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$\aleph_n$-free abelian group with no non-zero homomorphism to $\mathbb{Z}$

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ABSTRACT

We, for any natural n, construct an $\aleph_n$-free abelian groups which have few homomorphisms to $\mathbb{Z}$. For this we use "$\aleph_n$-free $(n+1)$-dimensional black boxes". The method is hopefully relevant to other constructions of $\aleph_n$-free abelian groups.

RESUMEN

Para cualquier natural n, contruimos un grupo abeliano libre $\aleph_n$ el cual tiene pocos homomorfismos hacia $\mathbb{Z}$. Para esto usamos $\aleph_n$ cajas negras libres $(n+i)$-dimensionales. El método es relevante para otras construcciones de grupos abelianos $\aleph_n$-libres.

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Annotated Content

0 Introduction

For regular $\theta = \aleph_n$ we look for a θ-free abelian group G with $\mathrm{Hom}(G, \mathbb{Z}) = \{0\}$. We first construct G and a pure subgroup $\mathbb{Z}z \subseteq G$ which is not a direct summand. If instead "not direct product" we ask "not free" so naturally of cardinality θ, we know much: see [1].

We can ask further questions on abelian groups, their endormorphism rings, similarly on modules; naturally questions whose answer is known when we demand $\aleph_1$-free instead $\aleph_n$-free; see [2] . But we feel those two cases can serve as a base for significant number of such problems (or we can immitate the proofs). Also this concentration is reasonable for sorting out the set theoretical situation. Why not $\theta = \aleph_\omega$ and higher cardinals? (there are more reasonable cardinals for which such results are not excluded), we do not fully know: note that also in previous questions historically this was harder.

Note that there is such an abelian group of cardinality $\aleph_1$, by [7, §4] and see more in Göbel-Shelah-Struüngman [3]. However, if MA then $\aleph_2 < 2^{\aleph_0} \Rightarrow$ any $\aleph_2$-free abelian group of cardinality $< 2^{\aleph_0}$ fail the question.

The groups we construct are in a sense complete, like ${}^\omega\mathbb{Z}$. They are close to the ones from [5, §5] but there $S = \{0, 1\}$ as there we are interested in Borel abelian groups. See earlier [8], see representations of [8] in [10, §3], [1].

However we still like to have $\theta = \aleph_\omega$, i.e. $\aleph_\omega$-free abelian groups. Concerning this we continue in [11].

We shall use freely the well known theorem saying

Theorem 0.1 A subgroup of a free abelian group is a free abelian group.

Definition 0.2 1) $\mathrm{Pr}(\lambda, \kappa)$: means that for some $\bar{G}$ we have:

(a) $\bar{G} = \langle G_\alpha : \alpha \leq \kappa + 1 \rangle$

(b) $\bar{G}$ is an increasing continuous sequence of free abelian groups

(c) $|G_{\kappa+1}| \leq \lambda$,

(d) $G_{\kappa+1}/G_\alpha$ is free for $\alpha < \kappa$,

(e) $G_0 = \{0\}$

(f) G_β/G_α is free if $\alpha \leq \beta \leq \kappa$

(g) some $h \in \mathrm{Hom}(G_\kappa; \mathbb{Z})$ cannot be extended to $\hat{h} \in \mathrm{Hom}(G_{\kappa+1}, \mathbb{Z})$.

2) We let $\mathrm{Pr}^-(\lambda, \theta, \kappa)$ be defined as above, only replacing "$G_{\kappa+1}/G_\alpha$ is free for $\alpha < \kappa$" by "$G_{\kappa+1}/G_\kappa$ is θ-free.

1 Constructing $\aleph_{k(*)+1}$-free abelian groups

Definition 1.1 1) We say $\mathbf{x}$ is a combinatorial parameter if $\mathbf{x} = (k, S, \Lambda) = (k^{\mathbf{x}}, S^{\mathbf{x}}, \Lambda^{\mathbf{x}})$ and they satisfy clauses (a)-(c)

(*a*) $k < \omega$

(*b*) S is a set (in [5], $S = \{0, 1\}$),

(*c*) $\Lambda \subseteq {}^{k+1}({}^{\omega}S)$ and for simplicity $|\Lambda| \geq \aleph_0$ if not said otherwise.

1A) We say $\mathbf{x}$ is an abelian group k-parameter when $\mathbf{x} = (k, S, \Lambda, \mathbf{a})$ such that (a),(b),(c) from part (1) and:

(*d*) $\mathbf{a}$ is a function from $\Lambda \times \omega$ to $\mathbb{Z}$.

2) Let $\mathbf{x} = (k^{\mathbf{x}}, S^{\mathbf{x}}, \Lambda^{\mathbf{x}})$ or $\mathbf{x} = (k^{\mathbf{x}}, S^{\mathbf{x}}, \Lambda^{\mathbf{x}}, \mathbf{a}^{\mathbf{x}})$. A parameter is a k-parameter for some k and $K^{\rm cb}_{k(*)}/K^{\rm gr}_{k(*)}$ is the class of combinatorial/abelian group $k(*)$-parameters.
3) We may write $\mathbf{a}^{\mathbf{x}}_{\bar\eta,n}$ instead $\mathbf{a}^{\mathbf{x}}(\eta, n)$. Let $w_{k,m} = w(k, m) = \{\ell \leq k : \ell \neq m\}$.
4) We say $\mathbf{x}$ is full when $\Lambda^{\mathbf{x}} = {}^{k(*)}({}^{\omega}S)$.
5) If $\Lambda \subseteq \Lambda^{\mathbf{x}}$ let $\mathbf{x} \restriction \Lambda$ be $(k^{\mathbf{x}}, S^{\mathbf{x}}, \Lambda)$ or $(k^{\mathbf{x}}, S^{\mathbf{x}}, \Lambda, \mathbf{a} \restriction (\Lambda \times \omega))$ as suitable. We may write $\mathbf{x} = (\mathbf{y}, \mathbf{a})$ if $\mathbf{a} = \mathbf{a}^{\mathbf{x}}, \mathbf{y} = (k^{\mathbf{x}}, S^{\mathbf{x}}, \Lambda^{\mathbf{x}})$.

Convention 1.2 If $\mathbf{x}$ is clear from the context we may write k or $k(*), S, \Lambda, \mathbf{a}$ instead of $k^{\mathbf{x}}, S^{\mathbf{s}}, \Lambda^{\mathbf{x}}, \mathbf{a}^{\mathbf{x}}$.

A variant of the above is

Definition 1.3 1) For $\bar S = \langle S_n : m \leq k \rangle$ we define when $\mathbf{x}$ is a $\bar S$-parameter: $\bar\eta \in \Lambda^{\mathbf{x}} \wedge m \leq k^{\mathbf{x}} \Rightarrow \eta_m \in {}^{\omega}(S_m)$.
2) We say $\bar\alpha$ is a $(\mathbf{x}, \bar\chi)$-black box or $\mathrm{Qr}(\mathbf{x}, \bar\chi)$ when:

(a) $\bar\chi = \langle \chi_m : m \leq k^{\mathbf{x}} \rangle$

(b) $\bar\alpha = \langle \bar\alpha_{\bar\eta} : \bar\eta \in \Lambda^{\mathbf{x}} \rangle$

(c) $\bar\alpha_{\bar\eta} = \langle \alpha_{\bar\eta,m,n} : m \leq k^{\mathbf{x}}, n < \omega \rangle$ and $\alpha_{\bar\eta,m,n} < \chi_m$

(d) if $h_m : \Lambda^{\mathbf{x}}_m \to \chi_m$ for $m \leq k^{\mathbf{x}}$ <u>then</u> for some $\bar\eta \in \Lambda^{\mathbf{x}}$ we have: $m \leq k^{\mathbf{x}} \wedge n < \omega \Rightarrow h_m(\bar\eta \upharpoonright \langle m, n \rangle) = \alpha_{\bar\eta,m,n}$, see Definition 1.4(a) below on "$\bar\eta \upharpoonright \langle m, n \rangle$ and $\Lambda^{\mathbf{x}}_m$.

2A) We may replace $\bar\chi$ by χ if $\bar\chi = \langle \chi_\ell : \ell \leq k^{\mathbf{x}} \rangle$. We may replace $\mathbf{x}$ by $\Lambda^{\mathbf{x}}$ (so say $\mathrm{Qr}(\Lambda^{\mathbf{x}}, \bar\chi)$ or say $\bar\alpha$ is a $(\Lambda, \bar\chi)$-black box).
3) We say a parameter $\mathbf{x}$ is $\bar S$-full when $\Lambda^{\mathbf{x}} = \prod_{m \leq k} {}^{\omega}(S_m)$.

Definition 1.4 For an $k(*)$-parameter $\mathbf{x}$ and for $m \leq k(*)$ let

(a) $\Lambda^{\mathbf{x}}_m = \Lambda_{\mathbf{x},m} = \{\bar\eta : \bar\eta = \langle \eta_\ell : \ell \leq k(*)\rangle$ and $\eta_m \in {}^{\omega>}S$ and $\ell \leq k(*) \wedge \ell \neq m \Rightarrow \eta_\ell \in {}^\omega S$ and for some $\bar\eta' \in \Lambda$ we have $n < \omega, \bar\eta = \bar\eta' \upharpoonright \langle m,n\rangle\}$ where $\bar\eta = \bar\eta' \upharpoonright \langle m,n\rangle$ means $\eta_m = \eta'_m \restriction n$ and $\ell \leq k(*) \wedge \ell \neq m \Rightarrow \eta_\ell = \eta'_\ell\}$

(b) $\Lambda^{\mathbf{x}}_{\leq k(*)}$ is $\cup\{\Lambda^{\mathbf{x}}_m : m \leq k(*)\}$

(c) $m(\bar\eta) = m$ if $\bar\eta \in \Lambda^{\mathbf{x}}_m$.

Definition 1.5 1) We say a combinatorial $k(*)$-parameter $\mathbf{x}$ is free when there is a list $\langle \bar\eta^\alpha : \alpha < \alpha(*)\rangle$ of $\Lambda^{\mathbf{x}}$ such that for every α for some $m \leq k(*)$ and some $n < \omega$ we have

$(*)$ $\bar\eta^\alpha_m \upharpoonright \langle m,n\rangle \notin \{\eta^\beta_m \upharpoonright \langle m,n\rangle : \beta < \alpha\}$.

2) We say a combinatorial k-parameter $\mathbf{x}$ is θ-free when $\mathbf{x} \restriction \Lambda = (k, S^{\mathbf{x}}, \Lambda)$ is free for every $\Lambda \subseteq \Lambda^{\mathbf{x}}$ of cardinality $< \theta$.

Remark 1) We can require in $(*)$ even $(\exists^\infty n)[\eta^\alpha_m(n) \notin \cup\{\eta^\beta_\ell(n') : \ell \leq k, \beta < \alpha, n' < \omega\}]$.

At present this seems an immaterial change.

