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# Tensor Differential Forms and Some Birational Invariants of Projective Manifolds

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**ABSTRACT**

Symmetry properties of tensors play an important role in physics. They correspond to the irreducible representations of the symmetric group, which can be described by Young tableaux $T$. The global $T$-symmetrical tensor differential forms on the projective manifold $Y$ define a birational invariant of $Y$. In the case of prime characteristic $char(K) = p > 0$ the pullback of the Frobenius provides an opportunity to define further discrete birational invariants of algebraic manifolds using the $p^s$-th powers $(df)^{p^s}$ instead of the differentials $df$. Using Sernesis result on infinitesimal deformations an explicit formula for the moduli space dimension of complete intersections is given. As an application among others a conjecture of Libgober and Wood will be confirmed concerning the existence of diffeomorphic three-dimensional complete intersections which lie in different dimensional components of the moduli space. Finally for arbitrary locally free sheaves $\mathcal{F}$ on $Y$ the Chern classes of the $T$-power $\mathcal{F}^T$ are calculated as polynomials in Chern classes of $\mathcal{F}$.

**RESUMEN**

Las propiedades simétricas de los tensores juegan un rol importante en física. Ellos corresponden a la representación irreducible del grupo simétrico, la cual puede ser descrita por las "Young tableaux" $T$. Las formas diferenciables del tensor global $T$-simétrico en las variedad proyectiva $Y$ define un invariante biracional de $Y$. En el caso de característica prime char$(K) = p > 0$ el pulback de el Frobenius provee una oportunidad para definir otras invariantes discretas biracionales de variedades algebraicas usando la $p^s$ -ésima potencia $(df)^{p^s}$ en vez de la diferenciable $df$ . Utilizando el resultado de Sernesis en deformaciones infinitesimal se da una fórmula explicita para la dimensión del espacio de moduli de intersecciones completas. Como una aplicación en medio de otras, una conjetura de Libgober y Word será confirmada con respecto a la existencia de tridimesional intersecciones diffeomorficas completas, las cuales están en diferentes componentes dimensionales del espacio de moduli. Finalmente, para localmente arbitrarias libre sheaves $\mathcal{F}$ en $Y$ las clases de Chern de la $T$-potencia $\mathcal{F}^T$son calculadas como polinomios en las clases de Chern $\mathcal{F}$.

**Math. Subj. Class.:** *14F10, 14M10*

# 1 Introduction

Let $K$ be an algebraically closed field (e.g. $K = \mathbb{C}$), and let $X$ and $Y$ be projective varieties defined over $K$ with fields $K(X)$ and $K(Y)$ of rational functions on $X$ resp. $Y$. We consider the following equivalence relations: $X$ and $Y$ are isomorphic if and only if there exist regular morphisms $\varphi : X \to Y$ and $\psi : Y \to X$ with $\psi = \varphi^{-1}$. Secondly, $X$ and $Y$ are birational isomorphic if and only if there exist such rational maps, i.e., if and only if the function fields $K(X)$ and $K(Y)$ are $K$-isomorphic [23]. Using the sheaf theory one can construct a lot of invariants. Let $\Omega^r$ be the sheaf of germs of alternating algebraic differential forms of degree $r$ on $X$. Then all cohomology groups $H^q(X, \Omega^r)$ are invariants with respect to isomorphisms. Since these groups are finite dimensional vector spaces over $K$ we have numerical invariants $\dim_K H^q(X, \Omega^r)$. In particular, if $X$ is smooth then the vector space $H^0(X, \Omega^r)$ of global differential forms is even a birational invariant of $X$ [40].
But there is no reason to confine oneself to alternating differential forms. This paper deals with general tensor differential forms "which hopefully will lead to a new birational geometry of algebraic varieties" [36].
Let $\Omega^1$ be the cotangential bundle on a smooth $n$-dimensional irreducible projective manifold $X$ and let $\Omega^r = \bigwedge^r \Omega^1$ , $S^r\Omega^1$ and $(\Omega^1)^{\otimes r}$ be its $r$-th alternating, symmetric and tensor power respectively. Then $\Omega^1$ ,$\Omega^r$ , $S^r\Omega^1$ and $(\Omega^1)^{\otimes r}$ are locally free sheaves on $X$, rank of which is equal to $n$ , $\binom{n}{r}$ , $\binom{n+r-1}{r}$ and $n^r$ respectively.
In physics symmetry properties of tensors play an important role. Actually the sheaf $(\Omega^1)^{\otimes r}$ decomposes to the direct sum $(\Omega^1)^{\otimes r} = \bigoplus_T \Omega^T$ where $T$ runs through the so called standard Young tableaux and where $\Omega^T$ denotes the sheaf of germs of $T$-symmetric tensor differential forms [20], [21], [43]. Above all, each of the vector

spaces $H^0(X, \Omega^T)$ is a birational invariant of $X$. Examples of complete intersections show that these very fine birational invariants $H^0(X, \Omega^T)$ are independent of each other. For instance, the plurigenus $P_f$ of $X$ is equal to $\dim_K H^0(X, \Omega^T)$ where $T$ denotes a rectangle with $n = \dim X$ rows and $f$ columns.
This paper is written in the language of coherent algebraic sheaves [38]. From a short exact sequence of coherent algebraic sheaves the corresponding long exact cohomology sequence ensues. In particular we will use the twist operation $\mathcal{F} \mapsto \mathcal{F}(t) = \mathcal{F} \otimes \mathcal{O}_X(t)$ $(t \in \mathbb{Z})$ on coherent algebraic sheaves $\mathcal{F}$ over a projective variety $X \subseteq \mathbb{P}^N$. This functor is exact and the sheaves $\mathcal{F}$ and $\mathcal{F}(t)$ are locally isomorphic. One has $H^q(X, \mathcal{F}(t)) = 0$ for $q > 0$ and sufficiently big $t \in \mathbb{Z}$. If $X$ is smooth then $H^q(X, \mathcal{F}(t)) = 0$ for $0 \leq q < \dim X$ and sufficiently small $t \in \mathbb{Z}$.

## 2 Young diagrams and Young tableaux

Remember some well known facts on representations of the symmetric group $S_r$. The equivalence classes of irreducible representations of $S_r$ correspond to the conjugacy classes of $S_r$, i.e., to the partitions $(l) : r = l_1 + l_2 + \cdots + l_d$ with $l_i \in \mathbb{Z}$ and $l_1 \geq l_2 \geq \cdots \geq l_d > 0$.
The partition $(l)$ can be described by a Young diagram with $r$ boxes and with the row lengths $l_1, l_2, \ldots, l_d$. The lengths of its columns are $d_j = \#\{i \in \mathbb{Z} : l_i \geq j\}$ with $j = 1, 2, \ldots, l_1$ ; $d = d_1$ ; $\sum d_j = \sum l_i = r$.
Set $l = l_1$ , $l_i = 0$ if $i > d$ and $d_j = 0$ if $j > l$. The box inside the $i$-th row and the $j$-th column of the Young diagram has its own "hook length" $h_{i,j} = l_i - j + d_j - i + 1$ and the degree of the corresponding irreducible representation of $S_r$ is equal to

$$\nu_{(l)} = \frac{r!}{\prod h_{i,j}} = \frac{r!}{d!} \cdot \prod_{i=1}^{d} \frac{i!}{(l_i + d - i)!} \cdot \prod_{1 \leq i < j \leq d} \left(\frac{l_i - l_j}{j - i} + 1\right) = r! \cdot \det\left(\left(\frac{1}{\Gamma(l_i + 1 - i + j)}\right)\right)_{i,j=1,2,\ldots,d}$$

[21].

A Young tableau $T$ to a given Young diagram with $r$ boxes is a numbering of these boxes by the integers $1, 2, \ldots, r$.
Assume $\mathrm{char}(K) = 0$ or $\mathrm{char}(K) = p > r$. Then for a given Young tableau $T$ an idempotent $e_T$ in the group algebra $KS_r$ is introduced:

$$e_T = \frac{\nu_{(l)}}{r!} \cdot \left(\sum_{q \in Q_T} \mathrm{sgn}(q) \cdot q\right) \cdot \left(\sum_{p \in P_T} p\right) \tag{1}$$

with the subgroups $P_T = \{p \in S_r : p$ preserves each row of $T\}$ and $Q_T = \{q \in S_r : q$ preserves each column of $T\}$.
A Young tableau $T$ is called a standard tableau if the sequence of box numbers inside any row or inside any column of $T$ is monotonically increasing. The number of standard tableaux to a given Young diagram is equal to the degree $\nu_{(l)}$ of the corresponding irreducible representation of $S_r$. We denote the set of all standard tableaux with $r$ boxes by $D(r)$.
Now let $K$ be an algebraically closed field and let $Y$ be a $n$-dimensional algebraic

variety defined over $K$. The symmetric group $S_r$ and therefore the group algebra $KS_r$ act on the sheaf $(\Omega^1)^{\otimes r}$ by permutations of the spots inside the tensor product

$$p(a_1 \otimes a_2 \otimes \cdots \otimes a_r) = a_{p^{-1}(1)} \otimes a_{p^{-1}(2)} \otimes \cdots \otimes a_{p^{-1}(r)} \quad \forall p \in S_r \qquad (2)$$

and the sheaf $(\Omega^1)^{\otimes r}$ decomposes to the direct sum

$$(\Omega^1)^{\otimes r} = \bigoplus_{T \in D(r)} \Omega^T \quad \text{with} \quad \Omega^T = e_T((\Omega^1)^{\otimes r}) \ \ (T-\text{power of } \Omega^1). \qquad (3)$$

If the Young tableaux $T$ and $\widetilde{T}$ correspond to the same Young diagram ( i.e. to the same partition $(l)$ ) then $\Omega^T$ and $\Omega^{\widetilde{T}}$ are isomorphic as $\mathcal{O}_Y$-moduls. If $Y$ is a smooth $n$-dimensional variety then $\Omega^T$ is locally free. In case $n < d =$ depth $T$ one has $\Omega^T = 0$ , otherwise rank $(\Omega^T) = \prod_{1 \le i < j \le n} (\frac{l_i - l_j}{j-i} + 1)$ $\quad (l_i = 0 \text{ if } i > d)$ [2].

In the same way one constructs the $T$-power $\mathcal{F}^T$ of any locally free sheaf $\mathcal{F}$ instead of $\Omega^1$.

Finally let $Y \subseteq \mathbb{P}^N$ be a smooth $n$-dimensional projective variety whose canonical line bundle $\omega_Y$ is isomorphic to $\mathcal{O}_Y(n_Y)$ with $n_Y \in \mathbb{Z}$. Then:

$$\Omega^T \cong \Omega^{T'} \otimes \omega_Y \cong \Omega^{T'}(n_Y) \ \text{ if } \ d = \text{ depth} T = \dim Y \ , \ l = \text{ length } T > 1 \qquad (4)$$

where the Young tableau $T'$ arises from $T$ by erasing the first column of $T$ [5] , [21].

