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Existence of asymptotically free solutions for quadratic nonlinear Schrödinger equations in 3d

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ABSTRACT

We study asymptotic behavior in time of solutions to nonlinear Schrödinger equations in three space dimensions with quadratic nongauge invariant nonlinearities. Due to the oscillation properties of such nonlinearities the time decay estimates are faster comparing with the case of gauge invariant nonlinearities. We prove the existence of solutions to the final problem and large time asymptotics.

RESUMEN

Estudiamos el comportamiento asintótico en el tiempo de las ecuaciones de Schrödinger en tres variables espaciales con no linealidades invariantes sin nivel (nongauge). Debido a las propiedades de oscilación de tales no linealidades, las estimaciones del decaimiento en el tiempo son más rápidas comparando el caso de no linealidades invariantes con nivel (gauge). Probamos la existencia de soluciones del problema final y estimaciones asintóticas de tiempo grande.

Key words and phrases: *Nonlinear Schrödinger equations, Large time asymptotics*
Math. Subj. Class.: *35Q55, 35B40*

1 Introduction

In this paper, we obtain the asymptotics in time of solutions to the nonlinear Schrödinger equations in three space dimensions

$$u(t) = u_1(t) + i\int_t^\infty \mathcal{U}(t-\tau)\mathcal{N}(u,\overline{u})(\tau)\,d\tau, (t,x) \in \mathbf{R}\times\mathbf{R}^3, \tag{1.1}$$

where $u_1(t) = \mathcal{U}(t)u_+$ is the solution of the free Schrödinger equation and $\mathcal{U}(t)$ is the free Schrödinger group defined by $\mathcal{U}(t) = e^{\frac{1}{2}it\Delta}$. The nonlinear term $\mathcal{N}(u,\overline{u})$ is a smooth function, satisfying the following conditions

$$\mathcal{N}(0,0) = \partial_u\mathcal{N}(0,0) = \partial_{\overline{u}}\mathcal{N}(0,0) = 0, \tag{1.2}$$

$$\partial_u^2\mathcal{N}(0,0) = \lambda_1,\ \partial_{\overline{u}}^2\mathcal{N}(0,0) = \lambda_2,\ \partial_u\partial_{\overline{u}}\mathcal{N}(0,0) = \lambda_3 \tag{1.3}$$

with $\lambda_1, \lambda_2, \lambda_3 \in \mathbf{C}$. Furthermore we assume that the higher order derivatives of the nonlinearity $\mathcal{N}(u,\overline{u})$ satisfy the estimates

$$\left|\partial_u^j\partial_{\overline{u}}^k\mathcal{N}(u,\overline{u})\right| \leq C\left(1 + |u|^{q-j-k}\right) \tag{1.4}$$

for all $u \in \mathbf{C}$, $j+k = 3,4,5,6$, where $q \geq 6$. Conditions (1.2)-(1.4) imply that the nonlinearity $\mathcal{N}(u,\overline{u})$ has a form

$$\mathcal{N}(u,\overline{u}) = \frac{\lambda_1}{2}u^2 + \frac{\lambda_2}{2}\overline{u}^2 + \lambda_3|u|^2 + \text{higher order terms}.$$

The integral equation (1.1) corresponds to the final problem for the nonlinear Schrödinger equation

$$\begin{cases} \mathcal{L}u = \mathcal{N}(u,\overline{u}), & (t,x) \in \mathbf{R}\times\mathbf{R}^3, \\ \mathcal{U}(-\infty)u(\infty) = u_+, \end{cases}$$

where $\mathcal{L} = i\partial_t + \frac{1}{2}\Delta$ is the linear Schrödinger operator. In [4], [5], [6], [7], [12], the sharp asymptotic behavior in time of the second approximation u_2 is studied in the case of the gauge invariant nonlinearities, namely $f\left(|u|^2\right)u$. In this paper we study the case of nongauge invariant nonlinearities. Due to the oscillation properties of such nonlinearities the time decay estimates are faster comparing with the case of gauge invariant nonlinearities.

We construct a wave operator in $\mathbf{L}^2$ to equation (1.1) for the final data $u_+ \in \mathbf{H}^3 \cap \mathbf{H}_1^3 \cap \mathbf{H}^{0,3}$ and show time decay estimates of solutions, where the weighted Sobolev spaces are defined by

$$\mathbf{H}_p^{m,s} = \left\{ u \in \mathcal{S}'; \|u\|_{\mathbf{H}_p^{m,s}} = \|\langle x\rangle^s \langle i\nabla\rangle^m u\|_{\mathbf{L}^p} < \infty \right\},$$

in addition $\mathbf{H}_p^{m,0} = \mathbf{H}_p^m$, $\mathbf{H}_2^{m,s} = \mathbf{H}^{m,s}$ and $\mathbf{H}^{m,0} = \mathbf{H}^m$. In order to state the results of the present paper we define the following function space