Definition 1.6 For $k(*) < \omega$ and an abelian group $k(*)$-parameter $\mathbf{x}$ we define an abelian group $G = G_{\mathbf{x}}$ as follows: it is generated by $\{x_{\bar\eta} : m \leq k(*)$ and $\bar\eta \in \Lambda^{\mathbf{x}}_m\} \cup \{y_{\bar\eta,n} : n < \omega$ and $\bar\eta \in \Lambda^{\mathbf{x}}\} \cup \{z\}$ freely except the equations:

$$\boxtimes_{\bar\eta,n} \quad (n!)y_{\bar\eta,n+1} = y_{\bar\eta,n} + \mathbf{a}^{\mathbf{x}}_{\bar\eta,n} z + \sum\{x_{\bar\eta \upharpoonright <m,n>} : m \leq k(*)\}.$$

Explanation 1.7 A canonical example of a non-free group is $(\mathbb{Q},+)$. Other examples are related to it after we divide by something. The y's here play the role of provided (hidden) copies of $\mathbb{Q}$. What about x's? For $\bar\eta \in \Lambda$ we consider $\langle y_{\bar\eta,n} : n < \omega\rangle$, as a candidate to represent $(\mathbb{Q},+), k(*)+1$ "chances", "opportunities" to avoid having $(\mathbb{Q},+)$ as a quotient, say by dividind K by a subgroup generated by some of the x's. This is used to prove $G_{\mathbf{x}}$ is not free even not $\aleph_{n+1}$-free which is necessary. But for each $m \leq k(*)$ if $\langle x_{\bar\eta \upharpoonright (m,n)} : n < \omega\rangle$ are not in K, or at least $x_{\eta \upharpoonright (m,n)}$ for n large enough then $\mathbb{Q}$ is not represented using $\langle y_{\bar\eta,n} : n < \omega\rangle$; so we have $k(*)+1$ "ways", "chances", "opportunities" to avoid having $\langle y_{\bar\eta,n} : n < \omega\rangle$ represents $(\mathbb{Q},+)$ in the quotient, one for each infinite cardinal $\leq \aleph_{k(*)}$. This helps us prove $\aleph_{k(*)}$-freeness. More specifically, for each $m(*) \leq k(*)$ if $H \subseteq G$ is the subgroup which is generated by $X = \{x_{\bar\eta \upharpoonright <m,n>} : m \neq m(*)$ and $\bar\eta \in {}^{k(*)+1}({}^\omega S)$ and $m \leq k(*)\}$, still in G/H the set $\{y_{\bar\eta,n} : n < \omega\}$ does not generate a copy of $\mathbb{Q}$, as witnessed by $\{x_{\bar\eta \upharpoonright <m(*),n>} : n < \omega\}$.

As a warm up we note:

Claim 1.8 *For $k(*) < \omega$ and $k(*)$-parameter* $\mathbf{x}$ *the abelian group $G_{\mathbf{x}}$ is an $\aleph_1$-free abelian group.*

Now systematically

Definition 1.9 Let $\mathbf{x}$ be a $k(*)$-parameter.
1) For $U \subseteq {}^\omega S$ let $G_U = G_U^{\mathbf{x}}$ be the subgroup of G generated by $Y_U = Y_U^{\mathbf{x}} = \{z\} \cup \{y_{\bar\eta,n} : \bar\eta \in \Lambda \cap {}^{k(*)+1}(U)$ and $n < \omega\} \cup \{x_{\bar\eta 1<m,n>} : m \leq k(*)$ and $\bar\eta \in \Lambda \cap {}^{(k(*)+1)}(U)$ and $n < \omega\}$. Let $G_U^+ = G_U^{\mathbf{x},+}$ be the divisible hull of G_U and $G^+ = G^+_{({}^\omega S)}$.
2) For $U \subseteq {}^\omega S$ and finite $u \subseteq {}^\omega S$ let $G_{U,u}$ be the subgroup [2] of G generated by $\cup\{G_{U \cup (u \setminus \{\eta\})} : \eta \in u\}$; and for $\bar\eta \in {}^{k(*) \geq} U$ let $G_{U,\bar\eta}$ be the subgroup of G generated by $\cup\{G_{U \cup \{\eta_k : k < \ell g(\bar\eta) \text{ and } k \neq \ell\}} : \ell < \ell g(\bar\eta)\}$.
3) For $U \subseteq {}^\omega S$ let $\Xi_U = \Xi_U^{\mathbf{x}} = \{$the equation $\boxtimes_{\bar\eta,n} : \bar\eta \in \Lambda \cap {}^{k(*)+1}U$ and $n < \omega\}$. Let $\Xi_{U,u} = \Xi^{\mathbf{x}}_{U,u} = \cup\{\Xi_{U \cup (u \setminus \{\beta\})} : \beta \in u\}$.

Claim 1.10 *Let $\mathbf{x} \in K_{k(*)}$.*
0) If $U_1 \subseteq U_2 \subseteq {}^\omega S$ __then__ $G^+_{U_1} \subseteq G^+_{U_2} \subseteq G^+$.
1) For any $n(*) < \omega$, the abelian group G_U^+ (which is a vector space over $\mathbb{Q}$), has the basis $Y_U^{n(*)} := \{z\} \cup \{y_{\bar\eta,n(*)} : \bar\eta \in \Lambda \cap {}^{k(*)+1}(U)\} \cup \{x_{\bar\eta 1<m,n>} : m \leq k(*), \bar\eta \in \Lambda \cap {}^{k(*)+1}(U)$ and $n < \omega\}$.
2) For $U \subseteq {}^\omega S$ the abelian group G_U is generated by Y_U freely (as an abelian group) except the set Ξ_U of equations.
3) If $m(*) < \omega$ and $U_m \subseteq {}^\omega S$ for $m < m(*)$ __then__ the subgroup $G_{U_0} + \ldots + G_{U_{m(*)-1}}$ of G is generated by $Y_{U_0} \cup Y_{U_1} \cup \ldots \cup Y_{U_{m(*)-1}}$ freely (as an abelian group) except the equations in $\Xi_{U_0} \cup \Xi_{U_1} \cup \ldots \cup \Xi_{U_{m(*)-1}}$.
3A) Moreover $G/(G_{U_0} + \ldots + G_{U_{m(*)-1}})$ is $\aleph_1$-free provided that

$\circledast$ *if $\eta_0, \ldots, \eta_{k(*)} \in \cup\{U_m : m < m(*)\}$ are such that*

$$(\forall \ell \leq k(*))(\exists m < m(*))[\{\eta_0, \ldots, \eta_{k(*)}\} \setminus \{\eta_\ell\} \subseteq U_m)$$

__then__ for some $m < m(*)$ we have $\{\eta_0, \ldots, \eta_{k(*)}\} \subseteq U_m$.

4) If $m(*) \leq k(*)$ and $U_\ell = U \setminus U'_\ell$ for $\ell < m(*)$ and $\langle U'_\ell : \ell < m(*)\rangle$ are pairwise disjoint __then__ $\circledast$ holds.
5) $G_{U,u} \subseteq G_{U \cup u}$ if $U \subseteq {}^\omega S$ and $u \subseteq {}^\omega S \setminus U$ is finite; moreover $G_{U,u} \subseteq_{pr} G_{U \cup u} \subseteq_{pr} G$.
6) If $\langle U_\alpha : \alpha < \alpha(*)\rangle$ is $\subseteq$-increasing continuous __then__ also $\langle G_{U_\alpha} : \alpha < \alpha(*)\rangle$ is $\subseteq$-increasing continuous.
7) If $U_1 \subseteq U_2 \subseteq U \subseteq {}^\omega S$ and $u \subseteq {}^\omega S \setminus U$ is finite, $|u| < k(*)$ and $U_2 \setminus U_1 = \{\eta\}$ and $v = u \cup \{\eta\}$ __then__ $(G_{U,u} + G_{U_2 \cup u})/(G_{U,u} + G_{U_1 \cup u})$ is isomorphic to $G_{U_1 \cup v}/G_{U_1,v}$.
8) If $U \subseteq {}^\omega S$ and $u \subseteq {}^\omega S \setminus U$ has $\leq k(*)$ members __then__ $(G_{U,u} + G_u)/G_{U,u}$ is isomorphic to $G_u/G_{\emptyset,u}$.

[2] note that if $u = \{\eta\}$ then $G_{U,u} = G_U$

Discussion 1.11 : For the reader we write what the group G_{x} is for the case $k(*) = 0$. So, omitting constant indexes and replacing sequences of length one by the unique entry we get that it is generated by $y_{\eta,n}$ (for $\eta \in {}^{\omega}S, n < \omega$) and x_ν (for $\nu \in {}^{\omega>}S$) freely as an abelian group except the equations $(n!)y_{\eta,n+1} = y_{\eta,n} + x_{\eta\restriction n}$.
Note that if K is the countable subgroup generated by $\{x_\nu : \nu \in {}^{\omega>}2\}$ then G/K is a divisible group of cardinality continuum hence G is not free. So G is $\aleph_1$-free but not free.

Now we have the abelian group version of freeness, see generally 1.13.

Claim 1.12 *The Freeness Claim Let* $\mathrm{x} \in K_{k(*)}$.
1) The abelian group $G_{U\cup u}/G_{U,u}$ *is free* <u>*if*</u> $U \subseteq {}^{\omega}S, u \subseteq {}^{\omega}S\backslash U$ *and* $|u| \le k \le k(*)$ *and* $|U| \le \aleph_{k(*)-k}$.
2) If $U \subseteq {}^{\omega}S$ *and* $|U| \le \aleph_{k(*)}$, <u>*then*</u> G_U *is free.*

Claim 1.13 *1) If* x *is a combinatorial* $k(*)$*-parameter* <u>*then*</u> x *is* $\aleph_{k(*)+1}$*-free.*
2) If x *is an abelian group parameter and* $(k^{\mathrm{x}}, S^{\mathrm{x}}, \Lambda^{\mathrm{x}})$ *is free,* <u>*then*</u> G_{x} *is free.*

Proof. 1) Easily follows by (2).
2) Similar and follows from 3.2 + Def 3.3 as easily G belongs to $\mathcal{G}_{k(*)}$.

Claim 1.14 *Assume* $\mathrm{x} \in K^{cb}_{k(*)}$ *is full (i.e.* $\Lambda^{\mathrm{x}} = {}^{k(*)+1}({}^{\omega}(S^{\mathrm{x}}))$*).*
1) If $U \subseteq {}^{\omega}S$ *and* $|U| \ge (|S| + \aleph_0)^{+(k(*)+1)}$, *the* $(k(*)+1)$*-th successor of* $|S| + \aleph_0$. <u>*Then*</u> G^{x}_U *is not free.*
2) If $|S^{\mathrm{x}}| \ge \aleph_{k(*)+1}$ <u>*then*</u> G_{x} *is not free.*
3) Assume $\mathrm{x} \in K^{cb}_{k(*)}, |S^{\mathrm{x}}_\ell| + \lambda_\ell < \lambda_{\ell+1}$ *for* $\ell < k(*)$ *and* $|\Lambda^{\mathrm{x}}| \ge \lambda_{k(*)}$ *and* $G \in \mathcal{G}_{\mathrm{x}}$ *(see §3)* <u>*then*</u> G *is not free.*

Proof. 1) Assume toward contradiction that G_U is free and let χ be large enough; for notational simplicity assume $|U| = \aleph_{\alpha,k(*)+1}$, this is O.K. as a subgroup of a free abelian group is a free abelian group where $\aleph_\alpha = |S|$. We choose N_ℓ by downward induction on $\ell \le k(*)$ such that

(a) N_ℓ is an elementary submodel [3] of $(\mathcal{H}(\chi), \in, <^*_\chi)$

(b) $\|N_\ell\| = |N_\ell \cap \aleph_{\alpha+k(*)}| = \aleph_{\alpha+\ell}$ and $\{\zeta : \zeta \le \aleph_{\alpha+\ell}\} \subseteq N_\ell$

(c) $\langle x_{\bar\eta} : \bar\eta \in \Lambda^{\mathrm{x}}_{\le k(*)}\rangle, \langle y_{\bar\eta,n} : \bar\eta \in \Lambda^{\mathrm{x}}$ and $n < \omega\rangle, U$ and G_U belong to N_ℓ and $N_{\ell+1}, \ldots, N_{k(*)} \in N_\ell$.

