CUBO A Mathematical Journal
Vol. 7, $N^{\underline{o}}$ 1, (159 - 167). April 2005.

A maximum principle for tensors on complete manifolds and its applications[1]

Lei Ni
Department of Mathematics
University of California, San Diego
la Jolla, CA 92093
lni @ math.ucsd.edu

ABSTRACT

In this expository paper we would like to discuss the maximum principle for the parabolic equations and its applications to the study of Kähler geometry and Ricci flow on complete manifolds.

RESUMEN

En este paper deseamos discutir el principio de maximización para las ecuaciones parabólicas y sus aplicaciones al estudio de geometria de Kähler y fluidos en variedades completas de Ricci.

Key words and phrases: *Maximum principle, complete noncompact manifolds, Ricci flow and parabolic equations.*
Math. Subj. Class.: *58J35 53C55*

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[1] Research partially supported by NSF grant DMS-0328624.

Recall first the classical maximum principle for the heat equation. Let Ω be a bounded domain in $\mathbb{R}^n$. If $u(x,t)$ is a smooth function satisfying

$$\left(\frac{\partial}{\partial t} - \Delta\right) u \geq 0$$

on $\Omega \times (0,T)$, and continuous on $\overline{\Omega} \times [0,T]$. Then

$$\min_{\overline{\Omega}\times[0,T]} u = \min_{(\partial\Omega\times(0,T))\cup(\Omega\times\{0\})} u. \tag{1}$$

For the Cauchy problem, the simplest form of the maximum principle reads

$$\min_{R^n\times[0,T]} u = \min_{\mathbb{R}^n} u(x,0), \tag{2}$$

assuming that u is a sup-solution, namely $\left(\frac{\partial}{\partial t} - \Delta\right) u \geq 0$, and $u(x,t)$ is bounded on $\mathbb{R}^n \times [0,T]$. If we do not assume the growth restriction on u, the maximum principle no longer holds. The simplest example can be found, say in the book by Fritz John [J], where a solution $u(x,t)$ to $\left(\frac{\partial}{\partial t} - \Delta\right) u = 0$ was constructed with the form

$$u(x,t) = \sum_{k=0}^{\infty} \frac{g^{(k)}(t)}{(2k)!} x^{2k}.$$

Here $g_0 = g(t)$, $g_1 = 0$, $g_j' = (j+2)(j+1)g_{j+2}$ and

$$g(t) = \begin{cases} \exp(-t^{-\alpha}), \text{ when } t > 0, \\ 0, \text{ when } t \leq 0. \end{cases}$$

It can be shown that the series is uniformly convergent and $|u(x,t)|$ is bounded by

$$U(x,t) = \begin{cases} \exp\left(\frac{1}{t}\left(\frac{x^2}{\theta} - \frac{1}{2}t^{1-\alpha}\right)\right), t > 0 \\ 0, t \leq 0 \end{cases}.$$

Here $0 < \theta < 1$.

The maximum principle (1) can be generalized to compact Riemannian manifolds with boundary. The generalization of (2) to the complete noncompact manifolds was carried in [K-L], where the author proved that (2) holds on a complete noncompact manifolds for the bounded sup-solution u if the manifold M satisfies

$$V_p(R) \leq e^{C(R^2+1)} \tag{3}$$

for some $C > 0$. Here $V_p(R)$ is the volume of $B_p(R)$. The result is sharp since a counter-example was given for a manifold with faster volume growth in [A]. Since the result was proved only for the bounded sup-solution, it is not very satisfactory in the view of the example given above. However, the following result was later proved in [L] using an improved argument from [K-L] and [L-T].

Theorem 1 *Let M be a complete Riemannian manifold. Let $u(x,t)$ be a smooth function on $M \times [0,T]$ such that $\left(\frac{\partial}{\partial t} - \Delta\right) u \geq 0$ whenever $u(x,t) \leq 0$. Assume that*

$$\int_0^T \int_M \exp(-ar^2(x)) u_-^2(x,s)\, dv\, ds < \infty \tag{4}$$

for some $a > 0$, where $r(x)$ is the distance function to a fixed point $o \in M$. Suppose $u(x,0) \geq 0$ for all $x \in M$. Then $u(x,t) \geq 0$ for all $(x,t) \in M \times [0,T]$.

The reader can consult [N-T1] for the proof of a slightly more general version of above result for family of metrics. The condition (4) is optimal by comparison with the example before.

