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Oscillation of second order differential equation with piecewise constant argument

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ABSTRACT

Two oscillation theorems are derived for a second-order differential equation with piecewise constant argument.

RESUMEN

Dos teoremas de oscilación son derivados para una ecuación diferencial de segundo orden con argumentos constantes por partes.

Key words and phrases: *delay differential equation, piecewise constant argument, oscillation criteria*
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First order delay differential equations with piecewise constant arguments were initiated by Cooke and Wiener [1] and Shah and Wiener [2]. As mentioned in [1-6], the strong interest in such equations is motivated by the fact that they represent a hybrid of continuous and discrete dynamical systems and combine the properties of both differential and difference equations. The oscillatory and nonoscillatory properties of first order delay differential equations with piecewise constant arguments have been the subject of many investigations (see, e.g. [1-15]), while those of higher order equations are relatively scarce.

However, there are reasons for studying higher order equations with piecewise constant arguments. Indeed, as mentioned in [10], a potential application of these equations is in the stabilization of hybrid control systems with feedback delay, where a hybrid system is one with a continuous plant and with a discrete (sampled) controller. As an example, suppose a moving particle with time variable mass $r(t)$ is subjected to a restoring controller $-\phi(x[t])$ which acts at sampled time $[t]$. Then the Newton's second law asserts that

$$(r(t)x'(t))' = -\phi(x[t]).$$

Since this equation is 'similar' to the harmonic oscillator equation

$$(r(t)x'(t))' + \kappa x(t) = 0,$$

we expect the well known oscillatory behavior of the later equation may also be found in the former equation, provided appropriate conditions on $r(t)$ and $f(x)$ are imposed.

In this paper we study a slightly more general second-order delay differential equations with piecewise constant argument

$$\left(r(t)\,x'(t)\right)' + f\left(t, x([t])\right) = 0, \quad t \geq 0, \tag{1}$$

where $r(t)$ is positive and continuous on $[0,\infty)$, $f(t,x)$ is continuous on $[0,+\infty) \times (-\infty,+\infty)$ such that $xf(t,x) > 0$ for $t \geq 0$ and $x \neq 0$, and $[\cdot]$ is the greatest-integer function. Two oscillation theorems for (1) will be obtained.

We will also assume the following property for f: there exist functions $p(t)$ and $\phi(x)$ such that $p(t)$ is continuous and nonnegative on $[0,\infty)$, $\phi(x)$ is continuously differentiable and nondecreasing on $(-\infty,+\infty)$, $x\phi(x) > 0$ for $x \neq 0$, and

$$f(t,x) \geq p(t)\,\phi(x), \quad x \neq 0, t \geq 0.$$

Note that when $\phi(x) = x$ and $p(t) = \kappa > 0$, the condition that $f(t,x) = \kappa x$ is just the Hooke's restoring force.

By a solution of (1) we mean a function $x(t)$ which is defined on $[0,+\infty)$ and which satisfies the condition (i) $x'(t)$ is continuous on $[0,\infty)$; (ii) $r(t)x'(t)$ is differentiable at each point $t \in [0,\infty)$, with the possible exception of the points $[t] \in [0,\infty)$ where one-sided derivatives exist; and (iii) substitution of $x(t)$ into Eq. (1) leads to an identity on each interval $[n, n+1) \subset [0,+\infty)$ with integral endpoints.

As is customary, a nontrivial solution $x(t)$ of (1) is said to be eventually positive (eventually negative) if there is some $T > 0$ such that $x(t) > 0$ (respectively $x(t) < 0$) for $t \geq T$; and $x(t)$ is said to be oscillatory if it is neither eventually positive nor eventually negative.

Lemma 1 *Let $x(t)$ be a solution of (1) such that there is some $T \geq 0$ and $x(t) > 0$ for $t \geq T$. If*

$$\int_0^{+\infty} \frac{1}{r(v)} dv = +\infty, \tag{2}$$

then $x'(t) \geq 0$ for $t \geq k$, where k is any integer $\geq T$.

Proof. Suppose to the contrary that there exists some integer $j \geq T$ such that $x'(j) < 0$. Let $x'(j) = -\delta$ where $\delta > 0$. In view of (1), for $t \in [j+i-1, j+i), i = 1, 2, ...,$ we have

$$(r(t) x'(t))' = -f(t, (x[t])) \leq -p(t) \phi(x([t])) \leq 0. \tag{3}$$

Hence, $r(t) x'(t)$ is nonincreasing in $[j+i-1, j+i)$. So,

$$x'(j+1) \leq \frac{r(j)}{r(j+1)} x'(j) = -\frac{r(j)}{r(j+1)} \delta < 0, \tag{4}$$