$$\mathbf{X}_T = \left\{ \phi \in \mathbf{C}\left([T,\infty);\mathbf{L}^2\left(\mathbf{R}^3\right)\right); \|\phi\|_{\mathbf{X}_T} < \infty \right\},$$

with the norm

$$\|\phi\|_{\mathbf{X}_T} = \sup_{t\in[T,\infty)} t^{\frac{3}{2}} \left\| \phi(t) - \sum_{j=1}^{4} u_j(t) \right\|_{\mathbf{H}^2}.$$

where $u_2(t)$, $u_3(t)$ and $u_4(t)$ are respectively the second, the third and the fourth approximations for solutions of (1.1) defined by

$$u_2(t) = i\int_t^\infty \mathcal{U}(t-\tau)\mathcal{N}(u_1,\overline{u_1})(\tau)\,d\tau,$$

$$u_3(t) = i\int_t^\infty \mathcal{U}(t-\tau)\left(\partial_u\mathcal{N}(u_1,\overline{u_1})u_2(\tau) + \partial_{\overline{u}}\mathcal{N}(u_1,\overline{u_1})\overline{u_2}(\tau)\right)d\tau$$

and

$$u_4(t) = i\int_t^\infty \mathcal{U}(t-\tau)\Big(\partial_u\mathcal{N}(u_1,\overline{u_1})u_3 + \partial_{\overline{u}}\mathcal{N}(u_1,\overline{u_1})\overline{u_3} + \frac{1}{2}\left(\partial_u^2\mathcal{N}(u_1,\overline{u_1})u_2^2 + 2\partial_u\partial_{\overline{u}}\mathcal{N}(u_1,\overline{u_1})|u_2|^2 + \partial_{\overline{u}}^2\mathcal{N}(u_1,\overline{u_1})\overline{u_2}^2\right)\Big)d\tau.$$

Our result in this paper is the following.

Theorem 1.1 *Let $u_+ \in \mathbf{H}^3 \cap \mathbf{H}_1^3 \cap \mathbf{H}^{0,3}$. Then for some time $T > 0$ there exists a unique solution $u \in \mathbf{X}_T$ of (1.1). Furthermore the time decay estimate*

$$\|u(t)\|_{\mathbf{L}^\infty} \le Ct^{-\frac{3}{2}}$$

is true for all $t > T$.

We organize our paper as follows. In the next section we prove some estimates for the approximate solutions $u_j(t)$. Section 3 is devoted to the proof of Theorem 1.1.

2 Preliminaries

Denote

$$u_{2,1}(t)=\frac{i}{2}\int_t^{\infty}\mathcal{U}(t-\tau)\left(\lambda_1 u_1^2(\tau)+\lambda_2\overline{u_1}^2(\tau)\right)d\tau.$$

By Lemma 2.3 from paper [2] we have

Lemma 2.1 *Let $u_+\in\mathbf{H}^3\cap\mathbf{H}_1^3\cap\mathbf{H}^{0,3}$. Then the estimate is true*

$$t^{\frac{5}{4}}\left\|u_{2,1}(t)\right\|_{\mathbf{H}^2}+t^{\frac{3}{2}}\left\|\nabla u_{2,1}(t)\right\|_{\mathbf{H}^2}\le C\rho^2\left(1+\rho^q\right)$$

for all $t>T>0$, where $\rho=\|u_+\|_{\mathbf{H}^3}+\|u_+\|_{\mathbf{H}_1^3}+\|u_+\|_{\mathbf{H}^{0,3}}$.

In the next lemma we estimate the second approximation

$$u_2(t)=i\int_t^{\infty}\mathcal{U}(t-\tau)\mathcal{N}(u_1,\overline{u_1})(\tau)\,d\tau.$$

Lemma 2.2 *Let $u_+\in\mathbf{H}^3\cap\mathbf{H}_1^3\cap\mathbf{H}^{0,3}$. Then the estimate is valid*

$$t^{\frac{1}{2}}\left\|u_2(t)\right\|_{\mathbf{H}^2}+t^{\frac{3}{2}}\left\|\nabla u_2(t)\right\|_{\mathbf{H}^2}\le C\rho^2\left(1+\rho^q\right)$$

for all $t>T>0$, where $\rho=\|u_+\|_{\mathbf{H}^3}+\|u_+\|_{\mathbf{H}_1^3}+\|u_+\|_{\mathbf{H}^{0,3}}$.

