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Some geometric properties of $\eta-$ Ricci solitons and gradient Ricci solitons on $(lcs)_n-$manifolds

S. K. Yadav[1], S. K. Chaubey[2] and D. L. Suthar[3]

[1] *Department of Mathematics*
Poornima college of Engineering,
ISI-6, RIICO, Institutional Area, Sitapura, Jaipur-302022, Rajasthan, India.
prof_sky16@yahoo.com
[2] *Section of Mathematics, Department of Information Technology,*
Shinas College of Technology,
Shinas, P.O. Box 77 Postal Code 324, Oman.
sk22_math@yahoo.co.in, sudhakar.chaubey@shct.edu.om
[3] *Department of Mathematics*
Wollo University,
P. O. Box: 1145, Dessie, South Wollo, Amhara Region, Ethiopia.
dlsuthar@gmail.com

ABSTRACT

In the context of para-contact Hausdorff geometry $\eta-$Ricci solitons and gradient Ricci solitons are considered on manifolds. We establish that on an $(LCS)_n-$manifold (M, ϕ, ξ, η, g), the existence of an $\eta-$Ricci soliton implies that (M, g) is quasi-Einstein. We find conditions for Ricci solitons on an $(LCS)_n-$manifold (M, ϕ, ξ, η, g) to be shrinking, steady and expanding. At the end we show examples of such manifolds with $\eta-$Ricci solitons.

RESUMEN

En el contexto de geometría para-contacto Hausdorff, consideramos $\eta-$Ricci solitones y Ricci solitones gradientes en variedades. Establecemos que en una $(LCS)_n-$variedad (M, ϕ, ξ, η, g), la existencia de un $\eta-$Ricci solitón implica que (M, g) es casi-Einstein. Encontramos condiciones para que los Ricci solitones en una $(LCS)_n-$variedad (M, ϕ, ξ, η, g) sean contractivos, estables o expansivos. Al concluir, mostramos ejemplos de dichas variedades con $\eta-$Ricci solitones.

Keywords and Phrases: $\eta-$Ricci solitons, gradient Ricci solitons, $(LCS)_n-$manifold.

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1 Introduction

In 2003, Shaikh [34] introduced the notion of Lorentzian concircular structure manifolds (briefly, $(LCS)_n-$manifolds) with an example, which generalize the notion of LP-Sasakian manifolds introduced by Matsumoto [27] and also by Mihai and Rosca [26]. Then Shaikh and Baishya ([32], [33]) investigated the application of $(LCS)_n-$manifolds to the general theory of relativity and cosmology. The $(LCS)_n-$manifolds are also studied by Atceken et al. ([1], [2], [21]), D. Narain and S. Yadav [30], S. Yadav et al. ([39]-[42]), Shaikh and his co-authors ([35], [36]) and many others. Ricci solitons represent a natural generalization of Einstein metrics on a Riemannian manifold, being generalized fixed points of Hamilton's Ricci flow $\frac{\partial}{\partial t}g = -2S$ [24]. The evolution equation defining the Ricci flow is a kind of nonlinear diffusion equation, an analogue of heat equation for metrics. Under Ricci flow, a metric can be improved to evolve into more canonical one by smoothing out its irregularities, depending on the Ricci curvature of the manifold, it will expand in the directions of negative Ricci curvature and shrink in the positive case. Ricci solitons have been studied in many contexts: on Kähler manifolds [18], on contact and Lorentzian manifolds ([3], [15], [25], [31], [38]), on Sasakian ([19], [20]), $\alpha-$Sasakian [25] and $K-$contact manifolds [31], on Kenmotsu ([4], [28]) and $f-$Kenmotsu manifolds [15] and by ([16], [43]) etc. In paracontact geometry, Ricci soliton firstly appeared in the paper of G. Calvoruso and D. Perrone [13]. C. L. Bejan and M. Crasmarean studied Ricci solitons on 3−dimensional normal paracontact manifolds [5].

A more general notion is that of $\eta-$Ricci soliton introduced by J. T. Cho and M. Kimura [6], which was treated by C. Calin and M. Crasmareanu on Hopf hypersurfaces in complex space forms [14].

2 $(LCS)_n-$manifolds (M, ϕ, ξ, η, g)

An $n-$dimensional Lorentzian manifold M is a smooth connected paracontact Hausdorff manifold with Lorentzian metric g, that is, M admits a smooth symmetric tensor field g of type $(0, 2)$ such that for each point $p \in M$ the tensor $g_p : T_pM \times T_pM \to$ Re is a non degenerate inner product of signature $(-, +, ..., +)$, where T_pM denotes the tangent space of M at p and Re is the real number space. A non-zero tangent vector $v \in T_pM$ is said to be timelike (resp., non-spacelike, null and spacelike) if it satisfies $g_p(v, v) < 0$ (resp., $< 0, =, > 0$) [29].

Definition 2.1. In a Lorentzian manifold (M, g) a vector field ρ defined by

$$g(X, \rho) = A(X),$$

for any $X \in \chi(M)$ is said to be a concircular vector field [44] if

$$(\nabla_X A)\ (Y) = \alpha \{g(X, Y) + \omega(X)A(Y)\},$$

where α is a non-zero scalar and ω is a closed $1-$form. Here ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g.

