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A Topological Characterization of the Beurling-Björck Space $\mathfrak{S}_w$ Using the Short-Time Fourier Transform

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ABSTRACT

In this article, we will use a previously obtained topological characterization of the Beurling-Björck space, to prove a topological characterization via the short-time Fourier transform. Our work builds on recent work by K. Gröchenig and G. Zimmermann.

RESUMEN

En este artículo usaremos una caracterización topológica del espacio Beurling-Björck, previamente obtenido, para probar una caracterización topológica via la transformada rápida de Fourier. Nuestros resultados se construyen a partir de trabajo reciente de K. Gröchenig y G. Zimmermann.

Key words and phrases: *Schwartz space, Beurling-Björck space, short-time Fourier transform, ultradistributions.*
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1 Introduction

It is well known that the Fourier series are a good tool to represent periodic functions. However, the Fourier series fail to represent nonperiodic functions accurately. To solve this problem, the short-time Fourier transform (STFT) was introduced by D. Gabor [8] in 1946, as one of the solutions. The short-time Fourier transform works by first cutting off the signal by multiplying it by another function called window (often compactly supported) then taking the Fourier transform. This technique maps a signal into a function of time and frequency.

The theory of test functions $\mathfrak{S}$ for tempered distributions $\mathfrak{S}'$ introduced by L. Schwartz ([13],[14]) was to provide a satisfactory frame work for the Fourier transform. In 1961, A. Beurling [2] presented his generalization of distributions published in ([3],[4]) which also provides a satisfactory frame work for the Fourier transform.

In [5], G. Björck studies the space $\mathfrak{S}_w$ of test functions for tempered ultradistributions $\mathfrak{S}'_w$ to extend work by L. Hörmander, with most theorems recognizable as counterparts in [12]. In [15], N. Teofanov shows the natural connection between the theory of tempered ultradistributions and the time-frequency analysis through the time-frequency representations and the modulation spaces.

The space $\mathfrak{S}_w$, as defined by Björck in [5], consists of C^∞ functions such that the functions and their Fourier transforms, jointly with all their derivatives decay ultra-rapidly at infinity. In [10], Gröchenig and Zimmermann obtained a characterization of the space $\mathfrak{S}_w$ via the short-time Fourier transform. This characterization imposes one condition on the growth of the short-time Fourier transform of the function without conditions on its derivatives and its Fourier transform, using the characterizations of the space $\mathfrak{S}_w$ proved by S.-Y. Chung, D. Kim and S. Lee in [6].

In this paper, we will obtain the results proved in [10] using the topological characterization of the space $\mathfrak{S}_w$ as stated in [1], without using the derivatives as in [10]. Moreover, a minor modification of the proof of the characterization of the space $\mathfrak{S}_w$ via the short-time Fourier transform shows that this topological equivalence can be given in terms of explicit linear estimates, which appropriately reflect the linearity of the problem.

This paper is organized in three sections. In Section 2, we include some preliminary definitions and results. We also discuss the relation between the Schwartz $\mathfrak{S}$ space and the Beurling-Björck space $\mathfrak{S}_w$ and their duals, by showing that the Beurling-Björck space is continuously and strictly included in the Schwartz space, with reverse strict inclusion between the duals. In Section 3, we will use the topological characterization of the Beurling-Björck space obtained in [1] to prove a topological characterization of the Beurling-Björck space via the short-time Fourier transform. This characterization then makes use jointly of time and frequency, whereas the characterization presented in [1] imposes separate conditions on the time domain and the frequency domain.

The notation we use is standard. The symbols C^∞, C_0^∞, L^p, etc., indicate the usual spaces of functions defined on $\mathbb{R}^n$, with complex values. We denote $|\cdot|$ the Euclidean norm on $\mathbb{R}^n$, while $\|\cdot\|_p$ indicates the p-norm in the space L^p, where $1 \leq p \leq \infty$. In general, we work on the Euclidean space $\mathbb{R}^n$ unless we indicate other than that as appropriate. Partial derivatives will be denoted ∂^α, where α is a multiindex $(\alpha_1, ..., \alpha_n)$ in $\mathbb{N}_0^n$. We will use the standard abbreviations $|\alpha| = \alpha_1 + ... + \alpha_n$, $x^\alpha = x_1^{\alpha_1}...x_n^{\alpha_n}$. The Fourier transform of a function f will be denoted $\mathcal{F}(f)$ or $\widehat{f}$ and it will be defined as $\int_{\mathbb{R}^n} e^{-2\pi i x\xi} f(x)\,dx$. The involution of a function f will be denoted $\widetilde{f}$ and it will be defined as $\widetilde{f}(x) = \overline{f(-x)}$. The letter C will indicate a positive constant, that may be different at different occurrences. If it is important to indicate that a constant depends on certain parameters, we will do so by attaching indexes to the constant.