Let $G_\ell = G_U \cap N_\ell$, a subgroup of G_U. Now

[3] $\mathcal{H}(\chi)$ is $\{x$: the transitive closure of x has cardinality $< \chi\}$ and $<^*_\chi$ is a well ordering of $\mathcal{H}(\chi)$

$(*)_0$ $G_U/(\Sigma\{G_\ell : \ell \le k(*)\})$ is a free (abelian) group [easy or see [6], that is: as G_U is free we can prove by induction on $k \le k(*)+1$ then $G/(\Sigma\{G_{k(*)+1-\ell} : \ell < k\})$ is free, for $k = 0$ this is the assumption toward contradiction, the induction step is by Ax VI in [6] for abelian groups and for $k = k(*)+1$ we get the desired conclusion.]

Now

$(*)_1$ letting U^1_ℓ be U for $\ell = k(*)+1$ and $\bigcap\limits_{m=\ell}^{k(*)} (N_m \cap U)$ for $\ell \le k(*)$; we have: U^1_ℓ has cardinality $\aleph_{\alpha+\ell}$ for $\ell \le k(*)+1$
[Why? By downward induction on ℓ. For $\ell = k(*)+1$ this holds by an assumption. For $\ell = k(*)$ this holds by clause (b). For $\ell < k(*)$ this holds by the choice of N_ℓ as the set $\bigcap\limits_{m=\ell+1}^{k(*)} (N_m \cap U)$ has cardinality $\aleph_{\alpha+\ell+1} \ge \aleph_\ell$ and belong to N_ℓ and clause (b) above.]

$(*)_2$ $U^2_\ell =: U^1_{\ell+1} \backslash (N_\ell \cap U)$ has cardinality $\aleph_{\alpha+1}$ for $\ell \le k(*)$
[Why? As $|U^1_{\ell+1}| = \aleph_{\ell+1} > \aleph_\ell = \|N_\ell\| \ge |N_\ell \cap U|$.]

$(*)_3$ for $m < \ell \le k(*)$ the set $U^3_{m,\ell} =: U^2_\ell \cap \bigcap\limits_{r=m}^{\ell-1} N_r$ has cardinality $\aleph_{\alpha+m}$
[Why? By downward induction on m. For $m = \ell - 1$ as $U^2_\ell \in N_m$ and $|U^2_\ell| = \aleph_{\alpha+\ell+1}$ and clause (b). For $m < \ell$ similarly.]

Now for $\ell = 0$ choose $\eta^*_\ell \in U^2_\ell$, possible by $(*)_2$ above. Then for $\ell > 0, \ell \le k(*)$ choose $\eta^*_\ell \in U^3_{0,\ell}$. This is possible by $(*)_3$. So clearly

$(*)_4$ $\eta^*_\ell \in U$ and $\eta^*_\ell \in N_m \cap U \Leftrightarrow \ell \ne m$ for $\ell, m \le k(*)$.
[Why? If $\ell = 0$, then by its choice, $\eta^*_\ell \in U^2_\ell$, hence by the definition of U^2_ℓ in $(*)_2$ we have $\eta^*_\ell \notin N_\ell$, and $\eta^*_\ell \in U^1_{\ell+1}$ hence $\eta^*_\ell \in N_{\ell+1} \cap \ldots \cap N_{k(*)}$ by $(*)_1$ so $(*)_4$ holds for $\ell = 0$. If $\ell > 0$ then by its choice, $\eta^*_\ell \in U^3_{0,\ell}$ but $U^3_{m,\ell} \subseteq U^2_\ell$ by $(*)_3$ so $\eta^*_\ell \in U^2_\ell$ hence as before $\eta^*_\ell \in N_{\ell+1} \cap \ldots \cap N_{k(*)}$ and $\eta^*_\ell \notin N_\ell$. Also by $(*)_3$ we have $\eta^*_\ell \in \bigcap\limits_{r=0}^{\ell-1} N_\ell$ so $(*)_4$ really holds.]

Let $\bar\eta^* = \langle \eta^*_\ell : \ell \le k(*) \rangle$ and let G' be the subgroup of G_U generated by $\{x_{\bar\eta 1<m,n>} : m \le k(*)$ and $\bar\eta \in {}^{k(*)+1}U$ and $n < \omega\} \cup \{y_{\bar\eta,n} : \bar\eta \in {}^{k(*)+1}U$ but $\bar\eta \ne \bar\eta^*$ and $n < \omega\}$. Easily $G_\ell \subseteq G'$ recalling $G_\ell = N_\ell \cap G_U$ hence $\Sigma\{G_\ell : \ell \le k(*)\} \subseteq G'$, but $y_{\bar\eta^*,0} \notin G'$ hence

$(*)_5$ $y_{\bar\eta^*,0} \notin \sum\{G_\ell : \ell \le k(*)\}$.

But for every n

$(*)_6$ $\bar{n}! y_{\bar{\eta}^*, n+1} - y_{\bar{\eta}^*, n} = \Sigma\{x_{\bar{\eta}^* 1 <m,n>} : m \le k(*)\} \in \Sigma\{G_\ell : \ell \le k(*)\}$.
[Why? $x_{\bar{\eta}^* 1 <m,n>} \in G_m$ as $\bar{\eta}^* \restriction (k(*)) + 1 \backslash \{m\}) \in N_m$ by $(*)_4$.]

We can conclude that in $G_U / \sum\{G_\ell : \ell \le k(*)\}$, the element $y_{\bar{\eta}^*, 0} + \sum\{G_\ell : \ell \le k(*)\}$ is not zero (by $(*)_5$) but is divisible by every natural number by $(*)_6$.
This contradicts $(*)_0$ so we are done.
2),3) Left to the reader. ∎

2 Black Boxes

Claim 2.1 *1) Assume $k(*) < \omega, \chi = \chi^{\aleph_0}$ and $\lambda = \beth_{k(*)}(\chi), S = \lambda, \Lambda_{k(*)} = {}^{k(*)+1}({}^\omega S)$ or just $S_\ell = \chi_\ell = \beth_\ell(\chi), \lambda_\ell^{\aleph_0} = \chi_\ell$ for $\ell \le k(*)$ and $\Lambda_{k(*)} = \prod_{\ell \le k(*)} {}^\omega(S_\ell)$ and $\mathbf{x}^{k(*)} = (k(*), \lambda, \Lambda_{k(*)})$ so $\mathbf{x}$ is a full combinatorial $\langle S_\ell : \ell \le k(*)\rangle$-parameter. Then Λ has a χ-black box, i.e. $\mathrm{Qr}(\Lambda_{\mathbf{x}^{k(*)}}, \chi)$, see Definition 1.3.*
2) Moreover, $\mathbf{x}$ has the $\langle \chi_\ell : \ell \le k(*)\rangle$-black box, i.e. for every $\bar{B} = \langle B_{\bar{\eta}} : \bar{\eta} \in \Lambda^{\mathbf{x}}_{\le k(*)}\rangle$ satisfying clause (c) below we can find $\langle h_{\bar{\eta}} : \bar{\eta} \in \Lambda\rangle$ such that:

(a) $h_{\bar{\eta}}$ is a function with domain $\{\bar{\eta} \upharpoonright \langle m, n\rangle : m \le k(*), 2 \le n < \omega\}$

(b) $h_{\bar{\eta}}(\bar{\eta} \upharpoonright \langle m, n\rangle) \in B_{\bar{\eta} 1 <m,n>}$

(c) $B_{\bar{\eta} 1 \langle m,n\rangle}$ is a set of cardinality $\beth_m(\chi)$

(d) if h is a function with domain $\Lambda^{\mathbf{x}}_{\le k(*)}$ such that $h(\bar{\eta} \upharpoonright \langle m, n\rangle) \in B_{(\bar{\eta} 1 <m,n>)}$ and $\alpha_\ell < \beth_\ell(\chi)$ for $\ell \le k(*)$ then for some $\bar{\eta} \in \Lambda^{\mathbf{x}}, h_{\bar{\eta}} \subseteq h$ and $\eta_\ell(0) = \alpha_\ell$ for $\ell \le k(*)$.

3) Assume $\chi_\ell = \lambda_\ell^{\aleph_0}, \chi_{\ell+1} = \chi_{\ell+1}^{\chi_\ell}$ for $\ell \le k(*)$. If $S_\ell = \lambda_\ell$ for simplicity $\ell \le k(*), \mathbf{x}$ is a full combinatorial $(\bar{S}, k(*))$-parameter, and $|B_{\bar{\eta} 1 <m,n>}| \le \chi_{k(*)}$ for $\bar{\eta} \in \Lambda^{\mathbf{x}}$ then we can find $\langle h_{\bar{\eta}} : \bar{\eta} \in \Lambda^{\mathbf{x}}\rangle$ as in part (2) replacing $\beth_\ell(\chi)$ by λ_ℓ, moreover such that:

(e) if $\bar{\eta} \in \Lambda$ then η_ℓ is increasing

(f) if λ_ℓ is regular then we can in clause (d) above add: if E_ℓ is a club of λ_ℓ for $\ell \le k(*)$ then we can demand: if $\bar{\eta} \in \Lambda^{\mathbf{x}}$ then for each ℓ for some $\alpha_\ell^* < \lambda_\ell$ we have $\eta_\ell \in {}^\omega(E_\ell \cup \{\alpha_\ell^*\})$

(g) if λ_ℓ is singular of uncountable cofinality, $\lambda_\ell = \Sigma\{\lambda_{\ell,i} : i < \mathrm{cf}(\lambda_\ell)\}$, $\mathrm{cf}(\lambda_{i,\ell}) = \lambda_{i,\ell}$ increasing with i we can add: if $u_\ell \subseteq \mathrm{cf}(\lambda_\ell)$ is unbounded, $E_{\ell,i}$ a club of $\lambda_{\ell,i}$ then $\eta_\ell \in {}^\omega(E_{i,\ell} \cup \{\alpha_\ell^*\})$ for some $i \in u_\ell$.

Proof. Part (1) follows form part (2) which follows from part (3), so let us prove part (3). To uniformize the notation in 2.1(1), i.e. 1.3 and 2.1(2),(3) we shall denote:

$\odot_1$ $h_{\bar\eta}(\bar\eta \restriction \langle m,n\rangle) = \alpha^{k(*)}_{\bar\eta,m,n}$.

Note that without loss of generality $B_{\bar\nu} = |B_{\bar\nu}|$ and we use $\alpha_{k(*),m,n} = h_{\bar\eta}(\bar\eta \restriction \langle m,n\rangle$ for $\bar\eta \in \Lambda_{\mathbf{x}}, m \le k(*)$ and $n < \omega$. We prove part (3) by induction on $k(*)$. Let $\Lambda_k = \Lambda^{\mathbf{x}}$ and without loss of generality $S_\ell = \lambda_\ell$.

Case 1: $k(*) = 0$.

By the simple black box, see [9, III,§4], or better [4, VI,§2], see below for details on such a proof.

Case 2: $k(*) = k+1$.

Let

$\odot_2$ $\alpha^k = \langle \alpha^k_{\bar\eta,m,n} : \bar\eta \in \Lambda_k, n < \omega, m \le k\rangle$ witness parts (2), (3) for k, i.e. for $\mathbf{x}^k$, hence no need to assume $\mathbf{x}^k$ is full.

So $\lambda = \lambda_{k(*)}, \chi = \chi_{k(*)}$ and let $\mathbf{H} = \{h : h$ is a function from Λ_k to $\chi\}$. So $|\mathbf{H}| \le (\lambda)^{\lambda_k^{\aleph_0}} = \chi$. By the simple black box, see below, we can find $\langle \bar h_\eta : \eta \in {}^\omega\lambda\rangle$ such that

$\odot_3$ (α) $\bar h_\eta = \langle h_{\eta,n} : n < \omega\rangle$ and $h_{\eta,n} \in \mathbf{H}$ for $\eta \in {}^\omega\lambda$

(β) if $\bar f = \langle f_\nu : \nu \in {}^{\omega>}\lambda\rangle$ and $f_\nu \in \mathbf{H}$ for every such ν and $\alpha < \lambda$ and $\rho \in {}^{\omega>}\lambda$ is increasing then for some increasing $\eta \in {}^\omega\lambda$ we have $\rho \triangleleft \eta$ and $n < \omega \Rightarrow h_{\eta,n} = f_{\eta\restriction n}$

(γ) if $\mathrm{cf}(\lambda) > \aleph_0$ and E is a club of λ then we can add $\cup\{\eta(n) : n < \omega\} \in E$.