Let M be a Lorentzian manifold admitting a unit timelike concircular vector field ξ, called the generator of the manifold. Then we have

$$g(\xi,\xi) = -1. \tag{2.1}$$

Since ξ is the unit concircular vector field, then there exists a non-zero $1-$form η such that for

$$g(X,\xi) = \eta(X), \tag{2.2}$$

the following equation holds

$$(\nabla_X \eta)\ (Y) = \alpha\{g(X,Y) + \eta(X)\eta(Y)\} \qquad (\alpha \neq 0), \tag{2.3}$$

that is,

$$\nabla_X \xi = \alpha\{X + \eta(X)\xi\},$$

for all vector fields X, Y on M, where α is a non-zero scalar function satisfies

$$\nabla_X \alpha = (X\alpha) = d\alpha(X) = \rho\eta(X), \tag{2.4}$$

ρ being a certain scalar function given by $\rho = -(\xi\alpha)$. If we put

$$\phi X = \frac{1}{\alpha}\nabla_X \xi, \tag{2.5}$$

then from (2.3) and (2.5), we have

$$\phi X = X + \eta(X)\xi, \tag{2.6}$$

from which it follows that ϕ is a $(1,\ 1)$ tensor and called the structure tensor of the manifold. Thus the Lorentzian manifold M together with the unit timelike concircular vector field ξ, its associated $1-$form η and $(1,\ 1)$ tensor field ϕ, is said to be a Lorentzian concircular structure manifold (briefly $(LCS)_n-$manifold) [34]. Specially, if we take $\alpha = 1$, then we can obtain the LP-Sasakian structure of Matsumoto [27]. In a $(LCS)_n-$manifold $(n > 2)$, the following relations hold ([32]-[35]):

$$a)\ \ \eta(\xi) = -1, \qquad b)\ \ \phi\xi = 0, \qquad c)\ \ \phi^2 X = X + \eta(X)\xi, \tag{2.7}$$

$$d)\ \ \eta(\phi X) = 0, \qquad e)\ g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y),$$

$$\eta(R(X,Y)Z) = (\alpha^2 - \rho)\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}, \tag{2.8}$$

$$R(X,Y)\xi = (\alpha^2 - \rho)\{\eta(Y)X - \eta(X)Y\}, \tag{2.9}$$

$$R(\xi,X)Y = (\alpha^2 - \rho)\{g(X,Y)\xi - \eta(Y)X\}, \tag{2.10}$$

$$R(\xi,X)\xi = (\alpha^2 - \rho)\{\eta(X)\xi + X\}, \tag{2.11}$$

$$(\nabla_X \phi)\ (Y) = \alpha\{g(X,Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X\}, \tag{2.12}$$

$$S(X,\xi) = (n-1)(\alpha^2 - \rho)\eta(X), \tag{2.13}$$

$$S(\phi X, \phi Y) = S(X,Y) + (n-1)(\alpha^2 - \rho)\eta(X)\eta(Y), \tag{2.14}$$

$$(X\rho) = d\rho(X) = \beta\eta(X), \tag{2.15}$$

for any vector fields X, Y, Z on M and $\beta = -(\xi\rho)$ is a scalar function, where R is the curvature tensor and S is the Ricci tensor of the manifold.

3 $\eta-$Ricci solitons on $(LCS)_n-$manifolds

Let (M, ϕ, ξ, η, g) be a $(LCS)_n-$manifold. We follow the equation

$$L_\xi g + 2S + 2\lambda g + 2\mu\eta \otimes \eta = 0, \tag{3.1}$$

where L_ξ is the Lie-derivative operator along the vector field ξ, S is the Ricci tensor field of the metric g, λ and μ are real constants. We write $L_\xi g$ in term of the Levi-Civita connection ∇, we obtain

$$(L_\xi g)(X, Y) = g(\nabla_Y \xi, X) + g(Y, \nabla_X \xi) = 2\alpha[g(X, Y) + \eta(X)\eta(Y)]. \tag{3.2}$$

In view of (3.1) and (3.2), we get

$$QX = -(\alpha + \lambda)X - (\alpha + \mu)\eta(X)\xi, \tag{3.3}$$

$$r = -n\lambda - (n - 1)\alpha + \mu, \tag{3.4}$$

$$S(X, Y) = -(\alpha + \lambda)g(X, Y) - (\alpha + \mu)\eta(X)\eta(Y), \tag{3.5}$$

$$S(X, \xi) = S(\xi, X) = (\mu - \lambda)\eta(X), \tag{3.6}$$

$$\mu - \lambda = (n - 1)(\alpha^2 - \rho), \tag{3.7}$$

for any $X, Y \in \chi(M)$. Here r is the scalar curvature and Q denotes the Ricci operator corresponding to S, that is, $S(X, Y) = g(QX, Y)$, for all X, Y on M. The structure (g, ξ, λ, μ) that follows the equation (3.1) is said to be an $\eta-$Ricci soliton to (M, g) [6]. In particular, if $\mu = 0$, (g, ξ, λ) is a Ricci soliton [24] and it is called shrinking, steady, or expanding according as λ is negative, zero or positive, respectively [12].