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2 Preliminary definitions and results

In this section we will introduce definitions and results that we will use. We start with the definition of the space of admissible functions.

Definition 2 *([10]) With $\mathcal{M}_c$ we indicate the space of functions $w : \mathbb{R}^n \to \mathbb{R}$ of the form $w(x) = \Omega(|x|)$, where*

1. $\Omega : [0, \infty) \to [0, \infty)$ *is increasing, continuous and concave,*
2. $\Omega(0) = 0$,
3. $\int_{\mathbb{R}} \frac{\Omega(t)}{(1+t^2)} dt < \infty$,
4. $\Omega(t) \geq a + b\ln(1 + t)$ *for some $a \in \mathbb{R}$ and some $b > 0$.*

Standard classes of functions w in $\mathcal{M}_c$ are given by

$$w(x) = |x|^d \text{ for } 0 < d < 1, \text{ and } w(x) = p\ln(1 + |x|) \text{ for } p > 0.$$

Remark 3 *Let us observe for future use that if $N > \frac{n}{b}$ is an integer, then*

$$C_N = \int_{\mathbb{R}^n} e^{-Nw(x)} dx < \infty \text{ for all } w \in \mathcal{M}_c,$$

where b is the constant in Condition 4 of Definition 2.

We now recall the following topological characterization of the Beurling-Björck space $\mathfrak{S}_w$ of test functions for tempered ultra-distributions, which we will take as the definition of $\mathfrak{S}_w$ in what follows.

Theorem 4 *([1]) Given $w \in \mathcal{M}_c$, the space $\mathfrak{S}_w$ can be described as a set as well as topologically by*

$$\mathfrak{S}_w = \left\{ \begin{array}{c} \varphi : \mathbb{R}^n \to \mathbb{C} : \varphi \text{ is continuous and for all} \\ k = 0, 1, 2, ..., \ q_{k,0}(\varphi) < \infty, \ q_{k,0} \circ \mathcal{F}(\varphi) < \infty \end{array} \right\},$$

where $q_{k,0}(\varphi) = \left\|e^{kw}\varphi\right\|_\infty$ *and* $q_{k,0} \circ \mathcal{F}(\varphi) = \left\|e^{kw}\widehat{\varphi}\right\|_\infty$.

Since $q_{k,0}(\varphi) < \infty$ for all $k = 0, 1, 2, \ldots$, φ is integrable, so $\widehat{\varphi}$ is well defined and the condition $q_{k,0} \circ \mathcal{F}(\varphi)$ makes sense for all $k = 0, 1, 2, \ldots$.
The Beurling-Björck space $\mathfrak{S}_w$ of test functions for tempered ultradistributions equipped with the family of semi-norms

$$S = \{q_{k,0}, q_{k,0} \circ \mathcal{F} : k \in \mathbb{N}_0\}$$

is a Fréchet space. Let us observe that $\mathfrak{S}_w$ becomes the Schwartz space $\mathfrak{S}$ when

$$w(x) = \ln(1 + |x|).$$

Remark 5 *A Fréchet space is a Hausdorff locally convex topological vector space that is metrizable and complete.*

Example 6 *As we see from Condition 2 in Definition 2,* $w(x) \leq |x|$ *for all* $w \in \mathcal{M}_c$. *So,* $g(x) = e^{-|x|^2} \in \mathfrak{S}_w$ *for all* $w \in \mathcal{M}_c$.

Lemma 7 *Given a measurable function* $\varphi : \mathbb{R}^n \to \mathbb{C}$, *the following statements are equivalent.*

1. $\|x^\alpha \varphi\|_\infty < \infty$, *for all* $\alpha \in \mathbb{N}_0^n$.
2. $\left\|(1 + |x|)^k \varphi\right\|_\infty < \infty$, *for all* $k \in \mathbb{N}_0$.

