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A Trigonometrical Approach to Morley's Observation

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ABSTRACT

Simple trigonometrical arguments verify that in a triangle the trisectors, proximal to sides respectively, meet at the vertices of an equilateral triangle by showing that the length of each side is 8R times the sines of the angles between the sides of the triangle and the trisectors that determine it, where R is the radius of the circumcircle of the triangle. The 27 meeting points of the trisectors, proximal to a side, determine 18 such equilaterals, which in pairs share a vertex having two collinear sides and the third parallel. Hence these points are located 6 by 6 on three triples of parallel lines.

RESUMEN

Argumentos trigonométricos simples verifican que en un triángulo los trisectores, próximos a los lados respectivamente, se encuentran en los vértices de un triángulo equilátero mostrando que la longitud de cada lado es 8R veces los senos de los ángulos entre los lados del triángulo y los trisectores que lo determinan, donde R es el radio del circuncírculo del triángulo. Los 27 puntos de encuentro de los trisectores, próximos a un lado, determinan 18 tales equiláteros, que a pares comparten un vértice teniendo dos lados colineales y el tercero paralelo. Luego estos puntos están ubicados 6 por 6 en tres triples de líneas paralelas.

Keywords and Phrases: Angle trisection, proximal trisector, triangle trisectors, Morley's theorem, Morley triangle, Morley's magic, Morley's miracle, Morley's mystery.

2010 AMS Mathematics Subject Classification: 00A05, 00A08

1. Introduction

One of the infamous contemporary problems in mathematics refers to the angle trisection in a triangle. The problem appeared suddenly around 1900, when Frank Morley made a shocking observation, known since then as Morley's theorem, usually expressed with the statement: *In a triangle, the trisectors of its angles, proximal to sides respectively, meet at the vertices of an equilateral triangle.*

The above theorem is considered among the most unexpected discoveries in mathematics, as strangely went unnoticed during the ages, even though it expresses a property for trisectors analog to bisectors. Ancient Greeks studied the triangle in depth and they could have discovered it, but simply ignored it. More than one hundred years since its discovery and a very respectable number of publications, several authored by distinguished mathematicians, we are still struggling to fully comprehend it. Words like magic, miracle, mystery, or paradox have appeared in titles of several articles. Notably, Morley's theorem has been included in the list of *The hundred greatest theorems.*[1]

Morley inferred it while studying the behavior of cardioids from an observation that plainly asserts: *In a triangle, the trisectors proximal to a side intersect on three sets of three parallel lines forming equilateral triangles... Thus, if we take the interior trisectors of the angles of a triangle, the points where those proximal to a side meet form an equilateral triangle.* [13, p.469]

In Fig.1 appear 27 equilaterals. Their placement reveals a structure with startling symmetry, where an impressive number of overlapping and interconnected equilaterals are arranged with common vertices and parallel or collinear sides. Their existence, in fact with arrangement, is interpreted as evidence of regularity in the behavior of angle trisectors in a triangle, like the incenter and the excenters of a triangle express regularity in the behavior of its angle bisectors.[11]

Visual inspection easily verifies that only 18 from the 27 triangles determined by the meeting points of trisectors of all three angles, proximal to sides respectively, are equilateral and they are called *Morley triangles.*

Fig.2 illustrates the trisectors of $\angle ABC = 3\beta$, $0 < \beta < 60^\circ$. The *proximal to side* BC trisector is the one (BT) which coincides with side BA after two rotations around B towards BA by $\angle CBT$. Note that the inclinations of the proximal trisectors to the corresponding proximal sides are β, $60^\circ - \beta$ and $60^\circ + \beta$.

The used formulation of Morley's theorem does not specify the type of trisectors, intersecting in pairs at the vertices of an equilateral. This becomes crucial in the presence of many triangles formed by the intersections of trisectors, proximal to sides respectively. However, several from these triangles - but not all - are equilateral. As the theorem can be valid for other types of trisectors this ambiguity may have been deliberately used, although most publications have focused only on the interior trisectors. In contrast to the elementary, sharp and clean statement of the theorem, approaches dealing with the observation remain abstract or in higher mathematics, where it was

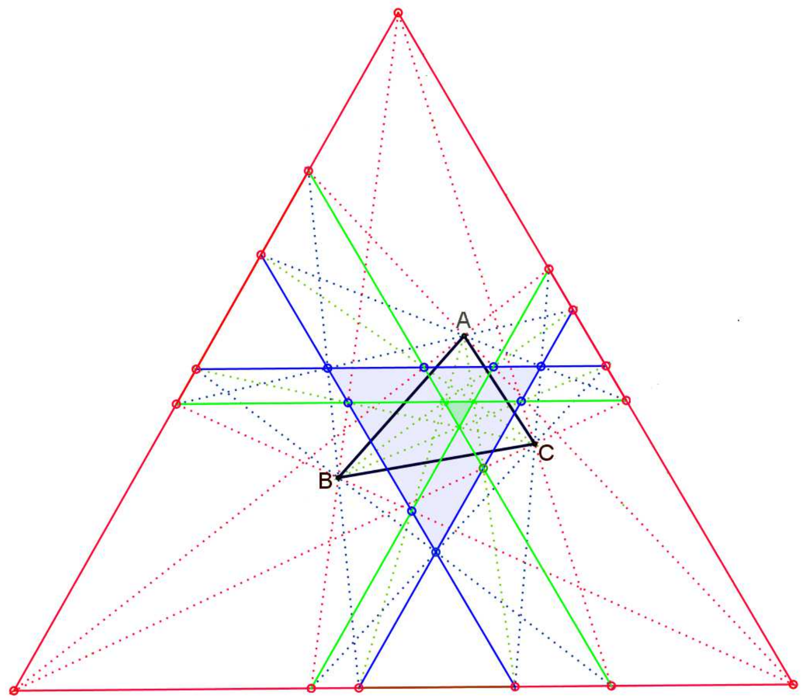

Figure 1: **Fig.1**: *The 27 intersections of trisectors, proximal to sides, determine 18 equilaterals*

discovered. [8],[12], [16]

In this presentation, we prove several instances of Morley's theorem yielding the 18 Morley triangles. In fact, it is shown: *The side length of a Morley triangle is 8R times the* sin *of the angles between the sides of the triangle and the trisectors that determine it, where R is the radius of the circumcircle of the triangle.* Then, utilizing the arrangement of these triangles, we establish Morley's observation about the alignment of the intersections of trisectors proximal to a side. The approach relies on the following trigonometrical property that combines the sine and cosine laws.