Assume the Young tableau $T$ has the lengths of columns $d_1, d_2, \ldots, d_l$ $(> 0)$ with $d_1 = d =$ depth $T \le n$. Let $T^*$ denote a Young tableau with the following lengths of columns: $d_j^* = n - d_{l+1-j} \quad \forall j \in \{1, 2, \ldots, l\}$.

Then $(T^*)^* = T$ if and only if depth $T < n$. By Serre duality [21] , [39] we have

$$Hom(\Omega^T(t), \mathcal{O}_Y) \cong \Omega^{T^*}(-t - l \cdot n_Y) \qquad (5)$$

$$\dim_K H^q(Y, \Omega^T(t)) = \dim_K H^{n-q}(Y, \Omega^{T^*}(-t + (1-l)n_Y)). \qquad (6)$$

# 3 The complex projective space $\mathbb{P}^N_{\mathbb{C}}$

Now set $K = \mathbb{C}$ and let $T$ be a Young tableau with $r$ boxes, the row lengths $l_i$ and the column lengths $d_j$. Additionally we set $t_i = r + l_i - i$ $(l_i = 0 \text{ if } i > \text{depth} T)$ and $\Delta(a_1, a_2, \ldots, a_m) = \prod_{1 \le i < j \le m} (a_i - a_j)$ (Vandermonde).

We calculate the dimensions of following cohomology groups of the $N$-dimensional complex projective space using the Bott theorem on homogeneous vector bundles:

**Theorem 3.1** *Assume* $d =$ depth $T \leq N$. *Then the Hilbert polynomial* $\chi(\mathbb{P}^N, \Omega^T(t))$ *has the in pairs different zeros* $t_1, t_2, \ldots, t_N$. *The fibration* $\Omega^T(t)$ *is acyclic if and only if* $t$ *is one of the zeros* $t_i$. *For fixed* $N, T, t$ *at most one of the cohomology groups* $H^q(\mathbb{P}^N, \Omega^T(t))$ $(q = 0, 1, \ldots)$ *may be nontrivial (cf. [4], [27]). Furthermore:*

$$\chi(\mathbb{P}^N, \Omega^T(t)) = \frac{\Delta(t, t_1, t_2, \ldots, t_N)}{\Delta(N, N-1, \ldots, 0)} = (\prod_{i=1}^{N} i!)^{-1} \cdot \prod_{1 \leq i < j \leq N} (t_i - t_j) \cdot \prod_{i=1}^{N} (t - t_i) \qquad (7)$$

$$H^q(\mathbb{P}^N, \Omega^T(t)) = 0 \quad \forall t \in \mathbb{Z} \text{ iff } q \neq 0, N, d_1, d_2, \ldots, d_l \quad (l = \text{ length } T) \qquad (8)$$

$$\begin{aligned} &H^0(\mathbb{P}^N, \Omega^T(t)) = 0 \text{ if and only if } t \leq t_1 \\ &\dim H^0(\mathbb{P}^N, \Omega^T(t)) = \chi(\mathbb{P}^N, \Omega^T(t)) \text{ if } t \geq t_1 \end{aligned} \qquad (9)$$

$$\begin{aligned} &H^N(\mathbb{P}^N, \Omega^T(t)) = 0 \text{ if and only if } t \geq t_N \\ &\dim H^N(\mathbb{P}^N, \Omega^T(t)) = (-1)^N \cdot \chi(\mathbb{P}^N, \Omega^T(t)) \text{ if } t \leq t_N \end{aligned} \qquad (10)$$

$$\begin{aligned} &\text{For each } i \in \{1, 2, \ldots, l\} \text{ with } 0 < d_i < N \text{ one has } t_{d_i+1} \leq t_{d_i} - 2 \\ &\text{and } H^{d_i}(\mathbb{P}^N, \Omega^T(t)) = 0 \text{ if and only if } t \leq t_{d_i+1} \text{ or } t \geq t_{d_i} \\ &\dim H^{d_i}(\mathbb{P}^N, \Omega^T(t)) = (-1)^{d_i} \cdot \chi(\mathbb{P}^N, \Omega^T(t)) \text{ if } t_{d_i+1} \leq t \leq t_{d_i} \end{aligned} \qquad (11)$$

**Remarks:**

*In the special case* $d_1 = r \leq N$, $0 = d_2 = d_3 = \ldots$ *one gets*
$\Omega^T = \Omega^r = \bigwedge^r \Omega^1, \chi(\mathbb{P}^N, \Omega^r(t)) = \binom{t-1}{r} \cdot \binom{t+N-r}{N-r}$ *and* (12)
$\dim H^q(\mathbb{P}^N, \Omega^r(t)) = \delta_{q,r} \cdot \delta_{t,0}$ *for* $0 < q < N$ *with the Kronecker-*$\delta$.

*In the case* $l_1 = r$, $0 = l_2 = l_3 = \ldots$ *we have* $\Omega^T = S^r\Omega^1$.
*Therefore* $\chi(\mathbb{P}^N, (S^r\Omega^1)(t)) = \frac{1}{N} \cdot \binom{r+N-1}{N-1} \cdot \binom{t+N-r}{N-1} \cdot (t - 2r + 1)$ (13)
*and* $H^q(\mathbb{P}^N, S^r\Omega^1(t)) = 0 \quad \forall t \quad \forall q$ *with* $1 < q < N$ *(cf. (8)).*

*The statements of the theorem remain true even in the case* $\Omega^T = \mathcal{O}_{\mathbb{P}^N}$, *i.e.,*
$l_i = 0 \ \forall i$. *Of course, that means* $\chi(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(t)) = \binom{t+N}{N}$ *and* (14)
$H^q(\mathbb{P}^N, \mathcal{O}(t)) = 0 \quad \forall t \quad \forall q$ *with* $0 < q < N$.

*By Serre duality (5), (6) we get* $Hom(\Omega^T(t), \mathcal{O}_{\mathbb{P}^N}) \cong \Omega^{T^*}(-t + l \cdot (N+1))$,
$\dim_K H^q(\mathbb{P}^N, \Omega^T(t)) = \dim_K H^{N-q}(\mathbb{P}^N, \Omega^{T^*}(-t + (l-1) \cdot (N+1)))$ *and* (15)
$\chi(\mathbb{P}^N, \Omega^T(t)) = (-1)^N \cdot \chi(\mathbb{P}^N, \Omega^{T^*}(-t + (l-1) \cdot (N+1)))$, *where* $T^*$ *denotes a Young tableau with the column lengths* $d_j^* = N - d_{l+1-j} \quad \forall j \in \{1, 2, \ldots, l\}$.

*It should be practicable to handle the Grassmannians instead of the projective space in similar way.* (16)

*We will see that the Hilbert polynomial* $\chi(\mathbb{P}^N, \Omega^T(t))$ *is independent of the ground field* $K$. *Since the calculation of dimensions of those cohomology groups essentially is a linear algebra problem it is natural to ask: Are these formulas true for the projective space defined over a field* $K$ *with* $char(K) = p > r$?. (17)

# 4 The $T$-power of an algebraic complex

Let $r$ be a positive integer and let $R$ be a commutative ring which contains the ground field $K$ and assume $\text{char}(K) = 0$ or $\text{char}(K) = p > r$. Let $\mathcal{K} : \mathcal{K}_0 \xrightarrow{d} \mathcal{K}_1 \xrightarrow{d} \mathcal{K}_2 \xrightarrow{d} \cdots$ $(d \circ d = 0)$ be an algebraic complex of $R$-modules. Then the $r$-th tensor power $P = \mathcal{K}^{\otimes r}$ of the complex $\mathcal{K}$ is defined as follows: $P = \mathcal{K}^{\otimes r} : P_0 \xrightarrow{\delta} P_1 \xrightarrow{\delta} P_2 \xrightarrow{\delta} \cdots$
with $P_s = \bigoplus_{s_1+\cdots+s_r=s} \mathcal{K}_{s_1} \otimes \mathcal{K}_{s_2} \otimes \cdots \otimes \mathcal{K}_{s_r}$ and with $\delta(a_1 \otimes a_2 \otimes \cdots \otimes a_r) =$
$\sum_{i=1}^{r} (-1)^{s_1+s_2+\cdots+s_{i-1}} \cdot a_1 \otimes a_2 \otimes \cdots \otimes a_{i-1} \otimes da_i \otimes a_{i+1} \otimes \cdots \otimes a_r$ if $a_j \in \mathcal{K}_{s_j}$ $\forall j$.
It is easy to show that $\delta \circ \delta = 0$.
The symmetric group $S_r$ and therefore the group algebra $KS_r$ act on the tensor power by permutation of the spots. Now another action on $P = \mathcal{K}^{\otimes r}$ is needed with the additional property $p \circ \delta = \delta \circ p$ $\forall p \in S_r$. This action can be defined as follows:

$$p(a_1 \otimes a_2 \otimes \cdots \otimes a_r) := (-1)^{\sigma(p;s_1,s_2,\ldots,s_r)} \cdot a_{p^{-1}(1)} \otimes a_{p^{-1}(2)} \otimes \cdots \otimes a_{p^{-1}(r)}$$
$$\text{with } a_j \in \mathcal{K}_{s_j} \ \ \forall j \in \{1,2,\ldots,r\} \text{ and } \sigma(p\,;s_1,s_2,\ldots,s_r) := \sum_{\substack{i<j \\ p(i)>p(j)}} s_i \cdot s_j. \tag{18}$$

Then one has $P_s = \bigoplus_{T \in D(r)} \mathcal{K}_s^T$ , $\mathcal{K}^{\otimes r} = \bigoplus_{T \in D(r)} \mathcal{K}^T$ , $H^*(\mathcal{K}^{\otimes r}) = \bigoplus_{T \in D(r)} H^*(\mathcal{K}^T)$ with $\mathcal{K}_s^T = e_T(P_s)$ and $\mathcal{K}^T = e_T(\mathcal{K}^{\otimes r}) : \mathcal{K}_0^T \xrightarrow{\delta} \mathcal{K}_1^T \xrightarrow{\delta} \mathcal{K}_2^T \xrightarrow{\delta} \cdots$ .
The complex $\mathcal{K}^T$ is called the $T$-power of the complex $\mathcal{K}$. If the Young tableaux $T$ and $\widetilde{T}$ correspond to the same Young diagram then we have $\mathcal{K}^T \cong \mathcal{K}^{\widetilde{T}}$.
Of course, the $T$-power of a complex $\mathcal{K} :\ \cdots \xrightarrow{d} \mathcal{K}_2 \xrightarrow{d} \mathcal{K}_1 \xrightarrow{d} \mathcal{K}_0$ can be defined in similar way.
In this paper these constructions are used only under the additional assumption $\mathcal{K}_2 = \mathcal{K}_3 = \cdots = 0$. That means if the product $a_1 \otimes a_2 \otimes \cdots \otimes a_r \in P_s$ is different from zero then $a_i \in \mathcal{K}_0$ or $a_i \in \mathcal{K}_1$ $\forall i$. Moreover, there are exactly $s$ spots $i_1, i_2, \ldots, i_s$ $(1 \le i_1 < i_2 < \cdots < i_s \le r)$ with $a_{i_1}, a_{i_2}, \ldots, a_{i_s} \in \mathcal{K}_1$.
Finally in this case $\sigma$ is equal to the number of inversions inside the sequence $(p(i_1), p(i_2), \ldots, p(i_s))$.