The proof of this lemma is based on the binomial theorem, it is quite straightforward and we will omit it.

Remark 8 *As a consequence of Lemma 7, the space* $\mathfrak{S}$ *can be described as a set as well as topologically by*

$$\mathfrak{S} = \left\{ \begin{array}{c} \varphi : \mathbb{R}^n \to \mathbb{C} : \varphi \text{ is continuous and for all} \\ k = 0, 1, 2, \ldots, \ p_{k,0}(\varphi) < \infty, \ p_{k,0} \circ \mathcal{F}(\varphi) < \infty \end{array} \right\},$$

where $p_{k,0}(\varphi) = \left\|(1 + |x|)^k \varphi\right\|_\infty$ *and* $p_{k,0} \circ \mathcal{F}(\varphi) = \left\|(1 + |x|)^k \widehat{\varphi}\right\|_\infty$.

We now prove that $\mathfrak{S}_w \subseteq \mathfrak{S}$ continuously, for all $w \in \mathcal{M}_c$, using the topological characterization for both $\mathfrak{S}_w$ and $\mathfrak{S}$.

Lemma 9 *([5])* $\mathfrak{S}_w \subseteq \mathfrak{S}$ *continuously, for all* $w \in \mathcal{M}_c$.

Proof. Fix $w \in \mathcal{M}_c$ and $\varphi \in \mathfrak{S}_w$. Then φ is continuous and for all $k = 0, 1, 2, ...,$ $q_{k,0}(\varphi) < \infty, q_{k,0} \circ \mathcal{F}(\varphi) < \infty$. To show that $\varphi \in \mathfrak{S}$ we need to show that for all $k = 0, 1, 2, ..., p_{k,0}(\varphi) < \infty, p_{k,0} \circ \mathcal{F}(\varphi) < \infty$. To do so we start with

$$\begin{aligned} p_{k,0}(\varphi) &= \left\| (1+|x|)^k \varphi \right\|_\infty \\ &= \left\| e^{k\ln(1+|x|)} \varphi \right\|_\infty \leq \left\| e^{k\frac{(w(x)-a)}{b}} \varphi \right\|_\infty \\ &\leq C \left\| e^{k'w} \varphi \right\|_\infty \leq C q_{k',0}(\varphi) < \infty. \end{aligned}$$

where a and b are the constants of Condition 4 in Definition 1. Note that we made use of Condition 4 in Definition 2.
Similarly, we can prove $p_{k,0} \circ \mathcal{F}(\varphi) < \infty$. Hence, $\varphi \in \mathfrak{S}$ and the inclusion $\mathfrak{S}_w \subseteq \mathfrak{S}$ is continuous. This completes the proof of Lemma 9. ∎

Remark 10 *As a consequence of Lemma 9, we have the reverse inclusion $\mathfrak{S}' \subseteq \mathfrak{S}'_w$, where $\mathfrak{S}'$, $\mathfrak{S}'_w$ are the dual spaces of $\mathfrak{S}$ and $\mathfrak{S}_w$ respectively.*

Example 11 *Let $w \in \mathcal{M}_c$ defined by $w(x) = \sqrt{|x|}$ and $f(x) = e^{-(1+|x|^2)^{\frac{1}{4}}}$. Then we can see that $f \in \mathfrak{S}$ and $f \notin \mathfrak{S}_w$. So $\mathfrak{S}_w \subsetneq \mathfrak{S}$. Moreover, if $h(x) = e^{2\sqrt{|x|}}$, then we can see that the distribution T_h defined by integration against h, belongs to $\mathfrak{S}'_w$. However $T_h \notin \mathfrak{S}'$. So, there are tempered ultra-distributions that are not necessarily tempered distributions.*
In general, the function $h(x) = e^{a|x|^b}$ defines a tempered ultra-distribution which is not necessarily a tempered distribution, for all $a > 0$ and $0 < b < 1$.

