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On the first eigenvalue for linear second order elliptic equations in divergence form

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ABSTRACT

Estimates of the first eigenvalue for linear second-order elliptic equations in divergence form are investigated and some qualitative properties in dependence of the coefficients of the equation are proved. As an application of new formulas for the first eigenvalue, its asymptotic with respect to the large drift is obtained.

RESUMEN

Se estudia la estimación del primer autovalor para la ecuación lineal elíptica de segundo orden en la forma divergente y se prueban algunas propiedades cualitativas con dependencia en los coeficientes de la ecuación. Como una aplicación de las fórmulas obtenidas para el primer valor propio, se obtiene su desarrollo asintótico respecto de grandes desviaciones.

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1 Introduction

Let L be a linear second-order uniformly elliptic operator in divergence form

$$Lu = -\left(a_j^k(x)u_{x_k} + a_j^0(x)u\right)_{x_j} + b^j(x)u_{x_j} + b^0(x)u \tag{1}$$

in Ω, where $a_j^k(x)\xi^j\xi^k \geq \mu|\xi|^2$ for every $x \in \overline{\Omega}$, $\xi \in R^n$, $\mu = const > 0$. Here Ω is a bounded domain in $\mathbf{R}^n$, $\partial\Omega \in C^{1,1}$,

$$a_j^k(x),\ a_j^0(x) \in W^{1,\infty}(\Omega),\ \ b^k(x),\ b^0(x) \in L^\infty(\Omega), \tag{2}$$

and under the repeating indices the summation convention is understood.

The paper is concerned with some new formulas for the first eigenvalue λ for the operator L with zero Dirichlet conditions on $\partial\Omega$

$$\begin{cases} Lu = \lambda u \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases} \tag{3}$$

The motivation of this study is the validity of the comparison and maximum principle for L. It is well-known, see [2], that the maximum principle for the operator L holds if and only if the first eigenvalue λ of L with zero Dirichlet data is positive. It is clear that the positiveness of the first eigenvalue λ is not easy checkable condition. However, there are some qualitative properties of λ which are used to find out lower and upper bounds for the first eigenvalue, see for example [2].

There are a general formulae for λ , see for example [2, 7, 12],

$$\lambda = \sup_{\phi>0} \inf_x (L\phi/\phi)\ ,\ \phi \in C^2(\Omega) \tag{4}$$

and there are results which are only sufficient for a wide class of equations. They are given, for example, in [5, 7, 12, 13] (see also the references there) and guarantee positiveness of the first eigenvalue and correspondingly the validity of the maximum principle for (1).

The aim of the paper is to obtain some new formulas for λ and to investigate precise dependence of λ on the coefficients a_j^k , a_j^0, b^j , b^0 .

The main results are in Section 3, in Theorem 3.1, where three equivalent formulas for the first eigenvalue λ for nonsymmetric operators are obtained. They are different from the well known results and are more convenient for lower and upper estimates for λ in many cases. Such estimates are shown in Theorem 3.2. Moreover λ is obtained as an extremum of the first eigenvalues of some explicitly given symmetric operators. This is the reason to present the well-known results for the first eigenvalue for symmetric operators, shortly, but from slightly different point of view in Section 2, see [12, 13].

Using the derived expressions for λ we study the dependence of λ on the coefficients of L in Propositions 4.1, 4.2 and the behavior of L with respect to the large parameter in the non symmetric terms, in Proposition 4.3, 4.4 in Section 4. As an application of the concavity result in Proposition 4.1 are shown in Corollary 4.1 some new sufficient conditions for comparison principle for quasilinear equations.

2 Preliminary notes and symmetric operators

We start with some notations and definitions. For matrix $P = \{P_{jk}(x)\}_{j,k=1}^n$, vector $q = \{q_j(x)\}_{j=1}^n$ and function $w(x)$ we'll use the notations:

$$Pq = \{\sum_{k=1}^{n} P_{jk}q_k\}_{j=1}^n, Pq^2 = \sum_{j,k=1}^{n} q_j P_{jk} q_k,$$

and $q\nabla w = \sum_{j=1}^n q_j w_{x_j}$. Let us denote $d = (a^0 + b)/2$, $a = (A + A^t)/2$, $\alpha = a^{-1}$, $c = (b - a^0)/2 + \partial' Q/2$ where $Q = (A - A^t)/2$, $A = \{a_j^k\}_{j,k=1}^n$ and ∂' means divergence in columns of Q and ^{t} means transposition. The operator L in (1) can be written down as $L = L_c = L_0 + N_c$, $L_0 = (L + L^*)/2$ and $N_c = (L - L*)/2$, correspondingly $L^* = L_c^* = L_0 + N_{-c} = L_{-c}$ where operators L_0 and N_c have the form

$$L_0 u = -\partial(a\nabla u + du) + d\nabla u + b^0 u, \quad N_c u = \partial(cu^2)/u. \tag{5}$$

So the operator L is represented as a sum of symmetric and skew-symmetric parts, i.e. $(u, L_0 v) = (L_0 u, v)$, $(u, N_c v) = -(N_c u, v)$ for $u, v \in H_0^1(\Omega)$, here $(u, w) = \int_\Omega uw dx$.

Further the first eigenvalue of the operator L with zero Dirichlet data in the domain Ω is denoted by $\lambda(L;\Omega)$ and shortly $\lambda(L)$ if the domain Ω is fixed.

It is interesting to write down some well-known operators with positive first eigenvalues - we'll use them partly in the future

$$\begin{aligned} M_g u &= -\partial(A^*\nabla u + 2gu) \\ M_g^* u &= -\partial A\nabla u + 2g\nabla u \\ M(d)u &= -\partial(A^*\nabla u + du) + d\nabla u + \alpha d^2 u, \end{aligned} \tag{6}$$

where A is a nonsymmetric operator in generally. The operator M_g corresponds to the operator A with coefficients $c = d = g$, $b^0 = 0$. The positiveness of its first eigenvalue follows from the formula (4) since

$$\lambda(M_g) = \sup_{u>0} \inf_x (M_g u/u) > M_g 1 = 0.$$

Our basic aim is to derive formulas and estimates for the first eigenvalue of the operator L_c in Ω connecting them with the first eigenvalues of suitably chosen symmetric operators. For this purpose let us note that operator L_c is invariant under every of the transfer couples

$$\begin{aligned}&\{d \to d + f;\ b^0 \to b^0 + \partial f\}\\ &\{A \to A + S;\ c \to c + \partial' S/2\},\ S^t = -S.\end{aligned} \tag{7}$$

Indeed $N_{c+\partial^t S/2}u = N_c u - \partial_j S_{jk} u_{x_k}$, since

$$\partial_j S_{jk} u_{x_k} = (\partial_j S_{jk}) u_{x_k} = (1/u)\partial_k(\partial_j S_{jk}/2)u^2.$$

So estimates and properties of the first eigenvalue of L should be preserved under these changes. Moreover an appropriate extremum over the admissible vectors f' s and skew-symmetric matrices S' s will lead to sharp estimates. The class of such vectors f and skew-symmetric matrices S is one and the same

$$F = \{f, S \in L^\infty(\Omega); \partial f, \partial' S \in L^\infty(\Omega)\}. \tag{8}$$

As a beginning let us recall the variational formula of the first eigenvalue $\lambda(L_0)$ for symmetric operator L_0

$$\lambda(L_0) = \inf_v B_{L_0}[v, v],\ v \in H_0^1(\Omega),\ \|v\|_{L^2} = 1, \tag{9}$$

where $B_{L_0}[u, v]$ is the bilinear form for L_0.