Proof. In view of conditions (1.2) - (1.4) by the Hölder and Sobolev inequalities we have

$$\begin{aligned}\left\|u_2(t)\right\|_{\mathbf{H}^2}&\le C\int_t^{\infty}\left\|\mathcal{N}(u_1,\overline{u_1})(\tau)\right\|_{\mathbf{H}^2}d\tau\\ &\le C\int_t^{\infty}\tau^{-\frac{3}{2}}\|u_+\|_{\mathbf{L}^1}\|u_+\|_{\mathbf{H}^2}\left(1+\tau^{-\frac{3}{2}q}\|u_+\|_{\mathbf{L}^1}^q\right)d\tau\\ &\le Ct^{-\frac{1}{2}}\|u_+\|_{\mathbf{L}^1}\|u_+\|_{\mathbf{H}^2}\left(1+\|u_+\|_{\mathbf{L}^1}^q\right).\end{aligned}$$

We write $u_2(t)$ as

$$\begin{aligned}u_2(t)&=i\int_t^{\infty}\mathcal{U}(t-\tau)\mathcal{N}(u_1,\overline{u_1})(\tau)\,d\tau\\ &=u_{2,1}(t)+u_{2,2}(t)+i\int_t^{\infty}\mathcal{U}(t-\tau)\mathcal{R}(u_1,\overline{u_1})(\tau)\,d\tau,\end{aligned}$$

where

$$u_{2,2}(t)=i\lambda_3\int_t^{\infty}\mathcal{U}(t-\tau)\left|u_1(\tau)\right|^2d\tau.$$

By using the identity $\nabla|u_1|^2=\frac{1}{it}\left(\overline{u_1}\mathcal{J}u_1-u_1\overline{\mathcal{J}u_1}\right)$ we have

$$\begin{aligned}\left\|\nabla u_{2,2}(t)\right\|_{\mathbf{H}^2}&\le C\int_t^{\infty}\left\|\nabla|u_1|^2\right\|_{\mathbf{H}^2}d\tau\\ &\le C\int_t^{\infty}\tau^{-1}\left\|\overline{u_1}\mathcal{J}u_1\right\|_{\mathbf{H}^2}d\tau\le C\rho^2\int_t^{\infty}\tau^{-\frac{5}{2}}d\tau\le C\rho^2t^{-\frac{3}{2}}.\end{aligned}\tag{2.1}$$

The assumptions (1.2) - (1.4) yield

$$\begin{aligned} \left\| \int_t^\infty \mathcal{U}(t-\tau)\mathcal{R}(u_1,\overline{u_1})(\tau)\,d\tau \right\|_{\mathbf{H}^3} &\le C\int_t^\infty \|\mathcal{R}(u_1,\overline{u_1})(\tau)\|_{\mathbf{H}^3}\,d\tau \\ \le\; C\int_t^\infty \left(1+\|u_1(\tau)\|_{\mathbf{L}^\infty}^q\right)\|u_1(\tau)\|_{\mathbf{L}^\infty}\|u_1(\tau)\|_{\mathbf{H}^1_\infty}\|u_1(\tau)\|_{\mathbf{H}^3}\,d\tau \\ \le\; Ct^{-2}\rho^2(1+\rho^q). \end{aligned} \tag{2.2}$$

By (2.1), (2.2) and Lemma 2.1 we get

$$\|\nabla u_2(t)\|_{\mathbf{H}^2} \le Ct^{-\frac{3}{2}}\rho^2(1+\rho^q).$$

Therefore the estimates of the lemma follow, and Lemma 2.2 is then proved. ∎

In the next lemma we estimate the third and the fourth approximations.

Lemma 2.3 *Let* $u_+ \in \mathbf{H}^3 \cap \mathbf{H}^3_1 \cap \mathbf{H}^{0,3}$. *Then*

$$\|u_3(t)\|_{\mathbf{H}^3} \le C\rho^3\left(1+\rho^{2q}\right)t^{-1}$$

and

$$\|u_4(t)\|_{\mathbf{H}^3} \le C\rho^4\left(1+\rho^{3q}\right)t^{-\frac{3}{2}},$$

where $\rho = \|u_+\|_{\mathbf{H}^3} + \|u_+\|_{\mathbf{H}^3_1} + \|u_+\|_{\mathbf{H}^{0,3}}$.

Proof. We have by Lemma 2.2

$$\begin{aligned} \|u_3(t)\|_{\mathbf{H}^3} &\le C\int_t^\infty \|\partial_u\mathcal{N}(u_1,\overline{u_1})u_2 + \partial_{\overline{u}}\mathcal{N}(u_1,\overline{u_1})\overline{u_2}\|_{\mathbf{H}^3}\,d\tau \\ \le\; C\int_t^\infty \left(1+\|u_1(\tau)\|_{\mathbf{H}^3_\infty}^q\right)\|u_2(\tau)\|_{\mathbf{H}^3}\|u_1(\tau)\|_{\mathbf{H}^3_\infty}\,d\tau \\ \le\; C\rho\int_t^\infty \|u_2(\tau)\|_{\mathbf{H}^3}\left(1+\tau^{-\frac{3}{2}q}\rho^q\right)\tau^{-\frac{3}{2}}d\tau \le C\rho^3\left(1+\rho^{2q}\right)t^{-1}. \end{aligned}$$