Remark 12 *As we see from the topological characterization of the Beurling-Björck space $\mathfrak{S}_w$ given in Theorem 4, the Fourier transform is a topological isomorphism of the Fréchet space $\mathfrak{S}_w$ onto itself. As a consequence, the Fourier transform is also a topological isomorphism from $\mathfrak{S}'_w$ onto itself defined as*

$$\widehat{T}(\varphi) = T(\widehat{\varphi}), \ T \in \mathfrak{S}'_w, \varphi \in \mathfrak{S}_w$$

Note that the weak topology is given to $\mathfrak{S}'_w$.

Remark 13 *([11], [7]) A wavelet function is defined to be a function $\psi \in L^2(\mathbb{R})$ such that*

$$\{\psi_{j,k} : \psi_{j,k}(x) = 2^{\frac{j}{2}}\psi(2^j x - k), \ j, k \in \mathbb{Z}\}$$

is an orthonormal basis of $L^2(\mathbb{R})$. It is well known that there is no wavelet function with compact support that belongs to $C^\infty(\mathbb{R}) \cap L^2(\mathbb{R})$. This is true as well for wavelet functions with exponential decay. As a consequence of the work of Dziubański and Hernández in [7], we can show that for each $0 < \varepsilon < 1$, there exists a C^∞ wavelet function ψ_ε such that $\widehat{\psi_\varepsilon}$ has compact support and $\psi_\varepsilon \in \mathfrak{S}_w(\mathbb{R})$, where $w(x) = |x|^{1-\varepsilon}$.

Lemma 14 *Consider the weight functions* $w : \mathbb{R}^n \to \mathbb{R}$ *and* $w' : \mathbb{R}^{2n} \to \mathbb{R}$ *such that* $w(x) = \Omega(|x|)$ *and* $w'(x,\xi) = \Omega(|(x,\xi)|)$. *Then, given* $F : \mathbb{R}^{2n} \to \mathbb{C}$ *continuous, the following statements are equivalent.*

1. $\left\| e^{mw'} F \right\|_\infty < \infty$, $\left\| e^{mw'} \widehat{F} \right\|_\infty < \infty$ for all $m \in \mathbb{N}_0$.

2. $\left\| e^{m(w(x)+w(\xi))} F \right\|_\infty < \infty$, $\left\| e^{m(w(x)+w(\xi))} \widehat{F} \right\|_\infty < \infty$ for all $m \in \mathbb{N}_0$.

The proof of this lemma is based on the subadditivity property of the weight functions and we will omit it. Using Lemma 14, we will denote $\mathfrak{S}_{w'}(\mathbb{R}^{2n})$ by $\mathfrak{S}_w(\mathbb{R}^{2n})$ instead.

Definition 15 *([9], [10]) The short-time Fourier transform (STFT) of a function or distribution* f *on* $\mathbb{R}^n$ *with respect to a non-zero window function* g *is formally defined as*

$$\nu_g f(x,\xi) = \int_{\mathbb{R}^n} f(t)\overline{g(t-x)} e^{-2\pi i t.\xi} dt = \widehat{(f\overline{T_x g})}(\xi) = < f, M_\xi T_x g > .$$

where $T_x g(t) = g(t-x)$ *is the translation operator and* $M_\xi g(t) = e^{2\pi i t.\xi} g(t)$ *is the modulation operator.*

The composition of T_x and M_ξ is the time-frequency shift

$$(M_\xi T_x g)(t) = e^{2\pi i t.\xi} g(t-x),$$

and its Fourier transform is given by

$$\widehat{M_\xi T_x g} = e^{2\pi i x.\xi} M_{-x} T_\xi \widehat{g}.$$

Remark 16 *Given* $w \in \mathcal{M}_c$, $g \in \mathfrak{S}_w \backslash \{0\}$ *and a function* f *with* $e^{-kw} f \in L^1$ *for some* $k \in \mathbb{N}$, *the STFT of* f *with respect to* g *is well defined and continuous. In fact,*

$$\begin{aligned} & \int_{\mathbb{R}^n} \left| f(t)\overline{g(t-x)} e^{-2\pi i t.\xi} \right| dt \\ \le & \int_{\mathbb{R}^n} e^{-kw(t)} |f(t)| \left| e^{kw(t)} g(t-x) \right| dt \\ \le & \left\| e^{-kw} f \right\|_1 \left\| e^{kw} g \right\|_\infty e^{kw(x)}. \end{aligned}$$

This shows that $\nu_g f(x,\xi)$ *is well defined on* $\mathbb{R}^{2n}$ *for each* $x, \xi \in \mathbb{R}^n$. *Moreover, the continuity of* $\nu_g f$ *follows by applying Lebesgue Dominated Convergence Theorem.*

We now recall the main properties of the short-time Fourier transform.

