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Construction of a stabilizing control and solution to a problem about the center and the focus for differential systems with a polynomial part on the right side

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ABSTRACT

The stationary differential systems with polynomial right sides are considered. Necessary and sufficient conditions are formulated when a given domain is a domain of asymptotic stability and the origin of coordinates is either the focus or the center. The problem of construction of a stabilizing control in a form of polynomial is studied.

RESUMEN

Consideramos los sistemas diferenciales estacionarios con lado derecho polinomial y formulamos condiciones necesarias y suficientes cuando el dominio dado es de estabilidad asintótica y el origen de coordenadas es el foco o el centro. Además se estudia el problema de la construcción de un control estabilizador de forma polinomial.

Key words and phrases: *Differential systems of n-th order, asymptotic stable systems, stabilizing control, domains of asymptotic stability.*
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1 Introduction

In the 19^{th} century the great French mathematician Henri Poincaré formulated the problem of finding stability conditions for differential systems without calculation being a solution. At the end of the 19^{th} and at the beginning of the 20^{th} centuries the great Russian mathematician A.M.Lyapunov developed the mathematical stabilization theory for differential systems. For this purpose he developed two methods. The first one is based on the characteristic numbers of the fundamental matrix. The second one is based on construction of special functions called Lyapunov's functions which have properties similar to distance from a considered point to the origin of coordinates (it is supposed that the zero solution is stationary). Lyapunov's methods received world recognition.

In this paper, differential systems with a right polynomial part $f(.)$ of

$$x = (x_1, x_2, ..., x_n) \in \Re^n$$

are considered i.e.

$$f(x) = (f_1(x), f_2(x), ..., f_n(x))^*,$$

where

$$f_p(x) = \sum_{l_1,l_2,...,l_n \in I_p} a^{(p)}_{l_1,l_2,...,l_n} x_1^{l_1} x_2^{l_2} ... x_n^{l_n},$$

$p \in 1 : n, l_1, l_2, ...l_n$ are non-negative integers and $*$ is the transposition sign, $a^{(p)}_{l_1,l_2,...,l_n}$ are real-valued numbers, I_p is a set of degrees of the polynomial $f_p(x)$.

To analyze the system

$$\dot{x} = f(x)$$

a method is suggested that is different from Lyapunov's methods and based on a system transformation idea, so that we are able to say something definite about stability.

From the technical point of view the important problem is finding a domain of asymptotic stability [1]-[5] and defining behaviour of trajectories of such a system in a small neighborhood of the origin of coordinates [6]. Conditions are given so that any domain including the origin of coordinates is one of them. Necessary and sufficient conditions are found when the origin of coordinates is either the focus or the center.

The case is considered when the coefficients $a^{(p)}_{l_1,l_2,...,l_n}$ depend on t. The sufficient conditions are given for the problems formulated above.

The systems considered are interesting because asymptotic stability of the zero solution is equivalent to the following statement, that any solution of our system starting from any point of some small region of the origin of coordinates tended to be zero; this is not correct in general case.

Further, we solve the problem of constructing a stabilizing control in any given domain of the origin of coordinates for a system

$$\dot{x} = f(x, u),$$

where $x \in \Re^n$ is a phase vector , $u \in R^r$ is a control, $f(x,u) = (f_1(x,u), f_2(x,u), \ldots, f_n(x,u))^*$ is a vector polynomial of x and u i.e.

$$f_p(x,u) = \sum_{l_1,l_2,\ldots,l_n,m_1,m_2,\ldots,m_r \in I_p} a^{(p)}_{l_1,l_2,\ldots,l_n,m_1,m_2,\ldots,m_r} x_1^{l_1} x_2^{l_2} \ldots x_n^{l_n} u_1^{m_1} u_2^{m_2} \ldots u_r^{m_r},$$

$p \in 1:n$, $l_1, l_2, ..., l_n, m_1, m_2, ..., m_r$ are non-negative integers, $a^{(p)}_{l_1,l_2,\ldots,l_n,m_1,m_2,\ldots,m_r}$ are real-valued numbers, I_p is a set of degrees of the polynomial $f_p(x,u)$. We assume that the zero vector $0 = (0, 0, ..., 0) \in \Re^n$ is a solution of the system i.e. $f(0,0) = 0$.

The theorem was proved that any domain including the origin of coordinates can be made a domain of asymptotic stability if we choose a suitable control $u(x)$. The control $u(x)$ is chosen as a polynomial with degree not higher than degree of the vector-function $f(x,u)$ as a function of x.

The formulated problem about asymptotic stability of a system with a right polynomial part is very important for practical applications in physics and technics and can be found in the book by V.I.Zubov "The lectures on the control theory", 1975. p.60.

2 Domains of asymptotic stability

Let us consider a differential system

$$\dot{x} = f(x), \tag{1}$$

where $x = (x_1, x_2, ..., x_n) \in \Re^n$ and

$$f(x) = (f_1(x), f_2(x), ..., f_n(x))^*.$$

The vector-polynomials $f_p(.)$ and their coefficients $a^p_{l_1,l_2,\ldots,l_n}$ satisfy the conditions in the Introduction.