This implies the first estimate of the lemma. In the same way we have by Lemma 2.2

$$\begin{aligned} \|u_4(t)\|_{\mathbf{H}^3} &\le C\int_t^\infty \left(\|\partial_u\mathcal{N}(u_1,\overline{u_1})\|_{\mathbf{H}^3_\infty} + \|\partial_{\overline{u}}\mathcal{N}(u_1,\overline{u_1})\|_{\mathbf{H}^3_\infty}\right)\|u_3\|_{\mathbf{H}^3}\,d\tau \\ +C\int_t^\infty \left(\|\partial_u^2\mathcal{N}(u_1,\overline{u_1})\|_{\mathbf{H}^3_\infty} + \|\partial_{\overline{u}}^2\mathcal{N}(u_1,\overline{u_1})\|_{\mathbf{H}^3_\infty}\right)\|u_2^2\|_{\mathbf{H}^3}\,d\tau \\ \le\; C\rho^4t^{-\frac{3}{2}}\left(1+\rho^{3q}\right) + C\left(1+\rho^q\right)\int_t^\infty \|\nabla u_2(\tau)\|_{\mathbf{H}^2}^{\frac{3}{2}}\|u_2(\tau)\|_{\mathbf{H}^3}^{\frac{1}{2}}\,d\tau \\ \le\; C\rho^4\left(1+\rho^{3q}\right)t^{-\frac{3}{2}}. \end{aligned}$$

This yields the second estimate of the lemma and Lemma 2.3 is then proved. ∎

Now we find a better time decay for the third approximation

$$u_3(t) = i\int_t^\infty \mathcal{U}(t-\tau)\left(\partial_u\mathcal{N}(u_1,\overline{u_1})u_2(\tau) + \partial_{\overline{u}}\mathcal{N}(u_1,\overline{u_1})\overline{u_2}(\tau)\right)d\tau.$$

Lemma 2.4 *Let $u_+ \in \mathbf{H}^3 \cap \mathbf{H}^3_1 \cap \mathbf{H}^{0,3}$. Then*

$$\|\nabla u_3(t)\|_{\mathbf{H}^1_6} + \|u_3(t)\|_{\mathbf{L}^\infty} \le C\rho^3\left(1+\rho^{2q}\right)t^{-\frac{7}{4}},$$

where $\rho = \|u_+\|_{\mathbf{H}^3} + \|u_+\|_{\mathbf{H}^3_1} + \|u_+\|_{\mathbf{H}^{0,3}}$.

Proof. We represent

$$u_3(t) = \sum_{j=1}^{4} u_{3,j}(t),$$

where

$$\begin{aligned} u_{3,1}(t) &= \frac{i}{2}\int_t^\infty \mathcal{U}(t-\tau)\left(\lambda_1 u_1 u_{2,1} + \lambda_2 \overline{u_1 u_{2,1}}\right)d\tau \\ &\quad + i\lambda_3 \int_t^\infty \mathcal{U}(t-\tau)\left(\overline{u_1}u_{2,1} + u_1\overline{u_{2,1}}\right)d\tau, \end{aligned}$$

$$u_{3,2}(t) = i\int_t^\infty \mathcal{U}(t-\tau)\,u_1\left(\frac{\lambda_1}{2}u_{2,2} + \lambda_3\overline{u_{2,2}}\right)d\tau,$$

$$u_{3,3}(t) = i\int_t^\infty \mathcal{U}(t-\tau)\,\overline{u_1}\left(\frac{\lambda_2}{2}\overline{u_{2,2}} + \lambda_3 u_{2,2}\right)d\tau$$

and

$$u_{3,4}(t) = i\int_t^\infty \mathcal{U}(t-\tau)\left(\mathcal{R}_2(u_1,\overline{u_1})\,u_2 + \mathcal{R}_3(u_1,\overline{u_1})\,\overline{u_2}\right)d\tau.$$

We have by Lemma 2.2

$$\begin{aligned} \|u_{3,1}(t)\|_{\mathbf{H}^3} &\le C\int_t^\infty \|u_1(\tau)\|_{\mathbf{H}^3_\infty}\|u_{2,1}(\tau)\|_{\mathbf{H}^3}\,d\tau \\ &\le C\rho^3\left(1+\rho^q\right)t^{-\frac{7}{4}}. \end{aligned} \tag{2.3}$$