In [H1], a maximum principle was derived for tensors satisfying certain heat equation on compact manifolds with nonnegative curvature. The tensor maximum principle on compact manifolds has proved to be useful in geometric evolution equations such as the study of Ricci flow [H3] and the mean curvature flow [Hu]. In [H2], in order to prove the Li-Yau-Hamilton inequality (or differential Harnack) for the Ricci flow on complete noncompact manifolds, Hamilton developed an argument basically proves a tensor maximum principle for bounded solution to certain heat equation (system) on complete Riemannian manifolds with bounded nonnegative curvature. One can refer [N-T2, Proposition 1.1] for an improved version, which include the tensor with pointwise growth control, following Hamilton's original argument in [H2]. In Hamilton's program towards geometrization, which was recently carried further in [P2] by Perelman, and may have solved Thurston's geometrization conjecture, one studies the singularity by dilations when approaching the singularity and studying the limit solutions, which obtained by some compactness theorems. The solutions obtained through this dilation procedure usually are noncompact. But they do have bounded curvatures if the dilation is chosen centered at points with curvature comparable with its nearby points. Therefore, the maximum principle for bounded tensor is sufficient for this purpose. However it does have serious drawback since in some other geometric applications the uniform boundedness assumption on the solution is not desirable. For instance, in study the geometry and topology of complete manifolds it is desirable to have smooth convex (plurisubharmonic) functions as indicated in, for example, [W]. Geometric construction usually only provides functions with little regularity. It is natural to use heat equation solution to approximate a continuous convex (plurisubharmonic) function. Since there is no pointwise control on the Hessian of the solution, one can not conclude that the convexity is preserved by the heat equation deformation if one only has Hamilton's tensor maximum principle for bounded tensors. This indeed becomes one of the major technical hurdles in the application of heat equation method on complete manifolds, as shown in [N-T2].

In a recent paper [N-T3], joint with Luen-Fai Tam, we proved an optimal tensor maximum principle for tensors, assuming a similar necessary growth condition as (4). The integral growth assumption can be verified for most cases in the applications. The result seems to be effective by proving several theorems on Kähler geometry and Ricci flow of complete manifolds. In order to explain the result we have to start with some notations.

Let M^m be a complete noncompact Kähler manifold of complex dimension m (real dimension $n = 2m$). We denote the Kähler metric by $g_{\alpha\bar{\beta}}$. The maximum principle is for Hermitian symmetric $(1,1)$ tensor η satisfying the complex Lichnerowicz heat equation:

$$\left(\frac{\partial}{\partial t} - \Delta\right)\eta_{\gamma\bar{\delta}} = R_{\beta\bar{\alpha}\gamma\bar{\delta}}\eta_{\alpha\bar{\beta}} - \frac{1}{2}\left(R_{\gamma\bar{p}}\eta_{p\bar{\delta}} + R_{p\bar{\delta}}\eta_{\gamma\bar{p}}\right). \tag{5}$$

Assume $\eta(x,t)$ is defined on $M \times [0,T]$ for some $T > 0$. We also assume that there exists a constant $a > 0$ such that

$$\int_M \|\eta\|(x,0)\exp\left(-ar^2(x)\right)\, dx < \infty \tag{6}$$

and

$$\liminf_{r\to\infty} \int_0^T \int_{B_o(r)} \|\eta\|^2(x,t)\exp\left(-ar^2(x)\right)\, dx\, dt < \infty. \tag{7}$$

Here $\|\eta\|$ is the norm of $\eta_{\alpha\bar{\beta}}$ with respect to the Kähler metric. By (6), we have

$$\int_{B_o(r)} \|\eta\|(x,0)\, dx \le \exp(ar^2)\cdot \mathcal{S} \tag{8}$$

where $\mathcal{S} = \int_M \|\eta\|(x,0)\exp\left(-ar^2(x)\right)\, dx$.

In the following, we always arrange the eigenvalues of η at a point in the ascending order.