We assume that $f(x) \neq 0$ for all $x \neq 0$ in some neighborhood of the origin of coordinates 0.

We will call the order $deg(f_p(.))$ of the polynomial $f_p(\cdot)$ the maximal degree of the polynomial $f_p(.)$ in the variables $x_j, j \in 1:n$, in totality i.e.

$$deg(f_p(x)) = \max_{l_1,l_2,\ldots,l_n \in I_p} (l_1 + l_2 + \ldots + l_n).$$

So if for $n = 2$ and $l_{ij} \neq 0$

$$f_1(x) = l_{11}x_1^2 + l_{12}x_2^2 + l_{13}x_1x_2, \quad f_2(x) = l_{21}x_1^3 + l_{22}x_2 + l_{23}x_3,$$

then the order of $f_1(x)$ is equal to two and the order of $f_2(x)$ is equal to three.

We will call the order of the function $f(.)$ the maximal degree of the polynomials $f_p(x), p \in 1:n$, with respect to the variables $x_j, j \in 1:n$, i.e.

$$deg(f(x)) = \max_{p \in 1:n} deg(f_p(x)).$$

Consequently, the order of the function $f(.)$ for the example written above is equal to $\max(2,3) = 3$.

We assume that the definitions of stability and asymptotic stability are known (see [1]-[4]).

Definition 2.1 *[1], [2] A domain $D, 0 \in intD$, consisting of whole trajectories of the system (1), is called a domain of asymptotic stability if the limit*

$$\| x(t, x_0, t_0) \| \to 0 \tag{2}$$

as $t \to \infty$ is fulfilled for any initial point $x_0 \in D$ and any solution $x(\cdot), x(t_0) = x_0$, of the system (1). In this case we will call the system (1) asymptotic stable in D.

The following problem is important for practical applications.

Problem 2.1 *([1], [2]) It is known that the zero solution of a system*

$$\dot{x} = Ax, \tag{3}$$

$A[n \times n], x \in \Re^n$, is asymptotic stable if all eigenvalues of the matrix A have negative real parts. In this case the system (3) is asymptotic stable in $\Re^n$.

Consider now a differential system

$$\dot{x} = Ax + \varphi(x) \tag{4}$$

where $\varphi(\cdot)$ is a vector polynomial with degree not less than two, $\varphi(0) = 0$.

What are the conditions on the coefficients of the vector- polynomial $\varphi(\cdot)$ from (4) for which the system (4) is asymptotic stable in a given domain $D, 0 \in intD$, i.e. for any solution of the system (4) and any initial point $x_0 \in D$ the limit (2) is true?

The considered circle of questions includes the problem about the center and the focus. This problem is formulated in the following way.

Problem 2.2 *Assume that the origin of coordinates (0,0) for the system (3), $n = 2$, $x \in \Re^2$ is the center. Necessary and sufficient conditions for this are that all eigenvalues of the matrix A are imaginary. By adding a vector polynomial $\varphi(\cdot)$ (system (4)) the center (0,0) can be the focus. Conditions on the coefficients of the polynomial $\varphi(\cdot)$ are needed to find that the point (0,0) was the center of the differential system (4).*

Let us rewrite the system (1) in an equivalent form

$$\dot{x} = A(x)x. \tag{5}$$

The elements $a_{ij}(x)$ of a matrix $A(x)[n \times n]$ are continuous polynomial functions of x. Conversion (4) to (5) is not unique. It can be done in an infinite number of ways. In fact,

$$a_{ij}(x) = \sum_{l_1 + l_2 + \ldots + l_n \leq deg(f)} \alpha^{(i)}_{j, l_1, l_2, \ldots, l_n}(x) x_1^{l_1} x_2^{l_2} \ldots x_j^{l_j - 1} \ldots x_n^{l_n}$$

if $l_j \geq 1$. We have the following correlation for the coefficients

$$\sum_j \alpha^{(i)}_{j,l_1,l_2,\ldots,l_n}(x) = a^{(i)}_{l_1,l_2,\ldots,l_n} \tag{6}$$

for all x from some region $D, 0 \in intD$.

For instance, let the system (1) have the right part

$$f(x) = \begin{pmatrix} x_1^2 x_2 + x_2 + 2x_1 x_2^2 \\ -x_1 + 3x_1^2 x_2 - 2x_1 x_2^2 \end{pmatrix}.$$

Then

$$A(x) = \begin{pmatrix} \alpha_1 x_1 x_2 + \alpha_2 x_2^2 & 1 + \beta_1 x_1^2 + \beta_2 x_1 x_2 \\ -1 + \gamma_1 x_1 x_2 + \gamma_2 x_2^2 & \delta_1 x_1^2 + \delta_2 x_1 x_2 \end{pmatrix}$$

where coefficients $\alpha_i, \beta_i, \gamma_i, \beta_i, i = 1,2$ such that

$$\alpha_1 + \beta_1 = 1, \quad \gamma_1 + \delta_1 = 3,$$

$$\alpha_2 + \beta_2 = 2, \quad \gamma_2 + \delta_2 = -2.$$

The system (4) for the vector function $f(\cdot)$ can be rewritten in the following form

$$\dot{x} = A_0 x + C(x)x \tag{7}$$

where

$$C(x) = \begin{pmatrix} \alpha_1 x_1 x_2 + \alpha_2 x_2^2 & \beta_1 x_1^2 + \beta_2 x_1 x_2 \\ \gamma_1 x_1 x_2 + \gamma_2 x_2^2 & \delta_1 x_1^2 + \delta_2 x_1 x_2 \end{pmatrix},$$

$$A_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Since eigenvalues of the matrix A_0 are $\lambda_{1,2} = \pm i$, the point (0,0) is the center for the linearized system. But it is question for the nonlinear system (7).