# 5 A free resolution of the sheaf $\Omega^T$ on $\mathbb{P}^N$

Let $T$ be a Young tableau with $r$ boxes and let $K$ be the ground field with $\text{char}(K) = 0$ or $\text{char}(K) = p > r$. The Hilbert polynomial $\chi(\mathbb{P}^N, \Omega^T(t))$ will be computed using the above techniques in this general case.

**Theorem 5.1** *Assume* $d =$ depth $T \le N$ *and set* $t_i = r + l_i - i$ $\forall i$ $(l_i = 0$ *if* $i > d)$. *Then the following sequence is exact:*

$$0 \longrightarrow \Omega^T \longrightarrow \mathcal{O}(-r)^{\oplus b_0} \longrightarrow \mathcal{O}(1-r)^{\oplus b_1} \longrightarrow \ldots \longrightarrow \mathcal{O}(d-r)^{\oplus b_d} \longrightarrow 0 \tag{19}$$

*with the integers* $b_0 = (\prod_{i=1}^{N} i!)^{-1} \cdot \Delta(t_1, t_2, \ldots, t_N, t_{N+1})$ *and for* $0 < s \leq d$ :
$b_s = (\prod_{i=1}^{N} i!)^{-1} \cdot \sum_{1 \leq i_1 < \ldots < i_s \leq d} \Delta(t_1, t_2, \ldots, t_{i_1} - 1, \ldots, t_{i_s} - 1, \ldots, t_N, t_{N+1}).$

**Corollary 1** *For ground fields* $K$ *with* $\operatorname{char}(K) = 0$ *or* $\operatorname{char}(K) > r$ *we have*

$$\chi(\mathbb{P}^N, \Omega^T(t)) = (\prod_{i=1}^{N} i!)^{-1} \cdot \prod_{1 \leq i < j \leq N} (t_i - t_j) \cdot \prod_{i=1}^{N} (t - t_i). \tag{20}$$

**Proof.** From the Euler sequence one has the short exact sequence

$$0 \longrightarrow \Omega^1 \longrightarrow \mathcal{O}(-1)^{\oplus N+1} \longrightarrow \mathcal{O} \longrightarrow 0.$$

The $r$-th tensor power of the complex $\mathcal{K} : \mathcal{K}_0 \longrightarrow \mathcal{K}_1 \longrightarrow 0$ with $\mathcal{K}_0 = \mathcal{O}(-1)^{\oplus N+1}$, $\mathcal{K}_1 = \mathcal{O}, \mathcal{K}_2 = \mathcal{K}_3 = \cdots = 0$ gives the exact sequence
$0 \longrightarrow \Omega^{\otimes r} \longrightarrow \mathcal{O}(-r)^{\oplus \alpha_0} \longrightarrow \mathcal{O}(1-r)^{\oplus \alpha_1} \longrightarrow \ldots \longrightarrow \mathcal{O}(-1)^{\oplus \alpha_{r-1}} \longrightarrow \mathcal{O} \longrightarrow 0$
with $\alpha_s = \binom{r}{s} \cdot (N+1)^{r-s}$. The $T$-power of the complex $\mathcal{K}$ is the exact sequence
$0 \longrightarrow \Omega^T \longrightarrow \mathcal{O}(-r)^{\oplus b_0} \longrightarrow \mathcal{O}(1-r)^{\oplus b_1} \longrightarrow \ldots \longrightarrow \mathcal{O}(-1)^{\oplus b_{r-1}} \longrightarrow \mathcal{O}^{\oplus b_r} \longrightarrow 0$
with suitable numbers $b_s$ and with $\mathcal{O}(s-r)^{\oplus b_s} = e_T(\mathcal{O}(s-r)^{\oplus \alpha_s})$.
Because of $e_T \in KS_r$ the coefficients of $e_T$ are constant. Therefore, it is possible to compute the ranks $b_s$ simply as dimensions of suitable $K$-vector spaces [10].
The exact sequence (19) and $\chi(\mathbb{P}^N, \mathcal{O}(t)) = \binom{t+N}{N}$ are independent of $K$, i.e., the corollary ensues from the special case $K = \mathbb{C}$ (cf. (7)). ∎

To consider cohomology groups we formulate the following technical

**Lemma 5.2** *Let* $0 \longrightarrow \mathcal{F}_0 \xrightarrow{\psi_0} \mathcal{F}_1 \xrightarrow{\psi_1} \ldots \xrightarrow{\psi_{m-1}} \mathcal{F}_m \longrightarrow 0$ *be an exact sequence of coherent sheaves on the algebraic variety* $Y$. *Then one has for each integer* $i \geq 0$:
*a)* $H^i(Y, \mathcal{F}_0) = 0$ *if* $H^{i-j+1}(Y, \mathcal{F}_j) = 0 \ \forall j \in \{1, 2, \ldots, \min\{i+1, m\}\}$
*b)* $H^i(Y, \mathcal{F}_m) = 0$ *if* $H^{i+j-1}(Y, \mathcal{F}_{m-j}) = 0 \ \forall j \in \{1, 2, \ldots, m\}$.

**Proof.** The lemma immediately ensues from the short exact sequences
$0 \longrightarrow \operatorname{im}\psi_{j-1} \longrightarrow \mathcal{F}_j \longrightarrow \operatorname{im}\psi_j \longrightarrow 0 \quad (j \in \{1, 2, \ldots, m-1\})$
and the corresponding long exact cohomology sequences [34], [38]. ∎

**Lemma 5.3** *Let* $T$ *be a Young tableau with* $r$ *boxes and with* $d =$ depth $T \leq N$.
*If* depth $T < q < N$ *then* $H^q(\mathbb{P}^N, \Omega^T(t)) = 0 \ \forall t \in \mathbb{Z}$. (21)
*If* $0 \leq q < N$ *and* $t < r - q$ *then* $H^q(\mathbb{P}^N, \Omega^T(t)) = 0$. (22)
$H^0(\mathbb{P}^N, \Omega^T(t)) = 0$ *if and only if* $t < r+$ length $T$. (23)

**Proof.** Let $l_1 \geq l_2 \geq \ldots$ be the row lengths of $T$. Because of $H^0(\mathbb{P}^N, \mathcal{O}(t)) = 0$ for all $t < 0$ and $H^q(\mathbb{P}^N, \mathcal{O}(t)) = 0 \ \forall q \in \{1, 2, \ldots, N-1\} \ \ \forall t \in \mathbb{Z}$ the statements (21) and (22) ensue via lemma 1 from the free resolution (19) of the sheaf $\Omega^T$.

(23) will be proven by induction on $N$:
In the case of $N = 1$ we obtain depth $T = 1$ , $r =$ length $T$ ,
$\Omega^T = S^r\Omega^1 = S^r(\mathcal{O}(-2)) = \mathcal{O}(-2r)$ and therefore, $H^0(\mathbb{P}^1, \Omega^T(t)) = 0$ if and only if $t < 2r = r+$ length $T$.
Now assume $N \geq 2$ and depth $T < N$. In this case one can show:
$H^0(\mathbb{P}^N, \Omega^T(t)) = 0$ if and only if $H^0(\mathbb{P}^{N-1}, \Omega^T_{\mathbb{P}^{N-1}})(t)) = 0$. In fact, if $\omega \neq 0$ is a global section of $\Omega^T(t)$ on $\mathbb{P}^N$ then a hyperplane $E = \mathbb{P}^{N-1}$ exists such that the restriction of $\omega$ onto $E$ is different from zero as well [5].
Conversely, let $E = \mathbb{P}^{N-1}$ be an arbitrary hyperplane, let $P \in \mathbb{P}^N \setminus E$ be any point, let $\psi : \mathbb{P}^N \setminus P \longrightarrow E$ be the projection from $P$ and let $\overline{\omega} \neq 0$ be a global section of $\Omega^T_E(t)$ . Then $\psi^*(\overline{\omega}) \neq 0$ is regular on $\mathbb{P}^N \setminus P$ with $N \geq 2$ , hence regular on the whole space $\mathbb{P}^N$. Actually, if $\psi^*(\overline{\omega})$ were not regular then a pole divisor of $\psi^*(\overline{\omega})$ through the point $P$ would exist. Therefore by induction condition: $H^0(\mathbb{P}^N, \Omega^T(t)) = 0$ if and only if $t < r+$ length $T$.
Now assume $N \geq 2$ and depth $T = N$ and $l_1 > l_N$. Then the Young tableau $T$ consists of a rectangle with $N$ rows and $l_N$ columns and a Young tableau $T'$ with $r' = r - N \cdot l_N$ boxes, with depth $T' < N$ and length $T' =$ length $T - l_N = l_1 - l_N$. Then one has $\Omega^T = \Omega^{T'}(-(N+1)\cdot l_N)$ (cf. (4)) and therefore $H^0(\mathbb{P}^N, \Omega^T(t)) = 0$ if and only if $t - (N+1)\cdot l_N < r'+$ length $T'$, in other words: $H^0(\mathbb{P}^N, \Omega^T(t)) = 0$ if and only if $t < r+$ length $T$.
Finally we assume that $N \geq 2$ and depth $T = N$ and $l_1 = l_N$. Then the Young tableau $T$ is a rectangle with $N$ rows and $l_1$ columns, i.e., $\Omega^T = \mathcal{O}(-(N+1)\cdot l_1)$ and $r = N \cdot l_1$. Therefore, $H^0(\mathbb{P}^N, \Omega^T(t)) = 0$ if and only if $t < r+$ length $T$. ∎