Before we state our result, let us first fix some notations. Let $\varphi : [0,\infty) \to [0,1]$ be a smooth function so that $\varphi \equiv 1$ on $[0,1]$ and $\varphi \equiv 0$ on $[2,\infty)$. For any $x_0 \in M$ and $R > 0$, let $\varphi_{x_0,R}$ be the function defined by

$$\varphi_{x_0,R}(x) = \varphi\left(\frac{r(x,x_0)}{R}\right).$$

Let $f_{x_0,R}$ be the solution of

$$\left(\frac{\partial}{\partial t} - \Delta\right) f = -f$$

with initial value $\varphi_{x_0,R}$. Then $f_{x_0,R}$ is defined for all t and is positive and bounded for $t > 0$. In fact

$$f_{x_0,R}(x,t) = e^{-t}\cdot \int_M H(x,y,t)\varphi_{x_0,R}(y)dy.$$

In [N-T3], we establish the following maximum principle.

Theorem 2 *Let M be a complete noncompact Kähler manifold with nonnegative holomorphic bisectional curvature. Let $\eta(x,t)$ be a Hermitian symmetric (1,1) tensor satisfying (5) on $M \times [0,T]$ with $0 < T < \frac{1}{40a}$ such that $\|\eta\|$ satisfies (6) and (7). Suppose at $t = 0$, $\eta_{\alpha\bar{\beta}} \ge -bg_{\alpha\bar{\beta}}$ for some constant $b \ge 0$. Then there exists $0 < T_0 < T$ depending only on T and a so that the following are true.*

(i) $\eta_{\alpha\bar\beta}(x,t) \geq -bg_{\alpha\bar\beta}(x)$ *for all* $(x,t) \in M \times [0,T_0]$.

(ii) *For any* $T_0 > t' \geq 0$, *suppose there is a point* x' *in* M^m *and there exist constants* $\nu > 0$ *and* $R > 0$ *such that the sum of the first* k *eigenvalues* $\lambda_1,\ldots,\lambda_k$ *of* $\eta_{\alpha\bar\beta}$ *satisfies*

$$\lambda_1 + \cdots + \lambda_k \geq -kb + \nu k\varphi_{x',R}$$

for all x *at time* t'. *Then for all* $t > t'$ *and for all* $x \in M$, *the sum of the first* k *eigenvalues of* $\eta_{\alpha\bar\beta}(x,t)$ *satisfies*

$$\lambda_1 + \cdots + \lambda_k \geq -kb + \nu k f_{x',R}(x, t-t').$$

The result has several applications in the study of the geometry and function theory of Kähler manifolds. For example, the following result was proved in [N-T3].

Theorem 3 *Let* M *be a complete noncompact Kähler manifold with nonnegative holomorphic bisectional curvature. Let* u *be a continuous plurisubharmonic function on* M. *Suppose that*

$$\limsup_{x\to\infty} \frac{u(x)}{\log r(x)} = 0.$$

Then u *must be a constant.*

One can see that Theorem 3 generalize the well-known results on $\mathbb{C}^m$ to general Kähler manifolds. The similar generalization for harmonic functions was done much earlier by Yau [Y], the case which is relatively easier since one is dealing with a solution to a linear differential equation (harmonic functions). The related differential equation related to the above Liouville theorem is the degenerated Monge-Amperé equation (See [N1]), which is fully nonlinear. The proof of Theorem 3 hinged on the study of the foliation defined by the degenerated Monge-Amperé equation and the implications of the maximum principle to the heat equation deformation of the studied plurisubharmonic functions. The reader can refer [N-T3] for more details.

Using Theorem 3 we obtain the following result, which can be viewed as a generalization of the splitting theorem of Cheeger-Gromoll.

Theorem 4 *Let* M^m *be a complete noncompact Kähler manifold with nonnegative holomorphic bisectional curvature. Suppose* f *is a nonconstant harmonic function on* M *such that*

$$\limsup_{x\to\infty} \frac{|f(x)|}{r^{1+\epsilon}(x)} = 0, \tag{9}$$

for any $\epsilon > 0$, *where* $r(x)$ *is the distance of* x *from a fixed point. Then* f *must be of linear growth and* M *splits isometrically as* $\widetilde{M} \times \mathbb{R}$. *Moreover the universal cover* $\widetilde{M}$ *of* M *splits isometrically and holomorphically as* $\widetilde{M'} \times \mathbb{C}$, *where* $\widetilde{M'}$ *is a complete Kähler manifold with nonnegative holomorphic bisectional curvature. Suppose that there exists a nonconstant holomorphic function* f *on* M *satisfying (9). Then* M *itself splits as* $\widetilde{M} \times \mathbb{C}$.