Proposition: *In a* ΔSTV *with* $\angle STV = \phi$, *if* $ST = p\sin\theta$ *and* $TV = p\sin\omega$, *where* $\phi+\theta+\omega = 180°$, *then* $SV = p\sin\phi$ *while* $\angle SVT = \theta$ *and* $\angle TSV = \omega$.

Proof. In ΔSTV by applying the law of cosines we get $(SV)^2 = (TS)^2+(TV)^2-2(TS)(TV)\cos\phi \;= p^2\sin^2\phi$ since $\sin^2\theta+\sin^2\omega-2\sin\theta\sin\omega\cos\phi = \sin^2\phi$ by the law of cosines in the triangle with sides $\sin\theta$, $\sin\omega$ and $\sin\phi$. Thus $SV = p\sin\phi$.

Now from the law of sines $ST/\sin(\angle SVT) = TV/\sin(\angle TSV) = SV/\sin\phi$. As $SV = p\sin\phi$, $ST = p\sin\theta$ and $TV = p\sin\omega$, we get $\sin\theta = \sin(\angle SVT)$, $\sin\omega = \sin(\angle TSV)$. Therefore, $\angle SVT = \theta$ or $\angle SVT = 180°-\theta$ and $\angle TSV = \omega$ or$\angle TSV = 180° - \omega$. Since $\phi+\theta+\omega = 180°$ and $\phi+\angle SVT+\angle TSV = 180°$, only the case $\angle SVT = \theta$ and $\angle TSV = \omega$ may hold. ∎

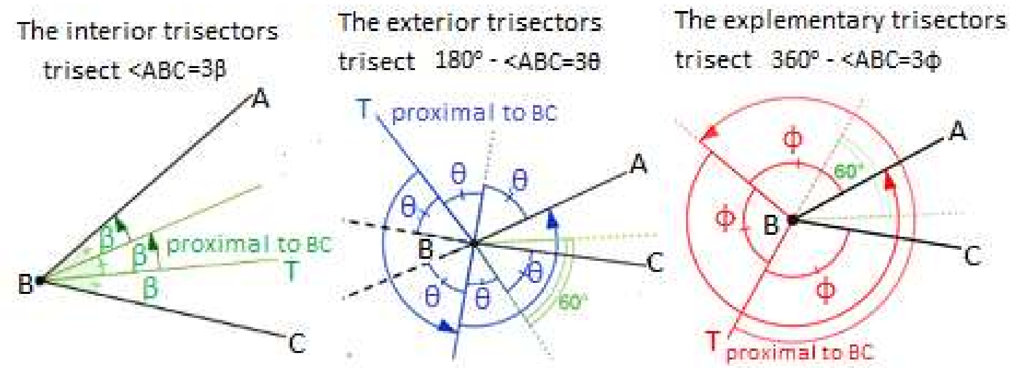

Figure 2: **Fig.2**: *The three types of trisectors*

Given a ΔABC, with $\angle A = 3\alpha$, $\angle B = 3\beta$ and $\angle C = 3\gamma$, where $\alpha+\beta+\gamma = 60°$, we establish that $\Delta A'B'C'$ is equilateral, where A', B' and C' are the intersections of trisectors proximal to sides of ΔABC respectively, by showing $A'B' = B'C' = C'A'$. To prove this we apply the above proposition to the adjacent triangles $\Delta A'CB', \Delta B'AC'$ and $\Delta C'BA'$, formed by a side of $\Delta A'B'C'$ and the trisectors that determine it. The lengths of the trisectors are found from the surrounding triangles $\Delta BA'C$, $\Delta CB'A$ and $\Delta AC'B$ after the sides are expressed by the formulas $AB = 2R\sin 3\gamma, BC = 2R\sin 3\alpha$, $AC = 2R\sin 3\beta$, obtained from the law of sines.

In the appearing expressions, it is convenient to represent angles $\theta + 60°$ and $60° - \theta$ as θ^+ and θ^- respectively. Hence $\theta^+ = 60° + \theta$ and $\theta^- = 60° - \theta$, while $\theta^{++} = 120° + \theta$ and $\theta^{-+} = 120° - \theta$. Also, the formula

$$\sin 3\theta = 4\sin(60° - \theta)\sin\theta\sin(60° + \theta) = 4\sin\theta^-\sin\theta\sin\theta^+$$

will be used to simplify expressions. As $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$ the above follows after factoring out the right hand side. In the article *Are These The Most Beautiful?* Hofstadter is quoted that would have given a very high score to Morley's theorem as it follows from this trigonometrical identity. [15]

In the sequel, Morley triangles are grouped according to the type of trisectors that determine them and only representatives of groups are examined. The groups are: The *primitive* triangles formed by the same type of trisectors, the *mix* triangles formed properly by one type of trisectors of an angle and another type of trisectors of the other two angles, and the *complete* triangles formed by trisectors of one distinct type for each angle. To avoid degenerate cases in which some of the considered triangles are not formed, in the sequel we assume that the angles of the given triangle are different and not multiples of 30°.

2. The 18 Morley triangles

In this section, we will prove that the 18 Morley triangles are indeed equilaterals by producing formulas for the lengths of their sides.

I. Primitive Morley triangles (3)

The primitive triangles are formed exclusively by the intersections of same type trisectors. Hence the two pairs of trisectors determining one of its sides form the same three angles with the corresponding sides of $\Delta \mathsf{ABC}$. In the sequel, we find an expression of the length for one side. As this is independent of the side then all sides have the same length and so the specific Morley triangle is equilateral.