We will consider all possible continuous matrices $A(x)$ whose elements are polynomial functions of x. The system (5) is equivalent to the system (1) . We will denote the set of all such matrices by $\mathcal{A}$.

Let us solve Problem 2.1.

Theorem 2.1 *In order that a domain D consisting of whole trajectories of the system (1) i.e. $x(\cdot, x_0, t_0) \in D, x_0 \in D$, for all $t > t_0$ is a domain of asymptotic stability it is necessary and sufficient that there is a matrix $A(\cdot) \in \mathcal{A}$ of the system (5) in the domain D whose eigenvalues have negative real parts at any point $x \in D, x \neq 0$.*

Proof. Necessity. Let the domain D be asymptotic stable. Consider any trajectory $x(\cdot, x_0, t_0), x(t_0) = x_0$. There is such a transformation $\xi = X(x)$ of $\Re^n$ in a neighborhood of a point $x_1 = x(t_1)$ that transforms the system (1) to a differential system

$$\dot{\xi} = B(\xi)\xi \tag{8}$$

where a matrix B has eigenvalues with negative real parts.

Indeed, for any point $x(t_1, x_0, t_0), t_1 > t_0$ there is a linear transformation T_x defined in a small neighborhood of the point $x(t_1, x_0, t_0)$ and that is the linear part of the transformation $X(.)$ in this neighborhood, so that the system (1) is transformed by this transformation to the differential system (8) and the vector

$$x_1 = \dot{x}\ (t_1, x_0, t_0)$$

is transformed to a vector

$$\xi_1 = \dot{\xi}\ (t_1, x_0, x_0).$$

Any differential equation can be defined by its current of tangent vectors. If we choose an asymptotic stable linear differential system like the system (3) current of tangent vectors of which is close to the current of tangent vectors of our system locally for each point x of the domain D, then eigenvalues of a matrix of this asymptotic stable linear differential system will have negative real parts. Consequently, the above mentioned transformation $X(x)$ exists.

As soon as the system (8) is transformed to the system

$$\dot{x} = T_x^{-1} B(\xi_1) T_x x$$

under the linear transformation

$$\xi = T_x\, x$$

in a neighborhood of the point ξ_1 and all locally linear transformations of the system (1) have the form (5), then a matrix $A(\cdot) \in \mathcal{A}$ exists for which

$$A(x(t_1, x_0)) = T_x^{-1} B(\xi_1) T_x \tag{9}$$

and eigenvalues of the matrix $A(x(t_1))$ are equal to eigenvalues of the matrix B.

Since the transformations T_x and T_x^{-1} are continuous with respect to x, the matrix $A(.)$ is also continuous with respect to $x \in D$.

Sufficiency. Let the system (1) admit such a transformation to the form (5) that all eigenvalues of a matrix $A(x)$ have negative real parts for all $x \in D, x \neq 0$. We prove that the domain D is a region of asymptotic stability.

Fist we remark that in general case not any solution of the system (1)

$$x(t) = x(t, x_0, t_0), \quad x(t_0) = x_0$$

can be represented in the form

$$x(t) = e^{\int_{t_0}^{t} A(x(\tau))d\tau} x_0 \tag{10}$$

where the integration is taken along the integral curve $x(.)$.

Instead of (10) we will use sequential approximations $\{x_k(t)\}$ of the system (1) on a segment $[t_0, t]$ having the form

$$x_k(t) = e^{A(x_{k-1})(t-t_{k-1})} x_{k-1}(t) \tag{11}$$

where $\{t_i\}$ is a subdivision set of the segment $[t_0, t]$ and

$$x_{i+1} = e^{A(x_i)(t_{i+1}-t_i)}x_i, \quad i \in 0:(k-1). \tag{12}$$

If we substitute (12) into (11), then we get

$$x_k(t) = e^{A(x_{k-1})(t-t_{k-1})}e^{A(x_{k-2})(t_{k-1}-t_{k-2})}\dots e^{A(x_0)(t_1-t_0)}x_0. \tag{13}$$

Since $A(x_i) \underset{\substack{x_i \to x(\tau)\\ t_i \to \tau}}{\longrightarrow} A(x(t))$, where $x(.)$ is a solution of (5), we have

$$x_k(\tau) \Longrightarrow x(\tau) \tag{14}$$

uniformly on $\tau \in [t_0, t]$ as $k \to \infty$ and

$$\max_{i\in 1:k} \mid t_i - t_{i-1} \mid \to_k 0.$$

It is obvious that all matrices $A(x_i), i \in 0:(k-1)$, have eigenvalues with negative real-valued parts for enough big k.