Proposition 3.1. *On a* $(LCS)_n-$*manifold* (M, ϕ, ξ, η, g) *the following relations hold*

$$(i)\ \eta(\nabla_X \xi) = 0,\ \ (ii)\ \nabla_\xi \xi = 0,\ \ (iii)\ \nabla\eta = \alpha\{g + \eta \otimes \eta\},\ \ (iv)\ \nabla_\xi \eta = 0,$$
$$(v)\ L_\xi \phi = 0,\ \ (vi)\ L_\xi \eta = 0,\ \ (vii)\ L_\xi(\eta \otimes \eta) = 0,\ \ (viii)\ L_\xi g = 2\alpha(g + \eta \otimes \eta),$$

where ∇ *is the Levi-Civita connection associated to* g. *Also* η *is closed, the distribution is involutive and tensor field of* ϕ *vanishes identically, i. e., the structure is normal.*

Proof. Since

$$(\nabla_X \phi)(Y) = \alpha\{g(X, Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X\},$$

this indicates that

$$\nabla_X \phi Y - \phi(\nabla_X Y) = \alpha\{g(X, Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X\}.$$

Taking $Y = \xi$ in above equation, we have $\phi(\nabla_X \xi) = \alpha\phi X$. Applying ϕ both sides, we get

$$\nabla_X \xi + \eta(\nabla_X \xi) = \alpha\{X + \eta(X)\xi\}.$$

Since $\nabla_X\xi = \alpha\phi X$ and $X(g(\xi,\xi)) = 2g(\nabla_X\xi,\xi)$, therefore $\eta(\nabla_X\xi) = 0$ and $\nabla_\xi\xi = 0$. Also $(\nabla_X\eta)(Y) = \alpha g(Y,\phi X) = \alpha\{g(X,Y)+\eta(X)\eta(Y)\}$, this implies that

$$\nabla\eta = \alpha\{g+\eta\otimes\eta\}, i.e., \nabla_\xi\eta = 0.$$

In view of definition of Lie-derivative, we get

$$(L_\xi\phi)(X) = [\xi,\phi X]-\phi([\xi,X]) = \nabla_\xi\phi X-\phi(\nabla_\xi X) = (\nabla_\xi\phi)(X) = 0,\ i.e., L_\xi\phi = 0.$$

Also,

$$(L_\xi\eta)(X) = \xi(\eta(X)-\eta([\xi,X]) = g(X,\nabla_\xi\xi)+g(\nabla_X\xi,\xi) = 0,\ \ i.e.,\ L_\xi\eta = 0.$$

Further we compute

$$(L_\xi(\eta\otimes\eta))(X,Y) = \xi(\eta(X)\eta(Y))-\eta([\xi,X])\eta(Y)-\eta(X)\eta([\xi,Y]),$$

which implies that

$$(L_\xi(\eta\otimes\eta))(X,Y) = \eta(X)g(Y,\nabla_\xi\xi)-\eta(Y)g(X,\nabla_\xi\xi) = 0,\ i.e.,\ L_\xi(\eta\otimes\eta) = 0.$$

Again $(L_\xi g)(X,Y) = \xi\, g(X,Y)-g([\xi,X],Y)-g(X,[\xi,Y])$, implies that

$$(L_\xi g)(X,Y) = \alpha[g(\phi X,Y)+g(X,\phi Y)].$$

Using (2.6), we get

$$L_\xi g = 2\alpha(g+\eta\otimes\eta).$$

At last $(d\eta)(X,Y) = X(\eta(Y))-Y(\eta(X))-\eta([X,Y])$, that implies

$$\begin{aligned}(d\eta)(X,Y) &= g(Y,\nabla_X\xi)-g(X,\nabla_Y\xi)\\ &= \alpha\{g(Y,X)+\eta(X)\eta(Y)\}-\alpha\{g(X,Y)+\eta(X)\eta(Y)\} = 0.\end{aligned}$$

Finally,

$$N_\phi(X,Y) = \phi^2[X,Y]+[\phi X,\phi Y]-\phi[\phi X,Y]-\phi[X,\phi Y].$$

This yields that

$$N_\phi(X,Y) = \phi^2(\nabla_X Y)-\phi^2(\nabla_Y X)-\phi(\nabla_X\phi Y)+\phi(\nabla_Y\phi X)$$

$$+\nabla_{\phi X}\phi Y-\phi(\nabla_{\phi X}Y)-\nabla_{\phi Y}\phi X+\phi(\nabla_{\phi Y}X) = 0.$$

Thus the structure is normal. □

In [17], S. Chandra et al. proved that a second order parallel symmetric tensor on a $(LCS)_n-$ manifold with $\alpha^2-\rho\neq 0$, is a constant multiple of the Ricci tensor. Thus we apply this concept for $\eta-$Ricci soliton and we prove the following result:

Theorem 3.2. *Let* (M, ϕ, ξ, η, g) *be a* $(LCS)_n-$*manifold. Assume that the symmetric* $(0, 2)$ *tensor field* $h = L_\xi g + 2S + 2\mu\eta \otimes \eta$ *is parallel with respect to the Levi-Civita connection associated to* g*, then* (g, ξ, λ) *yields an* $\eta-$*Ricci soliton on* M*.*

Proof. Since

$$h(\xi, \xi) = (L_\xi g)(\xi, \xi) + 2S(\xi, \xi) + 2\mu\eta(\xi)\eta(\xi) = 2\lambda,$$

implies that

$$\lambda = \frac{1}{2}h(\xi, \xi). \tag{3.8}$$

In [17], we have

$$h(X, Y) = -h(\xi, \xi)g(X, Y), \quad X, Y \in \chi(M). \tag{3.9}$$

Therefore, $L_\xi g + 2S + 2\mu\eta \otimes \eta = 2\lambda g$. Our theorem is proved. □

If $\mu = 0$, it follows that $L_\xi g + 2S + 2(n - 1)(\alpha^2 - \rho)g = 0$. Thus we conclude that

Corollary 3.3. *On a* $(LCS)_n-$*manifold* (M, ϕ, ξ, η, g) *with the property that the symmetric* $(0, 2)$ *tensor field* $h = L_\xi g + 2S$ *is parallel with respect to the Levi-Civita connection associated to* g*, then the equation* (3.1)*, for* $\mu = 0$ *and* $\lambda = -[(n - 1)(\alpha^2 - \rho)]$*, defines a Ricci soliton.*

As a consequence of the existence of $\eta-$Ricci soliton on a $(LCS)_n-$manifold. From (3.1), we state that

Corollary 3.4. *If the equation* (3.1) *define an* $\eta-$*Ricci soliton on a* $(LCS)_n-$ *manifold, then* (M, g) *is quasi-Einstein.*

Since the manifold is quasi-Einstein, if the Ricci tensor field S is a linear combination (with real scalar λ and μ, respectively, with $\mu \neq 0$) of g and the tensor product of a non-zero $1-$form η satisfying (2.2) and for an Einstein if S is co-linear with g ([13], [23]).

