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Characterizations of the Schwartz Space $\mathfrak{S}$ and the Beurling-Björck Space $\mathfrak{S}_w$

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ABSTRACT

In this expository article we present the characterizations proved by J. Chung, S.-Y. Chung and D. Kim, and by S.-Y. Chung, D. Kim and S. Lee, of the Schwartz space $\mathfrak{S}$ and of the Beurling-Björck space $\mathfrak{S}_w$. For the most part we follow the original proofs, only adjusting the estimates as necessary in order to prove not only set-theoretic equalities, but also topological equalities. These results show that the space $\mathfrak{S}_w$ is, as a set as well as topologically, a direct generalization of the space $\mathfrak{S}$. Minor modifications of the original arguments allow us to obtain explicit linear estimates.

1 Introduction

One of the great achievements of the theory of generalized functions devised by Laurent Schwartz was to provide a satisfactory framework for the Fourier transform ([17], [18]). The space $\mathfrak{S}$ of test functions and its topological dual $\mathfrak{S}'$, the space of tempered distributions, allow to formulate extensions of the classical definition and properties of the Fourier transform, including in a natural way results stated for L^p spaces.

The space $\mathfrak{S}$, as defined by Schwartz, consists of C^∞ functions that decay rapidly at infinity, jointly with all their derivatives. This means that each function and its derivatives of any order decay at infinity faster than the reciprocal of any polynomial. Jaeyoung Chung, Soon-Yeong Chung and Dohan Kim proved in [3] that $\mathfrak{S}$ can be described as the set of C^∞ functions that decay rapidly at infinity while their derivatives remain bounded. Their very nice proof uses induction and three steps that involve an L^2 estimate, an L^1 estimate, and an L^∞ estimate that resembles the proof of the Sobolev embedding theorem. In the same paper, they also give a second characterization of $\mathfrak{S}$ in terms of the rapid decay of the function and its Fourier transform. Soon-Yeong Chung, Dohan Kim and Sungjin Lee obtained in [4] a second shorter proof of the first characterization of $\mathfrak{S}$. It is based on Landau's inequality ([14], [5]) and it is also a very nice proof.

In [4], the authors formulate and prove by a similar argument a characterization of the Beurling-Björck space $\mathfrak{S}_w$, a subspace of $\mathfrak{S}$. The topological dual $\mathfrak{S}'_w$ of $\mathfrak{S}_w$ is a space of generalized functions, called tempered ultradistributions, that are not necessarily tempered distributions. The spaces $\mathfrak{S}_w$ and $\mathfrak{S}'_w$ were defined by Göran Björck in [2]. He took up and expanded the work of Arne Beurling ([1]), in order to extend work by Lars Hörmander ([10], [11], [12], [13]) and Avner Friedman ([6]), on various formulations of convexity, ellipticity, hypoellipticity, and existence of solutions.

One of the characterizations of $\mathfrak{S}$ and $\mathfrak{S}_w$ given in [3] and [4] respectively, allows us to see immediately that the space $\mathfrak{S}_w$ is an extension of the Schwartz space $\mathfrak{S}$. This characterization imposes separate conditions on the function and on its Fourier transform. This can be interpreted as treating time and frequency independently. Although we will not go into the details, we want to point out that Karlheinz Gröchenig and Georg Zimmermann have obtained in [7] and [8] joint time-frequency characterizations of $\mathfrak{S}$ and $\mathfrak{S}_w$ using the short-time Fourier transform.

The main purpose of our exposition is to discuss all the results presented in [3] and [4]. A careful analysis of the proofs shows that they produce non-linear recursive inequalities between appropriate norms. When these recursive inequalities are written in closed form, they give the topological equivalence of various sets of norms. A minor modification of one of the proofs shows that this topological equivalence can be given in terms of explicit linear estimates, which more appropriately reflect the linearity of the problem at hand. Certainly, in all the cases, an straightforward application of the Open Mapping Theorem for Fréchet spaces will give linear estimates, although without an explicit dependence of one set of norms on the other.

Our exposition is organized in three sections. In Section 2 we include some preliminary definitions and results. The characterizations of $\mathfrak{S}$ are presented in Section 3, while Section 4 is dedicated to the characterization of $\mathfrak{S}_w$.

The notation we use is standard. The symbols C^∞, C_0^∞, L^p, etc., indicate the usual spaces of functions defined on $\mathbb{R}^n$, with complex values. We denote $|\cdot|$ the Euclidean norm on $\mathbb{R}^n$, while $\|\cdot\|_{L^p}$ indicates the norm in the space L^p. When we do not work on the general Euclidean space $\mathbb{R}^n$, we will write $L^p(\mathbb{R})$, etc., as appropriate. Partial derivatives will be denoted ∂^α, where α is a multi-index $(\alpha_1, ..., \alpha_n)$. If it is necessary to indicate on which variables we are taking the derivative, we will do so by attaching sub-indexes. We will use the standard abbreviations $|\alpha| = \alpha_1 + ... + \alpha_n$, $x^\alpha = x_1^{\alpha_1}...x_n^{\alpha_n}$. With $\alpha \leq \beta$ we mean that $\alpha_j \leq \beta_j$ for every j. The Fourier transform of a function g will be denoted $\mathcal{F}(g)$ or $\widehat{g}$ and it will be defined as $\int_{\mathbb{R}^n} e^{-2\pi i x\xi} g(x)\,dx$. The inverse Fourier transform is then $\mathcal{F}^{-1}(g) = \int_{\mathbb{R}^n} e^{2\pi i x\xi} g(\xi)\,d\xi$. The letter C will indicate a positive constant, that may be different at different occurrences. If it is important to indicate that a constant depends on certain parameters, we will do so by attaching sub-indexes to the constant. We will not indicate the dependence of constants on the dimension n or other fixed parameters.