Denote by $\lambda_i(x_j)$ all eigenvalues of the matrices $A(x_j), i \in 1:n, j \in 0:k-1$. Then a number $C > 0$ exists that

$$\|e^{A(x_{k-1})(t-t_{k-1})}e^{A(x_{k-2})(t_{k-1}-t_{k-2})}\dots e^{A(x_0)(t_1-t_0)}x_0\| \le Ce^{\lambda(t)(t-t_0)}\|x_0\|$$

where

$$\lambda(t) = \lim_{k\to\infty} \max_{\substack{j\in 0:k-1,\\ i \in 1:n}} Re\, \lambda_i(x_j).$$

Since $\lambda(t) < 0$ for any $t > t_0$,

$$\lambda(t)(t-t_0) < 0.$$

Let

$$-a < \lambda(t)(t-t_0) < 0, a > 0.$$

This relation means that the trajectory $x(\cdot)$ can not go to a stationary point or be a stationary orbit. Indeed, if the trajectory $x(.)$ has a stationary point or is a stationary orbit then

$$\lim_{t\to\infty} \lambda(t)(t-t_0) = -\infty.$$

Consequently,

$$\| x(t, x_0, t_0) \| \longrightarrow 0$$

as $t \longrightarrow \infty$. We obtain the contradiction. Thus, the origin of coordinates can be only a stationary point. The sufficiency and the theorem are proved. □

Remark 2.1 *It follows from the Necessity of Theorem 2.1 that there is one-to- one correspondence between matrices $A(x) \in \mathcal{A}$ of the system (5) and locally linear transformations T_x of the system (1) defined in a neighborhood of the point x(see (9)). Indeed, there is a matrix $A(.)$ for any transformation T_x and, on the contrary, there is own coordinate system for any matrix $A(x) \in \mathcal{A}$, in which the equation (9) is true.*

Corollary 2.1 . *It follows from the Necessity of Theorem 2.1 that degree on x of any element of a matrix $A(x)$ that we denote by $a_{ij}(x), j \in 1:n$, that is a polynomial of x, does not exceed degree of the vector-function $f(\cdot)$.*

Proof The sum of $a_{ij}(x) \cdot x_j$ over j is equal to zero for those elements that do not belong to $f_i(\cdot)$. And on the contrary, the sum of the coefficients $\alpha^{(i)}_{j,l_1,l_2,\ldots,l_n}$ over j is not zero for those elements that are the terms of $f_i(\cdot)$. It follows from here that choosing $\alpha^{(i)}_{j,l_1,l_2,\ldots,l_n}$ from the system (6) we can only consider elements of the vector-polynomial $f(\cdot)$. □

Corollary 2.2 . *For asymptotic stability of a solution $x(.,x_0,t_0)$, $x(t_0)=x_0$, of the differential equation (1) it is sufficient that there was $\delta(t_0,\varepsilon)>0$ for any $t_0, \varepsilon > 0$ that for $\| x_0 \| < \delta$*

$$\| x(t,x_0,t_0) \| \longrightarrow 0$$

as $t \longrightarrow \infty$.

Proof. As soon as eigenvalues of the matrix $A(\cdot)$ have negative real parts in any neighborhood of the origin of coordinates from which any solution tends to zero-vector , then for any k we have from (13) that

$$\| x_k(t) \| \leq \| x_0 \| < \varepsilon$$

and in limit on k

$$\| x(t,x_0,t_0) \| \leq \| x_0 \| < \varepsilon$$

i.e. all solutions are in an ε -neighborhood of the origin of coordinates. The latter means stability. The Corollary is proved. □

3 To problem about the center and the focus

The following problem is interesting from technical point of a view: to recognize focus or center i.e. to give some conditions when trajectories turn around and go to the origin of coordinates or remain closed.

From the start we will give definitions of the focus and the center. Beforehand we define trajectories turning infinitely often around some ray with the initial point $0 \in \Re^n$.

Definition 3.1 *Let us say that a trajectory $x(t,x_0,t_0)$ turns around a ray $l \in \Re^n$ with the initial point $0 \in \Re^n$ infinitely often if radius vector $r(t) = x(t,x_0,t_0)$ forms with the ray l an angle $\varphi(t)$ taking all values from a segment $[\varphi_1, \varphi_2]$ infinite number of times.*

Definition 3.2 *We will call the point $0 \in \Re^n$ the focus of the system (1) if*

1) the point 0 is asymptotic stable ;

2) any solution $x(t,x_0,t_0)$ of the system (1) turns around a ray $l \in \Re^n$ with the initial point $0 \in \Re^n$ infinite number of times.

Definition 3.3 *We will call the point 0 the center of the system (1) if*

1) the point 0 is stable ;

2) all solutions $x(t, x_0, t_0)$ of the system (1) remain in some neighborhood of the point 0 and are closed loops.

Definition 3.4 *We will call the point $0 \in \Re^n$ the node of the system (1) if*

1) the point 0 is asymptotic stable ;

2) there are not a ray $l \in \Re^n$ with the initial point $0 \in \Re^n$ and a solution $x(t, x_0, t_0)$ of the system (1) turning around the ray l infinite number of times.