Also by assumptions (1.2) - (1.4) we get

$$\begin{aligned} \|u_{3,4}(t)\|_{\mathbf{H}^3} &\le \int_t^\infty \left(\|\mathcal{R}_2(u_1,\overline{u_1})\,u_2(\tau)\|_{\mathbf{H}^3} + \|\mathcal{R}_3(u_1,\overline{u_1})\,\overline{u_2}(\tau)\|_{\mathbf{H}^3}\right)d\tau \\ &\le C\rho^3\left(1+\rho^{2q}\right)t^{-\frac{5}{2}}. \end{aligned} \tag{2.4}$$

By the identity $\mathcal{U}(t)\,x\,\mathcal{U}(-t) = \mathcal{J} = x + it\nabla = itM\nabla\overline{M}$ with $M = e^{\frac{i|x|^2}{2t}}$ and Sobolev imbedding inequality we find

$$\begin{aligned} &\|u_{3,2}(t)\|_{\mathbf{H}^2_6} = \left\|\overline{M}u_{3,2}(t)\right\|_{\mathbf{L}^6} + \left\|\overline{M}\Delta u_{3,2}(t)\right\|_{\mathbf{L}^6} \\ \le\ & C\left\|\nabla\overline{M}u_{3,2}(t)\right\|_{\mathbf{L}^2} + C\left\|\nabla\overline{M}\Delta u_{3,2}(t)\right\|_{\mathbf{L}^2} \\ \le\ & Ct^{-1}\|\mathcal{J}u_{3,2}(t)\|_{\mathbf{H}^2} \le Ct^{-1}\int_t^\infty \left\|\mathcal{J}\left(\frac{\lambda_1}{2}u_1u_{2,2} + \lambda_3 u_1\overline{u_{2,2}}\right)\right\|_{\mathbf{H}^2} d\tau \\ \le\ & Ct^{-1}\int_t^\infty \left\|\left(\frac{\lambda_1}{2}u_{2,2} + \lambda_3\overline{u_{2,2}}\right)\mathcal{J}u_1\right\|_{\mathbf{H}^2} d\tau \\ & + Ct^{-1}\int_t^\infty \left\|u_1\left(\frac{\lambda_1}{2}\nabla u_{2,2} - \lambda_3\overline{\nabla u_{2,2}}\right)\right\|_{\mathbf{H}^2}\tau\,d\tau. \end{aligned}$$

Hence by applying Lemma 2.2

$$\begin{aligned}
\|u_{3,2}(t)\|_{\mathbf{H}_6^2} &\leq Ct^{-1}\int_t^\infty \left(\|u_{2,2}\mathcal{U}(\tau)xu_+\|_{\mathbf{H}^2} + \|\overline{u_{2,2}}\mathcal{U}(\tau)xu_+\|_{\mathbf{H}^2}\right)d\tau \\
&+Ct^{-1}\int_t^\infty \tau\|u_1\|_{\mathbf{H}_\infty^2}\|\nabla u_{2,2}\|_{\mathbf{H}^2}d\tau \\
&\leq Ct^{-1}\int_t^\infty \tau^{-\frac{1}{2}}\|u_+\|_{\mathbf{H}_1^2}\|\nabla u_2\|_{\mathbf{H}^2}d\tau \leq C\rho^3 t^{-1}\int_t^\infty \tau^{-2}d\tau \leq C\rho^3 t^{-2}. \quad (2.5)
\end{aligned}$$

We next consider $u_{3,3}(t)$. We have by Lemma 2.2

$$\begin{aligned}
\|\Delta u_{3,3}(t)\|_{\mathbf{H}^1} &\leq C\int_t^\infty \|u_1(\tau)\|_{\mathbf{H}_\infty^2}\|\nabla u_{2,2}\|_{\mathbf{H}^2}d\tau \\
&+\left\|\int_t^\infty \mathcal{U}(-\tau)\left(\lambda_3\overline{\Delta u_1}u_{2,2}+\frac{\lambda_2}{2}\overline{\Delta u_1 \overline{u_{2,2}}}\right)d\tau\right\|_{\mathbf{H}^1} \\
&\leq C\rho^3 t^{-2}+\left\|\int_t^\infty \mathcal{U}(-\tau)\left(\lambda_3\overline{\Delta u_1}u_{2,2}+\frac{\lambda_2}{2}\overline{\Delta u_1 \overline{u_{2,2}}}\right)d\tau\right\|_{\mathbf{H}^1}. \quad (2.6)
\end{aligned}$$