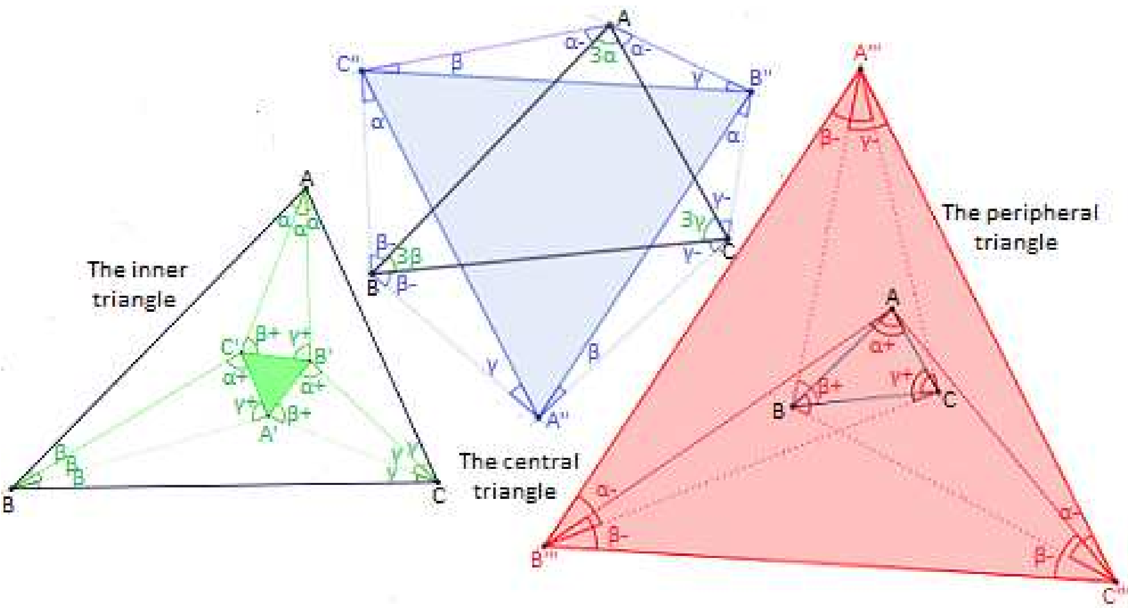

Figure 3: **Fig.3**: *The primitive Morley triangles*

Theorem 0.1. *In a triangle, the same type trisectors of its angles, proximal to sides respectively, meet at the vertices of a corresponding equilateral.*

Proof. Assume that the intersections of the interior, exterior and explementary trisectors, proximal to the sides, BC, CA and AB, meet at A', B' and C', $\mathsf{A}'', \mathsf{B}''$ and C'', $\mathsf{A}''', \mathsf{B}'''$ and C''', defining $\Delta \mathsf{A}'\mathsf{B}'\mathsf{C}'$, $\Delta \mathsf{A}''\mathsf{B}''\mathsf{C}''$ and $\Delta \mathsf{A}'''\mathsf{B}'''\mathsf{C}'''$, called *inner triangle*, *central triangle* and *peripheral triangle* respectively.

The proximal trisectors that determine one of their sides form angles with the corresponding sides of ΔABC equal to α, β and γ, for the inner, or α^-, β^- and γ^-, for the central, or α^+, β^+ and γ^+, for the peripheral triangle, regardless of the side.

a. The lengths of the interior trisectors determining side $B'C'$ are found from the law of sines in $\Delta AB'C$ and $\Delta BC'A$:

$$AB'/\sin\gamma = AC/\sin(180° - \alpha - \gamma) = 2R\sin 3\beta/\sin\beta^- = 8R\sin\beta\sin\beta^+;$$

so $AB' = 8R\sin\beta\sin\beta^+\sin\gamma$. Similarly, $AC' = 8R\sin\gamma\sin\gamma^+\sin\beta$.
Then in $\Delta B'AC'$, $\angle B'AC' = \alpha, AB' = p\sin\beta^+, AC' = p\sin\gamma^+$, where $p = 8R\sin\beta\sin\gamma$. As $\alpha+\beta^++\gamma^+ = 180°, B'C' = p\sin\alpha = 8R\sin\beta\sin\gamma\sin\alpha$ while $\angle AB'C' = \gamma^+$ and $\angle AC'B' = \beta^+$. We conclude that $\Delta A'B'C'$ is equilateral.

b. The lengths of the exterior trisectors determining side $B''C''$ are found from the law of sines in $\Delta AB''C$ and $\Delta BC''A$:

$$AB''/\sin\gamma^- = AC/\sin(180° - \alpha^- - \gamma^-) = 2R\sin 3\beta/\sin\beta^+ = 8R\sin\beta\sin\beta^-$$

and so $AB'' = 8R\sin\beta\sin\beta^-\sin\gamma^-$. Similarly, $AC'' = 8R\sin\gamma\sin\gamma^-\sin\beta^-$.
Then in $\Delta B''AC''$, $\angle B''AC'' = 2\alpha^-+3\alpha = \alpha^{++}, AB'' = p\sin\beta, AC'' = p\sin\gamma$, where $p = 8R\sin\beta^-\sin\gamma^-$. As $\alpha^{++}+\beta+\gamma = 180°$, $B''C'' = p\sin\alpha^{++} = 8R\sin\alpha^-\sin\beta^-\sin\gamma^-$ while $\angle AB''C'' = \gamma$ and $\angle AC''B'' = \beta$. We conclude that $\Delta A''B''C''$ is equilateral.

