Some remarks on topological entropy of a semigroup of continuous maps

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ABSTRACT

We investigate the notion of topological entropy of a semigroup of continuous maps and provide several of its basic properties.

RESUMEN

Investigamos la noción de entropía topológica de un semigrupo de aplicaciones continuas y damos varias de sus propiedades básicas.

Key words and phrases:	*topological entropy, entropy-like invariants, semigroup entropy.*
Math. Subj. Class.:	*37B40, 54H20, 37C85*

1   Introduction

The concept of entropy of a transformation plays a central role in topological dynamics. The notion of topological entropy was introduced by Adler, Konheim and McAndrew in [1] as an invariant of topological conjugacy. Later, Bowen [4] and Dinaburg [6] presented an equivalent approach to the notion of entropy in the case when the domain of the considered transformation is a metrizable space. The topological entropy $h(f)$, of an endomorphism f, measures the complexity of the transformation acting on a compact topological space in the sense that it shows the rate at which the action of the transformation disperses points.

Since the entropy appeared to be a very useful invariant in ergodic theory and dynamical systems, there were several attemps to find its suitable generalizations for other systems such as groups, pseudogroups, graphs, foliations. Among the others, Ghys, Langevin and Walczak in [9] proposed a definition of a topological entropy for finitely generated pseudogroups of continuous transformations. Biś and Walczak in [3] applied the notion of entropy of a group to hyperbolic groups in the sense of Gromov to study its geometry and dynamics.

Also, there have been attemps to introduce several entropy-like invariants for non-invertable maps. Langevin and Walczak in [12], Hurley in [10], Langevin and Przytycki [11], Nitecki and Przytycki ([15]) studied different entropy-like invariants. Nitecki in [14] investigated topological entropy and preimage structure of maps. Mihailescu and Urbański in [13] focused on inverse topological pressure and the Hausdorff dimension of the intersection between the local stable manifold and the basic set. Hurley ([10]) established relations between topological entropy, preimage relation entropy, preimage branch entropy and point entropy of a single transformation. The first author has introduced in ([2]) the concept of topological entropy of semigroups and generalised Hurley's results on entropies of a single transformations to the case of a finitely generated semigroup of transformations acting on a compact space.

In this paper we examine in detail the concept of this (introduced in [2]) topological entropy of semigroups. Our article is organized as follows. In Section 1, we recall the notion of topological entropy for a finitely generated semigroup and, for the convenience of the reader, provide some results. In the Sections 2 and 3 we formulate analogues of properties of topological entropy in the context of an action of any finitely generated semigroup of continuous maps on a compact metric space. In the last two sections we state some sufficient conditions for a finitely generated semigroup to have zero, positive or finite entropy. Ending this introduction we would like to add that the concept of entropy we consider in this paper is unrelated and incompatible with the notions of topological entropy dealt with in [7], [8], and [5].

2   Topological entropy of a semigroup

Many useful properties of the concept of entropy of a single transformation can be found in [16]. Let X be a compact metric space with a distance function d. Consider a semigroup G of continuous transformations of X into itself. The semigroup G is assumed to be finitely generated, e.g. there exists a finite set $G_1 = \{f_1, ..., f_k\}$ such that

$$G = \bigcup_{n \in N} G_n,$$

where

$$G_n = \{g_1 \circ ... \circ g_n : X \to X\}_{g_1,...,g_n \in G_1}.$$

We always assume that id_X, the identity map on X is in G_1. This implies that $G_m \subset G_n$ for all $m \leq n$. Following [9] we say that two points $p, q \in X$ are (n, ε)−separated by G (with respect to the metric d_{max}^n) if there exists $g \in G_n$ such that $d(g(p), g(q)) \geq \varepsilon$, e.g.

$$d_{max}^n(p, q) = \max\{d(g(p), g(q)) : g \in G_n\} \geq \varepsilon.$$

We say that a subset A of X is (n, ε)−separated if any two distinct points of A have this property. All (n, ε)−separated subsets of X are always finite, since X is compact. Therefore, we can write

$$s(n, \varepsilon, X) := \max\{card(A) : A \text{ is } (n, \varepsilon)\text{−separated subset of X}\}.$$

The following definition has appeared in [2].

