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Pseudo Almost Periodic Solutions to A Neutral Delay Integral Equation

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ABSTRACT

We give some sufficient conditions which do ensure the existence and uniqueness of pseudo almost periodic solutions to a neutral delay integral equation of advanced type introduced by T. A. Burton in the literature. We next make use of the previous result to characterize pseudo almost periodic solutions to the so-called logistic equation.

RESUMEN

Entregamos condiciones suficientes que aseguran la existencia y unicidad de una soluciones seudo casi periódicas al la ecuación integral con retraso neutral del tipo avanzado introducida en la literatura por T.A. Burton. Luego, utilizamos resultados previos para caracterizar las soluciones seudo casi periódicas de la llamada ecuación logística.

Key words and phrases: *pseudo almost periodic function; almost periodic function; neutral delay integral equation, logistic equation, integral equation of advanced type.*

Math. Subj. Class.: *43A60; 35B15; 47B55.*

1 Introduction

This paper is concerned with the existence and uniqueness of pseudo almost periodic solutions to the abstract integral equation of the form

$$u(t) = f(u(h_1(t))) + \int_t^\infty Q(s, u(s), u(h_2(s)))C(t-s)ds + g(t) \quad (1)$$

for each $t \in \mathbb{R}$, where $f, g, h_1, h_2, C : \mathbb{R} \mapsto \mathbb{R}$ are continuous functions with $h_i(\mathbb{R}) = \mathbb{R}$ for $i = 1, 2$, and $Q : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ is jointly continuous.

Setting, $h_1(t) = h_2(t) = t - p$ where $p > 0$ is a constant, in Eq. (1), one obtains the so-called neutral delay integral equation of advanced type

$$u(t) = f(u(t-p)) + \int_t^\infty Q(s, u(s), u(s-p)C(t-s)ds + g(t), \quad (2)$$

which was introduced in the literature by T. A. Burton [4] as an intermediate step while studying the existence and uniqueness of (periodic) bounded solutions to the logistic differential equation given by

$$u'(t) = au(t) + \alpha u'(t-p) - q(t, u(t), u(t-p)) \quad (3)$$

where $a > 0$, $0 \leq |\alpha| < 1$, and $p > 0$ are respectively constants.

Under some suitable assumptions, the existence and uniqueness of a pseudo almost periodic solution to Eq. (1) is obtained (Theorem 3.1). Next we make use of the previous result to prove the existence and uniqueness of a pseudo almost periodic solution to the logistic equation (Theorem 3.3).

Some contributions related to pseudo almost periodic solutions to abstract differential and partial differential equations have recently been made, among them are [1, 2, 3, 6, 7, 11]. However, the existence of pseudo almost periodic solutions to integral equations, especially those of the form Eq. (1) is an untreated topic and this is the main motivation of the present paper. In particular, we will make use of our result related to Eq. (1) to discuss the existence and uniqueness of pseudo almost periodic solutions to the logistic differential equation, that is, Eq. (3).

The existence of almost periodic, asymptotically almost periodic, and pseudo almost periodic solutions is among the most attractive topics in the qualitative theory of differential equations due to their applications, especially in biology, economics, and physics.

The concept of pseudo almost periodicity, which is the central issue in this paper, was introduced by C. Y. Zhang [14, 15, 16] in the early nineties. Since then, such a notion became of great interest to several mathematicians. The pseudo almost periodicity is a natural generalization of the classical almost periodicity in the sense of Bochner. Thus such a concept is welcome to implement another existing generalization of almost periodicity, the so-called asymptotically almost periodicity due to Fréchet, see, e.g., [12]. For more on the concepts of almost periodicity and pseudo almost periodicity and related issues, we refer to [5, 10, 12, 13] for both the almost periodicity and asymptotic almost periodicity, and to [1, 2, 3, 6, 7, 8, 9, 11, 14, 15, 16] for the pseudo almost periodicity.

2 Pseudo Almost Periodic Functions

Let $(\mathbb{X}, \|\cdot\|)$ be a Banach space and let $(BC(\mathbb{X}), \|\cdot\|_\infty)$ be the Banach space of bounded continuous functions from $\mathbb{R}$ into $\mathbb{X}$ endowed with the sup norm

$$\|\phi\|_\infty = \sup_{t\in\mathbb{R}} \|\phi(t)\|.$$

Let $(\mathbb{Y}, \|\cdot\|_{\mathbb{Y}})$ be another Banach space. If $\Omega \subset \mathbb{X}$ is an open subset, then $BC(\mathbb{R} \times \Omega, \mathbb{Y})$ denotes the vector space of bounded continuous functions $\Phi : \mathbb{R} \times \Omega \mapsto \mathbb{Y}$.