## 6 Lefschetz type theorem

**Theorem 6.1** *Let $X$ and $Y$ be smooth irreducible projective manifolds such that $Y$ is an ideal theoretically complete intersection of $X$ by algebraic hypersurfaces of multidegree $(m_1, m_2, \ldots, m_c)$ $(n = \dim Y$ , $N = \dim X$ , $c = N - n = \operatorname{codim}_X Y)$. Let $T$ be a Young tableau with $r$ boxes and with the column lengths $d_1 \geq d_2 \geq \ldots$ $(d_j = 0$ if $j >$ length $T)$ and let $\mu := \sum_{j=1}^{c} d_j$ denote the number of boxes inside the $c$ front columns of $T$. Assume $\mu \leq \dim Y$ and let $t_0 \in \mathbb{Z}$ be a fixed integer. If $m := \min\{m_1, m_2, \ldots, m_c\}$ is large enough then the restriction map*

$$\varphi^* : H^i(X, \Omega^T_X(t)) \longrightarrow H^i(Y, \Omega^T_Y(t))$$

- *is a monomorphism for all $t, i \in \mathbb{Z}$ with $t \leq t_0$ , $0 \leq i \leq \dim Y - \mu$ ,*
- *is an isomorphism for all $t, i \in \mathbb{Z}$ with $t \leq t_0$ , $0 \leq i < \dim Y - \mu$.*

**Proof.** These results are obtained from an exact sequence of sheaves which can be described as follows: Let $l =$ length $T$ be the number of columns of $T$ and let $d_1, d_2, \ldots, d_l$ $(> 0)$ be the column lengths. Assume $M(T)$ to be the set of all integer matrices $A = ((d_{i,j}))$ with $c+1$ rows, $l =$ length $T$ columns and with the properties:

- $d_{1,j} = d_j \quad \forall j \in \{1, 2, \ldots, l\}$,

- $d_{i,l} \geq d_{i+1,l} \geq 0 \quad \forall i \in \{1,2,\ldots,c\}$ ,
- $d_{i,j} \geq d_{i+1,j} \geq d_{i,j+1} \quad \forall i \in \{1,2,\ldots,c\} \ \forall j \in \{1,2,\ldots,l-1\}$.

The $i$-th row sum of such a matrix $A$ will be denoted by $\varrho_i(A) := \sum_{j=1}^{l} d_{i,j}$. In particular, we set $\varrho(A) := \varrho_{c+1}(A)$. Then $r - \mu \leq \varrho(A) \leq r$ for all $A \in M(T)$. Let $T'(A)$ denote the Young tableau with $\varrho(A)$ boxes and with the column lengths $d_{c+1,1}$ , $d_{c+1,2}$ , ..., $d_{c+1,l}$ , i.e., $T'(A)$ depends only on the last row of $A$. We set $M_j(T) := \{A \in M(T) \ : \ \varrho(A) = r - j\}$ for each $j \in \{0,1,\ldots,\mu\}$ and $t(A) = \sum_{\nu=1}^{c} (\varrho_{\nu+1}(A) - \varrho_\nu(A)) \cdot m_\nu$ .

Then the exact sequence is the following (cf. [10] , [11]):

$$0 \longrightarrow E_\mu^T \longrightarrow E_{\mu-1}^T \longrightarrow \ldots \longrightarrow E_2^T \longrightarrow E_1^T \xrightarrow{\gamma} \Omega_{X|Y}^T \longrightarrow \Omega_Y^T \longrightarrow 0. \tag{24}$$

with $E_j^T = \bigoplus_{A \in M_j(T)} \Omega_{X|Y}^{T'(A)}(t(A))$ , $\Omega_{X|Y}^{T'(A)} = \mathcal{O}_Y \otimes \Omega_X^{T'(A)}$ if $\varrho(A) > 0$ and $\Omega_{X|Y}^{T'(A)} = \mathcal{O}_Y$ in the case $\varrho(A) = 0$.

This exact sequence will be constructed as the $T$-power (cf. ch. 4 ) of the short exact sequence $0 \longrightarrow \bigoplus_{1\leq i\leq c} \mathcal{O}_Y(-m_i) \xrightarrow{\eta} \Omega_{X|Y}^1 \longrightarrow \Omega_Y^1 \longrightarrow 0$ $(\Omega_{X|Y}^1 = \mathcal{O}_Y \otimes \Omega_X^1)$, strictly speaking using the $T$-power of the complex $0 \longrightarrow \mathcal{K}_1 \longrightarrow \mathcal{K}_0$ with $\mathcal{K}_1 = \bigoplus_{1\leq i\leq c} \mathcal{O}_Y(-m_i)$ , $\mathcal{K}_0 = \Omega_{X|Y}^1$.

Furthermore, an exact sequence can be derived from the Koszul complex:

$$\begin{aligned} 0 \to \Omega_X^{T'}(-m_1 - \ldots - m_c) &\longrightarrow \bigoplus_{1\leq j\leq c} \Omega_X^{T'}(-m_1 - \ldots - \widehat{m_j} - \ldots - m_c) \longrightarrow \\ \ldots \longrightarrow \bigoplus_{1\leq j\leq c} \Omega_X^{T'}(-m_j) &\xrightarrow{\alpha} \Omega_X^{T'} \longrightarrow \Omega_{X|Y}^{T'} \longrightarrow 0. \end{aligned} \tag{25}$$

We consider corresponding exact cohomology sequences

$$H^i(X, (\mathrm{im}\alpha)(t)) \to H^i(X, \Omega_X^T(t)) \xrightarrow{\beta} H^i(Y, \Omega_{X|Y}^T(t)) \to H^{i+1}(X, (\mathrm{im}\alpha)(t)),$$

$$H^i(Y, (\mathrm{im}\gamma)(t)) \to H^i(Y, \Omega_{X|Y}^T(t)) \xrightarrow{\delta} H^i(Y, \Omega_Y^T(t)) \to H^{i+1}(Y, (\mathrm{im}\gamma)(t)).$$

It is a purely technical practice to prove via lemma 1 that under the given assumptions the groups $H^i(X, (\mathrm{im}\alpha)(t))$ , $H^{i+1}(X, (\mathrm{im}\alpha)(t))$ , $H^i(Y, (\mathrm{im}\gamma)(t))$ and $H^{i+1}(Y, (\mathrm{im}\gamma)(t))$ are trivial, i.e. $\varphi^* = \delta \circ \beta$ is an isomorphism. ∎

**Corollary 2** *If* $m = \min\{m_1, m_2, \ldots, m_c\}$ *is large enough then the restriction map* $\varphi^* : H^0(X, \Omega_X^T) \longrightarrow H^0(Y, \Omega_Y^T)$ *of the global sections is a monomorphism in the case* $\mu = \dim Y$ *and is an isomorphism in the case* $\mu < \dim Y$.

**Corollary 3** *Let* $X$ *and* $T$ *be as above and assume* $\dim X >$ depth $T + 1$. *Then there exists a smooth complete intersection* $Y \subset X$ *with* $\dim Y =$ depth $T + 1$ *and* $H^0(X, \Omega_X^T) \cong H^0(Y, \Omega_Y^T)$.

*Proof.* Because of the Bertini theorem there are projective hypersurfaces $H_1, \ldots, H_c$ with $c = \dim X -$ depth $T - 1$ such that for each $j \in \{1, \ldots, c\}$ the intersection $Y_j = X \cap H_1 \cap H_2 \cap \ldots \cap H_j$ becomes a smooth irreducible complete intersection in $X$ of dimension $\dim X - j$. One has to consider $Y_{j+1}$ as hypersurface of $Y_j$. Then one gets Corr. 3 step by step via theorem 3. ∎

## 7 The sheaf $\Omega^T$ on smooth complete intersections

**Theorem 7.1** *Let $Y \subseteq \mathbb{P}^N$ be a smooth irreducible projective ideal theoretically complete intersection by algebraic hypersurfaces of multidegree $(m_1, m_2, \ldots, m_c)$ $(c = \operatorname{codim} Y)$. Assume $2 \leq m_1 \leq m_2 \leq \ldots \leq m_c$. Let $T$ be a Young tableau with $r$ boxes and with the column lengths $d_1, d_2, \ldots$ $(0 < r \geq d_1 \geq d_2 \geq \ldots)$.*
*Let $\mu := \sum_{j=1}^{c} d_j$ be the number of boxes inside the $c$ front columns of $T$ $(\mu \leq r)$.*
*Assume again $\operatorname{char}(K) = 0$ or $\operatorname{char}(K) = p > r$.*
*If* depth $T < q < \dim Y - \mu$ *then* $H^q(Y, \Omega^T(t)) = 0 \quad \forall t \in \mathbb{Z}$. (26)

*If* $q < \dim Y - \mu$ *and* $t < r - q$ *then* $H^q(Y, \Omega^T(t)) = 0$. (27)

*If* $\mu < \dim Y$ *and* $t < r + \min\{\text{length } T, m_1 - 2\}$ *then* $H^0(Y, \Omega^T(t)) = 0$. (28)

**Corollary 4** *If $\mu < \dim Y$ then $H^0(Y, \Omega^T) = 0$, i.e., the smooth irreducible complete intersection $Y$ has no global $T$-symmetric tensor differential forms different from zero if the Young tableau $T$ has less than $\dim Y$ boxes inside its $\operatorname{codim} Y$ front columns.*

**Corollary 5** *If* $\operatorname{codim} Y < \dim Y$ then $H^0(Y, S^r\Omega^1) = 0$.

**Corollary 6** *If* $r < \dim Y$ *then* $H^0(Y, \bigwedge^r \Omega^1) = 0$.

**Corollary 7** *If* $r < \dim Y$ *then* $H^0(Y, (\Omega^1)^{\otimes r} = 0$.

**Corollary 8** *Let $Z \subset \mathbb{P}^N$ be a smooth irreducible algebraic hypersurface.*
*If* depth $T \neq \dim Z$ *then* $H^0(Z, \Omega^T) = 0$.
*If* $\deg Z < \dim Z + 2$ *and* $\operatorname{char}(K) = 0$ *then* $H^0(Z, (\Omega^1)^{\otimes r}) = 0 \quad \forall r > 0$ *and* $H^0(Z, \Omega^T) = 0$ *for each Young tableau $T$.*
*If* $\deg Z = \dim Z + 2$ *and* $\operatorname{char}(K) = 0$ *then* $H^0(Z, \Omega^T) \cong K$ *in the case that the Young tableau $T$ is a rectangle with exactly $\dim Z$ rows and $H^0(Z, \Omega^T) = 0$ otherwise. That means $P_f = 1 \; \forall f$.*