Lemma 17 *([9], [10]) For* $f, g \in \mathfrak{S}$, *the STFT has the following properties.*

1. (Inversion formula)
$$\int\int_{\mathbb{R}^n\times\mathbb{R}^n} \nu_g f(x,\xi)(M_\xi T_x g)(t)dxd\xi = \|g\|_2^2 f. \tag{1}$$

2. (STFT of the Fourier transforms)
$$\nu_{\widehat{g}}\widehat{f}(x,\xi) = e^{-2\pi i x.\xi}\nu_g f(-\xi,x).$$

3. (Fourier transform of the STFT)
$$\widehat{\nu_g f}(x,\xi) = e^{2\pi i x.\xi} f(-\xi)\overline{\widehat{g}(x)}. \tag{2}$$

The proof of this lemma uses straight forward computations and it will be omitted. We refer the reader to [10].
Now we will introduce two auxiliary results that we will use in the proof of the topological characterization of the Beurling-Björck space $\mathfrak{S}_w$ via the short-time Fourier transform.

Lemma 18 *([10]) Given $w \in \mathcal{M}_c$, let f and g be two nonnegative measurable functions. If $N > \frac{n}{b}$ is an integer, there exists $C > 0$ such that*
$$\left\|e^{kw}(f*g)\right\|_\infty \leq C\left\|e^{2(N+k)w}f\right\|_\infty \left\|e^{2(N+k)w}g\right\|_\infty,$$
for all $k = 0,1,2,\ldots$ The constant C does not depend on k, and b is the constant in Condition 4 of Definition 2.

Proof. First let us show
$$\int_{\mathbb{R}^n} e^{-2(N+k)w(t)}e^{-2(N+k)w(t-x)}dt \leq Ce^{-kw(x)}, \tag{3}$$
where C does not depend on k. If $|t-x| \leq \frac{|x|}{2}$, then $|t| \geq \frac{|x|}{2}$. This implies that $w(t) \geq w(\frac{x}{2}) \geq \frac{w(x)}{2}$ since $w(x) = \Omega(|x|)$ with Ω concave, increasing and $\Omega(0) = 0$. From here we obtain
$$-2(N+k)w(t) \leq -kw(x). \tag{4}$$

Now for $|t-x| \geq \frac{|x|}{2}$ we have
$$-2(N+k)w(t-x) \leq -kw(x). \tag{5}$$

Using (4) and (5), we can write
$$\begin{aligned} &\int_{\mathbb{R}^n} e^{-2(N+k)w(t)}e^{-2(N+k)w(t-x)}dt \\ \leq\ & e^{-kw(x)}\int_{|t-x|\leq\frac{|x|}{2}} e^{-2(N+k)w(t-x)}dt + e^{-kw(x)}\int_{|t-x|\geq\frac{|x|}{2}} e^{-2(N+k)w(t)}dt \\ \leq\ & 2e^{-kw(x)}\int_{\mathbb{R}^n} e^{-2(N+k)w(t)}dt = 2C_N e^{-kw(x)},\end{aligned}$$

where C_N is the constant in Remark 3.
Now, using (3)

$$\begin{aligned} |(f*g)(x)| &\leq \int_{\mathbb{R}^n} (|f(t)| * |g(t-x)|)dt \\ &\leq \left\|e^{2(N+k)w}f\right\|_\infty \left\|e^{2(N+k)w}g\right\|_\infty \int_{\mathbb{R}^n} e^{-2(N+k)w(t)}e^{-2(N+k)w(t-x)}dt \\ &\leq C\left\|e^{2(N+k)w}f\right\|_\infty \left\|e^{2(N+k)w}g\right\|_\infty e^{-kw(x)}. \end{aligned}$$

This completes the proof of Lemma 18. ∎

Corollary 19 *Given $f, g \in \mathfrak{S}_w$ for some $w \in \mathcal{M}_c$, we have $f * g \in \mathfrak{S}_w$.*

The proof of this corollary is immediate using Lemma 18.