Theorem 3.5. *If* (M, ϕ, ξ, η, g) *be a* $(LCS)_n-$*manifold and equation* (3.1) *define an* $\eta-$*Ricci soliton on* (M, g)*, then*
(i) $Q \circ \phi = \phi \circ Q$*,* *(ii)* Q *and* S *are parallel along* ξ*.*

Proof. The prove of (i) follows by direct computation. For (ii) we have

$$(\nabla_\xi Q)X = \nabla_\xi QX - Q(\nabla_\xi X)$$

and

$$(\nabla_\xi S)(X, Y) = \xi(S(X, Y)) - S(\nabla_\xi X, Y) - S(X, \nabla_\xi Y).$$

In view of (3.3) and (3.5), above equation leads the result. □

In a particular case if the manifold is $\phi-$Ricci symmetric, then $\phi^2 \circ \nabla Q = 0$, therefore we state the following proposition as:

Proposition 3.6. *If a* $(LCS)_n-$*manifold* (M, ϕ, ξ, η, g) *is* $\phi-$*Ricci symmetric and equation* (3.1) *leads to* $\eta-$*Ricci soliton, then* $\mu = -\alpha, \lambda = -[(n-1)(\alpha^2 - \rho) + \alpha]$ *and the manifold reduces to Einstein.*

Proof. We compute

$$(\nabla_X Q)Y = \nabla_X QY - Q(\nabla_X Y).$$

In view of (3.3) above equation takes the form

$$(\mu + \alpha)\{X + \eta(X)\xi\}\eta(Y) = 0, \quad \text{for} \quad X, Y \in \chi(M).$$

From above it follows that $\mu = -\alpha, \lambda = -[(n-1)(\alpha^2 - \rho) + \alpha]$, and $S = (n-1)(\alpha^2 - \rho)g$. □

As a weaker version of local symmetry, the notion of local-symmetric Sasakian manifold was introduced by Takahashi [37]. Chaubey ([7]- [11]) studied the properties of symmetric spaces in different extent. Shaikh et al. ([35], [36]) studied locally $\phi-$symmetric and locally $\phi-$recurrent $(LCS)_n-$manifolds. Hui [22] studied $\phi-$pseudosymmetric $(LCS)_n-$manifolds and obtained the form of Ricci tensor S as

$$S(X, Y) = \left\{\frac{\alpha(n-1)(\alpha^2 - \rho)}{\alpha + A(\xi)}\right\} g(X, Y) + \left\{\frac{(n-1)(\alpha^2 - \rho)A(\xi)}{\alpha + A(\xi)}\right\} \eta(X)\mu(Y), \tag{3.10}$$

provided $\alpha + A(\xi) \neq 0$.

Theorem 3.7. *If the tensor field* $L_\xi g + 2S$ *on a* $\phi-$*pseudo Ricci symmetric* $(LCS)_n-$ *manifold with* $\alpha^2 - \rho \neq 0$ *is parallel with respect to Levi-Civita connection associated to* g*, then for* $\mu = 0$ *the Ricci soliton* (g, ξ, λ) *is shrinking, steady and expanding according as* $\frac{(\alpha^2-\rho)\{A(\xi)-\alpha\}}{\alpha+A(\xi)} < 0$, $A(\xi) = \alpha$ *and* $\frac{(\alpha^2-\rho)\{A(\xi)-\alpha\}}{\alpha+A(\xi)} > 0$ *respectively.*

Proof. Let h is a $(0, 2)$ symmetric parallel tensor field on $(LCS)_n-$manifold. In view of (3.1), we obtain

$$h(X, Y) = (L_\xi g)(X, Y) + 2S(X, Y). \tag{3.11}$$

Using (3.2) and (3.10), equation (3.11) reduces to

$$\begin{aligned} h(X, Y) = 2\alpha[g(X, Y) + \eta(X)\eta(Y)] + 2\left\{\frac{\alpha(n-1)(\alpha^2-\rho)}{\alpha+A(\xi)}\right\} g(X, Y) \\ +2\left\{\frac{(n-1)(\alpha^2-\rho)A(\xi)}{\alpha+A(\xi)}\right\} \eta(X)\mu(Y). \end{aligned} \tag{3.12}$$

Replacing $X = Y = \xi$ in (3.12), we get

$$h(\xi, \xi) = \left\{\frac{2(n-1)(\alpha^2 - \rho)\{A(\xi) - \alpha\}}{\alpha + A(\xi)}\right\}. \tag{3.13}$$

In view of (3.8) and (3.13), we obtain

$$\lambda = \left\{\frac{(n-1)(\alpha^2 - \rho)\{A(\xi) - \alpha\}}{\alpha + A(\xi)}\right\}.$$

Since $n > 1$, $\alpha^2 - \rho \neq 0$ and $\alpha + A(\xi) \neq 0$, we conclude that $\lambda > 0$ if $\frac{(\alpha^2-\rho)\{A(\xi)-\alpha\}}{\alpha+A(\xi)} > 0$, $\lambda = 0$ if $A(\xi) = \alpha$ and $\lambda < 0$ if $\frac{(\alpha^2-\rho)\{A(\xi)-\alpha\}}{\alpha+A(\xi)} < 0$. Our theorem is proved. □