[Why? First assume $\chi = \lambda$. Let $\langle \bar g_\alpha = \langle g_{\alpha,\ell} : \ell < n_\alpha\rangle : \alpha < \lambda\rangle$ enumerate ${}^{\omega>}\mathbf{H}$ such that for each $\bar g \in {}^{\omega>}\mathbf{H}$ the set $\{\alpha < \lambda : \bar g_\alpha = \bar g\}$ is unbounded in λ. Now for $\eta \in {}^\omega\lambda$ and $n < \omega$ let $h_{\eta,n} = g_{\eta(k),n}$ for every k large enough if well defined and $g_{\eta\restriction(n+1),n}$ otherwise. So clause (α) of $\odot_3$ holds and as for clause (β) of $\odot_3$, let $\bar f = \langle f_\nu : \nu \in {}^{\omega>}\lambda\rangle$ be given, $f_\nu \in \mathbf{H}$.

Assume $\rho \in {}^{\omega>}\lambda$ is increasing. We choose α_n by induction on $n < \omega$ such that:

$\odot_4$ (α) $\alpha_n = \rho(n)$ if $n < \ell g(\rho)$

(β) $\alpha_n < \lambda$ and $\alpha_n > \alpha_m$ if $n = m+1$

(γ) if $n \ge \ell g(\rho)$ then α_n satisfies $\bar g_{\alpha_n} = \langle f_{\langle \alpha_\ell : \ell < m\rangle} : m \le n\rangle$.

Now $\eta =: \langle \alpha_n : n < \omega\rangle$ is as required in (β) of $\odot_3$; to get also (γ) of $\odot_3$ we should add in clause (β) of $\odot_4$ then $\alpha_n > \min(E \backslash \alpha_m)$.

Second, if $\chi > \lambda$ but still $\chi \le \lambda^{\aleph_0}$, let $\langle \bar g_\alpha : \alpha < \chi^{\aleph_0}\rangle$ list ${}^{\omega>}\mathbf{H}$, and let $h_n : \chi \to \lambda$ for $n < \omega$ be such [4] that $\alpha < \beta < \chi \Rightarrow (\forall^\infty n)(h_n(\alpha) \ne h_n(\beta))$ and let cd: $\lambda \to {}^{\omega>}\lambda$ be one to one onto. Now for $\eta \in {}^\omega\lambda$ and $n < \omega$ let $h_{\eta,n}$ be g_α where α is the unique ordinal $\alpha < \chi$ such that for every $k < \omega$ large enough $(\mathrm{cd}(\eta(k)))(n) = h_n(\alpha)$ so in

[4] recall $(\forall^\infty N)$ means "for every large enough $n < \omega$"

particular $\langle \ell g(\mathrm{cd}(\eta(k))) : k < \omega\rangle$ is going to infinity or $h_{\eta,n}$ is not well defined; in fact, we can use only the case $\ell g(\mathrm{cd}(\eta(k))) = k$; stipulating $h_{\eta,n} \in {}^{\omega}\{0\}$ when not defined. So we have defined $\langle h_{\eta,n} : \eta \in {}^{\omega}\lambda, n < \omega\rangle$. Now we immitate the previous argument: clause (β) of $\circledast_2$ holds.

Next we shall define $\bar{\alpha}^{k(*)} = \langle \alpha^{k(*)}_{\bar{\eta},m,n} : \bar{\eta} \in \Lambda_{k+1}, m \leq k(*), n < \omega\rangle$ as required; so let $\bar{\eta} = \langle \eta_\ell : \ell \leq k(*)\rangle \in \Lambda_{k(*)}$ we define $\bar{\alpha}^{k(*)}_{\bar{\eta}} = \langle \alpha^{k(*)}_{\bar{\eta},m,n} : m \leq k(*), n < \omega\rangle$ as follows:

$\odot_5$ if $\eta_{k(*)} \in {}^{\omega}\lambda$ and $\langle \eta_0, \ldots, \eta_{k(*)-1}\rangle \in \Lambda_k$ then for $m \leq k(*)$ and $n < \omega$

(α) if $m = k(*)$ then $\alpha^{k(*)}_{\bar{\eta},m,n} = h_{\eta_{k(*)},n}(\langle \eta_0, \ldots, \eta_{k(*)-1}\rangle) < \lambda_m$

(β) if $m < k(*)$, i.e. $m \leq k$ then $\alpha^{k(*)}_{\bar{\eta},m,n} = \alpha^{k}_{\bar{\eta}\restriction k(*),m,n} < \lambda_m$.

Clearly $\alpha^{k(*)}_{\bar{\eta},m,n} < \lambda_m$ in all cases, as required, (in clause (a),(b),(c) of 2.1(2) and (e) of 2.1(3). But we still have to prove that $\langle \bar{\alpha}^{k(*)}_{\bar{\eta},m,n} : \bar{\eta} \in \Lambda^{k+1}, m \leq k(*), n < \omega\rangle$ witness $\mathrm{Qr}(\mathbf{x}^{k(*)}, \chi)$, see Definition 1.3(2) this suffices for 2.1(2), little more is needed for 2.1(3); just using (γ) of $\odot_3$ and the induction hypothesis.

Why does this hold? Let h be a function with domain $\Lambda^{\mathbf{x}^{k(*)}}_{\leq k(*)}$ as in part (3) and $\alpha^*_\ell < \lambda_\ell$ for $\ell \leq k(*)$.

For $\nu \in {}^{\omega>}\lambda$ let $f_\nu : \Lambda_k \to \lambda = \lambda_{k(*)}$ be defined by: $f_\nu(\langle \eta_\ell : \ell \leq k\rangle) =: h(\langle \eta_\ell : \ell \leq k\rangle\hat{\ }\langle \nu\rangle)$. So by $\odot_3$ above for some increasing $\eta^*_{k(*)} \in {}^{\omega}\lambda$ we have $\eta^*_{k(*)}(0) = \alpha^*_{k(*)}$ and

$\odot_6$ $n < \omega \Rightarrow f_{\eta^*_{k(*)}\restriction n} = h_{\eta^*_{k(*)},n}$.

Now substituting the definition of $\bar{f}$ we have

$\odot_7$ $\langle \eta_0, \ldots, \eta_k\rangle \in \Lambda_k \wedge n < \omega \Rightarrow h_{\eta^*_{k(*)},n}(\eta_0, \ldots, \eta_k) = h(\langle \eta_0, \ldots, \eta_k, \eta^*_{\eta(*)\restriction n}\rangle)$.

Substituting the definition of $\bar{\alpha}^k$ we have

$\odot_8$ if $\langle \eta_0, \ldots, \eta_k\rangle \in \Lambda_k$ and $n < \omega$ then $\alpha^{k(*)}_{<\eta_0,\ldots,\eta_k,\eta^*_{k(*)}>} = h(\langle \eta_0, \ldots, \eta_k, \eta^*_{k(*)} \restriction n\rangle)$.

Now we define a function h' with domain $\Lambda^{\mathbf{x}^k}_{\leq k}$ by: if $\bar{\eta} \in \Lambda^{\mathbf{x}^k}_{\leq k}$ then $h'(\bar{\eta}) = h(\bar{\eta}\hat{\ }\langle \eta^*_{k(*)}\rangle)$.

So by the choice of $\bar{\alpha}^k$ in $\odot_2$ we can find $\langle \eta^*_0, \ldots, \eta^*_k\rangle \in \Lambda_k$ with no repetitions such that $\eta^*_\ell(0) = \alpha^*_\ell$ for $\ell \leq k$ and in $\odot_2$

$\odot_9$ $m \leq k \wedge n < \omega \Rightarrow \alpha^k_{\langle \eta^*_0,\ldots,\eta^*_k\rangle,m,\ell} = h'(\langle \eta^*_0, \ldots, \eta^*_k\rangle \restriction (m,n)\rangle)$.

Let $\bar{\eta}^* = \langle \eta^*_0, \ldots, \eta^*_k, \eta^*_{k+1}\rangle, \bar{\eta}' = \langle \eta^*_0, \ldots, \eta^*_i\rangle$.

Note that

$\odot_{10}$ if $m \leq k, n < \omega$ then $h'(\bar{\eta}' \restriction \langle k, m\rangle) = h((\bar{\eta}' \restriction \langle k, m\rangle)\hat{\ }\langle \eta^*_{k(*)}\rangle) = h(\bar{\eta}^* \restriction \langle k, m\rangle)$.

Now by $\odot_9 + \odot_{10}$ and $\odot_5(\beta)$ this means

$\odot_{11}$ if $m \leq k$ and $n < \omega$ then $\alpha^{k(*)}_{\bar{\eta}^*,m,n} = h(\bar{\eta}^* \restriction \langle k, m\rangle)$.

So by putting together $\odot_8 + \odot_{11}$ we are clearly done, i.e. we can check that $\langle \eta_0^*, \ldots, \eta_k^*, \eta_{k(*)}^* \rangle$ is as required. ∎

Conclusion 2.2 For every $k < \omega$ there is an $\aleph_{k+1}$-free abelian group G of cardinality $\beth_{k+1}$ and pure (non-zero) subgroup $\mathbb{Z}z \subseteq G$ such that $\mathbb{Z}z$ is not a direct summand of G.

Proof. Let $\chi = 2^{\aleph_0}$ and $\mathbf{x}$ be a combinatorial k-parmeter as guaranteed by 2.1. Now by 2.3(2) below we can expand $\mathbf{x}$ to an abelian group k-parameter, so $G_{\mathbf{x}}$ is as required.

Claim 2.3 *1) If* $\mathbf{x}$ *is a combinatorial k-parameter such that* $\mathrm{Qr}(\mathbf{x}, 2^{\aleph_0})$ *<u>then</u> for some* $\mathbf{a}, (\mathbf{x}, \mathbf{a})$ *is an abelian group k-parameter such that* $h \in \mathrm{Hom}(G_{\mathbf{x}}, \mathbb{Z}) \Rightarrow h(z) = 0$.
2) For every k there is an $\aleph_{k+1}$*-free abelian group G of cardinality* $\beth_{k+1}$ *and* $z \in G$ *a pure* $z \in G$ *as above.*

Proof. 1) Let $\bar{\alpha}$ witness $\mathrm{Qr}(\mathbf{x}, 2^{\aleph_0})$. We define Ord $\to \mathbb{Z}$ by $:\iota(\alpha)$ is α if $\alpha < \omega$, is $-n$ if $\alpha = \omega + n < \omega + \omega$ and zero otherwise. For each $\bar{\eta} \in \Lambda^{\mathbf{x}}$ we shall choose a sequence $\langle \mathbf{a}_{\bar{\eta},n} : n < \omega \rangle$ of integers such that for any $b \in \mathbb{Z}\backslash\{0\}$ for no $\bar{c} \in {}^{\omega}\mathbb{Z}$ do we have

$\boxtimes_{\bar{\eta}}$ for each $n < \omega$ we have

$$n! c_{n+1} = c_n + \mathbf{a}_{\bar{\eta},n} b + \Sigma\{\iota(\alpha_{\bar{\eta},m,n}) : m \leq k(*)\}.$$

This is easy: for each pair $(b, c_0) \in \mathbb{Z} \times \mathbb{Z}$ the set of sequences $\langle \mathbf{a}_{\bar{\eta},n} : n < \omega \rangle \in {}^{\omega}\mathbb{Z}$ there is a sequence $\langle c_0, c_1, c_2, \ldots \rangle$ of integers such that $\boxtimes_{\bar{\eta}}$ holds for them, so the choice of $\langle \mathbf{a}_{\bar{\eta},n} : n < \omega \rangle$ is possible.