Definition 2.1 *[5] A function $f \in BC(\mathbb{X})$ is called almost periodic if for each $\varepsilon > 0$, there exists $l_\varepsilon > 0$ such that every interval of length l_ε contains a number τ with the following property:*

$$\|f(t+\tau) - f(t)\| < \varepsilon, \quad \forall t \in \mathbb{R}.$$

The number τ above is then called an *ε-translation number* of f, and the collection of such functions will be denoted $AP(\mathbb{X})$.

Similarly,

Definition 2.2 *A function $F \in BC(\mathbb{R} \times \Omega, \mathbb{Y})$ is called almost periodic in $t \in \mathbb{R}$ uniformly in any $K \subset \Omega$ a bounded subset if for each $\varepsilon > 0$, there exists $l_\varepsilon > 0$ such that every interval of length $l_\varepsilon > 0$ contains a number τ with the following property:*

$$\|F(t+\tau, x) - F(t,x)\|_{\mathbb{Y}} < \varepsilon, \quad \forall t \in \mathbb{R},\ x \in K.$$

Here again, the number τ above is called an ε-translation number of F, and the class of such functions will be denoted $AP(\mathbb{R} \times \Omega, \mathbb{Y})$.

For more on $AP(\mathbb{X})$ (respectively, $AP(\mathbb{R} \times \Omega, \mathbb{Y})$) and related issues, we refer to [5, 10, 12, 13] and the references therein.

From now on, we suppose $\Omega = \mathbb{X}$ and set

$$AP_0(\mathbb{X}) := \{f \in BC(\mathbb{X}) : \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \|f(s)\| ds = 0\}.$$

Similarly, $AP_0(\mathbb{R} \times \mathbb{X}, \mathbb{Y})$ denotes the collection of functions $F \in BC(\mathbb{R} \times \mathbb{X}, \mathbb{Y})$ such that

$$\lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \|F(t,u)\|_{\mathbb{Y}} dt = 0$$

uniformly in $u \in \mathbb{X}$.

Definition 2.3 *A function $f \in BC(\mathbb{X})$ is called pseudo almost periodic if it can be expressed as $f = g + \phi$, where $g \in AP(\mathbb{X})$ and $\phi \in AP_0(\mathbb{X})$.*

The collection of such functions will be denoted by $PAP(\mathbb{X})$.

The functions g and ϕ appearing in Definition 2.3 are respectively called the *almost periodic* and the *ergodic perturbation* components of f. In addition, the decomposition given in Definition 2.3 is unique, see, e.g., [14, 15, 16].

We now equip $PAP(\mathbb{X})$ the collection of pseudo almost periodic functions from $\mathbb{R}$ into $\mathbb{X}$ with the sup norm. It is well-known that $(PAP(\mathbb{X}), \|.\|_\infty)$ is a Banach space, see details in [11].

Definition 2.4 *A function $f \in BC(\mathbb{R} \times \mathbb{X}, \mathbb{Y})$ is called pseudo almost periodic in $t \in \mathbb{R}$ uniformly in $x \in \mathbb{X}$ if it can be expressed as $f = g + \phi$, where $g \in AP(\mathbb{R} \times \mathbb{X}, \mathbb{Y})$ and $\phi \in AP_0(\mathbb{R} \times \mathbb{X}, \mathbb{Y})$.*

The collection of such functions will be denoted by $PAP(\mathbb{R} \times \mathbb{X})$.

3 The Main Result

Throughout the rest of the paper, we suppose that $\mathbb{X} = \mathbb{Y} = \mathbb{R}$ equipped the classical absolute value. Note however that when dealing with the pseudo almost periodicity of Q it would be more convenient to choose $\mathbb{X} = \mathbb{R} \times \mathbb{R}$, see **(H.3)**.