2 Preliminary definitions and results

In this section we will collect a few definitions and results that we will use later. We start with the definition of the space $\mathfrak{S}$ of Schwartz ([17], [18]).

Definition 2.1 ([17], [18])

$$\mathfrak{S} = \left\{\varphi : \mathbb{R}^n \to \mathbb{C} : \varphi \text{ is } C^\infty \text{ and } \left\|x^\alpha \partial^\beta \varphi\right\|_{L^\infty} < \infty \text{ for all } \alpha, \beta\right\} \tag{1}$$

We can give to $\mathfrak{S}$ an structure of Fréchet space by means of the countable family of norms

$$S = \left\{p_{k,m}(\varphi) = \sup_{|\alpha| \leq k, |\beta| \leq m} \left\|x^\alpha \partial^\beta \varphi\right\|_{L^\infty}\right\}_{k,m=0}^{\infty}. \tag{2}$$

Remark 2.2 *By a Fréchet space we mean a Hausdorff locally convex topological vector space that is metrizable and complete.*

Lemma 2.3 *Given a C^∞ function $\varphi : \mathbb{R}^n \to \mathbb{C}$, the following statements are equivalent.*

1. $\left\|x^\alpha \partial^\beta \varphi\right\|_{L^\infty} < \infty$, *for all* α, β.
2. $\left\|\partial^\beta (x^\alpha \varphi)\right\|_{L^\infty} < \infty$, *for all* α, β.
3. $\left\|\left(1 + |x|^2\right)^k \partial^\beta \varphi\right\|_{L^\infty} < \infty$, *for all* k, β.

The proof of this lemma, based on multi-variable versions of the Leibniz's rule and the binomial theorem, is quite straightforward and we will omit it.

Remark 2.4 *As a consequence of Lemma 2.3, the topology of the space $\mathfrak{S}$ can be described by other families of norms. Namely,*

$$\overline{S} = \left\{\overline{p}_{k,m}(\varphi) = \sup_{|\alpha| \leq k, |\beta| \leq m} \left\|\partial^\beta (x^\alpha \varphi)\right\|_{L^\infty}\right\}_{k,m=0}^{\infty}, \tag{3}$$

$$\overline{\overline{S}} = \left\{ \overline{\overline{p}}_{k,m}(\varphi) = \sup_{|\beta| \leq m} \left\| \left(1 + |x|^2\right)^k \partial^\beta \varphi \right\|_{L^\infty} \right\}_{k,m=0}^{\infty}. \tag{4}$$

The equivalence of the norms defined by (2), (3) and (4) can be given in terms of explicit linear estimates.

Corollary 2.5 *Given $\varphi \in \mathfrak{S}$, the function $\left(1 + |x|^2\right)^k \partial^\beta \varphi$ is integrable for every $k = 0, 1, 2, \ldots$ and every multi-index β. Moreover, the Fourier transform $\widehat{\varphi}$ is a C^∞ function.*

Proof. The proof of this corollary uses basic results on differentiation under the integral sign and it will be omitted. ∎

Before we define the space $\mathfrak{S}_w$ we need to introduce the space $\mathcal{M}_c$ of admissible functions w. The original definition of $\mathcal{M}_c$ is stated in [2] p. 363. However, it will be necessary for us to consider functions $w(x)$ that are radially non-decreasing. That is to say, $w(x) \leq w(y)$ when $|x| \leq |y|$. For this reason, we will use the slightly more restrictive definition stated in [16] p. 14.

Definition 2.6 ([16]) With $\mathcal{M}_c$ we indicate the space of functions $w : \mathbb{R}^n \to \mathbb{R}$ of the form $w(x) = \Omega(|x|)$, where

1. $\Omega : [0, \infty) \to [0, \infty)$ is increasing, continuous and concave,
2. $\Omega(0) = 0$,
3. $\int_{\mathbb{R}} \frac{\Omega(t)}{(1+t^2)} dt < \infty$,
4. $\Omega(t) \geq a + b \log(1 + t)$ for some $a \in \mathbb{R}$ and some $b > 0$.