In fact (9) is valid if the coefficients of L_0 satisfy

$$a,\ d,\ b^0 \in L^\infty(\Omega). \tag{10}$$

Let us note that positiveness of $\lambda(L_0)$ is sufficient for positiveness of $\lambda(L_c)$. Indeed, let ϕ is the first eigenfunction of L_c, i.e. $L_c\phi = \lambda(L_c)\phi$, $\phi \in H_0^1(\Omega)$, $\|\phi\|_{L^2} = 1$. Since $B_{L_0}[u, u] = B_{L_c}[u, u]$ for every $u \in H_0^1(\Omega)$, then $\lambda(L_0) = \inf_u B_{L_0}[u, u] \le B_{L_c}[\phi, \phi] = \lambda(L_c)$, so

$$\lambda(L_0) \le \lambda(L_c). \tag{11}$$

In the following proposition we formulate the qualitative properties of $\lambda(L_0)$ which we'll need further.

Proposition 2.1 *Let the coefficients of the operator L_0 satisfy (10). Then*

(i) λ_{L_0} is a continuous function of a, d, b^0 and Ω in the L^∞ norm;

(ii) λ_{L_0} is a monotone increasing function with respect to a, b^0, monotone decreasing on the domain inclusions and a concave one with respect to a, d, b^0.

The continuous dependence follows from the variational formula (9). The monotonicity of λ_{L_0} with respect to the domain Ω is well-known even under weaker assumptions. The concavity of the first eigenvalue with respect to the coefficient b^0 was proved for general nonsymmetric operators in Proposition 2.1 in [2].

As it is well-known, see for example [7], the infinum in (9) is attained for a positive function $u \in H_0^1(\Omega)$, which in the weak sense solves the equation

$$L_0 u = \lambda(L_0)u \text{ in } \Omega, u = 0 \text{ on } \partial\Omega.$$

If the coefficients of the operator L_0 satisfy smoothness conditions $a, d \in W^{1,\infty}(\Omega)$, $b^0 \in L^\infty(\Omega)$ the "max-min" representation formula, see (4) for the first eigenvalue $\lambda(L_0)$ holds

$$\lambda(L_0) = \sup_v \operatorname{ess}\inf_x (L_0 v/v), \ v \in W^{2,n}(\Omega), \ v > 0. \tag{12}$$

see [3], [9] and [10] for more details.

The integrand in (9), written as $J = (\nabla u, u)\begin{pmatrix} a & d \\ d & b^0 \end{pmatrix}\begin{pmatrix} \nabla u \\ u \end{pmatrix}$ can be estimated from below by $\inf_x(b^0 - \alpha d^2)u^2$, recall that $\alpha = a^{-1}$. Hence $\inf_x(b^0 - \alpha d^2) \leq \lambda(L_0)$ and the invariant change (7): $d \to d + f$, $b^0 \to b^0 + \partial f$, $f \in F$, leads to

$$\sigma(L_0) \leq \lambda(L_0). \tag{13}$$

Here it is denoted

$$\sigma(L_0) = \sup_f \inf_x \sigma(L_0; f) \text{ and } \sigma(L_0; f) = b^0 + \partial f - \alpha(d + f)^2. \tag{14}$$

Note that (13) is valid for the operator L_0 with bounded coefficients satisfying (10) and the imbedding theorems allow us to weaken once more the conditions (10) as $a \in L^\infty(\Omega)$, $d \in L^n(\Omega)$, $b^0 \in L^{n/2}(\Omega)$. Then $\sigma(L_0; f)$, $f \in F$ has the sense of essentially bounded from bellow function from $L^{n/2}$.

To get the estimate (13) for the first eigenvalue is the idea of Protter [12] which he developed further for some nonlinear problems. For completeness we'll add the proof that actually (13) is an equality:

Proposition 2.2 *Let the coefficients of operator L_0 satisfy (10). Then*

$$\lambda(L_0) = \sigma(L_0). \tag{15}$$

Proof. We will use a special choice of f for the operator $\bar{L}_0$ with smooth coefficients. Let us assume that $\bar{a}, \bar{d} \in W^{1,\infty}(\Omega_1)$ $b^0 \in L^\infty(\Omega_1)$ are extended in a wider smooth domain $\Omega_1 \supset \Omega$ preserving strong ellipticity . For every positive constant $\delta > 0$, there exists a smooth domain Ω_δ, $\Omega_1 \supset \Omega_\delta \supset \bar{\Omega}$ such that $\lambda(\bar{L}_0; \Omega_\delta) \geq \lambda(\bar{L}_0) - \delta$ see Chapter VI, Theorem 3 in [4]. Let u be the first eigenfunction of $\bar{L}_0$ in Ω_δ. From the Sobolev's imbedding theorems, see Theorem 5, Section 5.6.2 in [7], it follows $u \in W^{2,p}_{loc}(\Omega_1)$ for every finite p and hence $u \in C^1(\bar{\Omega})$. Since $u > 0$ in $\bar{\Omega}$ and $\bar{L}_0 u = \lambda(\bar{L}_0; \Omega_\delta)u$ easy calculations give us that $\bar{f} = -\bar{a}\nabla u/u - \bar{d} \in F$. Indeed

$$\partial \bar{f} = -\partial(\bar{a}\nabla u + \bar{d}u)/u + (\bar{a}(\nabla u)^2)/u^2 + \bar{d}\nabla u/u$$
$$= \lambda(\bar{L}_0; \Omega_\delta) + (\bar{a}(\nabla u)^2)/u^2 - b^0 \in L^\infty(\Omega)$$

and $\sigma(\bar{L}_0; f) = \bar{L}_0 u/u = \lambda(\bar{L}_0; \Omega_\delta) \geq \lambda(\bar{L}_0) - \delta$. This and (13) show that equality $\sigma(\bar{L}_0) = \lambda(\bar{L}_0)$ holds for an operator with smooth coefficients.

To prove this equality (15) for L_0, we choose for every $\varepsilon > 0$ an operator $\bar{L}_0$ such that $\bar{a} > a$, $(\bar{a} - a)^{-1}(\bar{d} - d)^2 < \varepsilon$. Then from Proposition 2.1 and (13) we receive

$$\sigma(\bar{L}_0) = \lambda(\bar{L}_0) \geq \lambda(L_0) + \lambda(\bar{L}_0 - L_0) \geq \lambda(L_0) + \sigma(\bar{L}_0 - L_0)$$

$$\geq \lambda(L_0) - \sup_x (\bar{a} - a)^{-1}(\bar{d} - d)^2 > \lambda(L_0) - \varepsilon.$$

Then for $\bar{a}^{-1} \to a^{-1}$ and $\bar{d} \to d$ in L^∞ for $\varepsilon \to 0$ we get $\sigma(\bar{L}_0) \to \sigma(L_0)$ and $\sigma(L_0) \geq \lambda(L_0)$. ∎

In fact $\sigma(L_0)$ gives a different expression for the first eigenvalue $\lambda(L_0)$ of a symmetric operator L_0. The advantage of the formula (14) in comparison with (12) is the possibility to find out a lower bound for the first eigenvalue $\lambda(L_0)$ by means of an appropriate choice of a vector f in (14) , instead of the choice of a scalar function in (12).

As a consequence of Propositions 2.1 and 2.2 the following monotonicity result holds for the first eigenvalue.

Corollary 2.1 *Let the coefficients of $L_0, \tilde{L}_0$ satisfy (10). If $a > \tilde{a}$ and $b^0 \geq \tilde{b}^0 - \partial f - (\tilde{a} - a)^{-1}(f + \tilde{d} - d)^2$ in Ω for some $f \in F$, then $\lambda(L_0) \geq \lambda(\tilde{L}_0)$.*

Finally, using Propositions 2.2 and Theorem 3.1 we show bellow that supremum in (12) is not attaining at a single $f \in F$. More precisely

Corollary 2.2 *Let M_{f+d} is the operator in (6) with $A = a$, then*

$$\lambda(L_0) = \sup_{f \in F}[\lambda(M_{f+d}) + \inf_x \sigma(L_0; f)]. \tag{16}$$

In particular, if $\sigma(L_0; f) \geq 0$ for some $f \in F$ then $\lambda(L_0) > 0$.