c. The lengths of the explementary trisectors determining side $B'''C'''$ are found from the law of sines in $\Delta AB'''C$ and $\Delta BC'''A$:

$$AB'''/\sin\gamma^+ = AC/\sin(180° - \alpha^+ - \gamma^+) = 2R\sin 3\beta/\sin\beta = 8R\sin\beta^+\sin\beta^-;$$

so $AB''' = 8R\sin\beta^+\sin\beta^-\sin\gamma^+$. Similarly, $AC''' = 8R\sin\gamma^+\sin\gamma^-\sin\beta^+$.
Then in $\Delta B'''AC'''$, $\angle B'''AC''' = 2\alpha^+ - 3\alpha = \alpha^{-+}, AB''' = p\sin\beta^-, AC''' = p\sin\gamma^-$, where $p = 8R\sin\beta^+\sin\gamma^+$. As $\alpha^{-+} + \beta^- + \gamma^- = 180°$,
$B'''C''' = p\sin\alpha^{-+} = 8R\sin\alpha^+\sin\beta^+\sin\gamma^+$ while $\angle AB'''C''' = \gamma^-$ and $\angle AC'''B''' = \beta^-$. We conclude that $\Delta A'''B'''C'''$ is equilateral. □

II. Mix Morley triangles (9)

The mix triangles are formed by the intersections of same type trisectors of an angle combined with a different type trisectors of the other two. Hence, they share a vertex with a primitive triangle. For a short statement of the next theorem we define the *mixable type* of explementary, interior and exterior trisectors to be the interior, exterior and explementary trisectors respectively.

Theorem 0.2. *In a triangle, the same type trisectors of an angle and the corresponding mixable type trisectors of the other two, proximal to sides respectively, meet at the vertices of an equilateral.*

Proof. **a.** The intersections of the explementary trisectors of an angle and the interior trisectors of the other two proximal to sides respectively, form three triangles, referred as *mix inner* triangles since each of them shares a vertex with the inner triangle. We will study only one representative from them. Consider the explementary trisectors of $\angle B$ and the interior trisectors of the other two.

Assume the interior trisectors proximal to AC meet at B' while the explementary and the interior proximal to BC and AB meet at B_C and B_A defining $\Delta B'B_AB_C$, called *B-inner triangle* as it shares vertex B' with the inner triangle. The angles between the trisectors that determine it and the corresponding sides are equal to β^+ (explementary) and α, γ (interior).

Find the lengths of the trisectors defining $\Delta B'B_AB_C$:

In ΔBB_CA, as $\angle BB_CA = 180° - \alpha - \beta^+ = \gamma^+$,

$$BB_C/\sin\alpha = AB_C/\sin\beta^+ = AB/\sin\gamma^+ = 2R\sin 3\gamma/\sin\gamma^+ = 8R\sin\gamma\sin\gamma^-$$

and so $BB_C = 8R\sin\gamma\sin\gamma^-\sin\alpha$ and $AB_C = 8R\sin\gamma\sin\gamma^-\sin\beta^+$.
Similarly, in ΔCB_AB, $BB_A = 8R\sin\alpha\sin\alpha^-\sin\gamma$ and $CB_A = 8R\sin\alpha\sin\alpha^-\sin\beta^+$.
In $\Delta AB'C$, as $\angle AB'C = 180° - \alpha - \gamma = \beta^{++}$,

$$AB'/\sin\gamma = CB'/\sin\alpha = AC/\sin\beta^- = 2R\sin 3\beta/\sin\beta^- = 8R\sin\beta\sin\beta^+$$

and so $AB' = 8R\sin\beta\sin\beta^+\sin\gamma$ and $CB' = 8R\sin\beta\sin\beta^+\sin\alpha$.

Now in $\Delta B_CAB'$, $AB' = 8R\sin\beta\sin\beta^+\sin\gamma = p\sin\beta$ and $AB_C = 8R\sin\gamma\sin\gamma^{++}\sin\beta^+ = p\sin\gamma^{++}$ where $p = 8R\sin\beta^+\sin\gamma$. As $\alpha+\beta+\gamma^{++} = 180°$, $B'B_C = 8R\sin\beta^+\sin\gamma\sin\alpha$ while $\angle B_CB'A = \gamma^{++}$. Similarly, in $\Delta B_ACB'$, $B'B_C = 8R\sin\beta^+\sin\gamma\sin\alpha$ while $\angle B_AB'C = \alpha^{++}$. Also, in ΔBB_AB_C, $\angle B_CBB_A = 2\beta^+ - 3\beta = \beta^{-+}$, $BB_A = p\sin\alpha^-, BB_C = p\sin\gamma^-$, where $p = 8R\sin\alpha\sin\gamma$. As $\beta^{-+}+\alpha^-+\gamma^- = 180°$, $B_AB_C = p\sin\beta^{-+} = 8R\sin\alpha\sin\gamma\sin\beta^{-+}$ while $\angle BB_CB_A = \alpha^-, \angle BB_AB_C = \gamma^-$. Since $\sin\beta^{-+} = \sin\beta^+$, we conclude that $\Delta B'B_AB_C$ is equilateral. □

Corollary 2a. *A mix inner and the inner triangle have two collinear sides and the third parallel.*

Proof. Consider for instance the B-inner $\Delta B'B_AB_C$. It was seen $\angle AB'B_C = \gamma^{++}$. As it is equilateral $\angle AB'B_A = \gamma^+$. Also, it was shown in the inner $\Delta A'B'C'$ that $\angle AB'C' = \gamma^+$. Thus, $B'B_A$ and $B'C'$ are collinear. □

b. The intersections of the interior trisectors of an angle and the exterior trisectors of the other two, proximal to sides respectively, form three triangles, referred as *mix central* triangles since each of them shares a vertex with the central triangle. We will study only a representative from them. Consider the interior trisectors of $\angle C$ and the exterior trisectors of the other two. Assume the exterior trisectors proximal to AB meet at C'' while the interior and the exterior proximal to BC and AC meet at C''_A and C''_B defining $\Delta C''C''_AC''_B$, called C-*central triangle* as it shares vertex C'' with the central triangle. The angles between the trisectors that determine it and the corresponding sides are equal to γ (interior) and α^-, β^- (exterior).