Theorem 3.1 *In order that the point $0 \in \Re^n$ is the focus of the system (1) with a right polynomial part $f(.), f(x) \neq 0$ for $x \neq 0$ it is necessary and sufficient that*

1. *There is such a continuous matrix $A(\cdot) \in \mathcal{A}$ of the system (5) that all its eigenvalues at any point $x, x \neq 0$, from some neighborhood D of the origin of coordinates, consisting from whole trajectories, have negative real parts and non-zero imaginary parts.*

2. *There is no matrix $A(\cdot) \in \mathcal{A}$ of the system (5) with negative real-valued eigenvalues at all points $x \in D, x \neq 0$.*

Proof. Necessity. Let the point 0 be the focus for the system (1). There is such a transformation $\xi = X(x)$ so that the system (1) can be rewritten in a neighborhood of some point $x_1, x_1 \neq 0$, in the form

$$\dot{\xi} = B(\xi)\xi$$

where $B(\xi)$ is a matrix whose eigenvalues have negative real parts and nonzero imaginary parts.

Consider any trajectory $x(\cdot, x_0, t_0), x(t_0) = x_0$. Then for any point $x_1 \in x(t, x_0, t_0)$, $x_1 \neq 0$, there is a transformation T_{x_1} in a neighborhood of the point x_1 that the system (1) can be rewritten in the form indicated above.

If trajectories of the system (1) do not turn around any ray $l \in \Re^n$ with the initial point $0 \in \Re^n$ infinite number of times then such system can be transformed by some continuous transformation that corresponds to some matrix $A(\cdot) \in \mathcal{A}$ that is a matrix of the system (5) whose eigenvalues are negative real-valued numbers at all points $x \in D$ (theorem 2.1). The latter is impossible according to condition 2 of the theorem.

Sufficiency. Let the conditions 1,2 of the theorem be true. Prove that the point 0 is the focus of the system (1).

Any solution $x(\cdot, x_0, t_0), x(t_0) = x_0$, is a limit of some sequence $x_k(t)$ on $k \to \infty$ obtained from (13). According to the conditions all matrices $A(x_j), j \in 1 : k$, have eigenvalues $\lambda_l(x_j), j \in 1 : k, l \in 1 : n$ with negative real-valued parts $a_l(\cdot)$ and nonzero imaginary parts $b_l(\cdot) : \lambda_l(x) = a_l(x) + ib_l(x), i^2 = -1, \ l \in 1 : n$, that are nonzero at any point $x \in D$. It follows from here that the point 0 is either the focus or the node.

We will prove that under the condition 2 all trajectories turn around a ray $l \in \Re^n$ with the initial point $0 \in \Re^n$ infinite number of times. If it is not true, then there is such a transformation which corresponds to a matrix $A(\cdot) \in \mathcal{A}$ whose eigenvalues are negative real-valued numbers for $x \in D, x \neq 0$. This contradicts the condition 2. The sufficiency and the theorem are proved. □

Corollary 3.1 *If instead of the condition 2 of Theorem 3.1 we require that there is a matrix $A(.) \in \mathcal{A}$ of the system (5) with eigenvalues $\lambda_i(x)$, $i \in 1:n$, described in condition 1 for which*

$$Im\,\lambda_i(x) > a, \quad a > 0$$

or

$$Im\,\lambda_i(x) < a, \quad a < 0,$$

where $Im\,\lambda_k(.)$ denotes an imaginary part of $\lambda_k(.)$, for $x \in D, x \neq 0$, then all trajectories turn around a ray $l \in \Re^n$ with the initial point $0 \in \Re^n$ infinite number of times.

Proof. From the representation of any solution of the system (1) as a limit of a sequence $\{x_j(.)\}$ from (13) it follows that a solution $x(\cdot, x_0, t_0)$ has a finite number of turns around any ray $l \in \Re^n$ with the initial point $0 \in \Re^n$ if and only if for any $i \in 1:n$ the sum

$$\sum_{j\in 1:k} Im\lambda_i(x_{j-1})(t_j - t_{j-1}),$$

where $x_j(.)$ was defined in (13), or in the limit on $k \to \infty$ the integral

$$\int_{t_0}^{\infty} Im\,\lambda_i(x(\tau))d\tau$$

is convergent. But under the condition of Corollary 3.1 this integral is divergent. The corollary is proved. □

Mathematicians were concerned about the problem of recognition of center ore focus for a long time. The theorem given below for two dimensional system with a right polynomial part has the point $0 = (0,0)$ as the center.

Theorem 3.2 *In order that the point $0 = (0,0)$ be the center for the two-dimensional system (1) it is necessary and sufficient that there was a matrix $A(.) \in \mathcal{A}$ of the system (5) whose eigenvalues are non-zero imaginary numbers in a neighborhood S of the origin of coordinates, consisting from whole trajectories, where $f(x) \neq 0$ for $x \neq 0, x \in S$.*

Proof. Necessity. Let the point $0 = (0,0)$ be the center of the system (1). Prove that the condition of the theorem is true.