Proof. From Theorem 3.1 it follows $\lambda(L_c) \leq \lambda(L_0 + \alpha c^2)$. Transferring b^0 to $b^0 - \alpha c^2$ we have $\lambda(L_c - \alpha c^2) \leq \lambda(L_0)$ for every $c \in L^\infty(\Omega)$. Since

$$L_{f+d} - \alpha(f + d)^2 = M_{f+d} + b^0 + \partial f - \alpha(f + d)^2 = M_{f+d} + \sigma(L_0; f)$$

and

$$\inf_x \sigma(L_0; f) < \lambda(M_{f+d}) + \inf_x \sigma(L_0; f) \leq \lambda(M_{f+d} + \sigma(L_0; f))$$
$$= \lambda(L_{f+d} - \alpha(f + d)^2) \leq \lambda(L_0)$$

then maximizing these inequalities in $f \in F$ we get (16).

Let's add that although $\lambda(M_{f+d})$ is strictly positive, it tends to 0 for f's such that $\inf_x \sigma(L_0; f)$ tends to $\lambda(L_0)$. ∎

3 Nonsymmetric operators

An equivalent definition of the first eigenvalue of L_c as in Proposition 2.2 by means of (15) is no more possible for general nonsymmetric operators (1). The corresponding expression for $\sigma(L_c)$ is more complicated. In this chapter we'll assume that the coefficients of L_c in Ω are sufficiently smooth, to ensure us that the corresponding first eigenfunction belongs to $C^1_{loc}(\Omega)$.

For $v \in H_0^1(\Omega)$, let us define the sets of functions

$$M(v;\Omega) = \{z \in W_{loc}^{1,2}(\Omega) : \int_\Omega a(\nabla z)^2 v^2 dx < \infty\},$$
$$M^*(v;\Omega) = \{h \in L_{loc}^2(\Omega) : \int_\Omega a h^2 v^2 dx < \infty\},$$
$$M_0^*(v;\Omega) = \{h \in M^*(v;\Omega) : N_h(v) = 0\}.$$

where the condition on the divergence in $N_h(v)$ is in "weak sense", i.e. $\int_\Omega (h\nabla z)v^2 dx = 0$, for every $z \in M(v,\Omega)$. The class $M_0^*(v,\Omega)$ - zero functionals on $M(v,\Omega)$ is obviously non-empty. It is easy to prove by induction that $h \in M_0^*(v,\Omega)$ *iff* $h_j = (\partial_k S_{jk} v^2)/v^2$ where $S = S_{jk}$ is bounded skew-symmetric matrix.

For example, let $\bar{z} = \ln(\phi/\psi), \bar{v} = \sqrt{\phi\psi}$, where the functions ϕ, ψ are the first eigenfunctions in Ω of L_c and L_c^* respectively. According to the forthcoming Lemma 3.1 $\bar{v} \in H_0^1(\Omega)$, $\bar{z} \in L^\infty(\Omega) \cap M(v,\Omega)$ and for $c \in L^\infty(\Omega)$ it holds $c - (1/2)a\nabla\bar{z} = h \in M_0^*(v,\Omega)$. Moreover if the coefficients of L_c are sufficiently smooth then $\bar{z} \in L^\infty(\Omega) \cap C_{loc}^1(\Omega)$.

Let us consider all operators L_c^z derived by L_c with a nondegenerate transformation $L_c^z u = e^{-z/2} L_c(ue^{z/2})$ for every $z \in C^{0,1}(\bar{\Omega})$ which preserve the first eigenvalue of L_c i.e. $\lambda(L_c) = \lambda(L_c^z)$. There exists a transformation with the extreme property such that the new transformed nonsymmetric operator L_c^z has the same first eigenvalue as its symmetric part $L_0^z = (L_c^z + L_c^{z,*})/2$. Thus Proposition 2.2 is applicable for L_0^z as well for L_c^z. The nonsymmetry of the operator $L_c = L_0 + N_c$ results from the vector c. We'll start with its representation in order to find a suitable transformation function z.

Lemma 3.1 *For every symmetric positively defined matrix $a \in W^{1,\infty}(\Omega)$ and for every $c \in L^\infty(\Omega)$, there exist $v \in H_0^1$, $v > 0$, $z \in L^\infty(\Omega) \cap M(v,\Omega)$, $h \in M_0^*(v,\Omega)$ such that*

$$c = (a\nabla z)/2 + h. \tag{17}$$

Proof. For an arbitrary $z \in C^{0,1}(\bar{\Omega})$ we denote $L_c^z u = e^{-z/2} L_c(ue^{z/2})$ and the computations show $L_c^z = L_0^z + f_c(\nabla z) + N_h = L_h^z + f_c(\nabla z)$ with $h = c - (a\nabla z)/2$ and $f_c(\xi) = c\xi - (a\xi^2)/4$. The first eigenvalues of L_c, L_c^*, L^z, $L^{z,*}$ are one and the same numbers and the corresponding first eigenfunctions are ϕ, ψ, $e^{-z/2}\phi$, $e^{-z/2}\psi$. Hence

$$\lambda(L_c) = \lambda(L_h + f_c(\nabla z)). \tag{18}$$

Since $\lambda(L_0) \le \lambda(L_c)$ according to (11) we get

$$\lambda(L_0 + f_c(\nabla z)) \le \lambda(L_c). \tag{19}$$

For $v = \sqrt{\phi\psi}$, $z = \ln(\phi/\psi)$ it holds $v \in H_0^1(\Omega)$ and $z \in M(v,\Omega)$. Moreover $(L_h + f_c(\nabla z))v = (L_h^* + f_c(\nabla z))v$. So $N_h v = 0$ and $(L_0^z + f_c(\nabla z))v = \lambda(L_c^z)v$. This proves the lemma and leads to the formulation of the main theorem. ∎

Define $\sigma(L_0 + f_c(\nabla z)) = \sup_g \inf_x \sigma(L_0 + f_c(\nabla z); g)$ and $\sigma(L_c) = \sup_z \sigma(L_0 + f_c(\nabla z))$.

Theorem 3.1 *Let the nonsymmetric operator L satisfies (2). Then*

(a) $\lambda(L_c) = \sup_z \lambda(L_0 + f_c(\nabla z)) = \sigma(L_c)$, $z \in C^{0,1}(\bar{\Omega})$;

(b) $\lambda(L_c) = \inf_u (B_{L_0}[u,u] + \beta(u^2))$, $u \in H_0^1(\Omega)$, $\|u\|_{L^2} = 1$, *where*

$$\beta(u^2) = \sup_z \frac{(\int_\Omega c\nabla z u^2 dx)^2}{\int_\Omega a(\nabla z)^2 u^2 dx} = \inf_h \int_\Omega \alpha(c-h)^2 u^2 dx,$$

$$z \in C^{0,1}(\bar{\Omega}) \ or \ z \in M(u;\Omega), h \in M_0^*(u;\Omega);$$

(c) $\lambda(L_c) = \inf_S \lambda(L_0 + M_0(S))$, $S^t = -S$, *is bounded matrix and* $M_0(S)$ *is the symmetric operator defined by*

$$B_{M_0(S)}[u,u] = \int_\Omega \alpha((c - \partial' S/2)u - S\nabla u)^2 dx,$$

where $(\partial' S)_j = \partial_l S_{jl}$

Proof. (a) Let us mention that the first equality in (a) can be derived directly from [4]. But nevertheless for completeness the proof is included. it should be: We proceed as in Section 2, for smoothly extended coefficients in some $\Omega_\delta \supset \bar{\Omega}$, such that $\lambda(L_c;\Omega_\delta) \geq \lambda(L_c) - \delta$ and use the representation (17) of c in Ω_δ. The corresponding z and v, $z \in C^{0,1}(\bar{\Omega})$, $v \in W^{2,p}(\Omega)$, $p < \infty$ and $\lambda(L_c;\Omega_\delta)v = (L_0 + f_c(\nabla z))v$ in Ω where $v > 0$. Then from (4) we get $\lambda(L_c;\Omega_\delta) \leq \lambda(L_0 + f_c(\nabla z))$, so $\lambda(L_c) - \delta \leq \lambda(L_0 + f_c(\nabla z)) \leq \sup_z \lambda(L_0 + f_c(\nabla z))$.