**Proof.** The arguments are almost the same as in the proof of theorem 3 with $\mathbb{P}^N$ instead of $X$, but in this case we need in addition the fact that for each matrix $A \in M(T)$ the Young tableau $T'(A)$ can be embedded into the top-left of $T$. This ensues from $d_{c+1,j} \leq d_j \; \forall j \in \{1, 2, \ldots, l\}$. Therefore, the number $\mu(T'(A))$ of boxes inside the $c$ front columns of $T'(A)$ is lower than or equal to $\mu = \mu(T)$. In particular,

we use depth $T'(A) = d_{c+1,1} \leq d_1 =$ depth $T$.
Corr.4 ensues from (28). One obtains Corr.5 and Corr.6 from Corr.4 because of $S^r\Omega^1 = \Omega^T$ with depth $T = 1$ and $\bigwedge^r \Omega^1 = \Omega^r = \Omega^{T'}$ with length $T' = 1$.
Corr.7 ensues from Corr.4 since $\mu \leq r$ and $(\Omega^1)^{\otimes r} = \bigoplus_{T \in D(r)} \Omega^T$.
Finally Corr.8 ensues from (28) with $\mu =$ depth $T$. That means $H^0(Z, \Omega^T) = 0$ if depth $T < \dim Z$. In the case of depth $T = \dim Z$ one has $\Omega^T \cong \Omega^{T'} \otimes \omega_Z \cong \Omega^{T'}(n_Z)$ with $n_Z = \deg Z - \dim Z - 2$ where the Young tableau $T'$ arises from $T$ by erasing the first column of $T$ (cf. (4)). ∎

Let $T$ be a Young tableau with $r$ boxes. We make use of (19), (20) , (24) and (25) to calculate the Euler-Poincare-characteristic $\chi(Y, \Omega_Y^T(t))$ for complete intersections $Y$. For that purpose to the Young tableau $T$ a symmetric polynomial will be assigned which is a generalization of the familiar Schur polynomial [16], [17]:
Let $\{y_0, y_1, \ldots, y_N; z_0, z_1, \ldots, z_c\}$ be a set of two kinds of variables which we consider as ordered in this way. Inscribing the elements of this set into the boxes of $T$ such that the elements inside each row and each column of $T$ are monotonously increasing and that the $y_i$'s in each column as well as the $z_j$'s in each row are strictly increasing one gets a so called standard scheme $B$. Let $M_B$ the monomial which is the product of all elements in the boxes and let $Q_{(T,N,c)} = \sum_B M_B$ the sum of these monomials over all standard schemes. Then the polynomial $Q_{(T,N,c)}(y_0, y_1, \ldots, y_N; z_0, z_1, \ldots, z_c)$ is symmetric in the $y_i$'s as well as in the $z_j$'s.
Finally we need the formal power series $P(x) = \sum_{s \in \mathbb{Z}} \binom{N+s}{N} \cdot x^s = \sum_{s \in \mathbb{Z}} \chi(\mathbb{P}^N, \mathcal{O}(s)) \cdot x^s$.
An elementary calculation proves the following

**Theorem 7.2** *Let $Y \subset \mathbb{P}^N$ be a $n$-dimensional smooth irreducible complete intersection of multidegree $m_1, m_2, \ldots, m_c$ ( $c =$ codim $Y = N - n$) .*
*Then for each $t \in \mathbb{Z}$ the Euler-Poincare-characteristic $\chi(Y, \Omega_Y^T(t))$ is equal to the coefficient of $x^t$ in the formal power series*

$$\prod_{i=1}^{c}(1 - x^{m_i}) \cdot P(x) \cdot Q_{(T,N,c)}(x, x, \ldots, x; -1, -x^{m_1}, -x^{m_2}, \ldots, -x^{m_c})$$

$$(y_0 = y_1 = \ldots = y_N = x \; ; \; z_0 = -1 \; , \; z_1 = -x^{m_1} \; , \; z_2 = -x^{m_2} \; , \; \ldots \; , \; z_c = -x^{m_c})$$

**Remark**: *It is possible to calculate completely the cohomology groups $H^q(Y, \Omega^r(t))$ of $n$-dimensional smooth irreducible projective complete intersections $Y$ with coefficients in the twisted sheaf $\Omega^r(t)$ of alternating differential forms even for arbitrary char(K) (cf. [6], [13]).*
Moreover $K$-bases and explicit formulas for the $K$-dimensions of these cohomology groups are known. It is shown that $\dim_K H^q(Y, \Omega^r(t))$ depends only on $q, r, t$ and on the dimension $n$, the codimension $c$ and the multidegree $m_1, m_2, \ldots, m_c$ of $Y$ and finally on char$(K)$ . If not one integer $m_j$ is divisible by char$(K)$ or if $t$ is not divisible by char$(K)$ then $\dim_K H^q(Y, \Omega^r(t))$ is given by the same formulas like in the complex case. In the general case the formula depends only on the fact, which of the integers $m_1, m_2, \ldots, m_c$ are divisible by char$(K)$ . A very simple

example is a 3-dimensional quadric $Y \subseteq \mathbb{P}^4$ for which one has $H^0(Y,\Omega^2(2)) = 0$ in case $\mathrm{char}(K) \neq 2$ and $H^0(Y,\Omega^2(2)) \cong K$ in case $\mathrm{char}(K) = 2$ . It is shown that the Hilbert-polynomials $\chi(Y,\Omega^r(t))$, the dimensions $h^q(Y,\Omega^r)$ (without twist), $h^q(Y,\mathcal{O}(t))$, $h^q(Y,\Omega^n(t))$, $h^0(Y,\Omega^1(t))$ and $h^n(Y,\Omega^{n-1}(t))$ are independent of $\mathrm{char}(K)$. The same is true for all dimensions $h^q(Y,\Omega^r(t))$ if $Y$ is a curve or surface. Note that for instance the birational invariants $\dim_K H^0(Y,\Omega^r \otimes \omega_Y^s) = \dim_K H^0(Y,\Omega^r(s \cdot n_Y))$ in general depend on $\mathrm{char}(K)$. For the vanishing of the higher cohomology groups $H^q(Y,\Omega^r(t))$ with $0 < q < n$ and $q + r \neq n$ we received a purely arithmetical necessary and sufficient condition.
Of course it was interesting to calculate all cohomology groups $H^q(Y,\Omega^T(t))$ of complete intersections $Y$ with coefficients in the twisted sheaf $\Omega^T(t)$ of $T$-symmetric tensor differential forms.

# 8 The moduli space of a complex projective complete intersection

Assume $K = \mathbb{C}$ . The vector spaces $H^0(Y,\Omega^T)$ of global tensor differential forms are important since they are birational invariants of $Y$ . Now we give a nice application of a higher cohomology group. For compact complex analytic manifolds $V$ with ample canonical bundle $\omega_V$ Kobayashi proved the finiteness of the group $\mathrm{Aut}(V)$ of analytic automorphisms [26] . Narasimhan and Simha [32] showed that the isomorphism classes of complex analytic structures on $V$ form a Hausdorff space given in a neighbourhood of a particular structure $V_t$ by the Kuranishi space of $V_t$ modulo $\mathrm{Aut}(V_t)$. Finally by Sernesis result on infinitesimal deformations follows that in the case of complete intersections the dimension $m(V_t)$ of this local moduli space is equal to $\dim H^1(V_t, T_{V_t})$ [41], [42].
Now let $Y \subseteq \mathbb{P}^N$ be a smooth complex $n$-dimensional projective complete intersection of the multidegree $d = (d_1, d_2, \ldots, d_c)$ with $n \geq 2$ and $d_i \geq 2$ $\forall i$ $(c = N - n)$. The multiplicity of the canonical class is $n_Y = \sum_{i=1}^{c} d_i - N - 1$ and one has $\omega_Y = \mathcal{O}_Y(n_Y)$.
As usual let $\sigma_i$ denote the $i$-th elementary symmetrical polynomial of the integers $d_1, d_2, \ldots, d_c$ and let $s_i = \sum_{j=1}^{c} {d_j}^i$ be the sum of the $i$-th powers of these integers. For these $\sigma_i$ and $s_i$ the Newton relations are known.
Because of Serre duality one has $m(V) = \dim H^1(V,T_V) = \dim H^{n-1}(V,\Omega_V^1(n_V))$. Libgober and Wood [29] gave the following formula for dimension of the moduli space component: If $V$ is not a K3-surface or a quadratic hypersurface, then
$m(V) = \dim H^1(V,T_V) = \dim H^{n-1}(V,\Omega_V^1(n_V)) =$
$$= 1 - (N+1)^2 + \sum_{i=1}^{c} \binom{N+d_i}{N} + \sum_{i=1}^{c} \sum_{j=1}^{c} (-1)^j \sum_{1 \leq k_1 < k_2 < \ldots < k_j \leq c} \binom{N+d_i-d_{k_1}-d_{k_2}-\ldots-d_{k_j}}{N}$$
where we set $\binom{k}{N} = 0$ if $k < N$ $(k \in \mathbb{Z})$.

**Theorem 8.1** *For each integer $r > 1$ there exist $r$ homeomorphic smooth complete intersection surfaces in the projective space $\mathbb{P}^{4r-2}$ belonging in the moduli space to* *components with in pairs different dimensions.*