Definition 2.7 ([2]) Given $w \in \mathcal{M}_c$, we denote by $\mathfrak{S}_w$ the space of functions $\varphi : \mathbb{R}^n \to \mathbb{C}$ such that $\varphi \in C^\infty$ and

$$q_{k,m}(\varphi) = \sup_{|\beta| \leq m} \left\| e^{kw} \partial^\beta \varphi \right\|_{L^\infty} < \infty, \tag{5}$$

$$q_{k,m} \circ \mathcal{F}(\varphi) = \sup_{|\beta| \leq m} \left\| e^{kw} \partial^\beta \widehat{\varphi} \right\|_{L^\infty} < \infty, \tag{6}$$

for all $k, m = 0, 1, 2, \ldots$.

We observe that (5) implies that $\varphi \in L^1$ and $\widehat{\varphi} \in C^\infty$. So, the formulation of (6) makes sense. The space $\mathfrak{S}_w$ is a Fréchet space with the topology defined by the family of norms $\{q_{k,m}, q_{k,m} \circ \mathcal{F}\}_{k,m=0}^{\infty}$. The two most important examples of functions w in $\mathcal{M}_c$ are $w(x) = \log(1 + |x|)$ and $w(x) = |x|^d$ for $0 < d < 1$. The conditions imposed on the function w assure that the space $\mathfrak{S}_w$ satisfies the properties expected from a space of testing functions. For instance, there is a suitable version of the space C_0^∞ that contains partitions of unity and it is dense in $\mathfrak{S}_w$, the operators of differentiation and multiplication by x^α are continuous from $\mathfrak{S}_w$ into itself, the space $\mathfrak{S}_w$ is a topological algebra under pointwise multiplication and convolution. We refer to [2], and [16] p.16 for discussions on the significance of each of the conditions 1. through 3. in Definition 2.7.

Remark 2.8 *When $w(x) = \log(1+|x|)$, the conditions $q_{k,m}(\varphi) < \infty$, $q_{k,m} \circ \mathcal{F}(\varphi) < \infty$ reduce to*

$$\sup_{|\beta| \le m} \left\| (1+|x|)^k \partial^\beta \varphi \right\|_{L^\infty} < \infty,$$
$$\sup_{|\beta| \le m} \left\| (1+|\xi|)^k \partial^\beta \widehat{\varphi} \right\|_{L^\infty} < \infty.$$

It is known from the theory of tempered distributions ([17],[18]), that these two sets of conditions are redundant, due to the very special role that the function $(1+|x|)$ plays with respect to the Fourier transform and its inverse. The characterizations of $\mathfrak{S}$ and $\mathfrak{S}_w$ proved in [3] and [4] avoid this problem, by formulating conditions that turn out to be the same for both spaces.

For future reference, we end Section 1 with a version due to Jacques Hadamard ([9]) of Landau's inequality ([14]), as it appears in [5].

Lemma 2.9 ([14], [5]) *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function with continuous derivatives of order ≤ 2. We assume that there exist $P, Q \ge 0$ so that*

$$|f(x)| \le P,$$
$$|f''(x)| \le Q,$$

for all $x \in \mathbb{R}$. Then

$$|f'(x)| \le \sqrt{2PQ}$$

for all $x \in \mathbb{R}$.

Proof. Given $h \in \mathbb{R}$, $h > 0$, we can write

$$f(x+h) = f(x) + f'(x)h + f''(y)\frac{h^2}{2},$$
$$f(x-h) = f(x) - f'(x)h + f''(z)\frac{h^2}{2}$$

where y and z are between x and $x+h$, and $x-h$ and x, respectively. Thus,

$$f(x+h) - f(x-h) = 2f'(x)h + (f''(y) - f''(z))\frac{h^2}{2},$$

or,

$$|f'(x)| \le \frac{P}{h} + \frac{hQ}{2}, \tag{7}$$

for every $x \in \mathbb{R}$, $h > 0$. If we assume momentarily that $P, Q > 0$, we can see that the right side of (7), as a function of h, has a global minimum at $h = \sqrt{\frac{2P}{Q}}$. Thus,

$$|f'(x)| \leq \sqrt{2PQ}, \tag{8}$$

which is also true when P or Q are zero.

This completes the proof of Lemma 2.9. ∎

Remark 2.10 *Although it will not be of importance to us, Hadamard showed that* $\sqrt{2}$ *is the best possible constant in (8). The original Landau's inequality ([14]) was proved by Edmund Landau ([14]) for functions defined on* $(0, \infty)$ *with best constant equal to* 2*. If we use in (8) the inequality* $\sqrt{PQ} \leq \frac{P+Q}{2}$*, we obtain a linear estimate of the form*

$$|f'(x)| \leq \frac{\sqrt{2}}{2}(P + Q).$$

This estimate can be deduced from (7) by choosing $h = \sqrt{2}$*. Of course any other positive value of* h *will also result in a linear estimate.*

3 Two characterizations of the Schwartz space $\mathfrak{S}$

In this section we will state and prove two characterizations of the space $\mathfrak{S}$, following the methods used in [3], [4]. We now state the first characterization.