Definition 1 *Let*

$$h(G, G_1, X) = \lim_{\varepsilon \to 0^+} \limsup_{n \to \infty} \frac{1}{n} \log(s(n, \varepsilon, X)).$$

The quantity $h(G, G_1, X)$ is called the topological entropy of a semigroup G generated by G_1.

As it was shown in [2] the topological entropy of a semigroup G depends on the generating set G_1. It may however still serve as a natural generalisation of the notion of the topological entropy of a continuous mapping $f : X \to X$. Indeed, Let $f : X \to X$ be a continuous transformation of a compact metric space X and $G(f)$ a semigroup generated by $G_1(f) = \{id_X, f\}$. Then, we get that

$$h(f) = h(G(f), G_1(f)),$$

where $h(f)$ is the entropy of f. We can also describe the entropy of a semigroup G generated by G_1 in terms of (n, ε)−spanning sets. Namely, a subset A of X is called (n, ε)−spanning if for every $x \in X$ there exists $a \in A$ such that

$$d_{max}^n(x, a) = \max\{d(g(x), g(a)) : g \in G_n\} < \varepsilon.$$

The minimum of cardinalities of all (n, ε)−spanning sets is denoted by $r(n, \varepsilon, X)$. The following characterization of the topological entropy of a semigroup G generated by a finite set G_1 has been established in [2].

Lemma 1 *For any semigroup G generated by a finite set G_1 the following equality holds*

$$h(G, G_1, X) = \lim_{\varepsilon \to 0^+} \limsup_{n \to \infty} \frac{1}{n} \log(r(n, \varepsilon, X)).$$

The notion of topological entropy of a semigroup of transformations shares many common features with the concept of the topological entropy of a single transformation. We examine them in detail in the following sections.

3   First results

A classical result concerning the entropy of a single transformation $f : X \to X$ states that for any integer $n \geq 1$ we have $h(f^n) = n \cdot h(f)$. A corresponding result holds for the entropy of a semigroup of transformations.

Theorem 3.1 *If (G, G_1) and (G^*, G_1^*) are finitely generated semigroups generated respectively by $G_1 = \{id_X, g_1, ..., g_k\}$ and $G_1^* = \{id_X, g_1^m, ..., g_k^m\}$, $m \in N$, then*

$$h(G^*, G_1^*) = m \cdot h(G, G_1).$$

Proof. Denote by (X, d_X) the compact metric space the semigroup G acts on. Consider two points $x, y \in X$ that are (n, ε)-separated by G^*. This means that there exists $p \in G_n^*$ such that $d_X(p(x), p(y)) \geq \varepsilon$. Since g is the composition $g_{i_1}^m \circ ... \circ g_{i_n}^m$ with some $g_1, ..., g_n \in G_1$, we have that $d_X(g_{i_1}^m \circ, ..., \circ g_{i_n}^m(x), g_{i_1}^m \circ, ..., \circ g_{i_n}^m(y)) \geq \varepsilon$. Thus, the points x, y are $(m \cdot n, \varepsilon)$-separated with respect to (G, G_1). So, $s(m \cdot n, \varepsilon, (G^*, G_1^*)) \leq s(m \cdot n, \varepsilon, (G, G_1))$ and taking the appropriate limit, we obtain that

$$h(G^*, G_1^*) \leq m \cdot h(G, G_1).$$

Starting to prove the opposite inequality, let $A \subset X$ be an $(m \cdot n, \varepsilon)$-spanning subset of X, with respect to (G, G_1), with minimal cardinality. Then, for any $x \in X$ there exists $a \in A$ such that for any $g \in G_{m \cdot n}$ we have $d_X(g(x), g(a)) < \varepsilon$. So, in particular, for any $x \in X$ there exists $a \in A$ such that for any n-tuple $g_1^m, ..., g_n^m$, where all elements g_i are in G_1, we have

$$d_X(g_{i_1}^m \circ, ..., \circ g_{i_n}^m(x), g_{i_1}^m \circ, ..., \circ g_{i_n}^m(a)) < \varepsilon.$$

Therefore, A is (n, ε)-spanning subset of X with respect to (G^*, G_1^*) and

$$card(A) = r(m \cdot n, \varepsilon, (G, G_1)) \geq r(n, \varepsilon, (G^*, G_1^*)).$$

Passing to the appropriate limit, we obtain

$$m \cdot h(G, G_1) \geq h(G^*, G_1^*),$$

which completes the proof. ∎

If semigroups (G, G_1) and (H, H_1) act respectively on compact metric spaces (X, d_1) and (Y, d_2), then $(G \times H, G_1 \times H_1)$ is a finitely generated semigroup, acting on the compact space $X \times Y$. We shall prove in this context the following.