Find the lengths of the trisectors defining $\Delta C''C''_AC''_B$:
In $\Delta CAC''_B$, as $\angle C''_BAC = 2\alpha^- + 3\alpha = \alpha^{++}$ and so $\angle CC''_BA = \beta$. Hence

$$AC''_B/\sin\gamma = CB''_A/\sin\alpha^{++} = AC/\sin\beta = 2R\sin 3\beta/\sin\beta = 8R\sin\beta^-\sin\beta^+$$

and so $AC''_B = 8R\sin\beta^-\sin\beta^+\sin\gamma$ and $CC''_B = 8R\sin\beta^-\sin\beta^+\sin\alpha^{++}$.
Similarly, in $\Delta CBC''_A$, $BC''_A = 8R\sin\alpha^-\sin\alpha^+\sin\gamma$ and $CC''_A = 8R\sin\alpha^-\sin\alpha^+\sin\beta^{++}$. Also in $\Delta AC''B$,

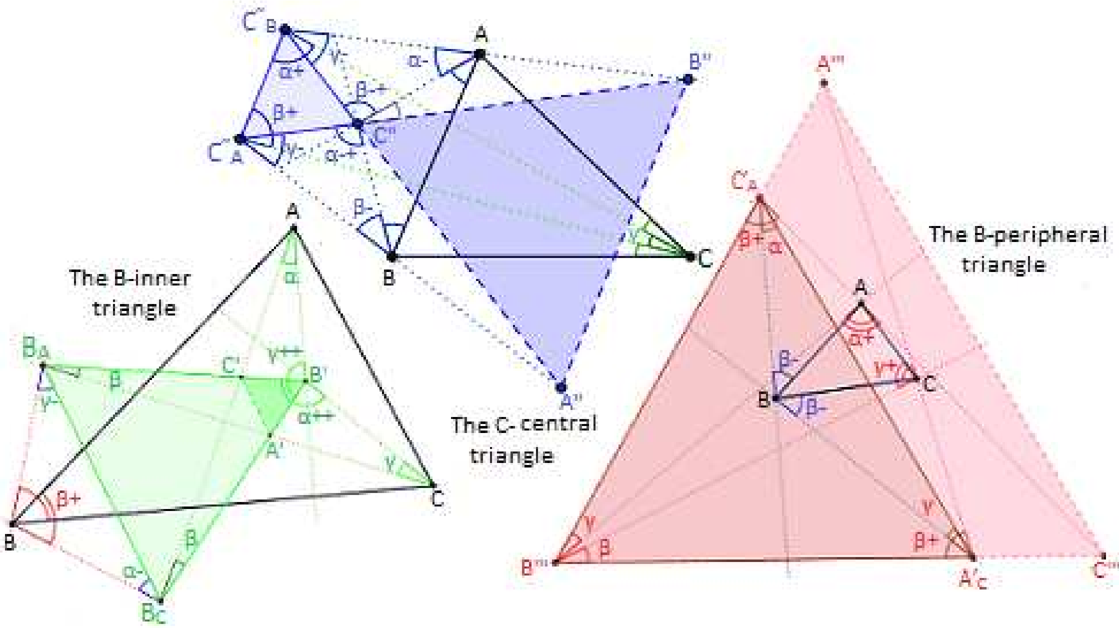

Figure 4: **Fig.4**: *Mix Morley triangles*

as $\angle AC''B = 180° - \beta^- - \alpha^- = \gamma^{-+}$,

$$AC''/\sin\beta^- = BC''/\sin\alpha^- = AB/\sin\gamma^{-+} = 8R\sin 3\gamma/\sin\gamma = 8R\sin\gamma\sin\gamma^-$$

and so $AC'' = 8R\sin\gamma\sin\gamma^-\sin\beta^-$ and $BC'' = 8R\sin\gamma\sin\gamma^-\sin\alpha^-$.

Then in $\Delta C''_B AC''$, $\angle C''_B AC'' = \alpha^-$, $AC''_B = p\sin\beta^{-+}, AC'' = p\sin\gamma^-$ where $p = 8R\sin\gamma\sin\beta^-$. As $\alpha^- + \beta^{-+} + \gamma^- = 180°$, $C''_B C'' = p\sin\alpha^- = 8R\sin\gamma\sin\beta^-\sin\alpha^-$, $\angle AC''_B C'' = \gamma^-$, $\angle AC''C''_B = \beta^{-+}$. Similarly in $\Delta C''_A BC''$, $C''_A C'' = 8R\sin\gamma\sin\beta^-\sin\alpha^-$ and $\angle BC''_A C'' = \gamma^-$, $\angle BC''C''_A = \alpha^{-+}$. Also in $\Delta C''_A CC''_B$, $\angle C''_A CC''_B = \gamma$, $CC''_B = p\sin\beta^+, CC''_A = p\sin\alpha^+$, where $p = 8R\sin\alpha^-\sin\beta^-$. As $\gamma + \beta^+ + \alpha^+ = 180°$, $C''_A C''_B = p\sin\gamma = 8R\sin\alpha^-\sin\beta^-\sin\gamma$ and $\angle CC''_A C''_B = \beta^+, \angle CC''_B C''_A = \alpha^+$. We conclude that $\Delta CC''_A C''_B$ is equilateral. □

Corollary 2b. *A mix central and the central triangle have two collinear sides and the third parallel.*

Proof. Consider for instance the mix central $\Delta C'' C''_A C''_B$. It was seen $\angle AC''C''_B = \beta^{-+}$. Also, it was shown that in the central $\Delta A''B''C''$, $\angle AC''B'' = \beta$. As it is equilateral, $\angle AC''C''_B + \angle AC''B'' + \angle B''C''A'' = 180°$. So $C''_B C''$ and $C''A''$ are collinear. □

c. The intersections of the exterior trisectors of an angle and the explementary trisectors of the other two, proximal to sides respectively, define three triangles, referred as *mix peripherals* since each of them shares a vertex with the peripheral triangle. We will study only one representative from them. Consider the exterior trisectors of $\angle B$ and the explementary trisectors of $\angle C$ and $\angle A$.