(b) Consider the sequence of inequalities, starting from the proved above

$$\lambda(L_c) \leq \sup_z \lambda(L_0 + f_c(\nabla z))$$

$$\leq \inf_u \{B_{L_0}[u,u] + \sup_z [(\int_\Omega c\nabla z u^2 dx)^2 / \int_\Omega a(\nabla z)^2 u^2 dx]\}$$

$$\leq \inf_u \{B_{L_0}[u,u] + \sup_z{}' [(\int_\Omega c\nabla z u^2 dx)^2 / \int_\Omega a(\nabla z)^2 u^2 dx]\}$$

$$\leq \inf_u \{B_{L_0}[u,u] + \inf_h \int_\Omega \alpha(c-h)^2 u^2 dx\} \leq B_{L_0}[u,u] + \int_\Omega \alpha(c-\bar{h})^2 v^2 dx.$$

Here $\sup_z$ is over $z \in C^{0,1}(\bar{\Omega})$; $\sup_z'$ is over $z \in M(u;\Omega)$; $\inf_h$ is over $h \in M_0^*(u;\Omega)$; $\bar{h} = c - (a\nabla\bar{z})/2$, $\bar{z} = \ln(\phi/\psi)$, $\bar{v} = \sqrt{\phi\psi}$ (normed), so $\bar{h} \in M_0^*(\bar{v};\Omega)$. The first inequality is due to the change of extremums and the majorization of $\int_\Omega f_c(k\nabla z)u^2 dx$ over k's ; the second inequality is due to the fact that $C^{0,1}(\bar{\Omega}) \subset M(u;\Omega)$; and the third inequality follows from

$$\int_\Omega c(\nabla z)u^2 dx = \int_\Omega (c-h)(\nabla z)u^2 dx \leq \sqrt{\int_\Omega \alpha(c-h)^2 u^2 dx}\sqrt{\int_\Omega a(\nabla z)u^2 dx}.$$

Easy computations show that the last expression in the chain above is $(\psi, L_c\phi) = \lambda(L_c)$ since for the chosen $\bar{z}$, $\bar{v}$ it holds

$$\int_\Omega \alpha(c-\bar{h})\bar{v}^2 dx = \frac{1}{4}\int_\Omega a(\nabla\bar{z})^2\bar{v}^2 dx = \int_\Omega c(\nabla\bar{z})^2\bar{v}^2 dx)^2 / \int_\Omega a(\nabla\bar{z})^2\bar{v}^2 dx$$

$$= \int_\Omega f_c(\nabla\bar{z})^2\bar{v}^2 dx = \int_\Omega (\psi c\nabla\phi - \phi c\nabla\psi)dx + \frac{1}{4}\int_\Omega a[(\nabla\phi/\phi) - (\nabla\psi/\psi)]\phi\psi dx.$$

(c) The proof is based on the representation of any $h \in M_0^*(u;\Omega)$ as $(\partial' Su^2)/2u^2$.

∎

Remark 3.1 Recall that $c = (b-a^0)/2+\partial' Q/2$ with $Q = (A-A^t)/2$. The divergence of Q in the expression of $\sigma(L_c)$ can be avoided. Indeed by the change $f \to f+Q\nabla z/2$ we'll get

$$\sigma(L_c) = \sup_{f,z}\inf_x[b^0 + \partial f - \alpha(f+d+Q\nabla z/2)^2 + f_{\bar{c}}(\nabla z)],$$

where $\bar{c} = (b-a^0)/2$. The same can be done in (c) if we take $S+Q$ instead of S.

Using Theorem 3.1 we can prove the next theorem.

Theorem 3.2 *The following inequalities hold*

$$\lambda(L_0) \leq \lambda(L_c) \leq \lambda(L_0 + \alpha c^2). \tag{20}$$

Moreover with ϕ_0, ϕ, ψ, the first eigenfunctions of L_0, L_c, L_c^* resp., we have

(i) $\lambda(L_c) = \lambda(L_0)$ iff $\phi = \phi_0$ or $\psi = \psi_0$ in Ω, iff $c \in M_0^*(v;\Omega)$ in Ω with $v = \phi_0$ or $v = \phi$ or $v = \psi$;

(ii) $\lambda(L_c) = \lambda(L_0 + \alpha c^2)$ iff $c = a\nabla p/2$ for some $p \in C^{0,1}(\Omega)$. In this case $p = \ln(\phi/\psi)$.

Remark 3.2 Recall that in (20) c has the form $c = \bar{c} + \partial' Q/2$ with $\bar{c} = (b-a^0)/2$, $Q = (A-A^t)/2$. Moreover, applying Theorem 3.1 (c), the divergence of Q can be avoided as was mention in the proof. For $S = 0$ in (c) we get $\lambda(L_c) \leq \lambda(\bar{L}_0 + \alpha\bar{c}^2)$ where $\bar{L}_0 u = -\partial(A^*\alpha A\nabla u + du + \bar{c}\alpha Qu) + (d+\bar{c}\alpha Q)\nabla u + b^0 u$ with $A = a + Q$.

Proof of Theorem 3.2. The inequalities (20) follow immediately from (11) , Theorem 3.1 (a) and the estimate

$$f_c(\nabla z) = c\nabla z - a(\nabla z)^2/4 \leq \alpha c^2. \tag{21}$$

(i) The first statement holds due to the uniqueness of the first eigenfunction up to multiplication with a constant and the following statement

$$L_c u = L_0 u \quad \textit{iff} \quad c \in M_0^*(u;\Omega), \text{ i.e. } N_c u = \partial(cu^2)/u = 0. \tag{22}$$

For instance $N_c\phi_0 = 0$ *iff* $L_c\phi_0 = L_0\phi_0 = \lambda_0\phi_0$ *iff* $\phi = \phi_0$ and

$$B_{L_0}[\phi_0,\phi_0] - B_{L_0}[\phi,\phi] = B_{L_0}[\phi_0,\phi_0] - B_{L_c}[\phi,\phi] = \lambda(L_0) - \lambda(L_c)$$

for normed ϕ, ϕ_0.

(ii) Suppose that c is a-potential vector i.e. $c = a\nabla p/2$, $p \in C^{0,1}(\bar{\Omega})$. Then since $f_c(\nabla p) = a(\nabla p)^2/4 = \alpha c^2$, Theorem 3.1 (a) gives

$$\lambda(L_c) \geq \lambda(L_0 + f_c(\nabla p)) = \lambda(L_0 + \alpha c^2). \tag{23}$$

The right hand side of (20) together with (23) leads to equality. Moreover, the representation (17) of c shows that correspondingly $h = a\nabla(p-z)/2$. But there exists $v > 0$, $v \in H_0^1(\Omega)$ such that $\int_\Omega h(\nabla q)v^2dx = 0$, in particular for $q = p - z \in M(v;\Omega)$, so $|\nabla(p-z)| = 0$ a.e. in Ω.