The following lemma is stated in [10]. We include a proof using the topological characterization of $\mathfrak{S}_w$ given in Theorem 4.

Lemma 20 *Let $g \in \mathfrak{S}_w$ be fixed and suppose that $F : \mathbb{R}^{2n} \to \mathbb{C}$ is a measurable function that has an ultra-rapid decay, i.e. for each $k = 0, 1, 2, \ldots$ there is a constant $C = C_k > 0$ satisfying $|F(x,\xi)| \leq Ce^{-k(w(x)+w(\xi))}$. Define*

$$f(t) = \int\int_{\mathbb{R}^{2n}} F(x,\xi)(M_\xi T_x g)(t)dxd\xi.$$

Then f is continuous and for each $k = 0, 1, 2, \ldots$ and $N > \frac{n}{b}$ integer

$$\left\|e^{kw}f\right\|_\infty \leq C\left\|e^{kw}g\right\|_\infty \left\|e^{(N+k)(w(x)+w(\xi))}F\right\|_\infty \tag{6}$$

and

$$\left\|e^{kw}\widehat{f}\right\|_\infty \leq C\left\|e^{kw}\widehat{g}\right\|_\infty \left\|e^{(N+k)(w(x)+w(\xi))}F\right\|_\infty. \tag{7}$$

In particular, $f \in \mathfrak{S}_w$.

Proof. First we show that f is continuous. To do so, fix $t_0 \in \mathbb{R}^n$ and let $\{t_j\}$ be any sequence in $\mathbb{R}^n$ converging to t_0 as $j \to \infty$. Since $F(x,\xi)(M_\xi T_x g)(t_j)$ converges to $F(x,\xi)(M_\xi T_x g)(t_0)$ pointwise as $j \to \infty$ and

$$|F(x,\xi)(M_\xi T_x g)(t_j)| \leq Ce^{-N(w(x)+w(\xi))} \in L^1(\mathbb{R}^{2n}),$$

where $C = C_N \|g\|_\infty$, $N > \frac{n}{b}$ integer, and b is the constant in Condition 4 of Definition 2, we can apply Lebesgue Dominated Convergence Theorem to obtain $f(t_j) \to f(t_0)$

as $j \to \infty$. This implies the continuity of f.
Now to prove that $f \in \mathfrak{S}_w$ we start with

$$\begin{aligned}
|(e^{kw}f)(t)| &= |\int\int_{\mathbb{R}^{2n}} (F(x,\xi)e^{kw(t)}(M_\xi T_x g)(t))dxd\xi| \\
&\leq \int\int_{\mathbb{R}^{2n}} |F(x,\xi)| \left|M_\xi T_x(e^{kw(t+x)}g)(t)\right| dxd\xi \\
&= \int\int_{\mathbb{R}^{2n}} |F(x,\xi)| \left|T_x(e^{kw(t+x)}g)(t)\right| dxd\xi \\
&\leq \left\|e^{kw}g\right\|_\infty \int\int_{\mathbb{R}^{2n}} e^{kw(x)} |F(x,\xi)| dxd\xi \\
&\leq \left\|e^{kw}g\right\|_\infty \left\|e^{(N+k)(w(x)+w(\xi))}F\right\|_\infty \int\int_{\mathbb{R}^{2n}} e^{-Nw(x)}e^{-Nw(\xi)}dxd\xi \\
&\leq C\left\|e^{kw}g\right\|_\infty \left\|e^{(N+k)(w(x)+w(\xi))}F\right\|_\infty
\end{aligned}$$

which shows (6).
Now let us show (7). From the definition of f we can write

$$\begin{aligned}
\widehat{f}(\tau) &= \int_{\mathbb{R}^n}(\int\int_{\mathbb{R}^{2n}} (F(x,\xi)(M_\xi T_x g)(t))dxd\xi)e^{-2\pi i\tau.t}dt \\
&= \int\int_{\mathbb{R}^{2n}} (F(x,\xi)(\widehat{M_\xi T_x g})(\tau))dxd\xi \\
&= \int\int_{\mathbb{R}^{2n}} (F(x,\xi)(M_{-x}T_\xi\widehat{g})(\tau))e^{2\pi i x.\xi}dxd\xi,
\end{aligned}$$

where we used that

$$(\widehat{M_\xi T_x g})(\tau) = (M_{-x}T_\xi\widehat{g})(\tau)e^{2\pi i x.\xi}.$$