Now toward contradiction assume that h is a homomorphism from $G_{\mathbf{x}}$ to $z\mathbb{Z}$ such that $h(z) = bz, b \in \mathbb{Z}\backslash\{0\}$. We define $h' : \Lambda^{\mathbf{x}}_{\leq k} \to \chi$ by $h'(\bar{\eta}) = n$ if $n < \omega$ and $h(x_{\bar{\eta}}) = nz$ and $h'(\bar{\eta}) = \omega + n$ if $n < \omega$ and $h(x_{\bar{\eta}}) = (-n)z$.

By the choice of $\bar{\alpha}$, for some $\bar{\eta} \in \Lambda^{\mathbf{x}}$ we have: $m \leq k \wedge n < \omega \Rightarrow h'(\bar{\eta} \upharpoonright \langle m, n \rangle) = \alpha_{\bar{\eta},m,n}$. Hence $h(x_{\bar{\eta}\upharpoonright\langle m,n\rangle}) = \iota(\alpha_{\bar{\eta},m,n})z$ for $m \leq k, n < \omega$.

Let $c_n \in \mathbb{Z}$ be such that $h(y_{\bar{\eta},n}) = c_n z$. Now the equation $\boxtimes_{\bar{\eta},n}$ in Definition 1.6 is mapped to the n-th equation in $\boxtimes_{\bar{\eta}}$, so an obvious contradiction.
2) By part (1) and 2.2. ∎

Remark 2.4 *1)*We can replace χ by a set of cardinality χ in Definition 1.3. Using $\mathbb{Z}z$ instead of χ simplify the notation in the proof of 2.3.
2) We have not tried to save in the cardinality of G in 2.3(2), using as basic of the induction the abelian group of cardinality $\aleph_0$ or $\aleph_1$.

Claim 2.5 *1) If* $\chi_0 = \chi_0^{\aleph_0}, \chi_{m+1} = 2^{\chi_m}$ *and* $\lambda_m = \chi_m$ *for* $m \leq k$ *<u>there is a</u>* $\bar{\chi}$*-full* $\mathbf{x}$ *such that* $(\mathbf{x}, \bar{\chi})$*-black box exist.*

Conclusion 2.6 Assume $\mu_0 < \ldots < \mu_{k(*)}$ are strong limit of cofinality $\aleph_0$ (or $\mu_0 = \aleph_0$), $\lambda_\ell = \mu_\ell^+, \chi_\ell = 2^{\mu_\ell}$.

Then in 2.1 for $\bar\eta \in \Lambda^{\mathbf{x}}$ we can let $h_{\bar\eta,m}$ has domain $\{\bar\nu \in \Lambda^{\mathbf{x}}_m : [\nu_\ell = \eta_\ell$ for $\ell = m+1, \ldots, k(*)\}$.

3 Constructing abelian groups from combinatorial parameters

Definition 3.1 1) We say F is a μ-regressive function on a combinatorial parameter $\mathbf{x} \in K^{\rm cb}_{k(*)}$ when: $S^{\mathbf{x}}$ is a set of ordinals and:

(a) $\mathrm{Dom}(F)$ is $\Lambda^{\mathbf{x}}$

(b) $\mathrm{Rang}(F) \subseteq [\Lambda^{\mathbf{x}} \cup \Lambda^{\mathbf{x}}_{\leq k(*)}]^{\leq \aleph_0}$

(c) for every $\bar\eta \in \Lambda^{\mathbf{x}}$ and $m \leq k(*)$ we [5] have $\sup \mathrm{Rang}(\eta_m) > \sup(\cup\{\mathrm{Rang}(\nu_n) : \bar\nu \in F(\bar\eta)\})$; note $\bar\nu_\ell \in \Lambda^{\mathbf{x}}$ or $\bar\nu \in \Lambda^{\mathbf{x}}_{\leq k(*)}$ as $F(\bar\eta)$ is a set of such objects.

1A) We say F is finitary when $F(\bar\eta)$ is finite for every $\bar\eta$.
1B) We say F is simple if $\eta_{k(*)}(0)$ determined $F(\bar\eta)$ for $\bar\eta \in \Lambda^{\mathbf{x}}$.
2) For $\mathbf{x}, F$ as above and $\Lambda \subseteq \Lambda^{\mathbf{x}}$ we say that Λ is free for $(\mathbf{x}, F)$ when: $\Lambda \subseteq \Lambda^{\mathbf{x}}$ and there is a sequence $\langle \bar\eta^\alpha : \alpha < \alpha(*)\rangle$ listing $\Lambda' = \Lambda \cup \bigcup\{F(\bar\eta) : \bar\eta \in \Lambda\}$ and sequence $\langle \ell_\alpha : \alpha < \alpha(*)\rangle$ such that

(*a*) $\ell_\alpha \leq k(*)$

(*b*) if $\alpha < \alpha(*)$ and $\bar\eta^\alpha \in \Lambda$ then $F(\bar\eta^\alpha) \subseteq \{\bar\eta^\beta, \bar\eta^\beta \upharpoonright \langle m, n\rangle : \beta < \alpha, n < \omega, m \leq k(*)\}$

(*c*) if $\alpha < \alpha(*)$ and $\bar\eta^\alpha \in \Lambda$ then for some $n < \omega$ we have $\bar\eta^\alpha \upharpoonright \langle \ell_\alpha, n\rangle \notin \{\bar\eta^\beta \upharpoonright \langle \ell_\alpha, n\rangle : \beta < \alpha, \eta^\beta \in \Lambda\} \cup \{\bar\eta^\beta : \beta < \alpha\}$.

3) We say $\mathbf{x}$ is θ-free for F is $(\mathbf{x}, F)$ is μ-free <u>when</u> $\mathbf{x}, F$ are as in part (1) and every $\Lambda \subseteq \Lambda^{\mathbf{x}}$ of cardinality $< \theta$ is free for $(\mathbf{x}, F)$.

Claim 3.2 *1) If $\mathbf{x} \in K^{cb}_{k(*)}$ and F is a regressive function on $\mathbf{x}$ <u>then</u> $(\mathbf{x}, F)$ is $\aleph_{k(*)+1}$-free provided that F is finitary or simple.*
2) In addition: if $k \leq k(*), \Lambda \subseteq \Lambda^{\mathbf{x}}$ has cardinality $\leq \aleph_k$ and $\bar u = \langle u_{\bar\eta} : \bar\eta \in \Lambda\rangle$ satisfies $u_{\bar\eta} \subseteq \{0, \ldots, k(*)\}, |u_\eta| > k$, <u>then</u> we can find $\langle \bar\eta^\alpha : \alpha < \aleph_k\rangle, \langle \ell_\alpha : \alpha < \aleph_k\rangle, \langle n_\alpha : \alpha < \aleph_k\rangle$ such that:

(*a*) $\Lambda \subseteq \{\bar\eta^\alpha : \alpha < \aleph_k\}$

(*b*) *if $\bar\eta_\alpha \in \Lambda^{\mathbf{x}}$ then $\ell_\alpha \in u_{\bar\eta^\alpha}, n_\alpha < \omega$*

[5] actually, suffice to have it for $\ell = k(*)$

(c) $\bar\eta^\alpha \restriction \langle \ell_\alpha, n_\alpha\rangle \notin \{\bar\eta^\beta \restriction \langle \ell_\alpha, n_\alpha\rangle : \beta < \alpha\} \cup \{\bar\eta^\beta : \beta < \alpha\}$.

Proof. 1) Follows by part (2) for the case $k = k(*), u_{\bar\eta} = \{0, \dots, k(*)\}$ for every $\bar\eta \in \Lambda$.

2) So we are assuming $\mathbf{x} \in K^{\rm cb}_{k(*)}, F$ is a regressive function on $\mathbf{x}, k \le k(*), \Lambda \subseteq \Lambda^{\mathbf{x}}$ has cardinality $\le \aleph_k$ and without loss of generality Λ is closed under $\bar\eta \mapsto F(\bar\eta) \cap \Lambda^{\mathbf{x}}$. We prove this by induction on k.

Case 1: $k = 0$.

Subcase 1A: Ignoring F.

Let $\langle \bar\eta^\alpha : \alpha < |\Lambda|\rangle$ list Λ with no repetitions (so $\alpha < |\Lambda| \Rightarrow \alpha < \aleph_k = \aleph_0$). Now $\alpha < |\Lambda| \Rightarrow u_{\bar\eta^\alpha} \ne \emptyset$ and let $\ell_\alpha = \min(u_{\bar\eta^\alpha}) \le k(*)$. Hence for each $\alpha < |\Lambda|$ we know that $\beta < \alpha \Rightarrow \bar\eta^\beta \ne \bar\eta^\alpha$, hence for some $n = n_{\alpha,\beta} < \omega$ we have $\bar\eta^\beta \restriction \langle \ell_\alpha, n_{\alpha,\beta}\rangle \ne \bar\eta^\alpha \restriction \langle \ell_\alpha, n_{\alpha,\beta}\rangle$.

Let $n_\alpha = \sup\{n_{\alpha,\beta} : \beta < \alpha\rangle$, it is $< \omega$ as $\alpha < \omega$. Now $\langle(\ell_\alpha, n_\alpha) : \alpha < |\Lambda|\rangle$ is as required.

Subcase 1B: $\bar\eta \in \Lambda \Rightarrow F(\bar\eta)$ is finite.

Let $\langle \eta^\alpha : \alpha < |\Lambda|\rangle$ list Λ, we choose w_j by induction on $j \le j(*), j(*) \le \omega$ such that:

(a) $w_j \subseteq |\Lambda|$ is finite

(b) $j \in w_{j+1}$

(c) if $\alpha \in w_j$ then $F(\bar\eta^\alpha) \cap \Lambda \subseteq \{\bar\eta^\alpha : \beta \in w_j\}$

(d) $w_{j(*)} = |\Lambda|$ and $w_0 = \emptyset$

(e) $w_j \subseteq w_{j+1}$ and $j(x) = w \Rightarrow w_{j(x)} = \cup\{w_j : j < j(x)\}$.

No problem to do this (for clause (c) use "F is regressive, the ordinals well ordered).

Now let $\langle \beta(j,i) : i < i^*_j\rangle$ list $w_{j+1} \backslash w_j$ such that: if $i_1, i_2 < i^*_j$ and $\bar\eta^{\beta(j,i_1)} \in F(\bar\eta^{\beta(j,i_2)})$ then $i_1 < i_2$; we prove existence by F being regressive. Let $\langle \bar\nu_{j,i} : i < i^{**}_j\rangle$ list $\cup\{F(\bar\eta^\alpha) : \alpha \in w_{j+1}\backslash w_j\}\backslash\Lambda^{\mathbf{x}}\backslash\{F(\bar\eta^\alpha) : \alpha \in w_j\}$.

Let $\alpha^*_j = \Sigma\{i^{**}_{j(1)} + i^*_{j(1)} : j(1) < j\}$. Now we choose $\bar\rho_\varepsilon$ for $\varepsilon < \alpha^*_j$ for $j < j(*)$ as follows:

(a) $\rho_{\alpha^*_j + i} = \nu_{j,i}$ if $i < i^{**}_j$

(b) $\bar\rho_{\alpha^*_j + i^{**}_j + i} = \bar\eta^{\beta(j,i)}$ if $i < i^*_j$.