Let now $\lambda(L_c) = \lambda(L_0 + \alpha c^2)$ and φ is the first eigenfunction of $L_0 + \alpha c^2$. We'll go back to the proof of Theorem 3.1 and denote by z_δ the corresponding component in an extended domain $\Omega_\delta \supset \bar{\Omega}$, so for small $\delta > 0$

$$\lambda(L_c) - \delta \le \lambda(L_0 + f_c(\nabla z_\delta)) = \inf_v B_{L_0+f_c(\nabla z_\delta)}[v,v]$$

$$\le B_{L_0}[\varphi,\varphi] + \int_\Omega f_c(\nabla z_\delta)\varphi^2dx = \lambda(L_0 + \alpha c^2) - \int_\Omega \alpha[c - a(\nabla z_\delta)^2/2]^2\varphi^2dx$$

Under the assumption above $\int_\Omega \alpha[c - a(\nabla z_\delta)^2/2]^2\varphi^2dx < \delta$ and for every smooth $\omega, \bar{\omega} \subset \Omega$ it holds $\int_\omega \alpha[c - a(\nabla z_\delta)^2/2]^2dx < \delta/m^2$ $m = \inf_\omega \varphi > 0$. So $\nabla z_\delta \to 2\alpha c$ in $L^2(\omega)$. According to the generalized Poincare inequality, see [11] there exist constants K_δ such that $z_\delta - K_\delta \to z$ in $L^2_{loc}(\omega)$. Note that these constants depend only on appriori and arbitrary fixed open set $\omega_0 \ne \emptyset$ if for all smooth domains in question $\omega \supset \bar{\omega}_0$. Hence $2\alpha c = \nabla z$ a.e. in Ω. ∎

Remark 3.3 In the case c is a-potential vector, $\nabla z = (\nabla\phi/\phi) - (\nabla\psi/\psi)$ is bounded in Ω, where ϕ, ψ are the first eigenfunction's of L_c, L_c^* respectively, $\phi = \psi = 0$ on $\partial\Omega$.

An open question is to characterize the conditions guaranteeing when $\lambda(L)$ coincides with a strictly interior point of the interval $(\lambda(L_0), \lambda(L_0 + \alpha c^2))$ i.e. $\lambda(L_c) = \lambda(L_0 + g)$ for some $0 \le g \le \alpha c^2$. However, by means of a family of equations having one and the same "maximal operator" $L_0 + \alpha c^2$ the following example illustrates that the first eigenvalue $\lambda(L_c)$ covers the whole interval.

Example 3.1 Consider $\Omega = B_1 \subset R^2$, $L_0 = -\Delta$, $\rho = |x|$, $p^2 + q^2 = 1$ and let $c = (pI + qS)x$, with $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Since $c = g + h$ with $g = px = \nabla(p\rho^2)/2$, then $\lambda(L_c) \le \lambda(L_0 + \alpha c^2) \le \lambda(L_0 + \rho^2)$ and from (18) $\lambda(L_c) = \lambda(L_h + p^2\rho^2) \ge \lambda(L_0 + p^2\rho^2)$, with $h = qSx$. If ϕ is the first eigenfunction of $L_0 + p^2\rho^2$ then $\phi = \phi(\rho)$ and $N_h\phi(\rho) = 2\phi' h\nabla\rho + \phi\partial h$, $h\nabla\rho = qSx(x/\rho) = 0$, $\partial h = q\mathrm{div}(x_2, -x_1) = 0$. So $h \in M_0^*(\phi;\Omega)$ and from Theorem 3.2 (i) the equality $\lambda(L_c) = \lambda(L_0 + p^2\rho^2)$.

4 Properties of the first eigenvalue

In this section we'll give some applications of Theorems 3.1, 3.2 in order to obtain some qualitative properties of the first eigenvalue of L.

4.1 Dependance of $\lambda(L_c)$ on the coefficients

Let us start with the following concavity result.

Proposition 4.1 *Let $L^{(k)}$ be operators with one and the same symmetric matrix a, $d^{(k)}$, $c^{(k)}$, $b^{0(k)}$ as coefficients and $\lambda(L^{(k)})$ are their's first eigenvalues, $k = 0, 1$. Denote $L^{(s)} = (1-s)L^{(0)} + sL^{(1)}$ then*

$$(1-s)\lambda(L^{(0)}) + s\lambda(L^{(1)}) \leq \lambda(L^{(s)} + s(1-s)\alpha(c^{(1)} - c^{(0)})^2). \tag{24}$$

Proof. The proposition follows from the concavity of $\lambda(L_c - \alpha c^2) = \sup_z \lambda(L_0 - \alpha(c - a\nabla z/2)^2)$ in the coefficients d and b^0, see Proposition 2.1. Indeed, for $\varepsilon > 0$ there exist $f^{(k)}$, $z^{(k)}$ such that for $k = 0, 1$

$$\begin{aligned} T^{(k)} &\equiv b^{0(k)} + \partial f^{(k)} - \alpha(f^{(k)} + d^{(k)})^2 - \alpha(c^{(k)} - a\nabla z^{(k)})^2 \\ &\geq \lambda(L^{(k)} - \alpha c^{(k)2}) - \varepsilon. \end{aligned} \tag{25}$$

The expressions in (25) are linear in $b^{0(k)}$ and concave in $f^{(k)}$, $d^{(k)}$, $c^{(k)}$, $\nabla z^{(k)}$. Denote by $g^{(s)} = (1-s)g^{(0)} + sg^{(1)}$ for vectors $g^{(k)}$, $k = 0, 1$ and by $T^{(s)}$ the expression as (25) with terms $g^{(s)}$ instead of $g^{(k)}$. Then we'll have

$$T^{(s)} \geq (1-s)T^{(0)} + sT^{(1)} \geq (1-s)\lambda(L^{(0)} - \alpha c^{(0)2}) + s\lambda(L^{(1)} - \alpha c^{(1)2}) - \varepsilon.$$

It proves the concavity of $\lambda(L_c - \alpha c^2)$ i.e.

$$\lambda(L^{(0)} - \alpha c^{(0)2}) + s\lambda(L^{(1)} - \alpha c^{(1)2}) \leq \lambda(L^{(s)} - \alpha c^{(s)2}). \tag{26}$$

We change $b^{0(k)}$ to $b^{0(k)} + \alpha c^{(k)2}$ and then add $(1-s)\alpha c^{(0)2} + s\alpha c^{(1)2}$ to the operator $L^{(s)} - \alpha c^{(s)2}$ on the right hand side of (26) . Thus the sum becomes $s(1-s)\alpha(c^{(1)} - c^{(0)})^2$ and (24) is proved. ∎

Example 4.1 Let us apply the Proposition 4.1 to two operators (1) with one and the same A. Namely the coefficients of $L^{(0)}$ are $d^{(0)} = c^{(0)} = b/(1-s)$, $b^{0(0)} = 0$ while the coefficients of $L^{(1)}$ are $d^{(1)} = -c^{(1)} = a^0/2s$, $b^{0(1)} = 0$, $s \in (0,1)$. These operators are the "positive" operators M_p, M_q^* in (6) with $p = b/(2-2s)$, $q = a^0/2s$. Then $L = L^{(s)}$ is the operator (1) with $b^{0(s)} = s(1-s)\alpha(p+q)^2 = \alpha(a^0/k + kb)^2/4$, where $k = \pm\sqrt{s/(1-s)} \in (-\infty, +\infty)$. So the non-negativeness of the matrix $J_0 = \begin{pmatrix} A & kb \\ a^0/k & b^0 \end{pmatrix}$ leads to $\lambda(L) > 0$.

With the invariant changes (7) of the coefficients of $L = L^{(s)}$ in (1) we get the matrix $J = \begin{pmatrix} A + S & k(b + f + \partial' S/2) \\ (a^0 + f - \partial' S/2)/k & b^0 + \partial f \end{pmatrix}$ and the conclusion is formulated below.