**Proof.** Taking e.g. the multidegrees $e = (26, 21, 16, 12)$ and $f = (24, 24, 14, 13)$ one has the same elementary symmetrical polynomials $\sigma_1 = 75, \sigma_2 = 2054$ of these integers and the same total degree $\sigma_4 = 2^7 \cdot 3^2 \cdot 7 \cdot 13$. Let $s > 0$ be a fixed integer and consider the composed multidegrees $d_{k,l} = (\underbrace{e, e, \ldots, e}_{k\ times}, \underbrace{f, f, \ldots, f}_{l\ times})$ $(k + l = s)$. Then obviously the multidegrees $d_{0,s}, d_{1,s-1}, \ldots, d_{s,0}$ have the same property.
Actually, for these multidegrees one has $\sigma_1 = 75 \cdot s$ , $\sigma_2 = \frac{s}{2} \cdot (75^2 \cdot s - 1517)$ and the same total degree $\sigma_{4s} = 2^{7s} \cdot 3^{2s} \cdot 7^s \cdot 13^s$.
Let $V_{(k,l)} \subseteq \mathbb{P}^{4s+2}$ be a smooth complete intersection surface of the multidegree $d_{k,l}$. The existence of this nonsingular surface follows from the Bertini theorem. Then for each fixed $s > 0$ the surfaces $V_{(0,s)}, V_{(1,s-1)}, \ldots, V_{(s,0)}$ have the same total degree $\deg V = \sigma_{4s}$ , the same multiplicity of the canonical class $n_V = 71 \cdot s - 3$, the same self-intersection number of the canonical class $c_1^2(V) = n_V^2 \cdot \deg V$ , the same topological Euler characteristic ( cf. [1, V.2 ])
$e(V) = [\binom{4s+3}{2} - (4s + 3)\sigma_1 + \sigma_1^2 - \sigma_2] \cdot \deg V = \frac{1}{2}\ (5041 s^2 + 1087 s + 6) \cdot \deg V$ and the same geometric genus $p_g(V) = \frac{1}{12}\ (\ c_1^2(V) + e(V)\ ) - 1$. The intersection form $q_V : H_2(V, \mathbb{Z}) \otimes H_2(V, \mathbb{Z}) \longrightarrow \mathbb{Z}$ of such a surface $V$ is an unimodular integral quadratic form. The signatur $\tau(V)$ of $q_V$ is equal to $\frac{1}{3}\ (c_1^2(V) - 2e(V))$ . $q_V$ is even if and only if $n_V$ is even , i.e. if and only if $s$ is odd [31]. The rank of any maximal subspace of $H_2(V, \mathbb{Z})$ on which $q_V$ is positiv definit is equal to $2p_g(V) + 1$. It follows that the intersection forms of the surfaces $V_{(0,s)}, V_{(1,s-1)}, \ldots, V_{(s,0)}$ have the same rank , the same signatur and the same parity , i.e. the intersection forms are isomorphic. Finally complete intersections are simply connected and by Freedman's result [19] we have that the surfaces $V_{(0,s)}, V_{(1,s-1)}, \ldots, V_{(s,0)}$ are homeomorphic ( cf. [18]).
On the other hand the dimensions of their moduli space components form a strictly monotonously increasing sequence. This easily can be proved using the formula given above because only a few binomial coefficients $\binom{N+d_i-d_{k_1}-d_{k_2}-\ldots-d_{k_j}}{N}$ are different from zero. Setting $r = s + 1$ one has $r$ complete intersection surfaces $V_{(0,s)}, V_{(1,s-1)}, \ldots, V_{(s,0)}$ in the projective space $\mathbb{P}^{4s+2} = \mathbb{P}^{4r-2}$ with the desired properties. ∎

In the same way, the following theorem about in pairs diffeomorphic three-dimensional manifolds can be proved.

**Theorem 8.2** *For each integer $r > 1$ there exist $r$ diffeomorphic three-dimensional complete intersections in the projective space $\mathbb{P}^{5r-2}$ belonging in the moduli space to components with in pairs different dimensions.*

**Proof.** In order to construct complete intersections $V_{(k,l)} \subseteq \mathbb{P}^{5k+5l+3}$ it is possible to use the multidegrees $e = (45, 35, 34, 21, 18)$ and $f = (42, 42, 27, 25, 17)$ having the same elementary symmetrical polynomials $\sigma_1, \sigma_2, \sigma_3, \sigma_5$.

This time the multidegrees $d_{0,s}, d_{1,s-1}, \dots, d_{s,0}$ have in pairs equal elementary symmetrical polynomials $\sigma_1, \sigma_2, \sigma_3$ and the same total degree $2^{2s} \cdot 3^{5s} \cdot 5^{2s} \cdot 7^{2s} \cdot 17^{s}$. Therefore the complete intersections $V_{(0,s)}, V_{(1,s-1)}, \dots, V_{(s,0)}$ are in pairs diffeomorphic [28] . Again the calculation of the integers $m(V_{(k,l)})$ shows the monotonicity of the corresponding sequences. Setting $r = s + 1$ one has $r$ three-dimensional complete intersections $V_{(0,s)}, V_{(1,s-1)}, \dots, V_{(s,0)}$ in the projective space $\mathbb{P}^{5s+3} = \mathbb{P}^{5r-2}$ with the desired properties. ∎

Fuquan and Klaus proved, that two 4-dimensional complete intersections are homeomorphic, if and only if they have the same total degree, Pontrjagin numbers and Euler number [22].

**Theorem 8.3** *For each integer $r > 1$ there exist $r$ homeomorphic complex four-dimensional nonsingular complete intersections in the projective space $\mathbb{P}_{\mathbb{C}}^{6r-2}$ isomorphism class of which lie in different dimensional components of the moduli space.*

**Proof.** Now we consider for instance the multidegrees $e = (91, 80, 70, 44, 43, 32)$ and $f = (88, 86, 64, 52, 35, 35)$. It is easy to check that the total degree $(= 2^{12} \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 43)$, the corresponding elementary symmetrical polynomials $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ and therefore the power sums $s_1, s_2, s_3, s_4$ agree. The same is true for the composed multidegrees $d_{0,s}, d_{1,s-1}, \dots, d_{s,0}$. Let $V_{(k,l)} \subseteq \mathbb{P}^{6s+4}$ be a complex 4-dimensional smooth complete intersection of the multidegree $d_{k,l}$. The existence of this nonsingular complete intersection follows from the Bertini theorem. Then for each fixed $s > 0$ the complete intersections $V_{(0,s)}, V_{(1,s-1)}, \dots, V_{(s,0)}$ have the same total degree $\deg V = 2^{12s} \cdot 5^{2s} \cdot 7^{2s} \cdot 11^{s} \cdot 13^{s} \cdot 43^{s}$ , the same Pontrjagin numbers $p_2 = (\binom{r'}{2} - r's'_2 + \frac{1}{2}({s'_2}^2 + s'_4)) \cdot \deg V$ and ${p_1}^2 = (r' - s'_2)^2 \cdot \deg V$ and the same Euler number $e = ( \binom{r'}{4} - \binom{r'}{3} s'_1 + \frac{1}{2}\binom{r'}{2}({s'_1}^2 + s'_2) - \frac{1}{6} r'({s'_1}^3 + 3s'_1 s'_2 + 2s'_3) + \frac{1}{24}({s'_1}^4 + 6{s'_1}^2 s'_2 + 8 s'_1 s'_3 + 3{s'_2}^2 + 6s'_4) ) \cdot \deg V$ with $r' = 6s + 5$ , $s'_i = s \cdot s_i$. Therefore these complete intersections are homeomorphic. Note that they are diffeomorphic up to connected sum with a homotopy 8-sphere [22]. On the other hand the dimensions $m(V_{(k,l)})$ of their moduli space components again form a strictly monotonously increasing sequence. Setting $r = s+1$ one has $r$ four-dimensional complete intersections $V_{(0,s)}, V_{(1,s-1)}, \dots, V_{(s,0)}$ in the projective space $\mathbb{P}^{6s+4} = \mathbb{P}^{6r-2}$ with the desired properties. ∎

**Remark**: *Note that in each of the three cases and for each fixed $s > 0$ the sequence of the birational invariants* $\left( \dim H^0(V_{(k,s-k)}, \Omega^1_{V_{(k,s-k)}}(n_{V_{(k,s-k)}})) \right)_{k=0,1,\dots,s}$ *is strictly monotonously increasing too. On the other hand these complete intersections have the same plurigenera $P_f$ $\forall f$ [10] .*

# 9 Frobenius pullback of locally free sheaves.

In the case of prime characteristic $p > 0$ there exists another operation to construct further birational invariants. Let $s \geq 0$ denote a fixed integer and set $q := p^s$. For any locally free algebraic sheaf $\mathcal{F}$ of rank $m$ on manifold $X$ the operation can be described as follows: We don't glue the free restrictions $\mathcal{F}_{U_\alpha}$ using the transition matrices $A_{\alpha,\beta} = ((f_{j,k}^{\alpha,\beta})) \in \mathrm{GL}_m(K(X))$ given by $\mathcal{F}$ , but we glue them using the transition matrices $A_{\alpha,\beta}^{(s)} := ((\ (f_{j,k}^{\alpha,\beta})^q\ ))$ taking the $q$-th power of each element of $A_{\alpha,\beta}$. From the compatibility conditions $A_{\alpha,\beta} \cdot A_{\beta,\gamma} = A_{\alpha,\gamma}$ we get via Frobenius automorphism the equations $A_{\alpha,\beta}^{(s)} \cdot A_{\beta,\gamma}^{(s)} = A_{\alpha,\gamma}^{(s)}$. Thus we have a locally free sheaf of $\mathcal{O}_Y$-modules with the same rank $m$ which we denote by $\mathcal{F}^{(s)}$ or $\mathcal{F}' = \mathcal{F}^{(1)}$ (Frobenius pullback of $\mathcal{F}$ ), $\mathcal{F}'' = \mathcal{F}^{(2)}$ and so forth. $\mathcal{F}^{(s)}$ is always a subsheaf of $S^q\mathcal{F}$. For line bundles $\mathcal{F} = \mathcal{L}$ ( i.e. $m = 1$ ) one has $\mathcal{L}^{(s)} = S^q\mathcal{L} = \mathcal{L}^q$. It is easy to see $(\mathcal{F} \oplus \mathcal{G})^{(s)} = \mathcal{F}^{(s)} \oplus \mathcal{G}^{(s)}$ , $(\mathcal{F} \otimes \mathcal{G})^{(s)} = \mathcal{F}^{(s)} \otimes \mathcal{G}^{(s)}$ , $(\bigwedge^r \mathcal{F})^{(s)} = \bigwedge^r \mathcal{F}^{(s)}$ , $(S^r\mathcal{F})^{(s)} = S^r\mathcal{F}^{(s)}$ , $(\mathcal{F}^T)^{(s)} = (\mathcal{F}^{(s)})^T$ , $\mathcal{F}^{(0)} = \mathcal{F}$ , $(\mathcal{F}^{(s)})^{(t)} = \mathcal{F}^{(s+t)}$ for locally free sheaves $\mathcal{F}$ and $\mathcal{G}$ on $X$.
In the case $p \leq r$ the sheaves $\mathcal{F}^{(s)}$ naturally appear in the treatment of the symmetric power $S^r\mathcal{F}$: If for instance $m = \mathrm{rank}(\mathcal{F}) = 2$ and $p = 2$ then $S^3\mathcal{F} = \mathcal{F} \otimes \mathcal{F}'$ and there exist short exact sequences $0 \longrightarrow \mathcal{F}' \longrightarrow S^2\mathcal{F} \longrightarrow \det \mathcal{F} \longrightarrow 0$ ,
$0 \longrightarrow \mathcal{G} \longrightarrow S^4\mathcal{F} \longrightarrow (\det \mathcal{F}) \otimes \mathcal{F}' \longrightarrow 0$ , $0 \longrightarrow \mathcal{F}'' \longrightarrow \mathcal{G} \longrightarrow (\det \mathcal{F})^2 \longrightarrow 0$
with a subsheaf $\mathcal{G}$ of $S^4\mathcal{F}$. If $m = 2$ and $p = 3$ then one has exact sequences
$0 \to \mathcal{F}' \longrightarrow S^3\mathcal{F} \longrightarrow (\det \mathcal{F}) \otimes \mathcal{F} \to 0$ , $0 \to \mathcal{F} \otimes \mathcal{F}' \longrightarrow S^4\mathcal{F} \longrightarrow (\det \mathcal{F})^2 \to 0$.
The general situation seems to be unknown.