Lastly, we choose $n_{\alpha_j + i} < \omega$ for $i < i^*_j$ as in case 1A.

Now check.

Subcase 1C: F is simple.

Note that $F(\bar\eta)$ when defined is determined by $\eta_{k(*)}(0)$ and is included in $\{\bar\nu \in \Lambda^{\mathbf{x}}_{\le k(*)} \cup \Lambda^{\mathbf{x}} :\ \sup \mathrm{Rang}(\nu_{k(*)}) < \eta_{k(*)}(0)\}$. So let $u = \{\eta_{k(*)}(0) : \bar\eta \in \Lambda\}$ and $u^* = u \cup \{\sup(u) + 1\}$ and for $\alpha \in u$ let $\Lambda_\alpha = \{\bar\eta \in \Lambda : \eta_{k(*)}(0) = \alpha\}$ and for $\alpha \in U^+$

let $\Lambda_{<\alpha} = \cup\{\Lambda_\alpha : \alpha \in u\}$. Now by induction on $\beta \in u^*$ we choose $\langle(\bar{\eta}^\varepsilon, \ell_\varepsilon) : \varepsilon < \varepsilon_\beta\rangle$ such that it is a required for $\Lambda_{<\alpha}$. For $\beta = \min(u)$ this is trivial and if $\text{otp}(u \cap \beta)$ is a limit ordinal this is obvious. So assume $\alpha = \max(u \cap \beta)$, we use Subcase 1A on Λ_α, and combine them naturally promising $\ell_\alpha = k(*) \Rightarrow n_\alpha > 1$.

Case 2: $k = k_* + 1$ and $|\Lambda| = \aleph_k$.

Let $\langle\Lambda_\varepsilon : \varepsilon < \aleph_k\rangle$ be $\subseteq$-increasing continuous with union $\Lambda, |\Lambda_{1+\varepsilon}| = \aleph_{k_*}, \Lambda_0 = \emptyset$, each Λ_ε closed enough, mainly:

$\circledast_1$ if $\bar{\eta}^i \in \Lambda_\varepsilon$ for $i < i(*) < \omega, \bar{\rho} \in \Lambda$ and $\{\rho_\ell : \ell \leq k(*)\} \subseteq \{\eta^i_\ell : \ell \leq k(*), i < i(*)\}$ then $\bar{\rho} \in \Lambda_\varepsilon$

$\circledast_2$ Λ_ε is closed under $\bar{\eta} \mapsto F(\bar{\eta}) \cap \Lambda^{\mathbf{x}}$.

Next

$\odot$ if $\varepsilon < \aleph_k, \bar{\eta} \in \Lambda_{\varepsilon+1} \backslash \Lambda_\varepsilon$ then $u'_{\bar{\eta}} = \{\ell \in u_{\bar{\eta}}$: for every or just some $n < \omega$ for some $\bar{\nu} \in \Lambda_\varepsilon$ we have $\bar{\eta} \upharpoonright \langle\ell, n\rangle = \bar{\nu} \upharpoonright \langle\ell, n\rangle\}$ has at most one member.

[Why? So assume toward contradiction that $\bar{\eta} \in \Lambda_{\varepsilon+1}$ and $\ell(1) \neq \ell(2)$ belong to $u'_{\bar{\eta}}$. Hence by the definition of $u'_{\bar{\eta}}$ there are $\bar{\nu}^1, \bar{\nu}^2 \in \Lambda_\varepsilon$ and $\eta_1, \eta_2 < \omega$ such that $\bar{\eta} \upharpoonright \langle\ell_1, \eta_1\rangle \in \bar{\nu}^1 \upharpoonright \langle\ell_1, m_1\rangle$ and $\bar{\eta} \upharpoonright \langle\ell_1, \eta_2\rangle = \bar{\nu}^2 \upharpoonright \langle\ell_2, \eta_2\rangle$. Now $m \leq k(*) \Rightarrow$ for some $i \in \{1, 2\}, m \leq \ell_i \Rightarrow \eta_m$ is $(\bar{\eta} \upharpoonright \langle\ell_i, n_i\rangle)_m \Rightarrow \eta_m \in \{\rho_\ell : \bar{\rho} \in \Lambda_\varepsilon$. Hence $\{\eta_\ell : \ell \leq k(*)\} \subseteq \{\rho_\ell : \ell \leq k(*)$ and $\bar{\rho} \in \Lambda_\varepsilon\}$. So by $\circledast_1$ we have $\bar{\eta} \in \Lambda_\varepsilon$, so we are done.]

Apply the induction hypothesis to $\Lambda_{\varepsilon+1} \backslash \Lambda_\varepsilon$ for each ε and get $\langle(\bar{\eta}^{\varepsilon,\alpha}, \ell_{\varepsilon,\alpha,n_{\varepsilon,\alpha}}) : \alpha < \alpha(\varepsilon)\rangle$ such that $\bar{\eta}^{\varepsilon,\alpha} \upharpoonright \langle\ell^\varepsilon_{\varepsilon,\ell}, n_{\varepsilon,\alpha}\rangle \notin \{\bar{\eta}^{\varepsilon,\beta} \upharpoonright \langle\ell_{\varepsilon,\beta}, n_{\varepsilon,\beta}\rangle : \beta < \alpha\rangle$.

Let $\alpha_* = \Sigma\{\alpha(\varepsilon) : \varepsilon < |\Lambda|\}$ and $\alpha = \Sigma\{\alpha(\zeta) : \zeta < \varepsilon\} + \beta, \alpha < \alpha(\varepsilon)$ let $\eta^\alpha = \eta^{\varepsilon,\beta}, \ell_\alpha = \ell_{\varepsilon,\beta}, \eta_\alpha = \eta_{\varepsilon,\beta}$. I.e. we combine but for $\Lambda_{\varepsilon+1} \backslash \Lambda_\varepsilon$ we use $\langle u_{\bar{\eta}} \backslash u'_{\bar{\eta}} : \bar{\eta} \in \Lambda_{\varepsilon+1} \backslash \Lambda_\varepsilon\rangle$, so $|u_{\bar{\eta}} \backslash u'_{\bar{\eta}}| \geq k - 1 = k_*$. ∎

Definition 3.3 For a combinatorial parameter $\mathbf{x}$ we define $\mathcal{G}_{\mathbf{x}}$, the class of abelian groups derived from $\mathbf{x}$ as follows: $G \in \mathcal{G}_{\mathbf{x}}$ if there is a simple (or finitary) regressive F on $\Lambda^{\mathbf{x}}$ and G is generated by $\{y_{\bar{\eta},n} : \eta \in \Lambda^{\mathbf{x}}, n < \omega\} \cup \{x_{\bar{\eta}} : \bar{\eta} \in \Lambda^{\mathbf{x}}_{\leq k(*)}\}$ freely except

$$\boxtimes_{\bar{\eta},n} \; (n!) y_{\bar{\eta},n+1} = y_{\bar{\eta},n} + b^{\mathbf{x}}_{\bar{\eta},n} z_{\bar{\eta},n} + \sum\{x_{\bar{\eta} \upharpoonright <m,n>} : m \leq k(*)\}$$

where

$\odot$ (a) $b_{\bar{\eta},n} \in \mathbb{Z}$

(b) $z_{\bar{\eta},n}$ is a linear combination of $\{x_{\bar{\nu}} : \bar{\nu} \in F(\bar{\eta}) \backslash \Lambda^{\mathbf{x}}\} \cup \{y_{\bar{\eta},n} : \bar{\eta} \in F(\bar{\eta}) \cap \Lambda^{\mathbf{x}}$ and $(\forall m \leq k(*))(\bar{\eta} \upharpoonright \langle m, n\rangle) \in F(\bar{\eta})\}$.

Claim 3.4 *If $\mathbf{x} \in K^{cb}_{k(*)}$ and $G \in \mathcal{G}_{\mathbf{x}}$ (i.e. G is an abelian group derived from $\mathbf{x}$), then G is $\aleph_{k(*)+1}$-free.*

Proof. We use claim 3.2. So let H be a subgroup of G of cardinality $\leq \aleph_{k(*)}$. We can find Λ such that

(∗) (a) $\Lambda \subseteq \Lambda^{\times}$ has cardinality $\leq \aleph_{k(*)}$

(b) every equation which $X_\Lambda = \{x_{\bar\eta \restriction <m,n>}, y_{\bar\eta,n} : m \leq k(*), n < \omega, \bar\eta \in \Lambda\}$ satisfies in G, is implied by the equations in $\Gamma_\Lambda = \cup\{\boxtimes_{\bar\eta,n} : \bar\eta \in \Lambda\}$

(c) $H \subseteq G_\Lambda = \langle x_{\bar\eta\restriction<m,n>}, y_{\bar\eta,n} : \bar\eta \in \Lambda, m \leq k(*), n < \omega\rangle_G$.

So it sufices to prove that G_Λ is a free (abelian) group.

Let the sequence $\langle(\bar\eta^\alpha, \ell_\alpha) : \alpha < \alpha(*)\rangle$ be as proved to exist in 3.2. Let $\mathcal{U} = \{\alpha < \alpha(*) : \bar\eta^\alpha \in \Lambda\} \cup \{\alpha(*)\}$ and for $\alpha \in \mathcal{U}$ let $X^0_\alpha = \{x_{\bar\eta^\beta\restriction<m,n>} : \beta \in \alpha \cap \mathcal{U}, m \leq k(*)$ and $n < \omega\}$ and $X^1_\alpha = X^0_\alpha \cup \{\bar\eta^\beta : \beta \in \alpha\backslash\mathcal{U}\}$. So for each $\alpha \in \mathcal{U}$ there is $\bar n_\alpha = \langle n_{\alpha,\ell} : \ell \in v_\alpha\rangle$ such that: $\ell_\alpha \in v_\alpha \subseteq \{0, \dots, k(*)\}, n_{\alpha,\ell} < \omega$ and $X^1_{\alpha+1}\backslash X^1_\alpha = \{x_{\bar\eta\restriction<\ell,n>} : \ell \in v_\alpha$ and $n \in [n_{\alpha,\ell}, \omega)\}$.

For $\alpha \leq \alpha(*)$ let $G_{\Lambda,\alpha} = \langle\{y_{\bar\eta^\beta,n}, x_{\bar\nu} : \beta \in \mathcal{U} \cap \alpha$ and $\bar\nu \in X^1_\beta\}\rangle_{G_\Lambda}$. Clearly $\langle G_{\Lambda,\alpha} : \alpha \leq \alpha(*)\rangle$ is purely increasing continuous with union G_Λ, and $G_{\Lambda,0} = \{0\}$. So it suffices to prove that $G_{\Lambda,\alpha+1}/G_{\Lambda,\alpha}$ is free. If $\alpha \notin \mathcal{U}$ the quotient is trivial by a free group, and if $\alpha \in \mathcal{U}$ we can use $\ell_\alpha \in v_\alpha$ to prove that is free giving a basis. ∎

Conclusion 3.5 For every $k(*) < \omega$ there is an $\aleph_{k(*)+1}$-free abelian group G of cardinality $\lambda = \beth_{k(*)+1}$ such that $\mathrm{Hom}(G, \mathbb{Z}) = \{0\}$.

Proof. We use $\mathbf{x}$ and $\langle h_{\bar\eta} : \bar\eta \in \Lambda^{\times}\rangle$ from 2.1(3), and we shall choose $G \in \mathcal{G}_{\mathbf{x}}$. So G is $\aleph_{k(*)+1}$-free by 3.4.