For the sake of the illustration, we outline a proof of Theorem 4 in the case f is of linear growth. With some calculation and applying the gradient estimate of Cheng-Yau [C-Y], one knows that $|\nabla f|$ is a bounded plurisubharmonic function. Then Theorem 3 implies that $|\nabla f|$ is a constant, from which it is an easy application of the Bochner formula to conclude the splitting of the manifold as in the simple analytic proof of Cheeger-Gromoll splitting theorem in [S-Y].

Applying Theorem 2 to the Busemann function we obtained the following result, which sharpens Perelman's soul theorem [P1] in the Kähler category.

Corollary 5 *(F.-Y. Zheng) Let M be a complete Kähler manifold with nonnegative sectional curvature. Then its universal cover is of the form $\tilde{M} = \mathbb{C}^k \times \tilde{N} \times \tilde{L}$ where $\tilde{N}$ is a compact Hermitian symmetric manifold, $\tilde{L}$ is Stein and $\tilde{L}$ contains no Euclidean factor. Moreover, there exists a discrete subgroup $\Gamma \subseteq I_h(\mathbb{C}^k)$ which acts freely on $\mathbb{C}^k$, and group homomorphisms $\rho : \Gamma \to I_h(\tilde{N})$, $\tau : \Gamma \to I_h(\tilde{L})$, such that M is holomorphically isometric to the quotient of $\tilde{M}$ by Γ which acts on $\tilde{M}$ as*

$$\gamma(x, y, z) = (\gamma(x), \rho(\gamma)(y), \tau(\gamma)(z))$$

for any $\gamma \in \Gamma$. In particular, M is a holomorphic and Riemannian fiber bundle with fiber $\tilde{N} \times \tilde{L}$ over the flat Kähler manifold $\mathbb{C}^k/\Gamma$. Here $I_h(X)$ denotes the group of isometric biholomorphisms of a Kähler manifold X. In particular, if M has positive bisectional curvature, M is diffeomorphic to $\mathbb{R}^{2m}$.

The corollary follows from some general splitting theorems on Kähler manifolds with nonnegative bisectional curvature, which in particular, imply that *any compact complex submanifold in a simply-connected complete Kähler manifold with nonnegative bisectional curvature must be an isometric factor.* This sharpen a conjectured picture, by Yau, on the structure of such manifolds.

Theorem 2 is also true for complete Riemannian manifolds with nonnegative sectional curvature. It becomes very useful when coupled with the Ricci flow. In particular it was proved in [N2]:

Theorem 6 *Let $(M, g_{ij}(x,t))$ be complete Riemannian metrics satisfying the Ricci flow with bounded nonnegative sectional curvature. Let $u(x)$ be a Lipschitz continuous convex function satisfying*

$$|u|(x) \le C \exp\left(ar^2(x)\right) \tag{10}$$

for some positive constants C and a. Let $v(x,t)$ be the solution to the time-dependent heat equation $\left(\frac{\partial}{\partial t} - \Delta\right) v = 0$. There exists $T_0 > 0$ depending only on a and there exists $T_0 > T_1 > 0$ such that the following are true.

(i) *For $0 < t \le T_0$, $v(\cdot, t)$ is a smooth convex function (with respect to $g_{ij}(x,t)$).*

(ii) *Let*

$$\mathcal{K}(x,t) = \{w \in T^{1,0}_x(M) |\ v_{ij}(x,t)w^i = 0, \text{ for all } j\}$$

be the null space of $v_{ij}(x,t)$. Then for any $0 < t < T_1$, $\mathcal{K}(x,t)$ is a distribution on M. Moreover the distribution is invariant in time as well as under the parallel translation.

The above result has the following consequences (cf. [N2]) on geometry and the Ricci flow on complete manifolds.

Corollary 7 *Let M be a complete simply-connected Riemannian manifold with bounded nonnegative curvature operator. Then M is a product of a compact Riemannian manifold with nonnegative curvature operator with a complete noncompact manfold which is diffeomorphic to $\mathbb{R}^k$. In the case of dimension three, the same result holds if one only assumes that the sectional curvature is nonnegative.*

Remarks. *The compact factor in the above result has been classified by Gallot and Meyer [Ga-M] (also in [Ch-Y] by Chow and Yang) to be the product of compact symmetric spaces, Kähler manifolds biholomorphic to the complex projective spaces and the manifolds homeomorphic to spheres.*

The above result was proved earlier in [No] by Noronha without assuming the boundedness of the curvature tensor. Our method here has this restriction since we have to use the short time existence result of Shi in [Sh2] on the Ricci flow. For dimension three, in [Sh1] the result was proved even for nonnegative Ricci curvature case. However, it relies on the previous deep results of Hamilton and Schoen-Yau.