If $\mathcal{F}$ is a sheaf of germs of differential forms (e.g. $\mathcal{F} = \Omega^r = \bigwedge^r \Omega^r$ , $\omega_Y$ , $S^r\Omega^1$ , $(\Omega^1)^{\otimes r}$ , $\Omega^T$ ) then $H^0(Y, \mathcal{F}^{(s)})$ is a birational invariant of $Y$ for each $s \geq 0$. Calculating those differential forms we have to work with the $q$-th powers $(df)^q$ of the differentials instead of $df$ and with the rules $(d(f_1 f_2))^q = f_1^q (df_2)^q + f_2^q (df_2)^q$ and $(g_1 df_1 + g_2 df_2)^q = g_1^q (df_1)^q + g_2^q (df_2)^q$.
For the sheaf $(\Omega^T)^{(s)}(t)$ on projective space or on smooth complete intersections one can prove some theorems and statements similar to the theorems 2-5 , similar to the corollaries 1-8 and similar to lemma 2 [14]. Instead of corollary 4 one has for instance the following generalization:
If $\mu < \dim Y$ then $H^0(Y, (\Omega^T)^{(s)}) = 0 \ \ \forall s \geq 0$.
The reason is that there exist exact sequences of sheaves like for instance on the projective space (cf. (18))
$0 \to (\Omega^T)^{(s)} \longrightarrow \mathcal{O}(-rq)^{\oplus b_0} \longrightarrow \mathcal{O}((1-r)q)^{\oplus b_1} \longrightarrow \ \dots \ \longrightarrow \mathcal{O}((d-r)q)^{\oplus b_d} \to 0$
or for complete intersections the exact sequence (24) with another twist
$t(A) = q \cdot \sum_{\nu=1}^{c} (\varrho_{\nu+1}(A) - \varrho_\nu(A)) \cdot m_\nu$ .

More comprehensively we consider the sheaf $(\Omega^r)^{(s)}(t)$ on smooth algebraic hypersurfaces $Y \subset \mathbb{P}^N$ [11]. The dimensions of the $K$-vector spaces $H^i(Y, (\Omega_Y^r)^{(s)}(t))$

and $H^i(\mathbb{P}^N, (\Omega^r_{\mathbb{P}^N})^{(s)}(t))$ are calculated. Some examples of algebraic hypersurfaces are given to show that the birational invariants $H^0(Y, \mathcal{F}^{(s)})$ are independent of the familiar invariants [11].

The ground field $K$ is assumed to be algebraically closed with $\operatorname{char}(K) = p > 0$ and $p >$ number of boxes of each Young tableau $T$ appearing in this chapter. Remember $q = p^s$ ($p, s$ fixed).

We start with the projective space and set $\binom{a}{b} = 0$ if $a < b$ or $b < 0$ $(a, b \in \mathbb{Z})$.

**Theorem 9.1** *Let $S$ be the polynomial ring $K[x_0, x_1, \dots, x_N]$ and let $I_s$ be the $\mathrm{PGL}_N(K)$-invariant and irrelevant ideal $(x_0^q, x_1^q, \dots, x_N^q)$ with $q = p^s$. Then for each $r$ with $0 < r < N$ the graduated $S$-modules $\bigoplus_{t\in\mathbb{Z}} H^r(\mathbb{P}^N, (\Omega^r)^{(s)}(t))$ and $S/I_s$ are isomorphic. The cohomology classes of the cocycles $\varrho^{(\alpha)}$ with $\varrho^{(\alpha)}_{j_0,\dots,j_r} = x_0^{\alpha_0} x_1^{\alpha_1} \dots x_N^{\alpha_N} \cdot (\omega_{j_0,j_1})^q \wedge (\omega_{j_1,j_2})^q \wedge \dots \wedge (\omega_{j_{r-1},j_r})^q$ and $0 \le \alpha_j < q$ $\forall j$, $\sum_{j=0}^{N} \alpha_j = t$, $\omega_{j,k} = \frac{x_j}{x_k} d\frac{x_k}{x_j}$ form a $K$-basis of $H^r(\mathbb{P}^N, (\Omega^r)^{(s)}(t))$. We get*

$$H^i(\mathbb{P}^N, (\Omega^r)^{(s)}(t)) = 0 \quad \text{if } 0 < i < N,\ 0 \le r \le N,\ i \ne r. \tag{29}$$

$$\dim H^0(\mathbb{P}^N, (\Omega^r)^{(s)}(t)) = \sum_{j=r+1}^{N+1} (-1)^{j-r-1} \cdot \binom{N+1}{j} \cdot \binom{t-j\cdot q+N}{N} \quad \text{for } 0 < r \le N, \tag{30}$$

$$\dim H^r(\mathbb{P}^N, (\Omega^r)^{(s)}(t)) = \dim\, [S/I_s]_t = \sum_{j=0}^{N+1} (-1)^j \cdot \binom{N+1}{j} \cdot \binom{t-j\cdot q+N}{N} =, \tag{31}$$

$$= \sum_{j=0}^{N+1} (-1)^{N+1-j} \cdot \binom{N+1}{j} \cdot \binom{j\cdot q-t-1}{N} \quad \text{for } 0 < r < N, \tag{32}$$

$$\dim H^N(\mathbb{P}^N, (\Omega^r)^{(s)}(t)) = \sum_{j=0}^{r} (-1)^{r-j} \cdot \binom{N+1}{j} \cdot \binom{j\cdot q-t-1}{N} \quad \text{for } 0 \le r < N, \tag{33}$$

$$\chi(\mathbb{P}^N, (\Omega^r)^{(s)}(t)) = \frac{1}{N!} \cdot \sum_{j=r+1}^{N+1} \left[(-1)^{j-r-1} \cdot \binom{N+1}{j} \cdot \prod_{k=1}^{N} (t - j\cdot q + k)\right] \quad \text{for } 0 < r, \tag{34}$$

$$\chi(\mathbb{P}^N, (\Omega^r)^{(s)}(t)) = \frac{1}{N!} \cdot \sum_{j=0}^{r} \left[(-1)^{r-j} \cdot \binom{N+1}{j} \cdot \prod_{k=1}^{N} (t - j\cdot q + k)\right] \quad \text{for } 0 \le r < N. \tag{35}$$

**Proof.** The theorem ensues from the Serre duality [39]
$\dim H^i(\mathbb{P}^N, (\Omega^r)^{(s)}(t)) = \dim H^{N-i}(\mathbb{P}^N, (\Omega^{N-r})^{(s)}(-t + (q-1)\cdot(N+1)))$
and from the following essentially self-dual short exact sequence of sheaves on $\mathbb{P}^N$:
$0 \longrightarrow (\Omega^r)^{(s)} \longrightarrow \bigoplus_{\binom{N+1}{r}} \mathcal{O}(-r\cdot q) \longrightarrow (\Omega^{r-1})^{(s)} \longrightarrow 0.$ ∎

**Corollary 9** *For $0 < r < N$ we get*
$H^0(\mathbb{P}^N, (\Omega^r)^{(s)}(t)) \ne 0$ *if and only if* $t \ge (r+1)\cdot q$ ,

$H^r(\mathbb{P}^N,(\Omega^r)^{(s)}(t)) \neq 0$ *if and only if* $0 \leq t \leq (N+1)\cdot(q-1)$ ,
$H^N(\mathbb{P}^N,(\Omega^r)^{(s)}(t)) \neq 0$ *if and only if* $t \leq r\cdot q - N - 1$.

**Remarks:**

- *For* $0 < r < N$ *the dimension* $\dim H^r(\mathbb{P}^N,(\Omega^r)^{(s)}(t))$ *is independent of* $r$.
- *The Hilbert function of the graduated S-module* $[S/I_s]$ *is equal to* $\sum_{t\in\mathbb{Z}} \dim\,[S/I_s]_t \cdot z^t = \left(\frac{1-z^q}{1-z}\right)^{N+1}$ . *Therefore*
  $\dim H^1(\mathbb{P}^N,(\Omega^1)^{(s)}(t)) = \sum_{j=0}^{q-1} \dim H^1(\mathbb{P}^{N-1},(\Omega^1)^{(s)}(t-j))$ *if* $N > 2$.
- *In the special case* $s = 0$ *(and also for* $char(K) = 0$*) we obtain the well known Bott formulas*
  $\dim H^0(\mathbb{P}^N,\Omega^r(t)) = \binom{t-1}{r}\cdot\binom{t+N-r}{N-r}$ *if* $0 < r \leq N$ ,
  $\dim H^i(\mathbb{P}^N,\Omega^r(t)) = \delta_{i,r}\cdot\delta_{t,0}$ *if* $0 < i < N$ , $0 \leq r \leq N$ ,
  $\dim H^N(\mathbb{P}^N,\Omega^r(t)) = \binom{r-t}{r}\cdot\binom{-t-1}{N-r}$ *if* $0 \leq r < N$

Now set $X = \mathbb{P}^N$ , $N \geq 2$ and let $Y \subset X$ be a smooth algebraic hypersurface defined by the equation $F = 0$ ( $\mathrm{Id}(Y) = (F)$ , $n = \dim Y = N-1$ , $m = \deg Y = \deg F$ , $\omega_Y = \mathcal{O}_Y(m-N-1)$ ).

**Theorem 9.2** *For any smooth hypersurface* $Y \subset \mathbb{P}^N$ *one has:*
*If* $(r < i$ , $i + r < n)$ *or* $(r > i$ , $i + r > n)$ *then* $H^i(Y,(\Omega^r_Y)^{(s)}(t)) = 0$.
*If* $0 < 2r < n$ *then the graduated S-module* $\underset{t\in\mathbb{Z}}{\oplus} H^r(Y,(\Omega^r_Y)^{(s)}(t))$ *is isomorphic to the graduated S-module* $[S/((F)+I_s)]$ *, that means,*
$\dim H^r(Y,(\Omega^r_Y)^{(s)}(t)) = \dim\,[S/((F)+I_s)]_t$ *for* $0 < 2r < n$ .
*If* $n < 2r < 2n$ *then* $\underset{t\in\mathbb{Z}}{\oplus} H^r(Y,(\Omega^r_Y)^{(s)}(t))$ *is isomorphic to the twisted graduated S-module* $[(I_s:(F))/I_s](mq-m)$ *, i.e.,*
$\dim H^r(Y,(\Omega^r_Y)^{(s)}(t)) = \dim\,[(I_s:(F))/I_s]_{t+mq-m}$ *for* $n < 2r < 2n$ .