Corollary 4.1 *If $J + J^t \geq 0$ for some choice of vector $f \in F$, skew-symmetric matrix $S \in F$, and $k \in R$, then $\lambda(L) > 0$.*

Let us describe Dirichlet problem for the quasilinear operator

$$M(u) = -\partial A(x, u, \nabla u) + b(x, u, \nabla u) \text{ in } \Omega. \qquad (27)$$

Define the operator in (1) with coefficients

$$a_j^k(x) = \int_0^1 \frac{\partial A_j}{\partial p_k}(x, R_t)dt, \ a_j^0(x) = \int_0^1 \frac{\partial A_j}{\partial u}(x, R_t)dt,$$
$$b^j(x) = \int_0^1 \frac{\partial B}{\partial p_j}(x, R_t)dt, \ b^0(x) = \int_0^1 \frac{\partial B}{\partial u}(x, R_t)dt, \qquad (28)$$

where $R_t = (v(x) + t(u(x) - v(x)), \nabla v(x) + t(\nabla u(x) - \nabla v(x)))$, and $u, v \in C^1(\bar{\Omega})$ are weak sub- and super- solutions of the Dirichlet problem for (27). With $f = f(x, u)$, $S = S(x, u)$ applying Corollary 4.1 we get a sufficient condition for the comparison principle. In this form it unifies and slightly generalizes the condition in [9].

Using Theorems 3.1, 3.2 we'll show some partial results about the monotonicity of $\lambda(L)$ with respect to the matrix a. There is no such monotonicity in general and to illustrate this we start with an example.

Example 4.2 Consider in Ω the operators $L_0 = -\Delta$, $\bar{L}_0 = -\tau^2\Delta$, $|\tau| < 1$ and let $c = (k, 0, \dots, 0)$, where $k = const.$ Denote $L_c = L_0 + N_c$, $\bar{L}_c = \bar{L}_0 + N_c$, $\bar{L}_{\tau c} = \bar{L}_0 + N_{\tau c}$ operators with a-potential and $\bar{a}$-potential c correspondingly. Hence $\lambda(L_c) = \lambda(L_0 + k^2) = \lambda(L_0) + k^2$, $\lambda(\bar{L}_c) = \lambda(\bar{L}_0 + k^2/\tau^2) = \tau^2\lambda(L_0) + k^2/\tau^2$, $\lambda(\bar{L}_{\tau c}) = \lambda(\bar{L}_0 + \tau^2k^2/\tau^2) = \tau^2\lambda(L_0) + k^2$. Then $\lambda(L_c) > \lambda(\bar{L}_{\tau c})$ but $\lambda(L_c) < \lambda(\bar{L}_c)$ if $k^2 > \lambda(L_0)\tau^2$. Moreover $\lambda(L_c) = \lambda(L_0 - \bar{L}_0) + \lambda(\bar{L}_{\tau c})$.

The following proposition concerns the general situation about monotonicity with respect to a.

Proposition 4.2 *Let the operators L_0, $\bar{L}_0$ in (5) have coefficients a, d, b^0 and $\bar{a}, \bar{d}, \bar{b}^0$ respectively, such that $\tau^2 a < \bar{a} < a$ for some τ, $|\tau| < 1$. Then with $c, \bar{c} \in L^\infty(\Omega)$ it holds*

$$\lambda(L_c) \geq \lambda(L_0 - \bar{L}_0) + \lambda(\bar{L}_{\bar{c}} - (\bar{a} - \tau^2 a)^{-1}(\bar{c} - \tau c)^2). \qquad (29)$$

Proof. From Theorem 3.1 (a) and Theorem 3.2 we receive

$$\lambda(L_c) \geq \lambda(L_0 + f_c(\tau\nabla z)) \geq \lambda(L_0 - \bar{L}_0) + \lambda(\bar{L}_0 + f_c(\tau\nabla z)). \qquad (30)$$

and

$$f_c(\tau\nabla z) = \bar{c}\nabla z - \bar{a}(\nabla z)^2/4 - (\bar{c} - \tau c)\nabla z + (\bar{a} - \tau^2 a)(\nabla z)^2/4$$
$$\geq \bar{f}_{\bar{c}}(\nabla z) - (\bar{a} - \tau^2 a)^{-1}(\bar{c} - \tau c)^2.$$

With supremum in z we obtain the assertion. ∎

Remark 4.1 Under (29) varying $c, \bar{c}$ several relations between $\lambda(L_c)$ and $\lambda(\bar{L}_{\bar{c}})$ can be obtained.

- If $\bar{c} = \tau c$, the Example 4.2 for $\lambda(L_c)$ and $\lambda(\bar{L}_{\tau c})$ shows that the equality in (29) is possible;

- If $\bar{c} = \bar{a}\alpha c/\tau$ then (29) becomes $\lambda(L_c) \geq \lambda(L_0 - \bar{L}_0) + \lambda(\bar{L}_{\bar{a}\alpha c/\tau} - \alpha\bar{a}\alpha c^2/\tau^2 + \alpha c^2)$ and for a-potential c then $\bar{a}\alpha c/\tau$ is $\bar{a}$-potential. So, when $\bar{L}_0 \to L_0$ in the sense of coefficients ($\bar{a} \to a, \bar{b}^0 \to b^0, \bar{d} \to d$) the two sides of the inequality (29) coincide;

- If $\bar{c} = c$ and for instance $\lambda(L_0 - \bar{L}_0) \geq (1-\tau)^2(\bar{a} - \tau^2 a)^{-1}c^2$ then $\lambda(L_c) \geq \lambda(\bar{L}_c)$.

Corollary 4.2 *The first eigenvalue $\lambda(L_c)$ continuously depends in L^∞ norm on the coefficients of L_c.*

Proof. Proceeding as in Section 3 the given operator L_c can be approximated with smooth operator $\bar{L}_c$ if the coefficients are such that $\tau^2 a < \bar{a} < a$ for some τ, $|\tau| < 1$ and $\bar{a} \to a$, $\bar{c} \to c$, $(a-\bar{a})^{-1}(d-\bar{d})^2 \to 0$, $(\bar{a} - \tau^2 a)^{-1}(\bar{c} - \tau c)^2 \to 0$ in L^∞ norm, with τ increasing to 1. ∎

The next example shows that the first eigenvalue can increase only due to the nonsymmetric part of A (determined by c) for the operator L.

Example 4.3 Let $\Omega \subset R^2$, $G \in C^2(\Omega)$, $\Delta G = 0$ and matrix $A = \begin{pmatrix} 1 & G \\ -G & 1 \end{pmatrix}$. So $Lu = -\Delta u + G_{x_2}u_{x_1} - G_{x_1}u_{x_2}$, where H is Cauchy-Riemann conjugate to G and $2c = (G_{x_2}, -G_{x_1}) = (H_{x_1}, H_{x_2})$. Then

$$\lambda(L) = \lambda(-\Delta + |\nabla H|^2/4) = \lambda(-\Delta + |\nabla G|^2/4) \geq \lambda(-\Delta) + \frac{1}{4}\inf_x |\nabla G|^2.$$

If $G = x_1^2 - x_2^2$, correspondingly $H = 2x_1x_2$, we get $\lambda(L) \geq \lambda(-\Delta) + \rho^2$, where $\rho = \operatorname{dist}(\Omega, (0,0))$. Hence $\lambda(L) \to \infty$ when $\rho \to \infty$.

4.2 Dependance of $\lambda(L_{Tc})$ on T

We'll study the behavior of the first eigenvalue $\lambda(L_{Tc})$ of the operator $L_{Tc} = L_0 + N_{Tc}$ for fixed c with respect to a large parameter T.

Proposition 4.3 *Let the operator L_{Tc} satisfies (1), (2). Then $\lambda(L_{Tc})$ is a concave monotone nondecreasing function of T^2 and*

(i) $\lambda(L_{Tc})$ is bounded iff there exists $u \in H_0^1(\Omega)$ such that $c \in M_0^*(u;\Omega)$. Moreover

$$\Lambda_c = \lim_{T\to\infty} \lambda(L_{Tc}) = \inf_{v\in V} B_{L_0}[v,v], \quad \text{where} \quad V = \{v : c \in M_0^*(v;\Omega)\},$$

(ii) if $\lambda(L_{Tc}) = const$ on some interval (T_0, T_1) then $\lambda(L_{Tc}) = \lambda(L_0)$ for all T and $c \in M_0^*(\phi_0;\Omega)$ where ϕ_0 is the first eigenfunction of L_0;

(iii) there exists $\lim_{T\to\infty}(\lambda(L_{Tc})/T^2) = K_c \in [0, \inf_x \alpha c^2]$ and for a-potential c it holds $K_c = \inf_x \alpha c^2$.