Theorem 3.2 *If (G, G_1) and (H, H_1) are finitely generated semigroups, then*

$$h(G \times H, G_1 \times H_1) = h(G, G_1) + h(H, H_1)$$

Proof. In the topological space $X \times Y$ we consider the metric $d_{X \times Y} = d_X + d_Y$ inducing the product topology. Thus, the finitely generated semigroup $(G \times H, G_1 \times H_1)$ acts on the compact metric space $(X \times Y, d_{X \times Y})$. Fix now an $(n, \varepsilon/2)$-separated set (with respect to (G, G_1)) $A = \{a_1, ..., a_p\} \subset X$ with maximal cardinality, and an $(n, \varepsilon/2)$-separated set $B = \{b_1, ..., b_r\} \subset Y$ with maximal cardinality. Then, for any two distinct elements $(a_{i_1}, b_{j_1}), (a_{i_2}, b_{j_2}) \in A \times B$ we get that $d_{X \times Y}((a_{i_1}, b_{j_1}), (a_{i_2}, b_{j_2})) \geq \varepsilon$. This means that the points $(a_{i_1}, b_{j_1}), (a_{i_2}, b_{j_2})$ are (n, ε)-separated with respect to $(G \times H, G_1 \times H_1)$. So, $card(A \times B) = card(A) \cdot card(B) = s(X, n, \varepsilon/) \cdot s(Y, n, \varepsilon/)$ and $s(X \times Y, n, \varepsilon) \geq card(A \times B) = s(X, n, \varepsilon) \cdot s(Y, n, \varepsilon)$. Passing to the appropriate limit, we get that

$$h(G \times H, G_1 \times H_1) \geq h(G, G_1) + h(H, H_1).$$

In order to prove the opposite inequality, we consider a set $C \subset X$ with minimal cardinality which is $(n, \varepsilon/2)$-spanning with respect to (G, G_1) and a set $D \subset Y$ with minimal cardinality which is $(n, \varepsilon/2)$-spanning with respect to (H, H_1). Then, $card(C \times D) = r(X, n, \varepsilon/2) \cdot r(Y, n, \varepsilon/2)$ and $C \times D$ is (n, ε)-spanning with respect to $(G \times H, G_1 \times H_1)$. Thus, $r(X \times Y, n, \varepsilon) \leq r(X, n, \varepsilon/2) \cdot r(Y, n, \varepsilon/2)$. Taking now the appropriate limit, we get that

$$h(G \times H, G_1 \times H_1) \leq h(G, G_1) + h(H, H_1).$$

The proof is complete.   ∎

Theorem 3.3 *If (G, G_1) is a finitely generated semigroup acting on a compact metric space (X, d), generated by $G_1 = \{id_X, g_1, ..., g_k\}$, and there exists a compact subset M of X such that for every $g_i \in G_1$*

$$g_i : M \to M$$

then,

$$h((G, G_1), X) \geq h((G^M, G_1^M), M),$$

where (G^M, G_1^M) is a semigroup acting on M, generated by

$$G_1^M = \{id_M, g_1|_M, ..., g_k|_M\}.$$

Proof. Denote by A an (n, ε)-separated subset of M (with maximal cardinality) with respect to (G^M, G_1^M). This means that for two distinct points $x, y \in A$ there exists $g \in G_n$ such that $d_M(g|_M(x), g|_M(y)) \geq \varepsilon$. But this eans that $d_X(g(x), g(y)) \geq \varepsilon$. Therefore, the set A is (n, ε)-separated with respect to (G, G_1), and consequently $s(n, \varepsilon, (G^M, G_1^M)) \leq s(n, \varepsilon, (G, G_1))$. We are thus done by passing the appropriate limit when $n \to \infty$.   ∎