Theorem 3.1 ([3], [4]) *The space* $\mathfrak{S}$ *defined by (1) can be described as a set as well as topologically by*

$$\mathfrak{S} = \left\{\varphi : \mathbb{R}^n \to \mathbb{C} : \varphi \text{ is } C^\infty \text{ and } \|x^\alpha \varphi\|_{L^\infty} < \infty, \left\|\partial^\beta \varphi\right\|_{L^\infty} < \infty \text{ for all } \alpha, \beta\right\} \tag{9}$$

We will present two proofs of this characterization, as given in [3] and [4]. These proofs are quite different, as shown by the type of estimates obtained from each of them. We will only make minor adjustments to the original proofs, with the purpose of showing that there is a topological equality as well as a set-theoretic equality, and to establish topological equivalences by means of linear estimates, whenever possible. We start by presenting the proof of Theorem 3.1 as given in [4].

Proof. For now, let us denote $\mathfrak{A}$ the family of functions introduced in (9). We can give to $\mathfrak{A}$ a structure of Fréchet space by means of the countable family of norms

$$A = \left\{p_{k,0}(\varphi) = \sup_{|\alpha| \leq k} \|x^\alpha \varphi\|_{L^\infty}, p_{0,m}(\varphi) = \sup_{|\beta| \leq m} \left\|\partial^\beta \varphi\right\|_{L^\infty}\right\}_{k,m=0}^{\infty}.$$

In order to prove the theorem, we need to show that $\mathfrak{S} = \mathfrak{A}$ as sets, and that the identity map is an isomorphism of Fréchet spaces between $(\mathfrak{S}, S)$ and $(\mathfrak{A}, A)$.

Since $A \subseteq S$, we can immediately deduce that $\mathfrak{S} \subseteq \mathfrak{A}$ and that the identity map is continuous from $(\mathfrak{S}, S)$ into $(\mathfrak{A}, A)$. To prove the other inclusion, we fix a function

$\varphi \in \mathfrak{A}$. We will show by induction that $\left\|x^\alpha \partial^\beta \varphi\right\|_{L^\infty} < \infty$ for every α, β. This assertion is clearly true for every α and for $\beta = 0$. We assume that it is true for every α and for $|\beta| \leq m$. We now prove that it is true for every α, for $|\beta| = m+1$. We fix a multi-index β with $|\beta| = m+1$. To simplify the notation, let us assume that $\beta = (\beta_1 + 1, \beta_2, ..., \beta_n)$, with $\beta_1 + \beta_2 + ... + \beta_n = m$, $m = 0, 1, 2, ...$ Let us indicate $\beta' = (\beta_1, \beta_2, ..., \beta_n)$, so $\partial^\beta \varphi = \partial_{x_1} \partial^{\beta'} \varphi$. We write $f_{x'}(x_1) = \partial^{\beta'} \varphi(x_1, x')$, with $x' = (x_2, ..., x_n)$ fixed. So, $\partial^\beta \varphi = f'_{x'}(x_1)$.

As in the proof of Lemma 2.9 we can write

$$f_{x'}(x_1 + h) = f_{x'}(x_1) + f'_{x'}(x_1) h + f''_{x'}(y_1) \frac{h^2}{2},$$

where $h \neq 0$, and y_1 is a number between x_1 and $x_1 + h$. We have

$$|f'_{x'}(x_1)| \leq \frac{|f_{x'}(x_1 + h)| + |f_{x'}(x_1)|}{|h|} + \frac{|h|}{2} |f''_{x'}(y_1)| .$$

Now given $N = 0, 1, 2, ...$, there exists $C = C_N > 0$ such that

$$\begin{aligned}\left(1 + |(x_1 + h, x')|^2\right)^N |f_{x'}(x_1 + h)| &= \left(1 + |(x_1 + h, x')|^2\right)^N \left|\partial^{\beta'} \varphi(x_1 + h, x')\right| \\ &\leq C_N \; p_{2N,m}(\varphi),\end{aligned}$$

and

$$\left(1 + |x|^2\right)^N |f_{x'}(x_1)| = \left(1 + |x|^2\right)^N \left|\partial^{\beta'} \varphi(x)\right| \leq C_N \; p_{2N,m}(\varphi) .$$

Moreover,

$$|f''_{x'}(y_1)| \leq p_{0,m+2}(\varphi) .$$

Selecting h with the same sign as x_1, we can write

$$|(x_1 + h, x')|^2 = |x|^2 + 2|x_1||h| + h^2 \geq |x|^2 .$$

Thus, we have

$$\left|\partial^\beta \varphi(x)\right| \leq \frac{1}{|h|} C_N \; p_{2N,m}(\varphi) \left(1 + |x|^2\right)^{-N} + \frac{|h|}{2} p_{0,m+2}(\varphi) . \tag{10}$$

Minimizing the right side of (10) as a function of $|h|$ we have

$$\left|\partial^\beta \varphi(x)\right| \leq C_N \sqrt{p_{2N,m}(\varphi) \, p_{0,m+2}(\varphi)} \left(1 + |x|^2\right)^{-N/2} .$$

