math. Soc., Providence RI, **10** (1967) 138–183.

[9] Hörmander, L., *On the L^2 continuity of pseudo-differential operators*, Comm. Pure Appl. Math., **24** (1971) 529–535.

[10] Kumano-go, H., *Pseudo-differential operators*, MIT Press, Cambridge MA and London UK, 1982.

[11] Li, C. Z. and Wang, R. H., *On the L^p-boundedness of several classes of pseudodifferential operators*, Chinese Ann. Math. Ser. A, **5** (1984) 193–213.

[12] Muramatu, T., and Nagase, M., *On sufficient conditions for the boundedness of pseudo-differential operators*, Proc. Japan Acad. Ser. A **55** (1979) 293–296.

[13] Nagase, M., *On some classes of L^p-bounded pseudo-differential operators*, Osaka J. Math., **23** (1986) 425–440.

[14] Nagase, M., *On L^p boundedness of a class of pseudodifferential operators. Harmonic analysis and nonlinear partial differential equations* (Japanese) (Kyoto, 2001). Sūrikaisekikenkyūsho Kōkyūroku, No. **1235** (2001), 45–53.

[15] Nagase, M. and Umeda, T., *On the essential selfadjointness of quantum Hamiltonians of relativistic particles in magnetic fields*, Sci. Rep., Col. Gen. Educ. Osaka Univ. **36** (1987) 1–6.

[16] Stein, E. M., *Singular integrals and differentiability properties of functions*, Princeton Univ. Press, Princeton NJ, 1970.

[17] Zygmund, A., *Trigonometric series*, Cambridge University Press, Cambridge UK, 1968.