Theorem 3.4 *Let (G, G_1) be a finitely generated semigroup acting on a compact metric space (X, d), generated by $G_1 = \{id_X, g_1, ..., g_k\}$. Assume that there exist compact subsets M_1 and M_2 of X such that $X = M_1 \cup M_2$ and that for every $j \in \{1, 2, \ldots, k\}$ $g_j(M_1) = M_1$ and $g_j(M_2) = M_2$ Then*

$$h((G, G_1), X) = \max\{h((G^{M_1}, G_1^{M_1}), M_1), h((G^{M_2}, G_1^{M_2}), M_2)\}.$$

Proof. Let $A_i \in M_i$, $i = 1, 2$, be (n, ε)–spanning sets with minimal cardinality, in the respective spaces $M_i, i = 1, 2$. Since

$$card(A_1 \cup A_2) \leq card(A_1) + card(A_2) \leq 2 \cdot \max\{card(A_1), card(A_2)\}$$

and since $A_1 \cup A_2$ forms an (n, ε)–spanning subset of X. We see that

$$r(n, \varepsilon, (G, G_1), X) \leq 2 \cdot \max\{card(A_1), card(A_2)\}$$
$$\leq 2 \max\{r(n, \varepsilon, (G^{M_1}, G_1^{M_1}), M_1), r(n, \varepsilon, (G^{M_2}, G_2^{M_2}), M_2)\}.$$

Hence, passing to the appropriate limit, we obtain

$$h(G, G_1, X) \leq \max\{h((G^{M_1}, G_1^{M_1}), M_1), h((G^{M_2}, G_2^{M_2})M_2)\}.$$

Since, by Theorem 3, we have

$$h((G, G_1), X) \geq \max\{h((G^{M_1}, G_1^{M_1})M_1), h((G^{M_2}, G_2^{M_2})M_2)\},$$

we are done. ∎

4   Positive entropy

As before, let (X, d_X) be a compact metric space. We consider continuous transformations of the space X into itself.

Theorem 4.1 *If $f_1, f_2 : X \to X$ are two surjective continuous maps of a compact metric space X and Y is a closed subset of X such that $f_1^{-1}(Y) \cap f_2^{-1}(Y) = \emptyset$ and $f_1^{-1}(Y) \cup f_2^{-1}(Y) \subset Y$, then $h(G, G_1) \geq \log 2 > 0$, where $G_1 = \{id_X, f_1, f_2\}$ and G is the semigroup generated by G_1.*

Proof. Since $f_1^{-1}(Y)$ and $f_2^{-1}(Y)$ are two disjoint compact sets, the distance δ between them is positive. Fix $\varepsilon \in (0, \delta)$. Since every map $g : X \to X$, $g \in G$ is surjective, one can select for every $g \in G$ exactly one point $z_g \in g^{-1}(Y)$. Now, for every $n \geq 0$ consider the set

$$A_n = \{z_g : g \in \hat{G}_n\}, \text{ where } \hat{G}_n = \{g_n \circ g_{n-1} \circ \ldots \circ g_2 \circ g_1 : g_1, \ldots, g_n \in \{f_1, f_2\}\}.$$

We shall show that A_n is an (n, ε)-separated set consisting of exactly 2^n elements. So, consider two arbitrary elements $g \neq h$ from $\hat{G}_n$. Write $g = g_n \circ g_{n-1} \circ \ldots \circ g_1$ and $h = h_n \circ h_{n-1} \circ \ldots \circ h_1$, where $g_j, h_j \in \{f_1, f_2\}$ for all $j = \{1, 2, \ldots, n\}$. Since

$g \neq h$, there exist $k \in \{1, 2, \ldots, n\}$ such that $g_1 = h_1$, $g_2 = h_2, \ldots, g_{k-1} = h_{k-1}$, and $g_k \neq h_k$. Hence

$$g_{k-1} \circ \ldots \circ g_1 \in g_{k-1} \circ \ldots \circ g_1((g_n \circ \ldots \circ g_1)^{-1}(Y)) \subset (g_n \circ \ldots \circ g_k)^{-1}(Y) \subset g_k^{-1}(Y)$$

and similarly

$$g_{k-1} \circ \ldots \circ g_1(z_h) \in g_{k-1} \circ \ldots \circ g_1((h_n \circ \ldots \circ h_1)^{-1}(Y))$$
$$= h_{k-1} \circ \ldots \circ h_1((h_n \circ \ldots \circ h_1)^{-1}(Y)) \subset h_k^{-1}(Y).$$

Hence, $d(g_{k-1} \circ \ldots \circ g_1(z_g), g_{k-1} \circ \ldots \circ g_1(z_h)) \geq \delta > \varepsilon$. Thus, the points z_g and z_h are (n, ε)-separated and, in particular $z_g \neq z_h$. This latter statement implies that the map $g \to z_g$, $g \in \hat{G}_n$ is bijective, and therefore $s(n, \varepsilon, X) \geq card(\hat{G}_n) = 2^n$. In consequence $h(G, G_1) \geq \log 2 > 0$. We are done.   ∎

As an immediate consequence of this theorem and Theorem 3.3, we get the following.