Corollary 3.8. *If the tensor field* $L_\xi g + 2S$ *on a* $\phi-$*pseudo Riccisymmetric* $(LCS)_n-$ *manifold with* $\alpha^2 - \rho \neq 0$ *is parallel with respect to Levi-Civita connection associated to* g*, then for* $\mu = 0$ *the Ricci soliton* (g, ξ, λ)*is shrinking and expanding according as* $\alpha^2 - \rho > 0$ *and* $\alpha^2 - \rho < 0$ *respectively.*

Let $(LCS)_n-$manifold admits a Ricci soliton defined by (3.1) for $\mu = 0$. It is known that $\nabla g = 0$. We consider λ constant, so $\nabla \lambda g = 0$. Thus $L_V g + 2S$ is parallel. Hence $L_V g + 2S$ is a constant multiple of metric tensors g, i.e. $L_V g + 2S = ag$, where a is constant. Thus $L_V g + 2S + 2\lambda g$ reduces to $(a + 2\lambda)g$, we get $\lambda = -\frac{a}{2}$. In view of above statement we state the result as the proposition.

Proposition 3.9. *In* $(LCS)_n-$*manifold the Ricci soliton* (V, ξ, λ) *is shrinking or expanding according as* a *is positive or negative.*

Theorem 3.10. *If in a* $(LCS)_n-$ *manifold, the metric* g *is a Ricci soliton and* V *is a point-wise co-linear with* ξ*, then* V *is a constant multiple of* g *provided* $\lambda = -(n-1)(\alpha^2 - \rho)$.

Proof. Suppose that V is pointwise colinear with ξ, i.e., $V = c\,\xi$, where c is a smooth function on (M, g). Then $(L_V g + 2S + 2\lambda g)(X, Y) = 0$ implies that

$$\begin{aligned} &cg(\nabla_X \xi, Y) + (Xc)\eta(Y) + cg(\nabla_Y \xi, X) + (Yc)\eta(X) \\ &+2S(X, Y) + 2\lambda g(X, Y) = 0. \end{aligned}$$

With the help of (2.5), the above equation takes the form

$$\begin{aligned} &c\alpha g(\phi X, Y) + (Xc)\eta(Y) + c\alpha g(\phi Y, X) + (Yc)\eta(X) \\ &+2S(X, Y) + 2\lambda g(X, Y) = 0. \end{aligned} \tag{3.14}$$

Substituting $Y = \xi$ in (3.14) and using (2.7) and (2.13) in it, we get

$$(Xc) = 2\left[\lambda + (n-1)(\alpha^2 - \rho)\right] \eta(X). \tag{3.15}$$

Since η is closed, i. e., $d\eta = 0$ on $(LCS)_n-$manifold. From (3.15) we yield $Xc = 0$, provided $\lambda = -(n-1)(\alpha^2 - \rho)$. Our theorem is proved □

Theorem 3.11. *If in a LP-Sasakian manifold the metric* g *is a Ricci soliton and* V *is a point-wise co-linear with* ξ*, then the manifold is an* $\eta-$*Einstein manifold provided* $c \neq -1$.

Proof. Particularly if $V = \xi$, then in view of that equation $(L_V g + 2S + 2\lambda g)(X, Y) = 0$, we have

$$\alpha g(\phi X, Y) + S(X, Y) + \lambda g(X, Y) = 0. \tag{3.16}$$

Putting $X = \xi$ in (3.16), we get $\lambda = -(n-1)(\alpha^2 - \rho)$. Since $n > 1$, $\alpha^2 - \rho \neq 0$. Therefore the Ricci soliton is shrinking or expanding as $\alpha^2 < \rho$ or $\alpha^2 > \rho$ respectively. Specially, if we take $\alpha = 1$, then (M, g) reduces to a LP-Sasakian structure of Matsumoto [27]. Then in view of (3.14) and (3.15), equation (3.16) reduces to

$$S(X,Y) = \left(\frac{2\lambda}{1+c}\right) g(X,Y) + \left(\frac{2}{1-c}\right)(-\lambda - (n-1)(1-\rho)\eta(X)\eta(Y), \tag{3.17}$$

provided $c \neq -1$. Our theorem is proved. □

Corollary 3.12. *If in a $(LCS)_n$−manifold, the metric g is a Ricci soliton (V, ξ, λ) and V is a point-wise co-linear with ξ, then the Ricci solution (V, ξ, λ) is shrinking or expanding according as $\alpha^2 - \rho > 0$ or $\alpha^2 - \rho < 0$.*

In [31], Sharma proved that a compact Ricci soliton of constant scalar curvature is Einstein. On contracting (3.17), we get $r = \left(\frac{2}{1+b}\right)[\lambda(n+1) + (1-\rho)]$ =constant. Thus we state the result as:

Corollary 3.13. *A LP-Sasakian manifold equipped with a compact Ricci soliton is an Einstein manifold.*

Theorem 3.14. *If $(LCS)_n$−manifold is η−Einstein of the form $S = \delta\, g + \gamma\, \eta \otimes \eta$ with $\delta, \gamma =$ constant, then the manifold is equipped a Ricci soliton $(g, \xi, -(\delta + \alpha))$.*