Now

$$\left|e^{kw(\tau)}\widehat{f}(\tau)\right| \leq \int\int_{\mathbb{R}^{2n}} |F(x,\xi)| \left|e^{kw(\tau)}(M_{-x}T_\xi\widehat{g})(\tau)\right| dxd\xi$$

and the proof of (7) follows the same argument as the one leading to the proof of (6).
This completes the proof of Lemma 20. ∎

3 The characterization of $\mathfrak{S}_w$ via the short-time Fourier transform.

We use the topological characterization of the space $\mathfrak{S}_w$ as stated in Theorem 4, so, we will not use the derivatives as in the original proofs in [9] and [10].

Theorem 21 *Given $w \in \mathcal{M}_c$ and $g \in \mathfrak{S}_w \backslash \{0\}$, the space $\mathfrak{S}_w$ can be described as a set as well as topologically by*

$$\mathfrak{S}_w = \{f : \mathbb{R}^n \longrightarrow \mathbb{C} : e^{-mw} f \in L^1 \text{ for some } m \in \mathbb{N} \text{ and } \pi_k(f) < \infty \forall k = 0, 1, 2, \ldots\},$$

$$\text{where } \pi_k(f) = \left\| e^{k\omega} \nu_g f \right\|_\infty \text{ and } \omega(x, \xi) = w(x) + w(\xi). \tag{8}$$

Proof. Let us indicate $\mathfrak{B}_w$ the space defined in (??). Observe that Remark 16 implies that $\left\| e^{k\omega} \nu_g f \right\|_\infty$ makes sense because $\nu_g f$ is continuous. We define in $\mathfrak{B}_w$ a structure of Fréchet space by means of the countable family of semi-norms

$$B = \{\pi_k : k = 0, 1, 2, \ldots\}.$$

We will show that $\mathfrak{B}_w = \mathfrak{S}_w$. To do so, let us first prove that $\mathfrak{B}_w \subseteq \mathfrak{S}_w$ continuously. If we fix $f \in \mathfrak{B}_w$, we need to show that $\left\| e^{kw} \widehat{f} \right\|_\infty$ and $\left\| e^{kw} f \right\|_\infty$ are finite, and f is continuous. Since $f \in \mathfrak{B}_w$, then $\pi_k(f) < \infty$ for all $k \in \mathbb{N}_0$, which implies that $\nu_g f$ has an ultra-rapid decay. Then, by the inversion formula given in Lemma 17, we can write

$$f(t) = \|g\|_2^{-2} \int \int_{\mathbb{R}^{2n}} (\nu_g f(x, \xi)(M_\xi T_x g)(t)) dx d\xi.$$

By (6) of Lemma 20 we deduce $\left\| e^{kw} f \right\|_\infty \leq C \pi_{N+k}(f)$; by (7) of the same lemma $\left\| e^{kw} \widehat{f} \right\|_\infty \leq C \pi_{N+k}(f)$ since $\widehat{g} \in \mathfrak{S}_w$. The continuity of f also follows from Lemma 20. Hence $f \in \mathfrak{S}_w$ and the last two inequalities show that the inclusion $\mathfrak{B}_w \subset \mathfrak{S}_w$ is continuous.

Conversely, let $f \in \mathfrak{S}_w$. It is clear that $e^{-mw} f \in L^1$ for some $m \in \mathbb{N}$. We need to show that $\pi_k(f) < \infty$ for all $k = 0, 1, 2, \ldots$. To show this let $k \in \mathbb{N}_0$ and write $\widetilde{g}(t) = \overline{g(-t)}$. Then

$$\begin{aligned} e^{2kw(x)} |\nu_g f(x, \xi)| &= e^{2kw(x)} \left| \int_{\mathbb{R}^n} f(t) \widetilde{g}(x - t) e^{-2\pi i \xi . t} dt \right| \\ &\leq e^{2kw(x)} \int_{\mathbb{R}^n} |f(t)| \, |\widetilde{g}(x - t)| \, dt \\ &= e^{2kw(x)} (|f| * |\widetilde{g}|)(x) \\ &\leq \| e^{2kw} (|f| * |\widetilde{g}|) \|_\infty . \end{aligned}$$