Corollary 1 *If for $f_1, f_2 \in Homeo(S^1)$ there exists a closed interval $I \subset S^1$ such that*

$$f_1^{-1}(I), f_2^{-1}(I) \subset I$$

and

$$f_1^{-1}(I) \cap f_2^{-1}(I) = \emptyset$$

then, the semigroup generated by id_X, f_1, f_2 has positive entropy.

Theorem 4.2 *If (G, G_1) is a semigroup generating by the set G_1 of Lipschitz transformations with a common Lipshitz constant L, acting on a compact Riemannian manifold M of dimension $m \geq 1$, then the entropy*

$$h(G, G_1) \leq m \log L < +\infty.$$

Proof. Denote by d the metric on the Riemannian compact manifold M. It follows from our assumptions that for any $g \in G_1$,

$$d(g(x), g(y)) \leq Ld(x, y).$$

Denote by A a maximal (n, ε)− separated subset of M. Then, for any distinct $a_1, a_2 \in A$ we obtain

$$\varepsilon \leq d_n^{max}(a_1, a_2) \leq L^n d(a_1, a_2).$$

Thus $d(a_1, a_2) \geq \varepsilon L^{-n}$ which means that A is a $(0, \varepsilon L^{-n})$−separated subset of M. Hence

$$s(n, \varepsilon) \leq s(0, \varepsilon L^{-n}) \leq \frac{vol M}{min_{x \in M} vol B(x, 2^{-1} \varepsilon L^{-n})}.$$

For an m-dimensional manifold M we have that a ball $B(x, r)$ centered at a point $x \in M$ and a radius r, satisfies the inequality

$$vol B(x, r) \geq cr^m$$

with some positive constant c independent of x and r. Thus,

$$\frac{volM}{min_{x \in M} \, volB(x, 2^{-1}\varepsilon L^{-n})} \leq \frac{volM}{c(2^{-1}\varepsilon L^{-n})^m},$$

and consequently

$$s(n, \varepsilon) \leq \frac{volM}{c(2^{-1}\varepsilon L^{-n})^m}.$$

Passing to the suitable limits we thus get that $h(G, G_1) \leq m \log L$, which finishes the proof.   ∎

5   Zero entropy and final remarks

Theorem 5.1 *Let (G, G_1) be a finitely generated semigroup acting on a compact metric space X. Assume that the family $\{g : X \to X\}_{g \in G}$ is equicontinuous. Then $h(G, G_1) = 0$.*

Proof. Denote by d the metric on the compact metric space X. Fix $\varepsilon > 0$. Since the semigroup G acts equicontinuously on X, there exists $\delta > 0$ such that if $x, y \in X$ and $d(x, y) < \delta$, then $d(g(x), g(y)) < \varepsilon$ for all $g \in G$. Coonsequently, if $A \subset X$ is δ-spanning (with respect to the metric d) subset of X, then A is (n, ε)-spanning. Hence $r(n, \varepsilon) \leq cardA < \infty$, (the latter inequality is true since X is compact) and therefore

$$h(G, G_1) \leq \lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \log(r(n, \varepsilon)) = 0.$$

We are done.   ∎

As an immediate consequence of this theorem, we get the following.

Corollary 2 *If (G, G_1) is a finitely generated semigroup of isometries acting on a compact metric space X, then $h(G, G_1) = 0$.*

Problem 5.2 *It is well known that for a homeomorphism $f : X \to X$ of a compact metric space X the equality $h(f) = h(f^{-1})$ holds. Is this also true in the case of semigroups generated respectively by homeomorphisms $id_X, f_1, ..., f_k$ and $id_X, f_1^{-1}, ..., f_k^{-1}$ of the space X?*

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