Our setting requires the following assumptions:

(H.1) The function $f, g : \mathbb{R} \mapsto \mathbb{R}$ are pseudo almost periodic and f satisfies,

$$|f(x) - f(y)| \leq \alpha . |x - y|, \qquad 0 \leq \alpha < 1,$$

for all $x, y \in \mathbb{R}$;

(H.2) The function $h_i : \mathbb{R} \mapsto \mathbb{R}$ is continuous, $h_i(\mathbb{R}) = \mathbb{R}$, and $u(h_i) \in PAP(\mathbb{R})$ $(i = 1, 2)$ whenever $u \in PAP(\mathbb{R})$;

(H.3) The function $Q : \mathbb{R} \times (\mathbb{R} \times \mathbb{R}) \mapsto \mathbb{R}$, $(t, x, y) \mapsto Q(t, x, y)$ is pseudo almost periodic in $t \in \mathbb{R}$ uniformly if $(x, y) \in \mathbb{R} \times \mathbb{R}$. Setting $Q = Q_1 + Q_2$ with $Q_1 \in AP(\mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and $Q_2 \in AP_0(\mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, we suppose that $Q_2(\cdot, v(\cdot), v(h_2(\cdot))) \in L^1(\mathbb{R})$ for each $v \in PAP(\mathbb{R})$ where h_2 is the function appearing in **(H.2)**. Furthermore, there exists $0 \leq k \leq 1$ such that

$$|Q(t, x, y) - Q(t, w, z)| \leq (k . |x - w| + (1 - k) . |y - z|)$$

for all $x, y, z, w \in \mathbb{R}$;

(H.4) $0 < \displaystyle\int_0^\infty |C(-s)|ds = C_0 < \infty.$

Our main result requires the following technical lemma:

Lemma 3.1 *Under assumptions* **(H.2)-(H.3)-(H.4)**, *the function defined by*

$$\Gamma u(t) := \int_t^\infty Q(s, u(s), u(h_2(s)))C(t - s)ds$$

maps $PAP(\mathbb{R})$ into itself.

Proof. Let $u \in PAP(\mathbb{R})$. First of all, note that $t \mapsto u(h_2(t))$ is pseudo almost periodic, by (**H.2**). Using (**H.3**) it follows that $s \mapsto Q(s, u(s), u(h_2(s)))$ is pseudo almost periodic, see, e.g., [3, 6].

Now write $Q = Q_1 + Q_2$ where $Q_1 \in AP(\mathbb{R} \times \mathbb{R}^2, \mathbb{R})$ and $Q_2 \in AP_0(\mathbb{R} \times \mathbb{R}^2, \mathbb{R})$. Consequently, $\Gamma u = \Gamma_1(u) + \Gamma_2(u)$ where

$$\Gamma_1 u(t) := \int_t^\infty Q_1(s, u(s), u(h_2(s)))C(t-s)ds$$

and

$$\Gamma_2 u(t) := \int_t^\infty Q_2(s, u(s), u(h_2(s)))C(t-s)ds.$$

To complete the proof, it remains to prove that $\Gamma_1 u \in AP(\mathbb{R})$ and $\Gamma_2 u \in AP_0(\mathbb{R})$.

Since $Q_1(\cdot, u(\cdot), u(h_2(\cdot))) \in AP(\mathbb{R})$, for each $\varepsilon > 0$, there exists $\delta > 0$ such that for all γ, there is $\tau \in [\gamma, \gamma + \delta]$ with

$$|Q_1(s+\tau, u(s+\tau), u(h_2(s+\tau))) - Q_1(s, u(s), u(h_2(s)))| < \frac{\varepsilon}{C_0} \tag{4}$$

for each $s \in \mathbb{R}$.

Note that $\Gamma_1 u(t+\tau)) = \displaystyle\int_t^\infty Q_1(r+\tau, u(r+\tau), u(h_2(r+\tau)))C(t-r)dr$, by setting $r = s - \tau$. Considering $\Gamma_1 u(t+\tau) - \Gamma_1 u(t)$ it easily follows that

$$|\Gamma_1 u(t+\tau) - \Gamma_1 u(t)| < \varepsilon, \quad \forall t \in \mathbb{R},$$

by Eq. (4) and (**H.4**), and hence $\Gamma_1(u) \in AP(\mathbb{R})$.