and

$$x'(j+2) \leq \frac{r(j+1)}{r(j+2)} x'(j+1) < 0.$$

It is easy to show that, for any positive integer i,

$$x'(j+i) \leq \frac{r(j+i-1)}{r(j+i)} x'(j+i-1) < 0. \tag{5}$$

Since $r(t) x'(t)$ is monotonic nonincreasing in $[j, j+1)$,

$$x'(t) \leq \frac{r(j)}{r(t)} x'(j), \quad t \in [j, j+1). \tag{6}$$

Integrating the above inequality, we have

$$x(t) \leq x(j) + r(j) x'(j) \int_j^t \frac{dv}{r(v)} \quad t \in [j, j+1).$$

Let $t \to (j+1)^-$, we get

$$x(j+1) \leq x(j) + r(j) x'(j) \int_j^{j+1} \frac{dv}{r(v)}. \tag{7}$$

so that

$$x(j+1) \leq x(j) - \delta r(j) \int_j^{j+1} \frac{dv}{r(v)}. \tag{8}$$

Repeating the same argument leads to

$$x(j+2) \leq x(j+1) + r(j+1) x'(j+1) \int_{j+1}^{j+2} \frac{dv}{r(v)}. \tag{9}$$

By (4), (8) and (9), we see that

$$\begin{aligned} x(j+2) &\leq x(j+1) + r(j+1)\frac{r(j)}{r(j+1)}x'(j)\int_{j+1}^{j+2}\frac{dv}{r(v)} \\ &= x(j+1) - \delta r(j)\int_{j+1}^{j+2}\frac{dv}{r(v)} \\ &\leq x(j) - \delta r(j)\int_{j}^{j+1}\frac{dv}{r(v)} - \delta r(j)\int_{j+1}^{j+2}\frac{dv}{r(v)} \\ &= x(j) - \delta r(j)\int_{j}^{j+2}\frac{dv}{r(v)}. \end{aligned}$$

By induction, for any positive integer i, we have

$$x(j+i) \leq x(j) - \delta r(j)\int_{j}^{j+i}\frac{dv}{r(v)}, \tag{10}$$

which is contrary to the assumption that $x(t) > 0$ for $t \geq T$, since by taking $i \to \infty$, the right hand side tends to $-\infty$ in view of (2). Therefore, $x'(k) \geq 0$ for any integer $k \geq T$. Because $r(t)\,x'(t)$ is nonincreasing in $[k, k+1)$, we have

$$x'(t) \geq \frac{r(k+1)\,x'(k+1)}{r(t)} \geq 0, \quad t \in [k, k+1).$$

Since $x'(t)$ is continuous on $[0, \infty)$, it is clear that $x'(t) \geq 0$ for $t \geq k$. The proof is complete. ■

Theorem 1. *Suppose (2) holds. Suppose further that*

$$\int_0^{+\infty} p(v)\,dv = +\infty. \tag{11}$$

Then every solution of (1) is oscillatory.

Proof. Suppose to the contrary that (1) has a nonoscillatory solution $x(t)$. Without loss of generality, we may assume that $x(t) > 0$ for $t \geq 0$. By Lemma 1, $x'(t) \geq 0$ for $t \geq 0$. Let

$$u(t) = \frac{r(t)\,x'(t)}{\phi(x([t]))}. \tag{12}$$

Then $u(t) \geq 0$ for $t \geq 0$ and $u(k^-) \geq 0$ for $k = 1, 2, \ldots$. In view of (1), we see that

$$u'(t) = -\frac{f(t, x(k-1))}{\phi(x(k-1))} \leq -p(t), \quad t \in [k-1, k). \tag{13}$$

Noting that $x\phi(x) > 0$ for $x \neq 0$ and that $\phi(x)$ is nondecreasing,

$$u(k) = \frac{r(k)\,x'(k)}{\phi(x(k))} \leq \frac{r(k)\,x'(k)}{\phi(x(k-1))} = u(k^-). \tag{14}$$

Integrating (13), we have

$$u(t) - u(k-1) \le -\int_{k-1}^{t} p(v)\,dv, \quad t \in [k-1, k). \tag{15}$$

Let $t \to k^-$, we get

$$u(k^-) - u(k-1) \le -\int_{k-1}^{k} p(v)\,dv. \tag{16}$$

In view of (14) and (16),

$$\begin{aligned} u(2^-) - u(0) &= u(2^-) - u(1) + u(1) - u(0) \le u(2^-) - u(1) + u(1^-) - u(0) \\ &\le -\int_1^2 p(v)\,dv - \int_0^1 p(v)\,dv = -\int_0^2 p(v)\,dv. \end{aligned}$$