For $|\alpha| \leq k$, $k = 1, 2, ...$we obtain

$$\left|x^\alpha \partial^\beta \varphi(x)\right| \leq C_{N,k} \sqrt{p_{2N,m}(\varphi) \, p_{0,m+2}(\varphi)} \left(1 + |x|^2\right)^{\frac{-N+k}{2}} .$$

If we choose $N = k$, we finally can write

$$p_{k,m+1}(\varphi) \leq C_k \sqrt{p_{2k,m}(\varphi) \, p_{0,m+2}(\varphi)}, \tag{11}$$

for $k, m = 1, 2, ...$ This estimate suffices to conclude by induction that $\mathfrak{A} \subseteq \mathfrak{S}$ as sets. To show that these two spaces are topologically equal, we observe that the recursive inequality (11) yields the following closed form.

$$p_{k,m+1}(\varphi) \leq C_k \left[p_{2^{m+1}k,0}(\varphi)\right]^{2^{-m-1}} \prod_{j=0}^{m} \left[p_{0,m+2-j}(\varphi)\right]^{2^{-j-1}},$$

which we can simplify if we observe that the norms $p_{0,m+2-j}$ are decreasing functions of j. So, we can write

$$p_{k,m+1}(\varphi) \leq C_k \left[p_{2^{m+1}k,0}(\varphi)\right]^{2^{-m-1}} \left[p_{0,m+2}(\varphi)\right]^{1-2^{-m-1}}. \tag{12}$$

This estimate is enough to conclude that the identity is continuous from $(\mathfrak{A}, A)$ onto $(\mathfrak{S}, S)$, thus completing the proof of Theorem 3.1. ∎

Remark 3.2 *We observe that the right side of (12) is a non-linear function of the norms in* $\mathfrak{A}$. *A quick way to get around the non-linearity of (12) is to use the Open Mapping Theorem. In fact, we know now that* $(\mathfrak{S}, S)$ *and* $(\mathfrak{S}, A)$ *are both Fréchet spaces for which the identity map from* $(\mathfrak{S}, S)$ *onto* $(\mathfrak{S}, A)$ *is continuous. Thus, the Open Mapping Theorem for Fréchet spaces (see for instance [15] p. 48), implies that the identity is an open map as well. The drawback of this approach is that it does not have an explicit quantitative formulation. Another way to obtain an explicit linear estimate is to proceed as in Remark 2.10. From (11), we can write*

$$p_{k,m+1}(\varphi) \leq C_k \left(p_{2k,m}(\varphi) + p_{0,m+2}(\varphi)\right).$$

Solving this linear recursive inequality we have

$$p_{k,m+1}(\varphi) \leq C_k \left(p_{2^{m+1}k,0}(\varphi) + \sum_{j=0}^{m} p_{0,m+2-j}(\varphi)\right).$$

If we observe that the norms $p_{0,m}$ *are non-decreasing functions of the parameter* m, *we finally have the linear estimate*

$$p_{k,m+1}(\varphi) \leq C_k \left(p_{2^{m+1}k,0}(\varphi) + (m+1)\, p_{0,m+2}(\varphi)\right).$$

We now present the second proof of Theorem 3.1 as given in [3]. It is also a very interesting proof, although it is a bit longer and less straightforward. A minor modification will allow us to obtain again a non-linear recursive inequality between appropriate norms.

Proof. We will keep the same notation used in the first proof. It is enough to show that $\mathfrak{A} \subseteq \mathfrak{S}$ and that the identity map from $(\mathfrak{A}, A)$ into $(\mathfrak{S}, S)$ is continuous. The proof will be accomplished in three steps, using induction. We fix a function $\varphi \in \mathfrak{A}$ and $m = 1, 2, ...$ The assertion $\left\|x^\alpha \partial^\beta \varphi\right\|_{L^\infty} < \infty$ is clearly true, regardless of the dimension n of $\mathbb{R}^n$, for every α, and for $\beta = 0$. We assume that it is true, regardless of the dimension n of $\mathbb{R}^n$, for every α and for $0 \le |\beta| \le m$. We now prove that it is true, regardless of the dimension n of $\mathbb{R}^n$, for every α and for $0 < |\beta| \le m+1$. Given $k = 1, 2, ...$ we fix multi-indexes α and β with $0 < |\alpha| \le k$, $0 < |\beta| \le m+1$. The first step in the proof is to obtain an L^2-estimate for $x^\alpha \partial^\beta \varphi$.