**Proof.** By Serre duality we get
$\dim H^i(Y,(\Omega^r_Y)^{(s)}(t)) = \dim H^{n-i}(Y,(\Omega^{n-r}_Y)^{(s)}(-t-(m-N-1)(q-1)))$ .
The theorem will be proved using the short exact sequences
$0 \longrightarrow (\Omega^r_X)^{(s)}(-m) \longrightarrow (\Omega^r_X)^{(s)} \longrightarrow (\Omega^r_{X|Y})^{(s)} \longrightarrow 0$ .
$0 \longrightarrow (\Omega^{r-1}_Y)^{(s)}(-mq) \longrightarrow (\Omega^r_{X|Y})^{(s)} \longrightarrow (\Omega^r_Y)^{(s)} \longrightarrow 0$
with $\omega^q_Y \cong (\Omega^n_Y)^{(s)} \cong (\Omega^N_{X|Y})^{(s)}(mq) \cong \mathcal{O}_Y((m-N-1)q)$ and $(\Omega^0_Y)^{(s)} \cong \mathcal{O}_Y$. ∎

To calculate the dimensions $\dim H^i(Y,(\Omega^r_Y)^{(s)}(t))$ completely we have to consider two cases, namely that $p = \mathrm{char}(K)$ divides or does not divide $m = \deg Y$. Let $[\frac{a}{2}]$ denotes the highest integer $\leq \frac{a}{2}$ for each $a \in \mathbb{Z}$.
In this paper we select some of the results [11]:

**Theorem 9.3** *Assume* $0 < r < n = \dim Y$ *and let* $p = char(K)$ *divide* $m = \deg Y$. *Then* $\dim H^i(Y, (\Omega_Y^r)^{(s)}(t)) = \dim [S/((F) + I_s)]_{t-[\frac{r-i}{2}]mq}$ *if* $(0 < i \leq r\ ,\ i + r < n\ ,\ i + r$ *even* ) *or* $(r < i < n\ ,\ i + r > n\ ,\ i + r$ *odd* ). *That means, under this condition the graduated S-module* $\underset{t \in \mathbb{Z}}{\oplus} H^i(Y, (\Omega_Y^r)^{(s)}(t))$ *is isomorphic to the twisted graduated S-module* $[S/((F) + I_s)](-[\frac{r-i}{2}]mq)$. $\dim H^i(Y, (\Omega_Y^r)^{(s)}(t)) = \dim [(I_s : (F))/I_s]_{t-[\frac{r-i-1}{2}]mq-m}$ *if* $(0 < i < r\ ,\ i + r < n\ ,\ i + r$ *odd* ) *or* $(r \leq i < n\ ,\ i + r > n\ ,\ i + r$ *even* ), *i.e.,* $\underset{t \in \mathbb{Z}}{\oplus} H^i(Y, (\Omega_Y^r)^{(s)}(t))$ *is isomorphic to* $[(I_s : (F))/I_s](-[\frac{r-i-1}{2}]mq - m)$.

**Theorem 9.4** *Let* $0 < r < n$ *and let* $p = \mathrm{char}(K)$ *does not divide* $m = \deg Y$. *Then* $\dim H^i(Y, (\Omega_Y^r)^{(s)}(t)) = \dim [(I_s : (F^{q-1}))/((F) + I_s)]_{t-[\frac{r-i}{2}]mq}$ *if* $(0 < i < r\ ,\ i + r < n\ ,\ i + r$ *even* ) *or* $(r < i < n\ ,\ i + r > n\ ,\ i + r$ *odd* ). *That means the graduated S-module* $\underset{t \in \mathbb{Z}}{\oplus} H^i(Y, (\Omega_Y^r)^{(s)}(t))$ *is isomorphic to the twisted graduated S-module* $[(I_s : (F^{q-1}))/((F) + I_s)](-[\frac{r-i}{2}]mq)$. $\dim H^i(Y, (\Omega_Y^r)^{(s)}(t)) = \dim [(I_s : (F))/((F^{q-1}) + I_s)]_{t-[\frac{r-i-1}{2}]mq-m}$ *if* $(0 < i < r\ ,\ i + r < n\ ,\ i + r$ *odd* ) *or* $(r < i < n\ ,\ i + r > n\ ,\ i + r$ *even* ), *i.e.,* $\underset{t \in \mathbb{Z}}{\oplus} H^i(Y, (\Omega_Y^r)^{(s)}(t)) \cong [(I_s : (F))/((F^{q-1}) + I_s)](-[\frac{r-i-1}{2}]mq - m)$.

**Example:** If $p = \mathrm{char}(K) \neq 3, 5$ then the following surfaces $Y_1, Y_2, Y_3 \subset \mathbb{P}^3$ are smooth: $Y_1 : x_0^5 + x_1^5 + x_2^5 + x_3^5 = 0$ , $Y_2 : x_0^4 \cdot x_1 + x_0 \cdot x_1^4 + x_2^5 + x_3^5 = 0$ , $Y_3 : x_0^4 \cdot x_1 + x_0 \cdot x_1^4 + x_2^4 \cdot x_3 + x_2 \cdot x_3^4 = 0$. Since these hypersurfaces have the same degree one cannot distinguish the one from the other using only conventional birational invariants. For instance $\dim H^i(Y_j, \Omega_{Y_j}^r(t))$ is independent of $j$ (cf. [6] , [13]) .
The multiplicity of the canonical class of $Y_j$ is 1 , i.e., for instance the dimension $h_j^0(t) := \dim H^0(Y_j, (\Omega_{Y_j}^r)^{(s)}(t)) = \dim H^0(Y_j, (\Omega_{Y_j}^r)^{(s)} \otimes \omega_{Y_j}^t)$ is a birational invariant of $Y_j$ for each integer $t \geq 0$. For $p = 7, s = 1, r = 1$ one has the values:

| $t$ | : | ... | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | ... |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $h_1^0(t)$ | : | ... | 0 | 0 | 0 | 1 | 16 | 60 | 136 | 235 | 350 | 480 | ... |
| $h_2^0(t)$ | : | ... | 0 | 0 | 1 | 5 | 24 | 68 | 140 | 236 | 350 | 480 | ... |
| $h_3^0(t)$ | : | ... | 0 | 1 | 4 | 10 | 32 | 76 | 145 | 239 | 351 | 480 | ... |

Our explicit formulas and the exact sequences give the opportunity to calculate global sections and elements of higher cohomology groups. For instance the differential form $x_0^8(x_2^3 - x_3^3) \cdot (d\frac{x_1}{x_0})^7 + x_0^7(x_0^3 - x_1^3) \cdot (x_2(d\frac{x_3}{x_0})^7 - x_3(d\frac{x_2}{x_0})^7)$ is a nonzero global section of $(\Omega_{Y_3}^1)^{(1)}(11) \cong (\Omega_{Y_3}^1)^{(1)} \otimes \omega_{Y_3}^{11}$.
In the cases $p = 11, 13, 17, \ldots$ we have similar situations. It would be reasonable to inspect complete intersections of hypersurfaces in the same way (cf. [6] , [13]).

# 10 Chern classes of the $T$-power of a locally free sheaf

Let $X$ be a $n$-dimensional projective manifold and let $T$ be a Young tableau with $r$ boxes and with the row lengths $l_1, l_2, \ldots$ . Then for every locally free sheaf $\mathcal{F}$ (vector bundle) of rank $m \geq \mathrm{depth}T$ its $T$-power $\mathcal{F}^T$ is a locally free sheaf rank of which is given by the formula $\mathrm{rk}(\mathcal{F}^T) = \prod_{1 \leq i < j \leq m} (\frac{l_i - l_j}{j-i} + 1)$.

Let $c_i(\mathcal{F})$ be the $i$-th Chern class and let $c_t(\mathcal{F}) = \sum_{i=0}^{m} c_i(\mathcal{F}) \cdot t^i$ denotes the Chern polynomial of $\mathcal{F}$. One sets $c_t(\mathcal{F}) = \prod_{j=1}^{m}(1+a_j \cdot t)$ with virtual variables $a_1, a_2, \ldots, a_m$, that means $c_i(\mathcal{F})$ is equal to the $i$-th elementary symmetrical polynomial of these variables $a_1, a_2, \ldots, a_m$ [23].
The Young tableau $T$ defines a numbering of its boxes by the integers $1, 2, \ldots, r$. Now one gets a so called Young scheme by inscription of the index numbers $1, 2, \ldots, m$ $(m = \mathrm{rk}(\mathcal{F}))$ into the boxes of $T$. This Young scheme is called a standard scheme if the entries inside any row of $T$ give a monotonously increasing sequence and the entries inside any column of $T$ are strictly monotonously increasing. It is well known that the number of standard schemes is equal to $\mathrm{rk}(\mathcal{F}^T)$ [21]. Let $M_T$ be the set of all these standard schemes $S_T$.
We set $a_{S_T} := \sum_{\nu=1}^{r} a_{j_\nu}$ with the virtual variables $a_1, a_2, \ldots, a_m$, where $j_\nu$ denotes the entry in the $\nu$-th box of the Young standard scheme $S_T$ $(j_\nu \in \{1, 2, \ldots, m\} \ \forall \nu)$. The following equation for the Chern polynomial of $\mathcal{F}^T$ can be proven making use of the so called splitting principle [15] , [23]:

**Theorem 10.1** $c_t(\mathcal{F}^T) = \prod_{S_T \in M_T} (1 + a_{S_T} \cdot t)$
*The coefficient of $t^j$ at the right side is a homogenous symmetrical polynomial of the virtual variables $a_j$ , i.e., the Chern class $c_j(\mathcal{F}^T)$ of $\mathcal{F}^T$ is a quasihomogenous polynomial of the Chern classes $c_i(\mathcal{F})$ of $\mathcal{F}$.*

This way we are able to calculate for instance the Euler-Poincare-characteristic $\chi(X, \mathcal{F}^T) = \deg\ (\ \mathrm{ch}(\mathcal{F}^T) \cdot \mathrm{td}(\mathcal{T})\ )_n$ (Riemann-Roch-Hirzebruch, [25]) with the exponential Chern character $\mathrm{ch}(\mathcal{F}^T)$ and with the Todd class $\mathrm{td}(\mathcal{T})$ of the tangent bundle $\mathcal{T}$ on $X$.
Primarily we are interested in the case of the cotangential bundle $\mathcal{F} = \Omega^1$ since for each Young tableau $T$ the dimension $\dim H^0(X, \Omega^T)$ is a birational invariant of the manifold $X$. For projective complete intersections $X$ of algebraic hypersurfaces the Chern classes of the tangent bundle $\mathcal{T}$ and with it of the cotangential bundle $\Omega^1$ on $X$ are defined by Hirzebruch [24].

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