By induction, for any positive integer k, we have

$$u(k^-) - u(0) \le -\int_0^k p(v)\,dv, \tag{17}$$

which is contrary to the fact that $u(k^-) \ge 0$ for $k = 1, 2, \ldots$, since by letting k tend to ∞, the right hand side tends to $-\infty$ in view of (11). The proof is complete. ∎

Theorem 2 *Suppose (2) holds. Suppose further that*

$$\int_{+\varepsilon}^{+\infty} \frac{dw}{\phi(w)} < +\infty, \int_{-\varepsilon}^{-\infty} \frac{dw}{\phi(w)} < +\infty \tag{18}$$

for some $\varepsilon > 0$, and

$$\int_0^{+\infty} p(v)\,dv < +\infty, \quad \int_0^{+\infty} \frac{ds}{r(s)} \int_{[s]+1}^{+\infty} p(v)\,dv = +\infty. \tag{19}$$

Then every solution of (1) is oscillatory.

Proof. Suppose to the contrary that (1) has a nonoscillatory solution $x(t)$. Without loss generality, we may assume that $x(t) > 0$ for $t \ge 0$. Lemma 1 shows that $x'(t) \ge 0$ for $t \ge 0$. So, $x(t)$ is nondecreasing on $[0, +\infty)$. In view of (1), for $k = 1, 2, \ldots$, we have

$$(r(t)\,x'(t))' = -f(t, x(k-1)) \le -p(t)\,\phi(x(k-1)), \quad t \in [k-1, k). \tag{20}$$

Hence

$$r(t)\,x'(t) - r(k-1)\,x'(k-1) \le -\phi(x(k-1)) \int_{k-1}^{t} p(v)\,dv, \quad t \in [k-1, k). \tag{21}$$

Let $t \to k^-$, we have

$$r(t)\,x'(t) - r(k-1)\,x'(k-1) \leq -\phi(x(k-1)) \int_{k-1}^{k} p(v)\,dv. \tag{22}$$

From the above inequality, we can see that

$$x'(k-1) \geq \frac{1}{r(k-1)}\phi(x(k-1)) \int_{k-1}^{k} p(v)\,dv + \frac{r(k)}{r(k-1)}x'(k) \quad k = 1, 2, ... \tag{23}$$

Integrating (20) from s to q, we have

$$r(q)\,x'(q) - r(s)\,x'(s) \leq -\phi(x(k-1)) \int_{s}^{q} p(v)\,dv, \tag{24}$$

where $k-1 \leq s < q < k$. Let $q \to k^-$, we get

$$r(k)\,x'(k) - r(s)\,x'(s) \leq -\phi(x(k-1)) \int_{s}^{k} p(v)\,dv, \quad s \in [k-1, k). \tag{25}$$

Noting that $x'(t) \geq 0$ for $t \geq 0$ and $x\phi(x) > 0$ for $x \neq 0$, by (23) and (25), for $s \in [k-1, k)$ we have

$$\begin{aligned}
x'(s) &\geq \frac{1}{r(s)}\phi(x(k-1)) \int_{s}^{k} p(v)\,dv + \frac{r(k)}{r(s)}x'(k) \\
&\geq \frac{r(k)}{r(s)}x'(k) \\
&\geq \frac{r(k)}{r(s)}\left(\frac{1}{r(k)}\phi(x(k)) \int_{k}^{k+1} p(v)\,dv + \frac{r(k+1)}{r(k)}x'(k+1)\right) \\
&\geq \frac{1}{r(s)}\phi(x(k)) \int_{k}^{k+1} p(v)\,dv + \frac{r(k+1)}{r(s)}x'(k+1).
\end{aligned}$$

From (23) and the above inequality, we may prove by induction that for any positive integer and $s \in [k-1, k)$,

$$\begin{aligned}
x'(s) \geq \frac{1}{r(s)}\Bigg(&\phi(x(k)) \int_{k}^{k+1} p(v)\,dv + \phi(x(k+1)) \int_{k+1}^{k+2} p(v)\,dv \\
&+ ... + \phi(x(k+n)) \int_{k+n}^{k+n+1} p(v)\,dv\Bigg).
\end{aligned} \tag{26}$$

Since $\phi(x)$ is nondecreasing, for any positive integer and $s \in [k-1, k)$, we have

$$\frac{x'(s)}{\phi(x(s))} \geq \frac{1}{r(s)} \int_{k}^{k+n+1} p(v)\,dv = \frac{1}{r(s)} \int_{[s]+1}^{k+n+1} p(v)\,dv. \tag{27}$$

Let $n \to +\infty$, yields

$$\frac{x'(s)}{\phi(x(s))} \geq \frac{1}{r(s)} \int_{[s]+1}^{+\infty} p(v)\, dv, \quad s \in [k-1, k). \tag{28}$$