Proof. Let (M, g) be an η−Einstein $(LCS)_n$−manifold, then

$$S(X,Y) = \delta g(X,Y) + \gamma\eta(X)\eta(Y), \tag{3.18}$$

where $\delta, \gamma =$ constants. Taking $V = \xi$ in (3.1) (for $\mu = 0$) and using (3.18), we get

$$(L_\xi g)(X,Y) + 2S(X,Y) + 2\lambda g(X,Y) = 2(\alpha + \delta + \lambda) + 2(\alpha + \gamma)\eta(X)\eta(Y). \tag{3.19}$$

It is clear from (3.19) that (M, g) admits a Ricci soliton (g, ξ, λ) if $\alpha + \delta + \lambda = 0$ and $\alpha + \gamma = 0$ it implies that $\gamma = -\alpha =$ constant. Also from (3.18) we have $\delta = -\alpha + (n-1)(\alpha^2 - 1) =$ constant. Thus $\lambda = -(\alpha + \delta) =$ constant. Our theorem is proved. □

Corollary 3.15. *If an η−Einstein $(LCS)_n$−manifold with the form $S = \delta\, g + \gamma\, \eta \otimes \eta$ admits a compact Ricci soliton $(g, \xi, -(\delta + \alpha))$ then it leads to an Einstein.*

4 Gradient Ricci solitons

In this section we consider gradient Ricci soliton on $(LCS)_n$− manifold and prove the following results

Theorem 4.1. *If an η−Einstein $(LCS)_n$−manifold equipped with a gradient Ricci soliton then manifold reduces to an Einstein provided $\lambda = (n-1)(\alpha^2 - \xi)$ within the frame field $\xi f = 0$.*

Proof. Let the vector field V be the gradient of a potential function f, is called gradient Ricci soliton. Thus (3.1) takes the form

$$\nabla\nabla f = S + \lambda g, \tag{4.1}$$

that implies

$$\nabla_Y Df = QY + \lambda Y, \tag{4.2}$$

where ∇ is the gradient operator of g. From above we notice that

$$g(R(\xi, Y)Df, \xi) = g((\nabla_\xi Q)Y, \xi) - g((\nabla_Y Q)\xi, \xi). \tag{4.3}$$

In view of (2.5) and (3.18), equation (4.3) yields

$$g(R(\xi, Y)Df, \xi) = 0. \tag{4.4}$$

Using (2.10) in (4.4), we get

$$Df = (\alpha^2 - \rho)\{-\eta(Df)\xi\} = -(\alpha^2 - \rho)(g(Df, \xi)\xi) = -(\alpha^2 - \rho)(\xi f)\xi. \tag{4.5}$$

From (4.2) and (4.5), we obtain

$$S(X, Y) + \lambda g(X, Y) = -Y(\xi f)(\alpha^2 - \rho)\eta(X) - (\xi f)(\alpha^2 - \rho)g(\phi Y, X). \tag{4.6}$$

Replacing $X = \xi$ in (4.6) and using (3.15), we yield

$$Y(\xi f)(\alpha^2 - \rho) = \{\lambda - (n-1)(\alpha^2 - \rho)\}\eta(Y).$$

It implies that if $\lambda = (n-1)(\alpha^2 - \rho)$ then ξf =constant and therefore from (4.5), we have

$$Df = -(\alpha^2 - \rho)(\xi f)\xi = \omega\xi, \ \ \omega = -(\alpha^2 - \rho)(\xi f).$$

If we consider $\xi f = 0$, then (4.5) implies that f = constant. Thus (4.1) yields that $S = (n-1)(\alpha^2 - \rho)g$. Our theorem is proved. □

5 Examples of an $\eta-$Ricci solition on $(LCS)_n-$manifolds

Example 5.1. *Let* $M = \left\{(x, y, z) \in \mathrm{Re}^3 : z \neq 0\right\}$ *be a* $3-$*dimensional smooth manifold, where* (x, y, z) *are the standard coordinates in* Re^3. *Let* $\{E_1, E_2, E_3\}$ *be linearly independent global frame on* M *given by*

$$E_1 = e^z\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right), \qquad E_2 = e^z\frac{\partial}{\partial y}, \qquad E_3 = e^{2z}\frac{\partial}{\partial z}.$$

Let g *be the Lorentzian metric defined by*

$$g(E_1, E_3) = g(E_2, E_3) = g(E_1, E_2) = 0, \ g(E_1, E_1) = g(E_2, E_2) = 1, \ g(E_3, E_3) = -1.$$

and η *be the* 1*-form defined by* $\eta(V) = g(V, E_3)$ *for any* $V \in \chi(M)$*. Let* ϕ *be the* $(1,1)$ *tensor field defined by* $\phi E_1 = E_1$, $\phi E_2 = E_2$, $\phi E_3 = 0$*. Then using the linearity of* ϕ *and* g *we have*

$$\eta(E_3) = -1, \quad \phi V = V + \eta(V)E_3, \quad g(\phi V, \phi W) = g(V, W) + \eta(V)\eta(W),$$

for any $V, W \in \chi(M)$. *Let* ∇ *be the Levi-Civita connection with respect to the Lorentzian metric* g *and* R *be the curvature tensor of* g*. Then we obtain*

$$[E_1, E_2] = -e^{z}E_2, \qquad [E_1, E_3] = -e^{2z}E_1, \qquad [E_2, E_3] = -e^{2z}E_2.$$

Taking $E_3 = \xi$ *and using Koszul's formula for the Lorentzian metric* g*, we have*

$$\nabla_{E_1}E_3 = -e^{2z}E_1, \qquad \nabla_{E_1}E_1 = -e^{2z}E_3, \qquad \nabla_{E_1}E_2 = 0,$$

$$\nabla_{E_2}E_3 = -e^{2z}E_2, \qquad \nabla_{E_3}E_2 = 0, \qquad \nabla_{E_2}E_1 = -e^{2z}E_2,$$

$$\nabla_{E_3}E_3 = 0, \qquad \nabla_{E_2}E_2 = -e^{2z}E_3 - e^{z}E_1, \qquad \nabla_{E_3}E_1 = 0.$$