Take such a small neighborhood S of the origin of coordinates where all trajectories starting in S are closed loops. Consider any trajectory $x(\cdot, x_0, t_0), x(t_0) = x_0$. Take a point

$$x_1 = x(t_1, x_0, t_0).$$

There is such a transformation T_x in a small enough neighborhood of the point $x_1, x_1 \neq 0$, that the system (1) can be rewritten in the form

$$\dot{\xi} = B(\xi)\xi$$

and the matrix $B(.)$ has imaginary eigenvalues for all $\xi \in S$. As was proved above (Theorem 2.1, the Proof of the Necessity) the matrix $B(\cdot)$ corresponds to some matrix $A(\cdot)$ that like the matrix $B(\cdot)$ has imaginary eigenvalues. The necessity is proved.

Sufficiency. Let the condition of the theorem be true. Prove that the point $0 = (0,0)$ is the center.

As soon as any transformation T_x is defined in a small neighborhood of the point x, then the above statement is not sufficient for the point $0 = (0,0)$ to be the center. The point $0 = (0,0)$ may happen to be the focus convergent (nonconvergent) with a finite number of turns around any ray $l \in \Re^n$ with the initial point 0. But it is impossible because from the limit of the sequence (13) we conclude that for any k

$$\| x_k(t) \| = \| x_0 \| .$$

From (14) it follows that solution $x(\cdot, x_0, t_0)$ can not go to the origin of coordinates.

A set of closed and not closed loops could occur. From the representation of integral curve $x(\cdot, x_0, t_0)$ in the forms (13) and (14) it follows that there is not a matrix $A(\cdot) \in \mathcal{A}$ with imaginary eigenvalues at all points of a small neighborhood of the origin of coordinates. The latter contradicts the condition of the theorem.

There is another case when for all $i \in 1:n$ the sum

$$\sum_{j \in 1:k} Im\lambda_i(x_{j-1})(t_j - t_{j-1})$$

or in the limit on $k \to \infty$ the integral

$$\int_{t_0}^{\infty} Im\ \lambda_i(x(\tau))d\tau$$

has a finite value where $\lambda_i(\cdot)$, $i \in 1:n$, are eigenvalues of a matrix $A(\cdot) \in \mathcal{A}$. In this case trajectory $x(\cdot, x_0, t_0)$ goes to some stationary point $\hat{x} \in S, \hat{x} \neq 0$. It means that $f(\hat{x}) = 0$. This contradicts the condition of the theorem. Consequently, the point $(0,0)$ is the center. The sufficiency and the theorem are proved. □

Corollary 3.2 *If a matrix $A(.) \in \mathcal{A}$ of the system (5) exists with imaginary eigenvalues $\lambda_i(x)$ for all $x \in S$ where S is a neighborhood of the origin of coordinates 0, consisting from whole trajectories, and*

$$Im\ \lambda_i(x) > 0$$

or

$$Im\ \lambda_i(x) < 0$$

for all $x \in S, x \neq 0$, and $i \in 1:n$, then the point 0 is the center of the n-dimensional system (1).

Proof. Indeed, if the conditions of Corollary 3.2 hold, then the conditions of Theorem 3.2 hold as well. □

Corollary 3.2 does not require that $f(x) \neq 0$ for all $x \neq 0$ from some neighborhood of the origin of coordinates.

Remark 3.1 *The theorem can be proved for any n-even-dimensional spaces. It is not difficult to see that for n dimensional spaces with n odd,* $n = 2k+1, k$ *is a natural number, the system (1) can not have the origin of coordinates as the center. Indeed, there is no imaginary number among eigenvalues of any matrix* $A(\cdot)$ *for odd n.*

Remark 3.2 *Let us consider the differential system*

$$\dot{x}= f(x,t) \tag{15}$$

where $x = (x_1, x_2, ..., x_n) \in \Re^n$ *and*

$$f(x,t) = (f_1(x,t), f_2(x,t), ..., f_n(x,t))^*.$$

The vector-function $f_p(.)$ *is a polynomial of x i.e.*

$$f_p(x) = \sum_{l_1,l_2,...,l_n \in I_p} a^{(p)}_{l_1,l_2,...,l_n}(t) x_1^{l_1} x_2^{l_2} ... x_n^{l_n},$$

$p \in 1:n, l_1, l_2, ...l_n$ *are non-negative integers,* $a^{(p)}_{l_1,l_2,...,l_n}(t)$ *are continuous real-valued functions and* $*$ *is the transposition sign,* I_p *is a finite set of indexes of the polynomial* $f_p(.)$. *We will assume that* $f(x,t) \neq 0$ *for all* $x \neq 0$ *in some neighborhood of the origin of coordinates* 0 *and* $t > t_0$.

In this case we will denote by $\lambda_i(x,t), i \in 1:n$, *eigenvalues of a matrix* $A(.) \in \mathcal{A}$ *of the system 5. If we demand that all cited above statements about eigenvalues* $\lambda_i(x,t), i \in 1:n$, *are true for all* $t > t_0$ *then we obtain sufficient conditions for the cited above theorems and corollaries.*

Problem 3.1 *For the system*

$$\dot{x}= f(x,t) \tag{16}$$

where $f(.,.)$ *is a polynomial of x and t it is required to find some conditions when the system (16) is asymptotic stable in a domain* $D, 0 \in intD$.