Assume that the explementary trisectors proximal to AC meet at B''' while the explementary and the exterior proximal to AB and BC meet at C'_A and A'_C defining $\Delta B'''C'_A A'_C$, called B-*peripheral triangle* as it shares vertex B''' with the peripheral. The angles between the trisectors that determine it and the corresponding sides are equal to β^- (exterior) and α^+, γ^+ (explementary).

The lengths of trisectors that determine $\Delta B'''C'_A A'_C$ are:
In $\Delta BC'_A A$, $\angle ABC'_A = \beta^-$, $\angle BAC' = 180° - \alpha^+ = \alpha^{-+}$ and so $\angle BC'_A A = \gamma^-$,

$$AC'_A/\sin\beta^- = BC'_A/\sin\alpha^{-+} = AB/\sin\gamma^- = 2R\sin 3\gamma/\sin\gamma^- = 8R\sin\gamma\sin\gamma^+;$$

thus $AC'_A = 8R\sin\gamma\sin\gamma^+\sin\beta^-$ and $BC'_A = 8R\sin\gamma\sin\gamma^+\sin\alpha^{-+}$.
Similarly, in $\Delta CA'_C B$, $CA'_C = 8R\sin\alpha\sin\alpha^+\sin\beta^-$ and $BA'_C = 8R\sin\alpha\sin\alpha^+\sin\gamma^{-+}$.
Also in $\Delta AB'''C$, as $\angle AB'''C = 180° - \alpha^+ - \gamma^+ = \beta$,

$$AB'''/\sin\gamma^+ = CB'''/\sin\alpha^+ = AC/\sin\beta = 2R\sin 3\beta/\sin\beta = 8R\sin\beta^+\sin\beta^-;$$

thus $AB''' = 8R\sin\beta^+\sin\beta^-\sin\gamma^+$ and $CB''' = 8R\sin\beta^+\sin\beta^-\sin\alpha^+$.

Then in $\Delta C'_A AB'''$, $\angle C'_A AB''' = 180° - 2\alpha^+ + 3\alpha = \alpha^+$, $AC'_A = p\sin\gamma$,
$AB''' = p\sin\beta^+$ where $p = 8R\sin\beta^-\sin\gamma^+$. As $\alpha^+ + \gamma + \beta^+ = 180°$,
$B'''C'_A = p\sin\alpha^+ = 8R\sin\beta^-\sin\gamma^+\sin\alpha^+$,$\angle AB'''C'_A = \gamma$, $\angle AC'_A B''' = \beta^+$. Similarly, in $\Delta CB'''A'_C$, $B'''A'_C = 8R\sin\beta^-\sin\alpha^+\sin\gamma^+$, $\angle CB'''A'_C = \alpha$, $\angle CA'_C B''' = \beta^+$.
Also, in $\Delta A'_C BC'_A$, as $\angle A'_C BC'_A = 2\beta^- + 3\beta = \beta^{++}$, $BC'_A = p\sin\gamma$, $BA'_C = p\sin\alpha$, where $p = 8R\sin\gamma^+\sin\alpha^+$. As $\beta^{++} + \alpha + \gamma = 180°$,
$C'_A A'_C = p\sin\beta^{++} = 8R\sin\gamma^+\sin\alpha^+\sin\beta^{++}$ while $\angle BA'_C C'_A = \gamma$, $\angle BC'_A A'_C = \alpha$. Since $\sin\beta^{++} = \sin\beta^-$, we conclude that $\Delta B'''C'_A A'_C$ is equilateral. □

Corollary 2c. *A mix peripheral and the peripheral triangle have two collinear sides and the third parallel.*

Proof. Consider for instance the mix peripheral $\Delta B'''C'_A A'_C$. It was seen $\angle AB'''C'_A = \gamma$. Also in the peripheral equilateral, it was shown $\angle AB'''C''' = \gamma^-$. Hence $\angle AB'''A''' = \gamma$. Thus, $B'''C'A$ and $B'''C'''$ are collinear. □

III. Complete Morley triangles (6)

The complete Morley triangles are formed by the intersections of the interior, exterior and explementary trisectors from each angle, proximal to sides respectively. Apparently, there are 3x2 such triangles. For example, the interior trisectors of $\angle C$ combined with the explementary trisectors proximal to CB or CA form two different complete triangles.

Theorem 0.3. *In a triangle, the trisectors of a distinct type from each angle, proximal to sides respectively, meet at the vertices of an equilateral.*

Proof. From the 6 triangles formed by the intersections of the interior, exterior and explementary trisectors from each angle, proximal to sides, we will study only a representative as the rest are

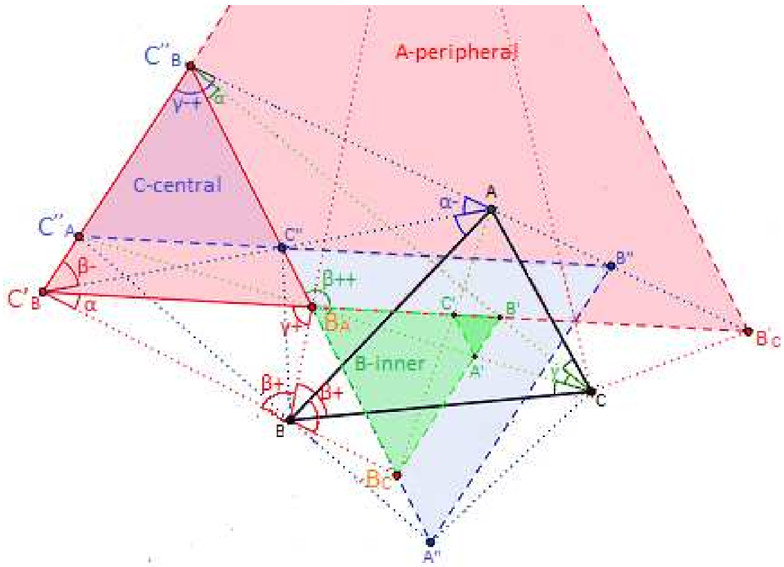

Figure 5: **Fig.5**: *The* B_A*-complete Morley triangle*

similar. Consider for instance the interior trisectors of $\angle C$, the exterior trisectors of $\angle A$ and the explementary trisectors of $\angle B$, proximal to sides respectively. Assume that the interior and the explementary proximal to CB meet at B_A, the interior with the exterior proximal to CA meet at C''_B and the exterior with the explementary meet at C'_B defining $\Delta B_A C''_B C'_B$ called B_A-*complete triangle* as it shares vertex B_A with the B-inner triangle. It also shares vertex C''_B with the C-central, and vertex C'_B with the A-peripheral triangle. The angles between the trisectors that determine it and the corresponding sides are equal to α^-(exterior), β^+ (explementary) and γ (interior).