Since $|\beta| > 0$, we can write $\partial^\beta \varphi(x) = \partial_j \partial^{\beta'} \varphi(x)$, with $|\beta'| \le m$ and $x' = (x_1, ...x_{j-1}, x_{j+1}, ..., x_n)$ for some $1 \le j \le n$. So,

$$\begin{aligned}\left\|x^\alpha \partial^\beta \varphi\right\|_{L^2}^2 &= \int_{\mathbb{R}^n} x^{2\alpha} \partial^\beta \varphi(x) \overline{\partial^\beta \varphi(x)} dx \\ &= \int_{\mathbb{R}^{n-1}} \lim_{M\to\infty} \int_{-M}^{M} x^{2\alpha} \partial^\beta \varphi(x) \overline{\partial^\beta \varphi(x)} dx_j dx' \\ &= \int_{\mathbb{R}^{n-1}} \lim_{M\to\infty} [x^{2\alpha} \partial^\beta \varphi(x) \overline{\partial^{\beta'} \varphi(x)}|_{-M}^{M} - \\ &\int_{-M}^{M} \partial_j \left(x^{2\alpha} \partial^\beta \varphi(x)\right) \overline{\partial^{\beta'} \varphi(x)} dx_j] dx'.\end{aligned}$$

By the inductive hypothesis, the function $x^{2\alpha} \overline{\partial^{\beta'} \varphi(x)}$ goes to zero as $|x_j| \to \infty$, uniformly on x'. Moreover, since $\varphi \in \mathfrak{A}$, $\partial^\beta \varphi$ is bounded in $\mathbb{R}^n$. Thus,

$$\lim_{M\to\infty} x^{2\alpha} \partial^\beta \varphi(x) \overline{\partial^{\beta'} \varphi(x)}|_{-M}^{M} = 0$$

uniformly on x'. So,

$$\left\|x^\alpha \partial^\beta \varphi\right\|_{L^2}^2 = -\int_{\mathbb{R}^n} \partial_j \left(x^{2\alpha} \partial^\beta \varphi(x)\right) \overline{\partial^{\beta'} \varphi(x)} dx.$$

We use repeated integration by parts to obtain

$$\left\|x^\alpha \partial^\beta \varphi\right\|_{L^2}^2 = (-1)^{|\beta|} \int_{\mathbb{R}^n} \partial^\beta \left(x^{2\alpha} \partial^\beta \varphi(x)\right) \overline{\varphi(x)} dx. \tag{13}$$

Using Leibniz's rule, we can write (13) as

$$\begin{aligned}&= \sum_{0 \le \gamma \le \beta} C_{\alpha,\beta,\gamma} \int_{\mathbb{R}^n} x^{2\alpha-\gamma} \partial^{2\beta-\gamma} \varphi(x) \overline{\varphi(x)} dx \\ &\le C_{k,m} p_{0,2|\beta|}(\varphi) p_{2k+n+1,0}(\varphi) \int_{\mathbb{R}^n} \left(1 + |x|^2\right)^{-\frac{n+1}{2}} dx.\end{aligned}$$

Finally, we obtain

$$\left\|x^\alpha \partial^\beta \varphi\right\|_{L^2} \le C_{k,\beta} \left[p_{0,2|\beta|}(\varphi)\right]^{\frac{1}{2}} \left[p_{2k+n+1,0}(\varphi)\right]^{\frac{1}{2}}. \tag{14}$$

This completes the first step in the proof. The second step consists of proving an L^1-estimate for the same function $x^\alpha \partial^\beta \varphi$. For $N \in \mathbb{N}$ to be chosen later, we write

$$\left\|x^\alpha \partial^\beta \varphi\right\|_{L^1} = \int_{\mathbb{R}^n} \left(1 + |x|^2\right)^{\frac{N}{2}} \left|x^\alpha \partial^\beta \varphi(x)\right| \left(1 + |x|^2\right)^{-\frac{N}{2}} dx.$$

Using Cauchy-Schwartz's inequality we can estimate this integral as

$$\leq \left(\int_{\mathbb{R}^n} \left(1+|x|^2\right)^N \left|x^\alpha \partial^\beta \varphi(x)\right|^2 dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} \left(1+|x|^2\right)^{-N} dx \right)^{\frac{1}{2}}.$$

For $N = \left[\frac{n}{2}\right] + 1$ we write

$$\left(\int_{\mathbb{R}^n} \left(1+|x|^2\right)^N \left|x^\alpha \partial^\beta \varphi(x)\right|^2 dx \right)^{\frac{1}{2}} \leq C \sum_{0 \leq |\gamma| \leq N} \left\| x^{\alpha+\gamma} \partial^\beta \varphi \right\|_{L^2}.$$

Thus, according to (14) we have

$$\left\| x^\alpha \partial^\beta \varphi \right\|_{L^1} \leq C_{k,\beta} \left[p_{0,2|\beta|}(\varphi) \right]^{\frac{1}{2}} \left[p_{2k+2n+3,0}(\varphi) \right]^{\frac{1}{2}}, \tag{15}$$

which completes the second step. The third and last step is to estimate $\left\| x^\alpha \partial^\beta \varphi \right\|_{L^\infty}$. Since $|\alpha| > 0$, the multi-index α must have at least one component that is positive. We fix one such component, say α_i, for some $1 \leq i \leq n$. Then we write

$$\left(x^\alpha \partial^\beta \varphi(x)\right)^2 = \int_0^{x_i} \partial_i \left[y^{2\alpha} \left(\partial^\beta \varphi(y)\right)^2 \right] dy_i,$$