The ideas stated above do not apply to the system (16) because the terms of $f(.,.)$ can be unbounded along some solution $x(\cdot, x_0, t_0)$ in D. If we require the terms of $f(.,.)$ to be bounded along trajectories of the system (16) and that $\|x(t, x_0, t_0)\|$ goes to zero uniformly on x_0 as $t \longrightarrow \infty$, then this system can be transformed to an equivalent stationary system. This idea will be developed in following articles.

4 Stabilizing control

Let us go to question of finding a stabilizing control.

Let us consider the differential system

$$\dot{x}= f(x,u) \tag{17}$$

where $x = (x_1, x_2, ..., x_n) \in \Re^n$ is a phase vector, $u = (u_1, u_2, ..., u_r) \in \Re^r$ is a control, $f(x,u) = (f_1(x,u), f_2(x,u), ..., f_n(x,u))^*$ is a vector-polinimial of x and u with constant real-valued coefficients, i.e.

$$f_p(x,u) = \sum_{l_1,l_2,...,l_n,m_1,m_2,...,m_r \in I_p} a^{(p)}_{l_1,l_2,...,l_n,m_1,m_2,...,m_r} x_1^{l_1} x_2^{l_2} ... x_n^{l_n} u_1^{m_1} u_2^{m_2} ... u_r^{m_r},$$

$p \in 1:n$, $l_1, l_2, ..., l_n, m_1, m_2, ..., m_r$ are non-negative integers and $a^{(p)}_{l_1,...,l_n,m_1,...,m_r}$ are real-valued numbers, I_p is a finite set of indexes of the polynomial $f_p(.)$.

Let us assume that the zero-vector $0 = (0, 0, ..., 0) \in \Re^n$ is a solution of the system (17) for $u = 0 \in \Re^r$.

Definition 4.1 *The control $u(x) = (u_1(x), u_2(x), ..., u_r(x)) \in \Re^r$ is called stabilizing in a domain $D \in \Re^n$, $0 \in intD$, for the system (17) if any solution of (17) $x(t) = x(t, x_0, t_0, u(x(t)))$ satisfies the limit*

$$\| x(t, x_0, t_0, u) \| \longrightarrow 0 \tag{18}$$

as $t \longrightarrow \infty$, $x(t_0) = x_0 \in D$.

As mentioned above condition (18) is sufficient for the system (17) to be asymptotic stable in D, i.e. the zero solution of the system (17) is asymptotic stable and the limit (18) is true for any initial point $x(t_0) = x_0 \in D$.

Problem 4.1 *It is required to find such a stabilizing control $u = u(x)$ in a given domain $D, 0 \in intD, x \in D$ that for any solution $x(t, x_0, t_0, u), x(t_0) = x_0$ of the system (17) the correlation (18) was true.*

For the linear system

$$\dot{x}= Ax + Bu, \tag{19}$$

where $B[n \times r]$ is a matrix of amplification coefficients, a stabilizing control $u \in \Re^r$ can be chosen in the form $u = Cx$ so that (18) is true.

Theorem 4.1 *[1],[4]. If the rank of the system*

$$B,\ AB,\ A^2B,\ ...,\ A^{n-1}B$$

is equal to n then we can always construct a stabilizing control in $\Re^n$ in the form

$$u = Cx,$$

where $C[r \times n]$ is some matrix.

Since the matrix B is defined by technical essence of a system, we can choose it by ourselves constructing this system. Therefore the condition of the theorem can be always satisfied.

The theorem was written for comparison with the results below.

Our goal is to construct a stabilizing control $u(x)$ that a domain D were a domain of asymptotic stability. In common case it is not always can be done. But if we change the system (17) a little bit the problem can be solved. Instead of equation (17) we will consider the equation

$$\dot{x}= f(x,u) + \varphi(u) \tag{20}$$

where $\varphi(\cdot)$ is a vector-polynomial $\varphi(u) = (\varphi_1(u), \varphi_2(u), ..., \varphi_n(u))^*$ with degree not bigger than degree of the function $f(z)$ as a function of $z = (x, u)$,

$$\varphi_p(u) = \sum_{i_1,i_2,...,i_r \in M_p} b^{(p)}_{i_1,i_2,...,i_r} u_1^{i_1} u_2^{i_2} ... u_r^{i_r}$$

where $p \in 1 : n$ and $b^{(p)}_{i_1,i_2,...,i_r}$ are constant real-valued numbers, $i_1, i_2, ..., i_r$ are non-negative integers, M_p is a finite set of indexes of the polynomial $\varphi_p(.)$.

In practice it is possible to construct $\varphi(\cdot)$ because we choose a control $u(\cdot)$ by ourselves.

We will look for a stabilizing control $u = u(x)$ in a form of polynomial in x.

Theorem 4.2 *For any domain $D, 0 \in intD$, vector-polynomials $u(.)$ and $\varphi(.)$ can be chosen such that*
1) D is a region of asymptotic stability for the differential system (20);
2) degree of $u(x)$ does not exceed degree of vector-polynomial $f(x,u)$ as a function of x;
3) degree of $\varphi(.)$ is not bigger than degree of $f(.)$ as a function of $z = (x, u)$.