The lengths of the trisectors that determine it are:

In $\Delta AC'_B B$, $\angle BAC'_B = \alpha^-$, $\angle C'_B BA = 180^\circ - \beta^+ = \beta^{-+}$ and so $\angle AC'_B B = \gamma^-$. Hence

$$AC'_B/\sin\beta^{-+} = C'_B/\sin\alpha^- = AB/\sin\gamma^- = 2R\sin 3\gamma/\sin\gamma^- = 8R\sin\gamma\sin\gamma^+;$$

thus $AC'_B = 8R\sin\gamma\sin\gamma^+\sin\beta^{-+}$ and $BC'_B = 8R\sin\gamma\sin\gamma^+\sin\alpha^-$.
Similarly, in $\Delta BB_A C$, as $\angle B_A BC = \beta^+$, $\angle BB_A C = \alpha^+$,

$$CB_A/\sin\beta^+ = BB_A/\sin\gamma = BC/\sin\alpha^+ = 2R\sin 3\alpha/\sin\alpha^+ = 8R\sin\alpha\sin\alpha^-;$$

thus $CB_A = 8R\sin\alpha\sin\alpha^-\sin\beta^+$ and $BB_A = 8R\sin\alpha\sin\alpha^-\sin\gamma$.
Also in $\Delta CC''_B A$, as $\angle ACC''_B = \gamma$ and $\angle C''_B AC = 2\alpha^- + 3\alpha = \alpha^{++}$, $\angle AC''_B C = \beta$,

$$CC''_B/\sin\alpha^{++} = AC''_B/\sin\gamma = AC/\sin\beta = 2R\sin 3\beta/\sin\beta = 8R\sin\beta^-\sin\beta^+;$$

thus $CC''_B = 8R\sin\beta^-\sin\beta^+\sin\alpha^{++}$ and $AC''_B = 8R\sin\beta^-\sin\beta^+\sin\gamma$.

Then, in $\Delta C''_B AC'_B$, $\angle C''_B AC'_B = \alpha^-$, $AC''_B = p\sin\beta^-$, $AC'_B = p\sin\gamma^+ = p\sin\gamma^{-+}$ where $p = 8R\sin\gamma\sin\beta^{-+}$. As $\alpha^- + \beta^- + \gamma^{-+} = 180^\circ$,
$C''_B C'_B = p\sin\alpha^- = 8R\sin\gamma\sin\beta^{-+}\sin\alpha^-$ while $\angle AC'_B C''_B = \beta^-$, $\angle AC''_B C'_B = \gamma^{-+}$.

Also in $\Delta C'_B B B_A$, $\angle C'_B B B_A = 180° - 2\beta^+ + 3\beta = \beta^+$, $BC'_B = p \sin\gamma^+, BB_A = p\sin\alpha$ where $p = 8R\sin\gamma\sin\alpha^-$. As $\beta^+ + \gamma^+ + \alpha = 180°$,
$C'_B B_A = p\sin\beta^+ = 8R\sin\gamma\sin\alpha^-\sin\beta^+$ while $\angle BB_A C'_B = \gamma^+$ and $\angle BC'_B B_A = \alpha$.
Furthermore, in $\Delta B_A CC''_B$, $\angle B_A CC''_B = \gamma$, $CC''_B = p\sin\beta^- = p\sin\beta^{++}$, $CB_A = p\sin\alpha$, where $p = 8R\sin\alpha^{++}\sin\beta^+$. As $\gamma + \beta^{++} + \alpha = 180°$,
$C''_B A_C = p\sin\gamma = 8R\sin\alpha^{++}\sin\beta^+\sin\gamma$ while $\angle C_A CC''_B = 120° + \beta = \beta^{++}$ and
$\angle CC''_B B_A = \alpha$. We conclude that $\Delta B_A C''_B C'_B$ is equilateral. □

Corollary 3. *A complete and a mix inner, or a mix central or a mix peripheral triangle, sharing a vertex, have two collinear sides and the third parallel.*

Proof. Consider for instance the complete $\Delta B_A C''_B C'_B$. This shares a vertex with the mix inner $\Delta B' B_A B_C$, the mix central $\Delta C'' C''_A C''_B$, and the mix peripheral $\Delta A''' B'_C C'_B$.

It was shown $\angle BB_A C'_B = \gamma^+$. However, in the B-inner $\Delta B' B_A B_C$, $\angle BB_A B_C = \gamma^-$. Since it is equilateral, $\angle B_C B_A B' = 60°$. Then $\angle C'_B B_A B' = \angle C'_B B_A B + \angle BB_A B_C + \angle B_C B_A B' = \gamma^+ + \gamma^- + 60° = 180°$ and so $C'_B B_A$ and $B_A B'$ are collinear. Hence, $\Delta B_A C''_B C'_B$ and $\Delta B' B_A B_C$ have two collinear sides with the third ones parallel.

It was shown $\angle AC''_B C'_B = \gamma^{-+}$. As it is equilateral $\angle AC''_B B_A = \gamma^-$. However, in the C-central, $\Delta C'' C''_A C''_B$ $\angle AC''_B C'' = \gamma^-$. Thus, $C''_B C''$ and $C''_B B_A$ are collinear. Hence, $\Delta B_A C''_B C'_B$ and $\Delta C'' C''_A C''_B$ have two collinear sides with the third parallel.