Proof. Recall Theorem 3.1 (b) and since $\beta_{Tc}(v^2) = T^2\beta_c(v^2)$ then

$$\lambda(L_{Tc}) = \inf_v\{B_{L_0}[v,v] + T^2\beta_c(v^2)\}, \tag{31}$$

and it is non-decreasing and concave in T^2.

(i) If for some $u \in H_0^1(\Omega)$, $c \in M_0^*(u;\Omega)$ then in (31) we can choose $v = u$ and h as c in $\int_\Omega a(c-h)^2u^2dx$. So $\lambda(L_{Tc}) \le B_{L_0}[u,u]$ and $\Lambda_c = \lim_{T\to\infty}\lambda(L_{Tc}) \le \inf_{v\in V} B_{L_0}[v,v]$.

Let now $\lambda(L_{Tc})$ be bounded and ϕ_{Tc} is normalized first eigenfunction of L_{Tc}. Then for $\varphi \in H_0^1(\Omega)$

$$\lambda(L_{Tc})(\varphi,\phi_{Tc}) = B_{L_0}[\varphi,\phi_{Tc}] + T\int_\Omega c(\phi_{Tc}\nabla\varphi - \varphi\nabla\phi_{Tc})dx. \tag{32}$$

If $\varphi = \phi_{Tc}$ in (32) we get $\Lambda_c \ge \lambda(L_{Tc}) = B_{L_0}[\phi_{Tc},\phi_{Tc}] \ge k_0\int_\Omega |\nabla\phi_{Tc}|^2dx - k_1$, for finite numbers $k_0, k_1 > 0$. So there exists $\phi \in H_0^1(\Omega)$ and a subsequence $\{\phi_{Tc}\}$ such that $\phi_{Tc} \to \phi$ and $\nabla\phi_{Tc} \rightharpoonup \nabla\phi$ weakly in $L^2(\Omega)$, $\phi \ge 0$, $\|\phi\|_{L^2} = 1$. Dividing (32) by T and letting $T \to \infty$ we come to

$$\int_\Omega c(\phi\nabla\varphi - \varphi\nabla\phi)dx = \lim_{T\to\infty}\frac{1}{T}\{\lambda(L_{Tc})(\varphi,\phi_{Tc}) - B_{L_0}[\varphi,\phi_{Tc}]\} = 0, \tag{33}$$

since $(\varphi,\phi_{Tc}) \to (\varphi,\phi)$ and $B_{L_0}[\varphi,\phi_{Tc}] \to B_{L_0}[\varphi,\phi]$. With $\varphi = z\phi$, $z \in C^{0,1}(\Omega)$ equality (33) can be written down as $\int_\Omega c(\nabla z)\phi^2dx = 0$, i.e. $c \in M_0^*(\phi;\Omega)$ and according to the first step $\Lambda_c \le B_{L_0}[\phi,\phi]$. Since there exists $\lim_{T\to\infty} B_{L_0}[\phi_{Tc},\phi_{Tc}] = \Lambda_c$, then

$$0 \le \lim_{T\to\infty}\int_\Omega a\nabla(\phi_{Tc}-\phi)^2dx = \lim_{T\to\infty} B_{L_0}[\phi_{Tc}-\phi,\phi_{Tc}-\phi]$$

$$= \lim_{T\to\infty}(B_{L_0}[\phi_{Tc},\phi_{Tc}] - 2B_{L_0}[\phi,\phi_{Tc}] + B_{L_0}[\phi,\phi] = \Lambda_c - B_{L_0}[\phi,\phi] \le 0.$$

So $\phi_{Tc} \to \phi$ in the norm of $H_0^1(\Omega)$ and $\Lambda_c = B_{L_0}[\phi,\phi]$.

(ii) Recall that in Theorem 3.1 (b) the extremums are attained at $v = \sqrt{\phi\psi}$, $z = \ln(\phi/\psi)$, $h = c - a\nabla z/2$ and the corresponding function $\beta_c(v^2) = \frac{1}{4}\int_\Omega a\nabla z^2v^2dx$. Let v_1, z_1, h_1 correspond to L_{T_1c}, then

$$\lambda(L_{T_1c}) = B_{L_0}[v_1,v_1] + \frac{T_0^2}{T_1^2}\int_\Omega a(T_1c-h_1)^2v_1^2dx + (1-\frac{T_0^2}{T_1^2})\int_\Omega a(T_1c-h_1)^2v_1^2dx$$

$$\ge \inf_{h,v}\{B_{L_0}[v,v] + \int_\Omega a(T_0c-h)^2v^2dx\} + \frac{1}{4}(1-\frac{T_0^2}{T_1^2})\int_\Omega a(\nabla z_1)^2v_1^2dx.$$

Since the infinum in the right hand side is equal to $\lambda(L_{T_0c}) = \lambda(L_{T_1c})$ then $\int_\Omega a(\nabla z_1)^2v_1^2dx = 0$, so $z_1 = \ln(\phi_{T_1}/\psi_{T_1}) = const$. Applying Theorem 3.2 we get $\phi_{T_1} = const\psi_{T_1} = const\phi_0$, hence $\lambda(L_{T_1c}) = \lambda(L_0)$ and $\lambda(L_{Tc}) = \lambda(L_0)$ for $T \in [0,T_1]$.

The result for $T > T_1$ is a consequence of the concavity, i.e. $\frac{\lambda(L_{Tc})-\lambda(L_0)}{T^2}$ is nonincreasing in T and nonnegative since $\lambda(L_{Tc}) \ge \lambda(L_0)$. So

$$0 \le \frac{\lambda(L_{Tc})-\lambda(L_0)}{T^2} \le \frac{\lambda(L_{T_1c})-\lambda(L_0)}{T_1^2} = 0.$$

(iii) The same inequality as above shows that there exist finite K_c where

$$K_c = \lim_{T\to\infty} \frac{\lambda(L_{Tc})}{T^2} = \lim_{T\to\infty} \frac{\lambda(L_{Tc} - \lambda(L_0)}{T^2} \geq 0.$$

We have

$$\frac{\lambda(L_{Tc})}{T^2} \leq \frac{1}{T^2} B_{L_0}[v,v] + \int_\Omega \alpha(c-h)^2 v^2 dx$$

for $v \in H_0^1(\Omega)$, $h \in M_0^*(v;\Omega)$. So $K_c \leq \int_\Omega \alpha(c-h)^2 v^2 dx$, choose $h = 0$ in ω and $h = c$ in $\Omega \setminus \omega$, then $h \in M_0^*(v;\Omega)$ for every $v \in H_0^1(\omega)$ and

$$K_c \leq \sup_{x\in\omega} \alpha c^2 \text{ for every } \omega \subset \Omega. \tag{34}$$

Minimizing (34) in $\omega \subset \Omega$ we reach the conclusion. ∎

Remark 4.2 In view of (i) the representation (b) in Theorem 3.1 can be written down as

$$\lambda(L_c) = \inf_h \Lambda_h(\alpha(c-h))^2, h \in \bigcup_{v\in H_0^1(\Omega)} M_0^*(v;\Omega)$$

and $\Lambda_h(g) = \lim_{T\to\infty} \lambda(L_{Th} + g)$. Note that

$$\Lambda_h = \sup_z \lambda(L_0 + h\nabla z) = \sup_f \lambda(L_{h+f} - \alpha f^2).$$

Indeed, from Theorem 3.1 (a) with the change $z \to z/T \in C^{0,1}(\Omega)$ we have

$$\sup_z \lambda(L_0 + h\nabla z) \geq \sup_z \lambda(L_0 + f_{Th}(\nabla z/T))$$

$$= \lambda(L_{Th}) \geq \lambda(L_0 + h\nabla z) - \frac{1}{4T^2} \sup_x a(\nabla z)^2$$

and

$$\lambda(L_{h+f} - \alpha f^2) = \sup_z \lambda(L_0 + h\nabla z - \alpha(f - a\nabla z/2)^2) \leq \sup_z \lambda(L_0 + h\nabla z),$$

$$\sup_f \lambda(L_{h+f} - \alpha f^2) \geq \sup_{f=a\nabla z} \lambda(L_{h+f} - \alpha f^2)$$

$$= \sup_{f=a\nabla z} \lambda(L_{h+f} + \alpha(h+f)^2 - \alpha h^2 - \alpha f^2) \geq \sup_z \lambda(L_0 + h\nabla z).$$

Here (18) is used in the sense that if g is a-potential then $\lambda(L_c) = \lambda(L_{c-g} + \alpha c^2 - \alpha(c-g)^2)$.