From the above it can be easily see that $E_3 = \xi$ *is a unit timelike concircular vector field and hence* (ϕ, ξ, η, g) *is a* $(LCS)_3$*-structure on* M*. Consequently* $M^3(\phi, \xi, \eta, g)$ *is a* $(LCS)_3$*-manifold with* $\alpha = -e^{2z} \neq 0$ *such that* $(X\alpha) = \rho\eta(X)$ *where* $\rho = 2e^{4z}$*. Using the above relations, we can easily calculate the non-vanishing components of the curvature tensor* R *and Ricci tensor* S *as follows:*

$$R(E_2, E_3)E_3 = e^{4z}E_2, \quad R(E_1, E_3)E_3 = e^{4z}E_1, \quad R(E_1, E_2)E_2 = \{e^{4z} - e^{2z}\}E_1,$$

$$R(E_2, E_3)E_2 = e^{4z}E_3, \quad R(E_1, E_3)E_1 = e^{4z}E_3, \quad R(E_1, E_2)E_1 = \{-e^{4z} - e^{2z}\}E_2,$$

$$S(E_1, E_1) = 0, \qquad S(E_2, E_2) = 0, \qquad S(E_3, E_3) = 2e^{4z}.$$

Also from (3.5)*, we calculated that*

$$S(E_1, E_1) = -(\alpha + \lambda), \qquad S(E_2, E_2) = -(\alpha + \lambda), \qquad S(E_3, E_3) = (\lambda - \mu).$$

Thus we conclude that from (3.5) *for* $\alpha = -e^{2z}, \lambda = e^{2z}$ *and* $\mu = e^{2z} - e^{4z}$*, the structure* (g, ξ, λ, μ) *is an* η*−Ricci soliton on* $M^3(\phi, \xi, \eta, g)$.

Example 5.2. *Let a* 3*−dimensional manifold* $M = \left\{(x, y, z) \in \mathrm{Re}^3 : z \neq 0\right\}$ *, where* (x, y, z) *are the standard coordinates in* Re^3 *. Let*$\{E_1, E_2, E_3\}$ *be linearly independent global frame on* M *given by*

$$E_1 = e^{-z}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right), \qquad E_2 = e^{-z}\frac{\partial}{\partial y}, \qquad E_3 = e^{-2z}\frac{\partial}{\partial z}.$$

Let g *be the Lorentzian metric defined by*

$$g(E_1, E_3) = g(E_2, E_3) = g(E_1, E_2) = 0, \; g(E_1, E_1) = g(E_2, E_2) = 1, \; g(E_3, E_3) = -1.$$

Let η *be the* 1*−form defined by* $\eta(V) = g(V, E_3)$ *for any* $V \in \chi(M)$*. Let* ϕ *be the* $(1,1)$ *tensor field defined by* $\phi E_1 = E_1$, $\phi E_2 = E_2$, $\phi E_3 = 0$*. Then using the linearity of* ϕ *and* g *we have*

$$\eta(E_3) = -1, \quad \phi V = V + \eta(V)E_3, \quad g(\phi V, \phi W) = g(V, W) + \eta(V)\eta(W),$$

for any $V, W \in \chi(M)$. *Let* ∇ *be the Levi-Civita connection with respect to the Lorentzian metric* g *and* R *be the curvature tensor of* g. *Then we obtain*

$$[E_1, E_2] = -e^{-z}E_2, \qquad [E_1, E_3] = -e^{-2z}E_1, \qquad [E_2, E_3] = -e^{-2z}E_2.$$

Taking $E_3 = \xi$ *and using Koszul's formula for the Lorentzian metric* g, *we have*

$$\nabla_{E_1}E_3 = e^{-2z}E_1, \qquad \nabla_{E_1}E_1 = e^{-2z}E_3, \qquad \nabla_{E_1}E_2 = 0,$$

$$\nabla_{E_2}E_3 = e^{-2z}E_2, \qquad \nabla_{E_3}E_2 = 0, \qquad \nabla_{E_2}E_1 = e^{-2z}E_2,$$

$$\nabla_{E_3}E_3 = 0, \qquad \nabla_{E_3}E_2 = e^{-2z}E_3 - e^{-z}E_1, \qquad \nabla_{E_3}E_1 = 0.$$

From the above it can be easily seen that $E_3 = \xi$ *is a unit timelike concircular vector field and hence* (ϕ, ξ, η, g) *is a* $(LCS)_3$*-structure on* M. *Consequently* $M^3(\phi, \xi, \eta, g)$ *is a* $(LCS)_3$*-manifolds with* $\alpha = e^{-2z} \neq 0$ *such that* $(X\alpha) = \rho\eta(X)$ *where* $\rho = 2e^{-4z}$ *Using the above relations, we can easily calculate the non-vanishing components of the curvature tensor* R *and Ricci tensor* S *as follows:*

$$R(E_2, E_3)E_3 = e^{-4z}E_2,\ R(E_1, E_3)E_3 = e^{-4z}E_1,\ R(E_1, E_2)E_2 = \{e^{-4z} - e^{-2z}\}E_1,$$

$$R(E_2, E_3)E_2 = e^{-4z}E_3,\ R(E_1, E_3)E_1 = e^{-4z}E_3,\ R(E_1, E_2)E_1 = \{-e^{-4z} - e^{-2z}\}E_2,$$

$$S(E_1, E_1) = 2e^{-4z} - e^{-2z}, \quad S(E_2, E_2) = 2e^{-4z} - e^{-2z}, \quad S(E_3, E_3) = 2e^{-4Z}.$$

Also from (3.5), *we calculated that*

$$S(E_1, E_1) = -(\alpha + \lambda), \quad S(E_2, E_2) = -(\alpha + \lambda), \quad S(E_3, E_3) = (\lambda - \mu).$$

We summaries that from (3.5) *for* $\alpha = e^{2z}, \lambda = -2e^{-4z}$ *and* $\mu = -4e^{-4z}$, *the data* (g, ξ, λ, μ) *admits an* $\eta-$*Ricci soliton on* $M^3(\phi, \xi, \eta, g)$.