The next step consists of showing that $\Gamma_2 u \in AP_0(\mathbb{R})$. It is clear that $s \mapsto \Gamma_2(u)(s)$ is a bounded continuous function. Thus, it remains to show that

$$\lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} |\Gamma_2 u(t)|\, dt = 0.$$

Clearly,

$$\lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} |\Gamma_2 u(t)|\, dt \le I + J,$$

where

$$I := \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} dt \left(\int_t^r |Q_2(s, u(s), u(h_2(s)))| . |C(t-s)| ds \right), \quad \text{and}$$

$$J := \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} dt \int_r^\infty |Q_2(s, u(s), u(h_2(s)))| . |C(t-s)| ds.$$

To show that $I = J = 0$, we make use of the following arguments:

(**A0**) $\displaystyle\int_{-r}^{s} |C(t-s)| dt = \int_0^{r+s} |C(-v)| dv \le C_0$ for all $r + s \ge 0$;

(A1) $Q_2(\cdot, u(\cdot), u(h_2(\cdot))) \in AP_0(\mathbb{R})$;

(A2) $Q_2(\cdot, u(\cdot), u(h_2(\cdot))) \in L^1(\mathbb{R})$.

Indeed, by changing the order of integration we obtain:

$$\begin{aligned} I &= \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} |Q_2(s, u(s), u(h_2(s)))|\, ds \left(\int_{-r}^{s} |C(t-s)| dt \right) \\ &\leq \lim_{r\to\infty} \frac{C_0}{2r} \int_{-r}^{r} |Q_2(s, u(s), u(h_2(s)))|\, ds \\ &= 0, \end{aligned}$$

by **(A0)**-**(A1)**.

Similarly,

$$\begin{aligned} J &= \lim_{r\to\infty} \frac{1}{2r} \int_{r}^{\infty} |Q_2(s, u(s), u(h_2(s)))|\, ds \int_{-r}^{r} |C(t-s)| dt \\ &= \lim_{r\to\infty} \frac{1}{2r} \int_{r}^{\infty} |Q_2(s, u(s), u(h_2(s)))| ds \int_{s-r}^{s+r} |C(-v)| dv \\ &= \lim_{r\to\infty} \int_{r}^{\infty} |Q_2(s, u(s), u(h_2(s)))| \,.\, \phi_r(s)\, ds, \end{aligned}$$

where $\phi_r(s) = \dfrac{1}{2r} \displaystyle\int_{s-r}^{s+r} |C(-v)| dv$.

Clearly, $\phi_r(s) \leq \dfrac{C_0}{2r}$, by $\displaystyle\int_{s-r}^{s+r} |C(-v)| dv \leq C_0$ for all $s \geq r$. And hence $\phi_r(s) \mapsto 0$ as $r \mapsto \infty$. Since $Q_2(\cdot, u(\cdot)\ u(h_2(\cdot))) \in L^1(\mathbb{R})$ it follows that

$$\lim_{r\to\infty} \int_{r}^{\infty} |Q_2(s, u(s), u(h_2(s)))| \,.\, \phi_r(s)\, ds = 0,$$

by **(A.2)** and the Lebesgue dominated convergence theorem. And therefore $\Gamma_2 u \in AP_0(\mathbb{R})$. ∎

Theorem 3.1 *Under assumptions* **(H.1)**-**(H.2)**-**(H.3)**-**(H.4)**, *Eq. (1) has a unique pseudo almost periodic solution whenever* $\alpha + C_0 < 1$.

Proof. Let $u \in PAP(\mathbb{R})$. Define the nonlinear operator

$$\Lambda(u)(t) := f(u(h_1(t))) + \int_{t}^{\infty} Q(s, u(s) u(h_2(s))) C(t-s) ds + g(t), \quad t \in \mathbb{R}.$$

First of all, let us mention that $f(u(h_1(\cdot))) \in PAP(\mathbb{R})$, which follows immediately from the composition theorem of pseudo almost periodic functions in [3, 6]. Thus, in view of the previous facts and Lemma 3.1, it easily follows that Λ maps $PAP(\mathbb{R})$ into itself and that $\Gamma_1 u$ and $\Gamma_2 u$ are respectively the almost periodic and ergodic perturbation components of $\Lambda(u)$.

To complete the proof, we must show that $\Lambda : PAP(\mathbb{R}) \mapsto PAP(\mathbb{R})$ has a unique fixed-point.