Using the identity $\overline{\Delta u_1} = 2i\partial_\tau \overline{u_1}$ and integrating by parts with respect to τ we obtain

$$\begin{aligned}
&\int_t^\infty \mathcal{U}(-\tau)\left(\lambda_3 u_{2,2}+\frac{\lambda_2}{2}\overline{u_{2,2}}\right)\overline{\Delta u_1}d\tau \\
=\; &-\mathcal{U}(-t)\left(2i\lambda_3 u_{2,2}+i\lambda_2\overline{u_{2,2}}\right)\overline{u_1} \\
&-\int_t^\infty \mathcal{U}(-\tau)\left(\lambda_3\Delta\left(\overline{u_1}u_{2,2}\right)+\frac{\lambda_2}{2}\Delta\left(\overline{u_1 u_{2,2}}\right)\right)d\tau \\
&-\int_t^\infty \mathcal{U}(-\tau)\left(2i\lambda_3\overline{u_1}\partial_\tau u_{2,2}+i\lambda_2\overline{u_1}\overline{\partial_\tau u_{2,2}}\right)d\tau.
\end{aligned}$$

Hence

$$\begin{aligned}
&2\int_t^\infty \mathcal{U}(-\tau)\left(\lambda_3\overline{\Delta u_1}u_{2,2}+\frac{\lambda_2}{2}\overline{\Delta u_1 \overline{u_{2,2}}}\right)d\tau \\
=\; &-\mathcal{U}(-t)\left(2i\lambda_3 u_{2,2}+i\lambda_2\overline{u_{2,2}}\right)\overline{u_1} \\
&-2\int_t^\infty \mathcal{U}(-\tau)\left(\lambda_3\left(\nabla\overline{u_1}\nabla u_{2,2}\right)+\frac{\lambda_2}{2}\left(\overline{\nabla u_1 \nabla u_{2,2}}\right)\right)d\tau \\
&-\int_t^\infty \mathcal{U}(-\tau)\left(\lambda_3\left(\overline{u_1}\Delta u_{2,2}\right)+\frac{\lambda_2}{2}\left(\overline{u_1}\overline{\Delta u_{2,2}}\right)\right)d\tau \\
&-\int_t^\infty \mathcal{U}(-\tau)\left(2i\lambda_3\overline{u_1}\partial_\tau u_{2,2}+i\lambda_2\overline{u_1}\overline{\partial_\tau u_{2,2}}\right)d\tau.
\end{aligned}$$

Then employing the identity $\partial_t u_{2,2}(t) = \frac{i}{2}\Delta u_{2,2}(t) - i\lambda_3 |u_1|^2$ we find

$$\left\| \int_t^\infty \mathcal{U}(-\tau)\left(\lambda_3 \overline{\Delta u_1} u_{2,2} + \frac{\lambda_2}{2}\overline{\Delta u_1 \overline{u_{2,2}}}\right) d\tau \right\|_{\mathbf{H}^1}$$
$$\leq C\|u_1(t)\|_{\mathbf{H}^1_\infty}\|u_{2,2}(t)\|_{\mathbf{H}^1} + C\int_t^\infty \|u_1(\tau)\|_{\mathbf{H}^2_\infty}\|\nabla u_{2,2}(\tau)\|_{\mathbf{H}^2}$$
$$+C\int_t^\infty \|u_1(\tau)\|^2_{\mathbf{H}^1_\infty}\|u_1(\tau)\|_{\mathbf{H}^1} \leq C\rho^3 t^{-2}$$

from which and (2.6) it follows that

$$\|\Delta u_{3,3}(t)\|_{\mathbf{H}^1} \leq C\rho^3 t^{-2}. \tag{2.7}$$

We also have

$$\begin{aligned} \|u_{3,3}(t)\|_{\mathbf{L}^2} &\leq \left\| \int_t^\infty \mathcal{U}(-\tau)\left(\lambda_3 \overline{u_1} u_{2,2} + \lambda_2 \overline{u_1 \overline{u_{2,2}}}\right) d\tau \right\|_{\mathbf{L}^2} \\ &\leq C\rho^3 t^{-1}. \end{aligned} \tag{2.8}$$

By (2.3), (2.4), (2.5), (2.7) and (2.8) we get

$$\begin{aligned} \|\nabla u_3(t)\|_{\mathbf{H}^1_6} &\leq C\left(\|u_{3,1}(t)\|_{\mathbf{H}^3} + \|u_{3,4}(t)\|_{\mathbf{H}^3} + \|u_{3,2}(t)\|_{\mathbf{H}^2_6}\right. \\ \left. + \|\Delta u_{3,3}(t)\|_{\mathbf{H}^1}\right) &\leq C\rho^3\left(1+\rho^{2q}\right) t^{-\frac{7}{4}} \end{aligned}$$

and

$$\|u_3(t)\|_{\mathbf{L}^\infty} \leq C\left(\|u_{3,1}(t)\|_{\mathbf{H}^2} + \|u_{3,4}(t)\|_{\mathbf{H}^2} + \|u_{3,2}(t)\|_{\mathbf{H}^2_6}\right.$$
$$\left. + \|u_{3,3}(t)\|^{\frac{1}{4}}_{\mathbf{L}^2}\|\Delta u_{3,3}(t)\|^{\frac{3}{4}}_{\mathbf{L}^2}\right) \leq C\rho^3\left(1+\rho^{2q}\right) t^{-\frac{7}{4}}.$$