Integrating the above inequality from $k-1$ to k, we have

$$\int_{k-1}^{k} \frac{ds}{r(s)} \int_{[s]+1}^{+\infty} p(v)\, dv \leq \int_{k-1}^{k} \frac{x'(s)}{\phi(x(s))} ds = \int_{x(k-1)}^{x(k)} \frac{dw}{\phi(w)}. \tag{29}$$

By (29), for any positive integer n,

$$\begin{aligned} \int_{0}^{n} \frac{ds}{r(s)} \int_{[s]+1}^{+\infty} p(v)\, dv &= \sum_{k=1}^{n} \int_{k-1}^{k} \frac{ds}{r(s)} \int_{[s]+1}^{+\infty} p(v)\, dv \\ &\leq \sum_{k=1}^{n} \int_{x(k-1)}^{x(k)} \frac{dw}{\phi(w)} = \int_{x(0)}^{x(n)} \frac{dw}{\phi(w)} \\ &\leq \int_{x(0)}^{+\infty} \frac{dw}{\phi(w)}, \end{aligned} \tag{30}$$

which is contrary to the conditions (18) and (19). The proof of Theorem 2 is complete. ■

As an example, consider the equation

$$(\exp(-t)x'(t))' + x([t]) \exp\left(t^2 + (x[t])^2\right) = 0, \ t \geq 0. \tag{31}$$

If we let $r(t) = \exp(-t)$, $\phi(x) = x$, $p(t) = \exp(t^2)$ and $f(t,x) = x\exp(t^2 + x^2)$, then it is easy to see that

$$f(t,x) \geq p(t)\phi(x), \ x \neq 0, t \geq 0,$$

$$\int_{0}^{\infty} \frac{1}{r(t)} dt = +\infty,$$

and

$$\int_{0}^{\infty} p(t) dt = +\infty.$$

In view of Theorem 1, every solution of (31) oscillates.

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References

[1] K. L. Cooke and J. Wiener, *Retarded differential equations with piecewise constant delays*, J. Math. Anal. Appl., **99**(1984), 265–297.

[2] S. M. Shah and J. Wiener, *Advanced differential equations with piecewise constant argument deviations.* Internal, J. Math. Math. Sci., **6**(1983), 671–703.

[3] E. C. Partheniadis, *Stability and oscillation of neutral delay differential equations with piecewise constant argument*, Differential Integral Equations, **1**(1988), 459-472.

[4] A. R. Aftabizadeh, J. Wiener and J. Xu, *Oscillatory and periodic solutions of delay differential equations with piecewise constant argument*, Proc. Amer. Math. Soc., **99**(1987), 673–679.

[5] A. R. Aftabizadeh and J. Wiener, *Oscillatory and periodic solutions of an equation alternately of retarded and advanced types*, Appl. Anal., **23**(1986), 219–231.

[6] J. Wiener and A. R. Aftabizadeh, *Differential equations alternately of retarded and advanced types*, J. Math. Anal. Appl., **129**(1988), 243–255.

[7] S. Busenberg and L. Cooke, *Models of vertically transmitted diseases with sequential-continuous dynamics*, in "Nonlinear Phenomena in Mathematical Sciences" (V. Lakshmikantham, E.D.), 179–187, Academic Press, New York, 1982.

[8] A. R. Aftabizadeh and J. Wiener, *Oscillatory and periodic solutions for systems of two first order linear differential equations with piecewise constant argument*, Appl. Anal., **26**(1988), 327–338.

[9] K. L. Cooke and J. Wiener, *An equation alternately of retarded and advanced type*, Proc. Amer. Math. Soc., **99**(1987), 726–732.

[10] K. L. Cooke and J. Wiener, *A survey of differential equation with piecewise constant argument*, in Lecture Notes in Mathematics, Vol. **1475**, pp. 1–15, Springer-Verlag, Berlin, 1991.

[11] K. Gopalsamy, M. R. S. Kulenovic and G. Ladas, *On a logistic equation with piecewise constant arguments*, Differential Integral Equations, **4**(1991), 215–223.

[12] L. C. Lin and G. Q. Wang, *Oscillatory and asymptotic behavior of first order nonlinear neutral differential equations with retarded argument* [*t*], Chinese Science Bulletin, **36**(1991), 889–891.

[13] Y. K. Huang, *Oscillations and asymptotic stability of solutions of first order netural differential equations with piecewise constant argument*, J. Math. Anal. Appl. **149**(1990), 70–85.

[14] G. Papaschinopoulos and J. Schinas, *Existence stability and oscillation of the solutions of first order neutral differential equations with piecewise constant argument*, Appl. Anal., **44**(1992), 99–111.

[15] J. H. Shen and I. P. Stavroulakis, *Oscillatory and nonoscillatory delay equations with piecewise constant argument*, J. Math. Anal. Appl., **248**(2000), 385–401.