Let $\mathcal{S} = \{\langle(a_i, \bar\eta_i) : i < i_1\rangle\hat{\ }\langle(b_j, \bar\nu_j, n_j) : j < j_1\rangle : i_1 < \omega, a_i \in \mathbb{Z}, \bar\eta_i \in \Lambda^{\times}_{\leq k(*)}$ and $j_1 < \omega, b_j \in \mathbb{Z}, \nu_j \in \Lambda^{\times}, n_j < \omega\}$ (actually $\mathcal{S} = \Lambda^{\times}_{\leq k(*)}$ suffice noting $\bar\nu_j = \langle \nu_{j,\ell} : \ell \leq k(*)\rangle$).

So $|S| = \lambda_{k(*)}$ and let $\bar p$ be such that:

(a) $\bar p = \langle p^\alpha : \alpha < \lambda\rangle$

(b) $\bar p$ lists $\mathcal{S}$

(c) $p^\alpha = \langle(a^\alpha_i, \bar\eta^\alpha_i) : i < i_\alpha\rangle\hat{\ }\langle(b^\alpha_j, \bar\nu^\alpha_j, n^\alpha_j) : j < j_\alpha\rangle$ so $\bar\nu^\alpha_j = \langle\nu^\alpha_{j,\ell} : \ell \leq k(*)\rangle$

(d) sup $\mathrm{Rang}(\eta^\alpha_{i,k(*)}) < \alpha$, sup $\mathrm{Rang}(\nu^\alpha_{j,k(*)}) < \alpha$ if $i < i_\alpha, j < j_\alpha$.

Now to apply Definition 3.3 we have to choose z_α (for Definition 3.3) as $\Sigma\{a^\alpha_i x_{\bar\eta_i} : i < i_\alpha\} + \Sigma\{b^\alpha_j y_{\bar\nu^\alpha_j, n^\alpha_j} : j < j_\alpha\}$ and $z_{\bar\eta} = z_{\eta_{k(*)}(0)}$ for $\bar\eta \in \Lambda^{\times}$ then for $\bar\eta \in \Lambda^{\times}$ we choose $\langle b_{\bar\eta,n} : n < \omega\rangle \in {}^{\omega}\mathbb{Z}$ such that:

⊛ there is no function h from $\{z_{\bar\eta}\} \cup \{y_{\bar\eta,n} : n < \omega\} \cup \{x_{\bar\eta\restriction<m,n>} : m \leq k(*), n < \omega\}$ into $\mathbb{Z}$ satisfying

⊛ (a) $h(z_{\bar\eta}) \neq 0$ and

(b) $h(x_{\bar\eta\restriction<m,n>}) = h_{\bar\eta}(\bar\eta \restriction \langle m, n\rangle)$ for $m \leq k(*), n < \omega$

(c) for every n

$$(*)_n \quad n!h(y_{\bar\eta,n+1}) = h(y_{\bar\eta,n}) + b_{\bar\eta,n}h(z_{\bar\eta}) + \Sigma\{(x_{\bar\eta\restriction<m,n>}) \colon m \le k(*)\}.$$

E.g. for each $\rho \in {}^{\omega}2$ we can try $b^{\rho}_n = \rho(n)$ and assume toward contradiction that for each $\rho \in {}^{\omega}2$ there is h_ρ as above. Hence for some $c \in \mathbb{Z}\backslash\{0\}$ the set $\{\rho \in {}^{\omega}2 : h_\rho(z_{\bar\eta}) = c\}$ is uncountable. So we can find $\rho_1 \neq \rho_2$ such that $h_{\rho_1} = c = h_{\rho_2}(x_\nu)$ and $\rho_1 \restriction (|c| + 7) = \rho_2 \restriction (|c| + 7)$. So for some $n \ge |c| + 7, \rho_1 \restriction n = \rho_2 \restriction n$ and $\rho_1(n) \neq \rho_2(n)$. Now consider the equation $(*)_n$ for $h_{\bar\rho_1}$ and $h_{\bar\rho_2}$, subtract them and get $(\rho_1(n) - \rho_2(n))c$ is divisible by $n!$, clear contradiction.

So $G \in \mathcal{G}_{\mathbf{x}}$ is well defined and is $\aleph_{k(*)+1}$-free by 3.4. Suppose $h \in$ Hom$(G, \mathbb{Z})$ is non-zero, so for some $\alpha < \lambda_{k(*)}, h(z_\alpha) \neq 0$ (actually as $G^1 = \langle\{x_{\bar\nu} : \bar\nu \in \Lambda^{\mathbf{x}}_{\le k(*)}\}\rangle_G$ is a subgroup such that G/G^1 is divisible necessarily $h \restriction G^1$ is not zero hence in 2.1(2) for some $\bar\nu \in \Lambda^{\mathbf{x}}_{\le k(*)}$ we have $h(x_{\bar\nu}) \neq 0$. Let $\mathbf{y} = \{\bar\nu\}$ and so by the choice of $\langle h_{\bar\eta} : \bar\eta \in \Lambda\rangle$ for some $\bar\eta \in \Lambda^{\mathbf{x}}, \eta_{k(*)}(0) = \alpha$ and we have $h_{\bar\eta} = h \restriction \{x_{\bar\eta\restriction<m,n>} : m \le k(*), n < \omega\}$. By ⊛ we clearly get a contradiction. ∎

Remark We can give more details as in the proof of 2.3.

Conclusion 3.6 *°rm For every $n \le m < \omega$ there is a purely increasing sequence $\langle G_\alpha : \alpha \le \omega_n + 1\rangle$ of abelian groups, $G_\alpha, G_\beta/G_\alpha$ are free for $\alpha < \beta \le \omega_n$ and $G_{\omega_n+1}/G_{\omega_n}$ is $\aleph_n$-free and for some $h \in$ Hom$(G_\kappa, \mathbb{Z})$ has no extension in Hom$(G_{\omega_n+1}, \mathbb{Z})$.*

Proof. Let G, z be as in 2.2. So also $G/\mathbb{Z}z$ is $\aleph_n$-free. Let $G_\alpha = \langle\{z\}\rangle_G$ for $\alpha \le \omega_2, G_{\omega_n+1} = G$.

4 Appendix 1

Notation 4.1 If $\bar\eta^* \in \Lambda^{\mathbf{x}}_m$ and $\bar\eta = \bar\eta^* \restriction \{\ell \le k(*) : \ell \neq m\}$ and $\nu = \eta^*_m$ then let $x_{m,\bar\eta,\nu} := x_{\bar\eta^*}$. (See proof of 1.12).

Proof of 1.8. Let $U \subseteq {}^{\omega}S$ be countable (and infinite) and define G'_U like G restricting ourselves to $\eta_\ell \in U$; by the Löwenheim-Skolem argument it suffices to prove that G'_U is a free abelian group. List $\Lambda \cap {}^{k(*)+1}U$ without repetitions as $\langle\bar\eta_t : t < t^* \le \omega\rangle$, and choose $s_t < \omega$ by induction on $t < \omega$ such that $[r < t \,\&\, \bar\eta_r \restriction k(*) = \bar\eta_t \restriction k(*) \Rightarrow \emptyset = \{\eta_{t,k(*)} \restriction \ell : \ell \in [s_t, \omega)\} \cap \{\eta_{r,k(*)} \restriction \ell : \ell \in [s_r, \omega)\}]$.

Let

$$Y_1 = \{x_{m,\bar\eta,\nu} : m < k(*), \bar\eta \in {}^{k(*)+1\backslash\{m\}}U \text{ and } \nu \in {}^{\omega>}2\}$$

$$Y_2 = \Big\{x_{m,\bar\eta,\nu} : \; m = k(*), \bar\eta \in {}^{k(*)}U \text{ and for no } t < t^* \text{ do we have } \bar\eta = \bar\eta_t \restriction k(*) \,\&\, \nu \in \{\eta_{t,k(*)} \restriction \ell : s_t \le \ell < \omega\}\Big\}$$

$Y_3 = \{y_{\bar\eta_t,n} : t < t^*$ and $n \in [s_t, \omega)\}$. Now

$(*)_1$ $Y_1 \cup Y_2 \cup Y_3 \cup \{z\}$ generates G'_U.

[Why? Let G' be the subgroup of G'_U which $Y_1 \cup Y_2 \cup Y_3$ generates. First we prove by induction on $n < \omega$ that for $\bar\eta \in {}^{k(*)}U$ and $\nu \in {}^nS$ we have $x_{k(*),\bar\eta,\nu} \in G'$. If $x_{k(*),\bar\eta,\nu} \in Y_2$ this is clear; otherwise, by the definition of Y_2 for some $\ell < \omega$ (in fact $\ell = n$) and $t < \omega$ such that $\ell \geq s_t$ we have $\bar\eta = \bar\eta_t \restriction k(*), \nu = \eta_{t,k(*)} \restriction \ell$.
Now

(a) $y_{\bar\eta_{t,\ell+1}}, y_{\bar\eta_{t,\ell}}$ are in $Y_3 \subseteq G'$

(b) $x_{m,\bar\eta_t \restriction \{i \leq k(*): i \neq m\},\nu}$ belong to $Y_1 \subseteq G'$ if $m < k(*)$.

Hence by the equation $\boxtimes_{\bar\eta,n}$ in Definition 1.6, clearly $x_{k(*),\bar\eta,\nu} \in G'$. So as $Y_1 \subseteq G' \subseteq G'_U$, all the generators of the form $x_{m,\bar\eta,\nu}$ with each $\eta_\ell \in U$ are in G'.

Now for each $t < \omega$ we prove that all the generators $y_{\bar\eta_t,n}$ are in G'. If $n \geq s_t$ then clearly $y_{\bar\eta_t,n} \in Y_3 \subseteq G'$. So it suffices to prove this for $n \leq s_t$ by downward induction on n; for $n = s_t$ by an earlier sentence, for $n < s_t$ by $\boxtimes_{\bar\eta,n}$. The other generators are in this subgroup so we are done.]

$(*)_2$ $Y_1 \cup Y_2 \cup Y_3 \cup \{z\}$ generates G'_U freely.
[Why? Translate the equations, see more in [5, §5].]

■

Proof of 1.10 0), 1) Obvious.
2),3),4) Follows.
5) Let $\langle \eta_\ell : \ell < m(*)\rangle$ list $u, U_\ell = U \cup (u \backslash \{\eta_\ell\})$ so $G_{U,u} = G_{U_0^+} \dots + G_{U_{m(*)-1}}$. First, $G_{U,u} \subseteq G_{U \cup u}$ follows by the definitions. Second, we deal with proving $G_{U,u} \subseteq_{\rm pr} G_{U \cup u}$. So assume $z^* \in G, a^* \in \mathbb{Z}$ and $a^* z^*$ belongs to $G_{U_0} + \dots + G_{U_{m(*)}}$ so it has the form $\Sigma\{b_i x_{\bar\eta^i \restriction < m_i, n_i>} : i < i(*)\} + \Sigma\{c_j y_{\bar\eta_j, n_j} : j < j(*)\} + az$ with $i(*) < \omega, j(*) < \omega$ and $a^*, b_i, c_j \in \mathbb{Z}$ and $\nu_i, \bar\eta^i, \bar\eta_j$ are suitable sequences of members of $U_{\ell(i)}, U_{\ell(i)}, U_{k(j)}$ respectively where $\ell(i), k(j) < m(*)$. We continue as in [5].
6) Easy.
7) Clearly $U_1 \cup v = U_2 \cup u$ hence $G_{U_1 \cup u} \subseteq G_{U_1 \cup v} = G_{U_2 \cup u}$ hence $G_{U,u} + G_{U_1 \cup u}$ is a subgroup of $G_{U,u} + G_{U_2 \cup u}$, so the first quotient makes sense.