The following result in [N2] is the first example of such.

Corollary 8 *For $n \geq 4$, there are complete Riemannian manifolds on which the Ricci flow does not preserve the nonnegativity of the sectional curvature.*

The result suggests that the Ricci flow in high dimension may not be as tractable as in dimension three, when one can have nice pinching estimates([H3]), which holds one of the keys to the study of the singularity, as illustrated in [H3, P2].

Received: October 2003. Revised: March 2004.

References

[A] R. Azencott, *Behavior of diffusion semi-groups at infinity* Bull. Soc. Math. France **102** (1974), 193–240.

[C-Y] S. Y. Cheng and S. T. Yau , *Differential equations on Riemannian manifolds and their geometric applications* Comm. Pure Appl. Math. **28** 1975, 333–354

[Ch-Y] B. Chow and Deane Yang , *Rigidity of nonnegatively curved compact quaternionic-Kähler manifolds* J. Differential Geom. **29** 1989, 361–372

[Ga-M] S. Gallot and D. Meyer , *Opérateur de courbure et Laplacian des formes differentielles d'une variété Riemanniene* J. Math. Pures Appl. 197554, 285–304

[H1] R. S. Hamilton, *Four-manifolds with positive curvature operator* J. Differential Geom.**24** 1986, 153–179

[H2] R. S. Hamilton, *The Harnack estimate for the Ricci flow* J. Differential Geom.**37** 1993, 225–243

[H3] R. S. Hamilton, *The formation of singularities in the Ricci flow* Surveys in Differential Geom.**2** 1995, 7–136

[Hu] G. Huisken, *Flow by mean curvature of convex surfaces into spheres* J. Differential Geom.**20** 1984, 237–266

[J] F. John, *Partial Differential Equations* fourth edition, Springer-Verlag, New York, 1982

[K-L] K. Karp and P. Li , *The heat equation on complete Riemannian manifolds* unpublished, 1982

[L] P. Li, *Lectures on heat equations* 1991-1992 at UCI

[L-T] G. Liao and L.-F. Tam, *On the heat equation for harmonic maps from non-compact manifolds* Pacific J. Math **153** 1984, 279–301

[N1] L. Ni , *Vanishing theorems on complete Kähler manifolds and their applications* J. Differential Geom.**50** 1998, 89–122

[N2] L. Ni , *Ricci flow and nonnegativity of sectional curvature* Math. Res. Lett, **11** (2004), 883-904

[No] M. Noronha , *A splitting theorem for complete manifolds with nonnegative curvature operator* Proceedings of AMS. **105** 1989 , 979–985

[N-T1] L. Ni and L.-F.Tam, *Kähler Ricci flow and Poincaré-Lelong equation* Comm. Anal. Geom. **12**(2004), 111-141.

[N-T2] L. Ni and L.-F. Tam, *Plurisubharmonic functions and the Kähler-Ricci flow* Amer. J. Math. **125** 2003 , 623–645

[N-T3] L. Ni and L.-F.Tam, *Plurisubharmonic functions and the structure of complete Kähler manifolds with nonnegative curvature* J. Differential Geom. **64**(2003), 457-524.

[P1] G. Perelman, *Proof of the soul conjecture of Cheeger and Gromoll* J. Differential Geom. **40** 1994 , 209–212

[P2] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications* arXiv: math.DG/0211159

[Sh1] W. X. Shi, *Complete noncompact three-manifolds with nonnegative Ricci curvature* J. Differential Geom. **29** 1989, 353–360

[Sh2] W. X. Shi, *Deforming the metric on complete Riemannian manifolds* J. Differential Geom.**30** 1989, 223–301

[S-Y] R. Schoen and S.T. Yau , *Lectures on Differential Geometry* Conference Proceedings and Lecture Notes in Geometry and Topology**1** 1994

[W] H. Wu, *An elementary methods in the study of nonnegative curvature* Acta. Math.**142** 1979 , 57–78

[Y] S. T. Yau, *Harmonic functions on complete Riemannian manifolds* Comm. Pure Appl. Math.**28** 1975 , 201–228