This completes the proof of the lemma. ∎

3 Proof of Theorem 1.1

To apply the contraction mapping principle we consider the linearized problem associated with (1.1)

$$u(t) = u_1(t) + i\int_t^\infty \mathcal{U}(t-\tau)\mathcal{N}(v,\overline{v})(\tau)\,d\tau \tag{3.1}$$

in the closed ball of a radius $2\rho^3$ in $\mathbf{X}_T$, where $v \in \mathbf{X}_T$, $\|v\|_{\mathbf{X}_T} \leq 2\rho^3$, where $\rho = \|u_+\|_{\mathbf{H}^3} + \|u_+\|_{\mathbf{H}^3_1} + \|u_+\|_{\mathbf{H}^{0,3}}$. Denote $w(t) = u_1(t) + u_2(t) + u_3(t) + u_4(t)$. Then from (3.1) we obtain for the difference $u(t) - w(t)$

$$u(t) - w(t) = i\int_t^\infty \mathcal{U}(t-\tau)\left(\mathcal{N}(v,\overline{v}) - \mathcal{N}(u_1,\overline{u_1})\right.$$
$$-\left(\partial_u\mathcal{N}(u_1,\overline{u_1})(u_2+u_3) + \partial_{\overline{u}}\mathcal{N}(u_1,\overline{u_1})(\overline{u_2}+\overline{u_3})\right)$$
$$\left.-\frac{1}{2}\left(\partial_u^2\mathcal{N}(u_1,\overline{u_1})u_2^2 + 2\partial_u\partial_{\overline{u}}\mathcal{N}(u_1,\overline{u_1})|u_2|^2 + \partial_{\overline{u}}^2\mathcal{N}(u_1,\overline{u_1})\overline{u_2}^2\right)\right) d\tau.$$

By the Taylor expansion we have

$$\begin{aligned}
&\mathcal{N}(v,\overline{v})-\mathcal{N}(u_1,\overline{u_1})-\partial_u\mathcal{N}(u_1,\overline{u_1})(u_2+u_3)+\partial_{\overline{u}}\mathcal{N}(u_1,\overline{u_1})(\overline{u_2}+\overline{u_3})\\
&-\frac{1}{2}\left(\partial_u^2\mathcal{N}(u_1,\overline{u_1})u_2^2+2\partial_u\partial_{\overline{u}}\mathcal{N}(u_1,\overline{u_1})|u_2|^2+\partial_{\overline{u}}^2\mathcal{N}(u_1,\overline{u_1})\overline{u_2}^2\right)\\
=\;&(1+|u_1|^q)\left(O(r^2)+O(rw)+O(u_1u_4)+O((u_2+u_3+u_4)(u_3+u_4))\right),
\end{aligned}$$

where $r=v(t)-w(t)$. Then by applying the Hölder inequality and estimates of Lemmas 2.2 - 2.4 it follows that

$$\begin{aligned}
&\|u(t)-w(t)\|_{\mathbf{H}^2}\le C(1+\rho^q)\int_t^\infty d\tau\left(\|r\|_{\mathbf{H}^2}^2\right.\\
&+\|r\|_{\mathbf{H}^2}\left(\|u_1\|_{\mathbf{H}^2_\infty}+\|u_2\|_{\mathbf{L}^\infty}+\|\nabla u_2\|_{\mathbf{H}^1}\right.\\
&\left.+\|\nabla u_3\|_{\mathbf{H}^1_6}+\|u_3\|_{\mathbf{L}^\infty}+\|u_4\|_{\mathbf{H}^2}\right)+\|u_1\|_{\mathbf{H}^2_\infty}\|u_4\|_{\mathbf{H}^2}\\
&+\left(\|u_2\|_{\mathbf{L}^6}+\|\nabla u_2\|_{\mathbf{H}^1}+\|\nabla u_3\|_{\mathbf{H}^1_6}+\|u_3\|_{\mathbf{L}^\infty}+\|u_4\|_{\mathbf{H}^2}\right)\\
&\left.\times\left(\|u_3\|_{\mathbf{L}^3}+\|\nabla u_3\|_{\mathbf{H}^1_6}+\|u_3\|_{\mathbf{L}^\infty}+\|u_4\|_{\mathbf{H}^2}\right)\right)\\
\le\;&C\rho^4\left(1+\rho^{6q+4}\right)t^{-\frac{7}{4}}.
\end{aligned}$$

Therefore there exists a sufficiently large time T such that

$$\sup_{t\in[T,\infty)}t^{\frac{3}{2}}\|u(t)-w(t)\|_{\mathbf{H}^2}\le C\rho^4\left(1+\rho^{6q+4}\right)T^{-\frac{1}{4}}\le 2\rho^3. \tag{3.2}$$