where $y = (x_1, ...x_{i-1}, y_i, x_{i+1}, ..., x_n)$. So,

$$\begin{aligned} \left| \left(x^\alpha \partial^\beta \varphi(x)\right)^2 \right| &= \left| \int_0^{x_i} \partial_i \left[y^{2\alpha} \left(\partial^\beta \varphi(y)\right)^2 \right] dy_i \right| \\ &\leq \int_{-\infty}^{\infty} \left| \partial_i \left[y^{2\alpha} \left(\partial^\beta \varphi(y)\right)^2 \right] \right| dy_i \\ &\leq 2\alpha_i \int_{-\infty}^{\infty} \left| y^{2\alpha - e^i} \left(\partial^\beta \varphi(y)\right)^2 \right| dy_i \\ &+ 2 \int_{-\infty}^{\infty} \left| y^{2\alpha} \left(\partial^\beta \varphi(y)\right) \left(\partial^{\beta+e^i} \varphi(y)\right) \right| dy_i, \end{aligned} \tag{16}$$

where $e^i = \left(0, ..., 0, \underset{i}{1}, 0, ..., 0\right)$. If we indicate $x' = (x_1, ..., x_{i-1}, x_{i+1}, ..., x_n)$, $\alpha' = (\alpha_1, ..., \alpha_{i-1}, \alpha_{i+1}, ..., \alpha_n)$ and $\beta' = (\beta_1, ..., \beta_{i-1}, \beta_{i+1}, ..., \beta_n)$, we can estimate (16) as

$$\begin{aligned} &\leq 2\alpha_i (x')^{2\alpha'} \left\| \partial_i^{\beta_i} \left(\partial^{\beta'} \varphi\right) \right\|_{L^\infty(\mathbb{R})} \left\| y_i^{2\alpha_i - 1} \partial_i^{\beta_i} \left(\partial^{\beta'} \varphi\right) \right\|_{L^1(\mathbb{R})} \\ &+ 2 (x')^{2\alpha'} \left\| \partial_i^{\beta_i+1} \left(\partial^{\beta'} \varphi\right) \right\|_{L^\infty(\mathbb{R})} \left\| y_i^{2\alpha_i} \partial_i^{\beta_i} \left(\partial^{\beta'} \varphi\right) \right\|_{L^1(\mathbb{R})}. \end{aligned}$$

If $0 < \beta_i \leq m+1$, using (15) in the ith variable, we obtain

$$\left|\left(x^{\alpha}\partial^{\beta}\varphi\left(x\right)\right)^{2}\right| \leq C_{\alpha_i,\beta_i} p_{0,m+2}\left(\varphi\right)\left\{\sup_{\substack{0\leq v\leq 2\beta_i \\ y_i\in\mathbb{R}}}\left|\partial_i^{v}\left(\partial^{\beta'}\varphi\right)\left(y_i\right)\right|\right\}^{\frac{1}{2}}$$
$$\times\left\{\left(x'\right)^{4\alpha'}\sup_{\substack{0\leq w\leq 4\alpha_i+5 \\ y_i\in\mathbb{R}}}\left|y_i^{w}\left(\partial^{\beta'}\varphi\right)\left(y_i\right)\right|\right\}^{\frac{1}{2}}.$$

So, when $\beta_i > 0$ we have

$$\left|\left(x^{\alpha}\partial^{\beta}\varphi\left(x\right)\right)\right| \leq C_{k,m}\left[p_{0,m+2}\left(\varphi\right)\right]^{\frac{1}{2}}\left[p_{0,2(m+1)}\left(\varphi\right)p_{4k+5,m}\left(\varphi\right)\right]^{\frac{1}{4}}. \quad (17)$$

If $\beta_i = 0$, then we can write

$$\begin{aligned}\left|\left(x^{\alpha}\partial^{\beta}\varphi\left(x\right)\right)^{2}\right| &\leq 2\alpha_i\left(x'\right)^{2\alpha'}\left\|\partial^{\beta'}\varphi\right\|_{L^{\infty}(\mathbb{R})}\left\|y_i^{2\alpha_i-1}\partial^{\beta'}\varphi\right\|_{L^{1}(\mathbb{R})} \\ &\quad + 2\left(x'\right)^{2\alpha'}\left\|\partial_i\left(\partial^{\beta'}\varphi\right)\right\|_{L^{\infty}(\mathbb{R})}\left\|y_i^{2\alpha_i}\partial^{\beta'}\varphi\right\|_{L^{1}(\mathbb{R})} \\ &\leq C_{\alpha_i}p_{0,m+2}\left(\varphi\right)\left\{\left(x'\right)^{2\alpha'}\sup_{\substack{0\leq w\leq 2\alpha_i+1 \\ y_i\in\mathbb{R}}}\left|y_i^{w}\left(\partial^{\beta'}\varphi\right)\left(y_i\right)\right|\right\}.\end{aligned}$$

So, when $\beta_i = 0$ we have

$$\left|\left(x^{\alpha}\partial^{\beta}\varphi\left(x\right)\right)\right| \leq C_{k}\left[p_{0,m+2}\left(\varphi\right)\right]^{\frac{1}{2}}\left[p_{2k+1,m}\left(\varphi\right)\right]^{\frac{1}{2}}. \quad (18)$$