For $u, v \in PAP(\mathbb{X})$,

$$|\Lambda(u)(t) - \Lambda(v)(t)|$$

$$\leq \alpha\|u - v\|_\infty + \int_t^\infty |(Q(s, u(s), u(h_2(s))) - Q(s, v(s), v(h_2(s))))| \, . \, |C(t-s)|ds$$

$$\leq \alpha\|u - v\|_\infty + \int_t^\infty [k|u(s) - v(s)| + (1-k)|u(h_2(s) - v(h_2(s))] \, . \, |C(t-s)|ds$$

$$\leq \alpha\|u - v\|_\infty + \|u - v\|_\infty \, . \int_t^\infty |C(t-s)|ds,$$

and hence

$$\|\Lambda(u) - \Lambda(v)\|_\infty \leq (\alpha + C_0) \, . \, \|u - v\|_\infty.$$

Therefore, by the Banach fixed-point principle, the operator Λ has a unique fixed point whenever $\alpha + C_0 < 1$, which obviously is the only pseudo almost periodic solution to Eq. (1).

■

Setting $h_1(t) = h_2(t) = t - p$, one can easily see that **(H.2)** holds, and hence the next corollary is a straightforward consequence of Theorem 3.1. (In assumption **(H.3)**, we suppose that the ergodic component Q_2 of Q is given such that $Q_2(\cdot, v(\cdot), v(\cdot - p)) \in L^1(\mathbb{R})$ for each $v \in PAP(\mathbb{R})$.)

Corollary 3.2 *Under assumptions* **(H.1)-(H.3)-(H.4)**, *Eq. (2) has a unique pseudo almost periodic solution whenever* $\alpha + C_0 < 1$.

The rest of this paper is devoted to the existence and uniqueness of pseudo almost periodic solutions to Eq. (3). In what follows we define the function

$$\tilde{q}(t, x, y) := q(t, x, y) - a\alpha y$$

for all $t \in \mathbb{R}$ and $x, y \in \mathbb{R}$, where a, α and q are respectively the constants and the function appearing in Eq. (3).

We require the following assumption:

(H.5) The function $\tilde{q} : \mathbb{R} \times (\mathbb{R} \times \mathbb{R}) \mapsto \mathbb{R}$, $(t, x, y) \mapsto \tilde{q}(t, x, y)$ is pseudo almost periodic in $t \in \mathbb{R}$ uniformly if $(x, y) \in \mathbb{R} \times \mathbb{R}$. Setting $\tilde{q} = q_1 + q_2$ where $q_1 \in AP(\mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and $q_2 \in AP_0(\mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, we suppose that $q_2(\cdot, v(\cdot), v(\cdot - p))) \in L^1(\mathbb{R})$ for each $v \in PAP(\mathbb{R})$. Furthermore, suppose that $\alpha, a > 0$, and

$$|q(t, x, y) - q(t, w, z)| \leq (1 - a\alpha)|x - w|$$

for all $t, x, y, z, w \in \mathbb{R}$.

Theorem 3.3 *Under assumption* **(H.5)**, *the logistic equation, Eq. (3), has a unique pseudo almost periodic solution whenever* $\alpha + \dfrac{1}{a} < 1$.

Proof. One follows along the same lines as in [4]. We are interested in bounded solutions only. Thus if u is a bounded solution to Eq. (3), then

$$\frac{d}{dt}[(u(t) - \alpha u(t-p))\, e^{-at}] = [a\alpha u(t-p) - q(t, u(t), u(t-p))]e^{-at}.$$

Clearly,

$$u(t) = \alpha u(t-p) + \int_t^{\infty} [q(s, u(s), u(s-p)) - a\alpha u(s-p)]e^{a(t-s)}ds, \qquad (5)$$

for each $t \in \mathbb{R}$, by $\lim_{t\to\infty}[(u(t) - \alpha u(t-p))\, e^{-at}] = 0$ (u is bounded).

To complete the proof, in Eq. (1), take $f(t) = \alpha t$, $h_1(t) = h_2(t) = t-p$, $C(t) = e^{at}$, $g(t) = 0$, and

$$Q(t, u(t), u(t-p)) = \tilde{q}(t, u(t), u(t-p)), \quad \forall t \in \mathbb{R},$$

and follow along the same lines as in the proof of Theorem 3.1. ∎

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