Proof. We will prove that a vector polynomial $u(x)$ can always be chosen satisfying the following conditions
1) degree of $u(x)$ does not exceed degree of the vector-polynomial $f(x, u)$ as a function of x;
2) $u(x)$ is a stabilizing control for (20) in the domain D.

We will use the results obtained before (Theorem 2.1).

Let us substitute in (20) control $u = u(x)$ in a form of vector-function of x. The system (20) is rewritten as

$$\dot{x}= \hat{f}(x) \tag{21}$$

or

$$\dot{x}= \hat{A}(x)x.$$

We can write conditions for the matrix $\hat{A}(x)$ to have eigenvalues with negative real parts.

There is no difficulty calculating degrees of functions $f(x,u)$ and $\varphi(u)$ as functions of x after substituting $u = u(x)$.

Let us denote by l_x and l_u degrees of the function $f(x,u)$ in the variables x and u correspondently. Then after substituting $u = u(x)$ degree of the function $f(.,.)$ as a function of x is not bigger than $l_x + l_u l_x$ and degree of the function $\varphi(.)$ as a function of x is not bigger than $l_x(l_x + l_u)$. It is easy to see that

$$l_x + l_x l_u \leq l_x(l_x + l_u).$$

Having chosen the function $\varphi(\cdot)$, the coefficients of the vector polynomial $\hat{f}(x)$ can be chosen so that the matrix $\hat{A}(x)$ has eigenvalues with negative real parts. The theorem is proved. □

In an analogous way it can be proved the following theorem.

Theorem 4.3 *For the differential system (20) with even n vector-polynomials $u(\cdot)$ and $\varphi(\cdot)$ can be chosen such that*
1) the origin of coordinates 0 is the center for the system (20);
2) degree of $u(x)$ does not exceed degree $f(x,u)$ as a function of x
3) degree of $\varphi(.)$ is not bigger than degree of $f(.)$ as a function of $z = (x,u)$.

This theorem can be used in physics of plasma specially for stabilization of plasma in reactor.

Remark 4.1 *In the case when the coefficients $a^{(p)}_{l_1,l_2,\ldots,l_n,m_1,m_2,\ldots,m_r}$ depend on t the coefficients $b^{(p)}_{i_1,i_2,\ldots,i_r}$ will depend on t as well. We can try to find these coefficients that the conditions of the theorems of the previous section were true for all $t > t_0$.*

5 One aspect of application

We will study for instance how to choose a control $u(\cdot)$ so that a given domain D consisting from whole trajectories of a differential system were a region of asymptotic stability. For that we have to solve the following optimization problem.

Let us substitute a vector-control $u(\cdot)$ in a form of polynomial of x with degree m not bigger than degree of the vector-polynomial $f(x,u)$ as a function of x in equation (20) and rewrite (20) in the form (21). Denote eigenvalues of a matrix $\hat{A}(\cdot)$ in (21) by $\lambda_j(x, \alpha_{ij}, d^{(l)}_{i_1,i_2,\ldots,i_n}, b^{(p)}_{i_1,i_2,\ldots,i_r})$ where α_{ij} and $d^{(l)}_{i_1,i_2,\ldots,i_n}$ are the coefficients of the matrix $\hat{A}(\cdot)$ and the vector-function $u(\cdot)$ correspondingly, $b^{(p)}_{i_1,i_2,\ldots,i_r}$, $p \in 1:n$, are the coefficients of the vector-function $\varphi(.)$. Then our problem is reduced to the following optimization problem: to find out such continuous functions $\alpha_{ij}(x), i,j \in 1:n$, $x \in D$ and the numbers $d^{(l)}_{i_1,i_2,\ldots,i_n}$, $b^{(p)}_{i_1,i_2,\ldots,i_r}$, $p \in 1:n$, that the following correlation

$$Re\,\lambda_j(x, \alpha_{ij}(x), d^{(l)}_{i_1,i_2,\ldots,i_n}, b^{(p)}_{i_1,i_2,\ldots,i_r}) < 0 \tag{22}$$

was true for all $x \in D, x \neq 0$, where $\alpha_{ij}(x), i,j \in 1:n$, are connected with each other by linear equations for each $x \in D, x \neq 0$, by other words, we should solve the following problem

$$\inf_{\substack{d^{(l)}_{i_1,i_2,\dots,i_n},\, l \in 1:r \\ b^{(p)}_{i_1,i_2,\dots,i_r},\, p \in 1:n}} \; \sup_{\substack{j \in 1:n \\ x \in D \setminus B^n_\delta(0)}} \; \inf_{c_{ij}} Re\,\lambda_j(x, \alpha_{ij}, d^{(l)}_{i_1,i_2,\dots,i_n}, b^{(p)}_{i_1,i_2,\dots,i_r}) < 0$$

$\forall\, \delta > 0$,where $B^n_\delta(0) = \{z \in \Re^n \mid \| z \| \leq \delta\}$, $\delta-$ is any sufficient small number for which $D \supset B^n_\delta(0)$.

It is obvious that the inequalities (22) can be replaced by an equivalent system of inequalities for coefficients of characteristic polynomial of the matrix $\hat{A}(.)$. To solve this system it is easier than to find eigenvalues of the matrix $\hat{A}(.)$.

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