We let

$$u^{(j)}=u_1(t)+i\int_t^\infty\mathcal{U}(t-\tau)\mathcal{N}\left(v^{(j)},\overline{v^{(j)}}\right)(\tau)\,d\tau,$$

for $v^{(j)}\in\mathbf{X}_T$. Then we have

$$\left\|v^{(j)}\right\|_{\mathbf{L}^\infty}\le C\left\|v^{(j)}-w\right\|_{\mathbf{L}^\infty}+\sum_{j=1}^4\|u_j\|_{\mathbf{L}^\infty}\le C\rho\left(1+\rho^{3q+3}\right)t^{-\frac{3}{2}} \tag{3.3}$$

and

$$\left\|v^{(j)}\right\|_{\mathbf{H}^2}\le C\left\|v^{(j)}-w\right\|_{\mathbf{H}^2}+\sum_{j=1}^4\|u_j\|_{\mathbf{H}^2}\le C\rho\left(1+\rho^{3q+3}\right). \tag{3.4}$$

Hence

$$\begin{aligned}
&\left\|u^{(1)}(t)-u^{(2)}(t)\right\|_{\mathbf{L}^2}\\
\le\;&C\int_t^\infty\left\|\mathcal{N}\left(v^{(1)},\overline{v^{(1)}}\right)(\tau)-\mathcal{N}\left(v^{(2)},\overline{v^{(2)}}\right)(\tau)\right\|_{\mathbf{L}^2}d\tau\\
\le\;&C\rho\left(1+\rho^{3q+3}\right)t^{-\frac{1}{2}}\left(\sup_{t\in[T,\infty)}\left\|v^{(1)}(t)-v^{(2)}(t)\right\|_{\mathbf{L}^2}\right)
\end{aligned}$$

from which it follows that there exists a large $T > 0$ such that

$$\sup_{t\in[T,\infty)} \left\| u^{(1)}(t) - u^{(2)}(t) \right\|_{\mathbf{L}^2} \leq \frac{1}{2} \left(\sup_{t\in[T,\infty)} \left\| v^{(1)}(t) - v^{(2)}(t) \right\|_{\mathbf{L}^2} \right) \tag{3.5}$$

By (3.2) and (3.5) applying the contraction mapping principle we see that there exists a unique solution $u \in \mathbf{X}_T$ of (1.1). Time decay estimate follows from (3.3). This completes the proof of the theorem. ∎

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References

[1] N. Hayashi, T. Mizumachi and P. I. Naumkin, *Time decay of small solutions to quadratic nonlinear Schrödinger equations in 3D,* Differential Integral Equations **16** (2003), 159-179.

[2] N. Hayashi, Y.Kawahara and P. I. Naumkin, *Lower bounds of asymptotics in time of solutions to Nonlinear Schrödinger Equations in 3d* , Nonlinear Anal. **65** (2006), N° 7, 1394–1410.

[3] Y.Kawahara, *Global existence and asymptotic behavior of small solutions to nonlinear Schrödinger equations in 3D,* Differential Integral Equations **18** (2005), 169-194.

[4] N. Kita, *Sharp L^r asymptotics of the small solutions to the nonlinear Schrödinger equations,* Nonlinear Anal. **52** (2003), 1365-1377.

[5] N. Kita and T. Ozawa, *Sharp asymptotic behavior of solutions to nonlinear Schrödinger equations with repulsive interactions,* Commun. Contemp. Math. **7** (2005) , N° 2, 167–176.

[6] N. Kita and T. Wada, *Sharp asymptotic behavior of solutions to nonlinear Schrödinger equations in one space dimension,* Funkcial. Ekvac. **45** (2002), 53-69.

[7] N. Kita and T. Wada, *Sharp asymptotics of the small solutions to the nonlinear Schrödinger equations of derivative type,* Differential Integral Equations **15** (2002), 367-384.

[8] S. Klainerman and G. Ponce, *Global, small amplitude solutions to nonlinear evolution equations,* Comm. Pure Appl. Math. **36** (1983), 133-141.

[9] K. Nakanishi and T. Ozawa, *Remarks on scattering for nonlinear Schrödinger equations*, NoDEA Nonlinear Differential Equations Appl. **9** (2002), 45-68.

[10] J. Shatah, *Global existence of small solutions to nonlinear evolution equations*, J. Differential Equations **46** (1982), 409-425.

[11] W. A. Strauss, *Nonlinear scattering thory at low energy*, J. Funct. Anal. **41** (1981), 110-133.

[12] T. Wada, *Asymptotic expansion of the solution to the nonlinear Schrödinger equation with nonlocal interaction*, J. Funct. Anal. **180** (2001), 11-30.