Remark 4.3 For the operator $\Gamma_{Tc}u = -\Delta u + Tc\nabla u, \mathrm{div} c = 0$ in [1] it was shown that $\lambda(\Gamma_{Tc})$ is bounded *iff* there exists $w \in H_0^1(\Omega), w \neq 0$ such that $c\nabla w = 0$ a.e. in Ω. It was proved in [8] that $K_c = \inf_u \beta_c(u^2)$ and $K_c > 0$ *iff* there exists $z \in C^{0,1}(\Omega)$ such that $c\nabla z > 0$ in $\bar{\Omega}$, a sufficient condition was proved in [6].

We'll give an analogue of Proposition 4.3 in the selfadjoint case. If c is a-potential then $\lambda(L_{Tc}) = \lambda(L_0 + T^2\alpha c^2)$ so it is necessery some information about the behavior of $\lambda(L_0 + T^2 g)$ with respect to g.

Proposition 4.4 *Let the operator L_0 satisfies (2) and $g \in L^\infty(\Omega)$. Then $\lambda(L_0+T^2g)$ is a concave function of T^2 and nondecreasing if $g \geq 0$ and*

(i) for $g \geq 0$, $\lambda(L_0 + T^2 g)$ is bounded iff *the set $G_0 = \{x \in \Omega : g(x) = 0\}$ has a positive measure. In this case $\lim_{T\to\infty} \lambda(L_0 + T^2 g) = \lambda(L_0; G_0)$.*

(ii) for every $g \in L^\infty(\Omega)$ it holds

$$\lim_{T\to\infty} \frac{\lambda(L_0 + T^2 g)}{T^2} = \inf_x g.$$

Proof. (i) If $meas(G_0) > 0$, the monotony in sets gives $\lambda(L_0 + T^2 g) \leq \lambda(L_0 + T^2 g; G_0) = \lambda(L_0; G_0)$. Let $\lambda(L_0 + T^2 g)$ is bounded and ϕ_T is the normed first eigenfunction of $L_0 + T^2 g$, then with some positive and finite constants k_0, k_1 it holds

$$\lambda(L_0 + T^2 g) = B_{L_0}[\phi_T, \phi_T] + T^2 \int_\Omega g\phi_T^2 dx \geq k_0 \int_\Omega |\nabla\phi_T|^2 dx - k_1 + T^2 \int_\Omega g\phi_T^2 dx.$$

Then there exists $\phi \in H_0^1(\Omega)$, $\|\phi\| = 1$, $\phi \geq 0$ and a subsequence $\{\phi_T\}$ such that for $T \to \infty$

$$\phi_T \to \phi \text{ in } L^2, \nabla\phi_T \rightharpoonup \nabla\phi \text{ weakly }, \int_\Omega g\phi_T^2 dx \leq \frac{\lambda(L_0 + T^2 g) + k_1}{T^2} \to 0.$$

So $\int_\Omega g\phi^2 dx = 0$ and $g = 0$ in $\{x \in \Omega : \phi > 0\}$, obviously $measG_0 > 0$. Further,

$$\lambda(L_0 + T^2 g)(\phi, \phi_T) = B_{L_0}[\phi, \phi_T] + T^2 \int_\Omega g\phi\phi_T dx = B_{L_0}[\phi, \phi_T]$$

and hence

$$\lim_{T\to\infty} \lambda(L_0 + T^2 g) = B_{L_0}[\phi, \phi] \geq \inf_{u\in H_0^1(G_0)} B_{L_0}[u, u] = \lambda(L_0; G_0).$$

(ii) It holds $\lambda(L_0 + T^2 g) \geq \lambda(L_0) + T^2 \inf_\Omega g$ and

$$\lambda(L_0 + T^2 g) \leq \lambda(L_0 + T^2 g; \omega) \leq \lambda(L_0; \omega) + T^2 \sup_\omega g$$

for $\omega \subset \Omega$. So

$$\inf_\Omega g \leq \lim_{T\to\infty} \frac{\lambda(L_0 + T^2 g)}{T^2} \leq \inf_{\omega\subset\Omega} \sup_\omega g = \inf_\Omega g.$$

∎

The following representation of K_c as a consequence of Proposition 4.3 can be derived.

Corollary 4.3 *K_c has the representation*

$$K_c = \inf_{L_0}(\lambda(L_c) - \lambda(L_0)). \tag{35}$$

where infinum is over all operators L_0 with arbitrary d, b^0 and the same a and with L_c corresponding to L_0, $L_c = L_0 + N_c$.

Proof. From the concavity in T^2 in Proposition 4.3

$$K_c = \inf_T \frac{\lambda(L_{Tc}) - \lambda(L_0)}{T^2} \leq \lambda(L_c) - \lambda(L_0)$$

and from Remark 4.3, $K_c = \inf_u \beta_c(u^2)$, $u \in H_0^1(\Omega)$ and $\|u\| = 1$. So K_c doesn't depend on the coefficients d, b^0 in L_0 and the inequality in (35) from above is obtained.

Further, we'll choose d and $b^0 = \alpha d^2$, then

$$\lambda(L_0) = \inf_u B_{L_0}[u,u] = \inf_u \int_\Omega \alpha(a\nabla u + ud)^2 dx > 0$$

and according to Theorem 3.1, with $u \in H_0^1(\Omega)$ and $\|u\| = 1$, $h \in M_0^*(u;\Omega)$ $\lambda(L_c) \leq B_{L_0}[u,u] + \beta_c(u^2) \leq B_{L_0}[u,u] + \int_\Omega \alpha(c-h)^2 u^2 dx$. So

$$\lambda(L_c) - \lambda(L_0) \leq \lambda(L_c) \leq \int_\Omega \alpha(a\nabla u + ud)^2 dx + \int_\Omega \alpha(c-h)^2 u^2 dx. \tag{36}$$

From the formula for K_c in Remark 4.3, for $\varepsilon > 0$, there exist $u \in H_0^1(\Omega)$, $u > 0$ and $h \in M_0^*(u;\Omega)$ such that $\int_\Omega \alpha(c-h)^2 u^2 dx \leq K_c + \varepsilon$. There as well exists $\delta_\varepsilon > 0$ such that $\int_{0<u\leq\delta_\varepsilon} a(\nabla u)^2 dx < \varepsilon$. With $v = \max(u, \delta_\varepsilon) \geq \delta_\varepsilon > 0$ and $d = -a\nabla v/v$ the left hand side of (36) with such d and the same u, h becomes

$$\int_{0<u\leq\delta_\varepsilon} a(\nabla u)^2 dx + \int_\Omega \alpha(c-h)^2 u^2 dx < K_c + 2\varepsilon.$$

which proves the rest inequality in (35) from below. ∎

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