Using Lemma 18 we get the following estimate

$$\begin{aligned} e^{2kw(x)} |\nu_g f(x, \xi)| &\leq \| e^{2kw} (|f| * |\widetilde{g}|) \|_\infty \\ &\leq C \left\| e^{2(N+2k)w} f \right\|_\infty \left\| e^{2(N+2k)w} \widetilde{g} \right\|_\infty \\ &\leq C \left\| e^{2(N+2k)w} f \right\|_\infty \\ &\leq C q_{2N+4k,0}(f). \end{aligned} \tag{9}$$

Moreover, by Lemma 17 we can write $\nu_g f(x,\xi) = e^{-2\pi i\xi\cdot x}\nu_{\widehat{g}}\widehat{f}(\xi,-x)$ so that

$$e^{2kw(\xi)}\left|\nu_g f(x,\xi)\right| \le e^{2kw(\xi)}\left|\nu_{\widehat{g}}\widehat{f}(\xi,-x)\right|.$$

An argument similar to the one leading to (9) produces

$$e^{2kw(\xi)}\left|\nu_g f(x,\xi)\right| \le Cq_{2N+4k,0}\left(\widehat{f}\right). \tag{10}$$

Combining (9) and (10) we have that

$$e^{2k\omega(x,\xi)}\left|\nu_g f(x,\xi)\right|^2 \le Cq_{2N+4k,0}(f)q_{2N+4k,0}\left(\widehat{f}\right)$$

This implies that

$$\pi_k(f) \le C(q_{2N+4k,0}(f) + q_{2N+4k,0}\left(\widehat{f}\right)). \tag{11}$$

So, $f \in \mathfrak{B}_w$. Hence $\mathfrak{S}_w \subset \mathfrak{B}_w$ and the inclusion is continuous. This completes the proof of Theorem 21. ■

Remark 22 *As we see in the proof of Theorem 21 the equivalence between the family of semi-norms $A = \{q_{k,0}\circ\mathcal{F}, q_{k,0} : k \in \mathbb{N}_0\}$ and $B = \{\pi_k : k \in \mathbb{N}_0\}$ is formulated by means of explicit linear estimates.*

Corollary 23 ([9]) Given $g \in \mathfrak{S}\backslash\{0\}$, the space $\mathfrak{S}$ can be described as a set as well as topologically by

$$\mathfrak{S}_w = \{f : \mathbb{R}^n \to \mathbb{C} : (1+|x|)^{-m} f \in L^1 \text{for some } m \in \mathbb{N} \text{ and } \pi_k(f) < \infty, \forall k = 0,1,\dots\},$$

where $\pi_k(f) = \| (1+|x|)^k(1+|\xi|)^k\nu_g f \|_\infty$.

Corollary 24 ([9]) Let $g \in \mathfrak{S}_w\backslash\{0\}$ be fixed. Then for $f \in \mathfrak{S}_w(\mathbb{R}^n)$, we have $\nu_g f \in \mathfrak{S}_w(\mathbb{R}^{2n})$.

Proof. By Lemma 14 it is enough to prove

$$\left\|e^{k\omega}\nu_g f\right\|_\infty < \infty, \quad \left\|e^{k\omega}\widehat{\nu_g f}\right\|_\infty < \infty$$

for $k = 0,1,2,\dots$, where $\omega(x,\xi) = w(x)+w(\xi)$. By Theorem 21 $\pi_k(f) = \left\|e^{k\omega}\nu_g f\right\|_\infty < \infty$. Then it is enough to show that $\left\|e^{k\omega}\widehat{\nu_g f}\right\|_\infty < \infty$. Using Lemma 17, we can write

$$\left|\widehat{\nu_g f}(x,\xi)\right| = |f(-\xi)|\left|\overline{\widehat{g}(x)}\right|.$$

Then

$$\begin{aligned} \left|e^{kw(x,\xi)}\widehat{V_g f}(x,\xi)\right| &= \left|e^{kw(\xi)}f(-\xi)\right|\left|e^{kw(x)}\overline{\widehat{g}(x)}\right| \\ &\le \left\|e^{kw}f\right\|_{\infty}\left\|e^{kw}\widehat{g}\right\|_{\infty}. \end{aligned}$$

This completes the proof of Corollary 24. ∎

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