Hence $(G_{U,u} + G_{U_2 \cup u})/(G_{U,u} + G_{U_1 \cup u})$ is isomorphic to $G_{U_2 \cup u}/(G_{U_2 \cup u} \cap (G_{U,u} + G_{U_1 \cup u}))$. Now $G_{U_1,v} \subseteq G_{U_1 \cup v} = G_{U_2 \cup v} \subseteq G_{U,u} + G_{U_2,u}$ and $G_{U_1,v} \subseteq G_{U,v} = G_{U,v \backslash U} = G_{U,u} \subseteq G_{U,u} + G_{U_2,u}$. Together $G_{U_1,v}$ is included in their intersection, i.e.

$G_{U_2 \cup u} \cap (G_{U,u} + G_{U_1 \cup u})$ include $G_{U_1,v}$ and using part (1) both has the same divisible hull inside G^+. But as $G_{U_1,v}$ is a pure subgroup of G by part (5) hence of $G_{U_1 \cup v}$. So necessarily $G_{U_1 \cup u} \cap (G_{U,u} + G_{U_1,u}) = G_{U_1,v}$, so as $G_{U_2 \cup u} = G_{U_1 \cup v}$ we are done.
8) See [5].

■

Proof of 1.12 1) We prove this by induction on $|U|$; without loss of generality $|u| = k$ as also $k' = |u|$ satisfies the requirements.

Case 1: U is countable.

So let $\{\nu_\ell^* : \ell < k\}$ list u be with no repetitions, now if $k = 0$, i.e. $u = \emptyset$ then $G_{U\cup u} = G_U = G_{U,u}$ so the conclusion is trivial. Hence we assume $u \neq \emptyset$, and let $u_\ell := u \backslash \{\nu_\ell^*\}$ for $\ell < k$.

Let $\langle \bar\eta_t : t < t^* \leq \omega\rangle$ list with no repetitions the set $\Lambda_{U,u} := \{\bar\eta \in \Lambda^\times \cap {}^{k(*)+1}(U \cup u)$: for no $\ell < k$ does $\bar\eta \in {}^{k(*)+1}(U \cup u_\ell)\}$. Now comes a crucial point: let $t < t^*$, for each $\ell < k$ for some $r_{t,\ell} \leq k(*)$ we have $\eta_{t,r_{t,\ell}} = \nu_\ell^*$ by the definition of $\Lambda_{U,u}$, so $|\{r_{t,\ell} : \ell < k\}| = k < k(*) + 1$ hence for some $m_t \leq k(*)$ we have $\ell < k \Rightarrow r_{t,\ell} \neq m_t$ so for each $\ell < k$ the sequence $\bar\eta_t \restriction (k(*) + 1 \backslash \{m_t\})$ is not from $\{\langle \rho_s : s \leq k(*)$ and $s \neq m_t\rangle : \rho_s \in {}^\omega(U \cup u_\ell)$ for every $s \leq k(*)$ such that $s \neq m_t\}$.

For each $t < t^*$ we define $J(t) = \{m \leq k(*) : \{\eta_{t,s} : s \leq k(*) \,\&\, s \neq m\}$ is included in $U \cup u_\ell$ for no $\ell \leq k\}$. So $m_t \in J(t) \subseteq \{0, \dots, k(*)\}$ and $m \in J(t) \Rightarrow \bar\eta_t \restriction \{j \leq k(*) : j \neq m\} \notin {}^{k(*)+1\backslash\{m\}}(U \cup u_\ell)$ for every $\ell \leq k$. For $m \leq k(*)$ let $\bar\eta'_{t,m} := \bar\eta_t \restriction \{j \leq k(*) : j \neq m\}$ and $\bar\eta'_t := \bar\eta'_{t,m_t}$. Now we can choose $s_t < \omega$ by induction on t such that

$(*)$ if $t_1 < t, m \leq k(*)$ and $\bar\eta'_{t_1,m} = \bar\eta'_{t,m}$, then $\eta_{t,m} \restriction s_t \notin \{\eta_{t_1,m} \restriction \ell : \ell < \omega\}$.

Let $Y^* = \{x_{m,\bar\eta} \in G_{U\cup u} : x_{m,\bar\eta} \notin G_{U\cup u_\ell}$ for $\ell < k\} \cup \{y_{\bar\eta,n} \in G_{U\cup u} : y_{\bar\eta,n} \notin G_{U\cup u_\ell}$ for $\ell < k\}$.
Let

$$Y_1 = \{x_{m,\bar\eta,\nu} \in Y^* : \text{ for no} t < t^* \text{ do we have } m = m_t \,\&\, \bar\eta = \bar\eta'_t\}.$$

$$Y_2 = \{x_{m,\bar\eta,\nu} \in Y^* : x_{m,\bar\eta} \notin Y_1 \text{ but for no } t < t^* \text{ do we have } m = m_t \;\&\; \bar\eta = \bar\eta'_t \;\&\; \eta_{t,m_t} \restriction s_t \trianglelefteq \nu \triangleleft \eta_{t,m_t}\}$$

$$Y_3 = \{y_{\bar\eta,n} : y_{\bar\eta,n} \in Y^* \text{ and } n \in [s_t, \omega) \text{ for the } t < t^* \text{ such that } \bar\eta = \bar\eta_t\}.$$

Now the desired conclusion follows from

$(*)_1$ $\{y + G_{U,u} : y \in Y_1 \cup Y_2 \cup Y_3\}$ generates $G_{U\cup u}/G_{U,u}$

$(*)_2$ $\{y + G_{U,u} : y \in Y_1 \cup Y_2 \cup Y_3\}$ generates $G_{U\cup u}/G_{U,u}$ freely.

Proof of $(*)_1$. It suffices to check that all the generators of $G_{U\cup u}$ belong to $G'_{U\cup u} =: \langle Y_1 \cup Y_2 \cup Y_3 \cup G_{U,u}\rangle_G$.

First consider $x = x_{m,\bar\eta,\nu}$ where $\eta \in {}^{k(*)+1}(U \cup u), m < k(*)$ and $\nu \in {}^nS$ for some $n < \omega$. If $x \notin Y^*$ then $x \in G_{U,u_\ell}$ for some $\ell < k$ but $G_{U\cup u_\ell} \subseteq G_{U,u} \subseteq G'_{U\cup u}$ so we are done, hence assume $x \in Y^*$. If $x \in Y_1 \cup Y_2 \cup Y_3$ we are done so assume $x \notin Y_1 \cup Y_2 \cup Y_3$. As $x \notin Y_1$ for some $t < t^*$ we have $m = m_t \& \bar\eta = \eta'_t$. As $x \notin Y_2$, clearly for some t as above we have $\eta_{t,m_t} \restriction s_t \trianglelefteq \nu \triangleleft \eta_{t,m_t}$. Hence by Definition 1.6 the equation $\boxtimes_{\bar\eta_t,n}$ from Definition 1.6 holds, now $y_{\bar\eta_t,n}, y_{\bar\eta_t,n+1} \in G'_{U\cup u}$. So in order to deduce from the equation that $x = x_{\bar\eta'_t 1<m_t,n>}$ belongs to $G_{U\cup u}$, it suffices to show

that $x_{\bar\eta'_{t,j}\restriction<j,n>} \in G'_{U\cup u}$ for each $j \le k(*), j \ne m_t$. But each such $x_{\bar\eta'_{t,j}\restriction<j,n>}$ belong to $G'_{U\cup u}$ as it belongs to $Y_1 \cup Y_2$.
[Why? Otherwise necessarily for some $r < t^*$ we have $j = m_r, \bar\eta'_{t,j} = \bar\eta'_{r,m_r}$ and $\eta_{r,m_r} \restriction s_r \trianglelefteq \eta_t \restriction n \triangleleft \eta_{r,m_r}$ so $n \ge s_r$ and as said above $n \ge s_t$. Clearly $r \ne t$ as $m_r = j \ne m_t$, now as $\bar\eta'_{t,m_r} = \bar\eta'_{r,m_r}$ and $\bar\eta_t \ne \bar\eta_r$ (as $t \ne r$) clearly $\eta_{t,m_r} \ne \eta_{r,m_r}$. Also $\neg(r < t)$ by $(*)$ above applied with r, t here standing for t_1, t there as $\eta_{r,m_r} \restriction s_r \trianglelefteq \eta_{t,j} \restriction n \triangleleft \eta_{r,m_r}$. Lastly for if $t < r$, again $(*)$ applied with t, r here standing for t_1, t there as $n \ge m_t$ gives contradiction.]
So indeed $x \in G'_{U\cup u}$.

Second consider $y = y_{\bar\eta,n} \in G_{U\cup u}$, if $y \notin Y^*$ then $y \in G_{U,u} \subseteq G'_{U\cup u}$, so assume $y \in Y^*$. If $y \in Y_3$ we are done, so assume $y \notin Y_3$, so for some $t, \bar\eta = \bar\eta_t$ and $n < s_t$. We prove by downward induction on $s \le s_t$ that $y_{\bar\eta,s} \in G'_{U\cup u}$, this clearly suffices. For $s = s_t$ we have $y_{\bar\eta,s} \in Y_3 \subseteq G'_{U\cup u}$; and if $y_{\bar\eta,s+1} \in G'_{U\cup u}$ use the equation $\boxtimes_{\bar\eta_t,s}$ from 1.6, in the equation $y_{\bar\eta,s+1} \in G'_{U\cup u}$ and the x's appearing in the equation belong to $G'_{U\cup u}$ by the earlier part of the proof (of $(*)_1$) so necessarily $y_{\bar\eta,s} \in G'_{U\cup u}$, so we are done.

Proof of $(*)_2$ We rewrite the equations in the new variables recalling that $G_{U\cup u}$ is generated by the relevant variables freely except the equations of $\boxtimes_{\bar\eta,n}$ from Definition 1.6. After rewriting, all the equations disappear.

Case 2: U is uncountable.
As $\aleph_1 \le |U| \le \aleph_{k(*)-k}$, necessarily $k < k(*)$.
Let $U = \{\rho_\alpha : \alpha < \mu\}$ where $\mu = |U|$, list U with no repetitions. Now for each $\alpha \le |U|$ let $U_\alpha := \{\rho_\beta : \beta < \alpha\}$ and if $\alpha < |\mathcal{U}|$ then $u_\alpha = u \cup \{\rho_\alpha\}$. Now

$\odot_1$ $\langle (G_{U,u} + G_{U_\alpha\cup u})/G_{U,u} : \alpha < |U|\rangle$ is an increasing continuous sequence of subgroups of $G_{U\cup u}/G_{U,u}$.
[Why? By 1.10(6).]

$\odot_2$ $G_{U,u} + G_{U_0\cup u}/G_{U,u}$ is free.
[Why? This is $(G_{U,u} + G_{\emptyset\cup u})/G_{U,u} = (G_{U,u} + G_u)/G_{U,u}$ which by 1.10(8) is isomorphic to $G_u/G_{\emptyset,u}$ which is free by Case 1.]

Hence it suffices to prove that for each $\alpha < |U|$ the group $(G_{U,u} + G_{U_{\alpha+1}\cup u})/(G_{U,u} + G_{U_\alpha\cup u})$ is free. But easily

$\odot_3$ this group is isomorphic to $G_{U_\alpha\cup u_\alpha}/G_{U_\alpha,u_\alpha}$.
[Why? By 1.10(7) with $U_\alpha, U_{\alpha+1}, U, \rho_\alpha, u$ here standing for U_1, U_2, U, η, u there.]

$\odot_4$ $G_{U_\alpha\cup u_\alpha}/G_{U_\alpha,u_\alpha}$ is free.
[Why? By the induction hypothesis, as $\aleph_0 + |U_\alpha| < |U| \le \aleph_{k(*)-(k+1)}$ and $|u_\alpha| = k + 1 \le k(*)$.]

2) If $k(*) = 0$ just use 1.8, so assume $k(*) \geq 1$. Now the proof is similar to (but easier than) the proof of case (2) inside the proof of part (1) above.

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