Example 5.3. *We consider the* $4-$*dimensional manifold* $M = \left\{(x, y, z, u) \in \mathrm{Re}^4 : u \neq 0\right\}$, *where* (x, y, z, u) *are the standard coordinates in* Re^4 . *Let*$\{E_1, E_2, E_3, E_4\}$ *be linearly independent global frame on* M *given by*

$$E_1 = u\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right), \quad E_2 = u\frac{\partial}{\partial y}, \quad E_3 = u\left(\frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right), \quad E_4 = (u)^3\frac{\partial}{\partial u}.$$

Let g *be the Lorentzian metric defined by*

$$g(E_1, E_1) = g(E_2, E_2) = g(E_3, E_3) = 1, g(E_4, E_4) = -1, g(E_i, E_j) = 0, i \neq j, i, j = 1, 2, 3, 4.$$

Let η *be the* $1-$*form defined by* $\eta(V) = g(V, E_4)$, $\xi = (u)^4\frac{\partial}{\partial u}$ *for any* $V \in \chi(M)$. *Let* ϕ *be the* $(1, 1)$ *tensor field defined by* $\phi E_1 = E_1$, $\phi E_2 = E_2$, $\phi E_3 = E_3, \phi E_4 = 0$. *Then using the linearity of* ϕ *and* g *we have*

$$\eta(E_4) = -1, \quad \phi V = V + \eta(V)E_3, \ g(\phi V, \phi W) = g(V, W) + \eta(V)\eta(W),$$

for any $V, W \in \chi(M)$. *Let* ∇ *be the Levi-Civita connection with respect to the Lorentzian metric* g *and* R *be the curvature tensor of* g. *Then we obtain*

$$[E_1, E_2] = -uE_2, \quad [E_1, E_4] = -(u)^4 E_1, \quad [E_2, E_4] = -(u)^4 E_2, \ [E_3, E_4] = -(u)^4 E_3.$$

Taking $E_4 = \xi$ *and using Koszul's formula for the Lorentzian metric* g, *we have*

$$\nabla_{E_1} E_4 = -(u)^2 E_1, \quad \nabla_{E_2} E_1 = uE_3, \quad \nabla_{E_1} E_1 = -(u)^4 E_4,$$

$$\nabla_{E_3} E_4 = -(u)^4 E_3, \quad \nabla_{E_3} E_3 = -(u)^4 E_4, \quad \nabla_{E_2} E_2 = -(u)^2 E_4 - uE_1.$$

From the above it can be easily seen that the structure (ϕ, ξ, η, g) *is an* $(LCS)_4-$ *structure on* M. *Consequently* $M^4(\phi, \xi, \eta, g)$ *is an*$(LCS)_4$*-manifold with* $\alpha = -(u)^4 \neq 0$ *such that* $(X\alpha) = \rho\eta(X)$ *where* $\rho = 2(u)^4$. *Using the above relations, we can easily calculate the non-vanishing components of the curvature tensor* R *and Ricci tensor* S *as follows:*

$$R(E_1, E_4)E_1 = (u)^4 E_4, \ \ R(E_2, E_4)E_2 = (u)^4 E_4, \ \ R(E_3, E_4)E_3 = (u)^4 E_4,$$

$$R(E_1, E_3)E_3 = (u)^4 E_1, \ \ R(E_1, E_3)E_1 = -(u)^4 E_3, \ \ R(E_2, E_3)E_2 = -(u)^4 E_3,$$

$$R(E_1, E_4)E_4 = (u)^4 E_1, \ \ R(E_2, E_4)E_4 = (u)^4 E_2, \ \ R(E_1, E_2)E_2 = [(u)^4 - (u)^2]E_1,$$

$$R(E_2, E_3)E_3 = (u)^4 E_2, \ \ R(E_3, E_4)E_4 = (u)^4 E_3, \ \ R(E_1, E_2)E_1 = -[(u)^4 - (u)^2]E_2,$$

$$S(E_1, E_1) = 3(u)^4 - (u)^2, S(E_2, E_2) = 3(u)^4 - (u)^2, S(E_3, E_3) = 3(u)^4, S(E_4, E_4) = 3(u)^4.$$

Also from (3.5), *we calculated that*

$$S(E_1, E_1) = -(\alpha + \lambda), \ S(E_2, E_2) = -(\alpha + \lambda), \ S(E_3, E_3) = (\lambda - \mu), \ S(E_4, E_4) = (\lambda - \mu).$$

We conclude that from (3.5) *for* $\alpha = -(u)^4, \lambda = -3(u)^4 + 2(u)^2$ *and* $\mu = -6(u)^4 + 2(u)^2$ *the data* (g, ξ, λ, μ) *admits an* $\eta-$*Ricci soliton on* $M^4(\phi, \xi, \eta, g)$.

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