It was shown $\angle AC'_B C''_B = \beta^-$. Since is $\Delta B_A C''_B C'_B$ equilateral, $\angle AC'_B B_A = \beta$. However, in the B-peripheral, $\angle BC'_A A'_C = \alpha$. Symmetrically, in the A-peripheral $\Delta A''' B'_C C'_B$, $\angle AC'_B B'_C = \beta$. Hence $\angle AC'_B B_A = \angle AC'_B B'_C$ and so $C'_B B_A$ and $C'_B B'_C$ are collinear. Thus $\Delta B_A C''_B C'_B$ and $\Delta A''' B'_C C'_B$ have two collinear sides with the third parallel. □

From the above we infer that a precise formulation of the general Morley's theorem yielding 18 equilaterals is: *In a triangle, the same type trisectors of the three angles, the same type trisectors of an angle with its mixable type trisectors of the other two, and the trisectors of a distinct type from each angle, proximal to sides respectively, meet at the vertices of an equilateral.*

3. Arrangement of Morley triangles and alignment of intersections of proximal trisectors

The trisectors of a triangle, proximal to one of its sides, meet at 27 points. Each of them is a common vertex of two Morley triangles which are arranged with two sides collinear and the third parallel.

From corollary 3, sides $C''_B B_A$ and $C''_B C'_B$ of the complete $\Delta B_A C''_B C'_B$ and the mix central $\Delta C'' C''_B C''_A$ are collinear. Hence, C''_B and B_A lie on the line determined by the side $C'' A''$ of the central triangle $\Delta A''B''C''$. Thus, on the line determined by a side of the central triangle lie 6 intersections of trisectors proximal to a side, two of the interior with the exterior trisectors, two of the interior with the explementary trisectors and two between the exterior trisectors of the central triangle.

Also, sides $B_AC'_B$ and B_AB of the complete $\Delta B_AC''_BC'_B$ and the mix inner are collinear. Hence, C''_B and B_A lie on the line which is determined by a side of the inner triangle. Thus, on the line determined by a side of the inner triangle lie 6 intersections of trisectors proximal to a side, two of the exterior with the explementary trisectors, two of interior with the explementary trisectors and two intersections between interior trisectors of the inner triangle.

Finally, sides $C'_BC''_B$ and C'_BA''' of the complete $\Delta B_AC''_BC'_B$ and the mix peripheral triangle $\Delta A'''B'_CC'_B$ are collinear. As the A-peripheral and the peripheral have sides $A'''C'_B$ and $A'''B'''$ collinear, C''_B and C'_B lie on side $A'''B'''$. Symmetrically C''_B and C'_B lie on side $A'''B'''$ as well. Thus, on a side of the peripheral triangle lie 6 intersections of trisectors proximal to a side, two of the interior with the exterior trisectors, two of the explementary with the exterior trisectors and two between the explementary trisectors of the peripheral triangle.

Conclude that the intersections of trisectors proximal to a side lie 6 by 6 on three triples of parallel lines intersecting with $60°$ angles.

4. Open Problems

Next there are three basic questions stemming out from this work inviting further exploration.
1. How many equilaterals do the intersections of trisectors in a triangle determine?
2. Are there lines or circles, beyond Morley's 3 triples of parallel lines, on which the intersections of trisectors lie?
3. Do theorems exist, analog to Morley's theorem regarding angle trisectors, for the side or perpendicular trisectors?

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References

[1] Paul and Jack Abad, *The Hundred Greatest Theorems*, in Actes de la confrence JFLA2008.

[2] B. Bollobás, *The Art of Mathematics*, Cambridge University Press, (2006), 126-127.

[3] Alain Connes, *A new proof of Morley's theorem*, Publications Mathmatiques de l' IHÉS, (1998), 43-46.

[4] John Conway, The power of mathematics, In Power, Cambridge University Press, (2006), 36-50.

[5] John Conway, *On Morley's Trisector Theorem*, The Mathematical Intelligencer, 36, no. 3(2014), 3-3

[6] Edsger W. Dijkstra, *A Collection of Beautiful Proofs*, Selected Writings on Computing: A personal Perspective, Springer, (1982), 174-183.

[7] Edsger W. Dijkstra, *On the design of a simple proof for Morley's Theorem*, Programming and Mathematical Method, Volume 88 of the NATO ASI Series, Springer (1992) 3-9.

[8] W. J. Dobbs, *Morley's triangle*, The Mathematical Gazette, 1938 JSTOR.

[9] Richard K. Guy, The Lighthouse Theorem *A Budget of Paradoxes*, Amer. Math. Monthly, 114 (2007), 97-141

[10] Clark Kimberling, *Central points and central lines in the plane of a triangle*, Mathematics Magazine, 67 (1994), 163-187.

[11] Spiridon A. Kuruklis, *Trisectors like Bisectors with Equilaterals instead of Points*, CUBO 16, 2 (2014), 71-110.

[12] Henri Lebesgue, Sur les n-sectrices d'un triangle [En memoire de Frank Morley (1860-1930)], Enseingnement Math., 38 (1940), 39-58.

[13] Frank Morley, *Extensions of Clifford's Chain-Theorem*, American J. of Math., 51 (1929), 465-472.

[14] Roger Penrose, *Morley's trisector theorem*, Eureka The Archimedeans Journal, Cambridge, 16 (1953), 6-7.

[15] David Wells, *Are These The Most Beautiful?* The Mathematical Intelligencer, 12 (1990), 37-41.

[16] Glanville F. Taylor and L.W. Marr, *The six trisectors of each of the angles of a triangle*, Proceedings of the Edinburgh Mathematical Society 33 (1913-14), 119-131.

[17] Alexander Bogomolny, *Morley's Miracle*, Interactive Mathematics Miscellany and Puzzles, `http://www.cut-the-knot.org/triangle/Morley/`