As a consequence we can write

$$\begin{aligned}p_{k,m+1}\left(\varphi\right) &\leq C_{k,m}\left[p_{0,m+2}\left(\varphi\right)\right]^{\frac{1}{2}}\left\{\left[p_{2k+1,m}\left(\varphi\right)\right]^{\frac{1}{2}}\right. \\ &\quad \left.+\left[p_{0,2(m+1)}\left(\varphi\right)p_{4k+5,m}\left(\varphi\right)\right]^{\frac{1}{4}}\right\}. \quad (19)\end{aligned}$$

According to the inductive hypothesis, this estimate implies that $\left|\left(x^{\alpha}\partial^{\beta}\varphi\left(x\right)\right)\right| < \infty$, thus showing that $\mathfrak{A} \subseteq \mathfrak{S}$ as sets. Since the other inclusion is obvious, we can invoke again the Open Mapping Theorem to conclude that the identity map from $(\mathfrak{S}, A)$ onto $(\mathfrak{S}, S)$ is continuous. In principle we could try to write (19) in a closed form, providing an explicit estimate that shows that the identity map from $(\mathfrak{A}, A)$ into $(\mathfrak{S}, S)$ is continuous. But it would be a fairly complicated formula, so we will not do it.

This completes the second proof of Theorem 3.1. ∎

Remark 3.3 *We do not know if there is a modification of this proof that will give explicit linear estimates. We see that the first proof we presented is shorter and does provide explicit linear estimates. In this sense we would say that it is a better proof.*

We now present the second characterization of the space $\mathfrak{S}$.

Theorem 3.4 ([3]) *The space $\mathfrak{S}$ defined by (1) can be described as a set as well as topologically by*

$$\mathfrak{S} = \left\{ \begin{array}{c} \varphi : \mathbb{R}^n \to \mathbb{C} : \varphi \text{ is continuous and for} \\ \text{all } k = 0, 1, 2, ..., \; p_{k,0}(\varphi) < \infty, p_{k,0} \circ \mathcal{F}(\varphi) < \infty \end{array} \right\} \tag{20}$$

Proof. For now, we denote $\mathfrak{B}$ the space defined in (20). We observe that the condition $p_{k,0}(\varphi) < \infty$ for all $k = 0, 1, 2, ...$ implies that $\varphi \in L^1$, so the formulation of the condition $p_{k,0} \circ \mathcal{F}(\varphi) < \infty$ makes sense for all $k = 0, 1, 2, ...$ and implies that $\widehat{\varphi} \in L^1$ also. Furthermore, φ and $\widehat{\varphi}$ are C^∞ functions. We can give to $\mathfrak{B}$ a structure of Fréchet space by means of the countable family of norms

$$B = \left\{ \, p_{k,0}(\varphi) < \infty, p_{k,0} \circ \mathcal{F}(\varphi) \right\}_{k=0}^{\infty}.$$

Using Lemma 2.3, Corollary 2.5, and Theorem 3.1, we will prove that $\mathfrak{S} = \mathfrak{B}$ and that the identity map from $\left(\mathfrak{S}, \overline{\overline{S}}\right)$ to $(\mathfrak{B}, B)$ is an isomorphism of Fréchet spaces, with $\overline{\overline{S}}$ defined by (4).

Given $\varphi \in \mathfrak{S}$, if we fix $k = 0, 1, 2, ...$ and a multi-index α with $|\alpha| \leq k$, we can write

$$\begin{aligned} |x^\alpha \varphi(x)| &\leq C_\alpha \left(1 + |x|^2\right)^{-n} \left(1 + |x|^2\right)^{\frac{k}{2}+n} |\varphi(x)| \leq C_k \overline{\overline{p}}_{k+n,0}(\varphi) \left(1 + |x|^2\right)^{-n} \\ &\leq C_k \overline{\overline{p}}_{k+n,0}(\varphi). \end{aligned}$$

In particular, these inequalities show that the function φ is indeed integrable. Moreover,

$$\begin{aligned} |\xi^\alpha \widehat{\varphi}(\xi)| &= \left| (-2\pi i)^{|\alpha|} (\partial^\alpha \varphi)^\wedge(\xi) \right| \\ &\leq C_\alpha \left\| \left(1 + |x|^2\right)^{-n} \right\|_{L^1} \left\| \left(1 + |x|^2\right)^n \partial^\alpha \varphi \right\|_{L^\infty} \\ &\leq C_k \overline{\overline{p}}_{n,k}(\varphi). \end{aligned}$$

Conversely, if we fix $\varphi \in \mathfrak{B}$, we observe that $\xi^\beta \widehat{\varphi}$ is integrable for every multi-index β. Then, we can write

$$\partial^\beta \varphi(x) = \int_{\mathbb{R}^n} e^{2\pi i x\xi} (2\pi i \xi)^\beta \, \widehat{\varphi}(\xi) \, d\xi.$$

Thus, the function $\partial^\beta \varphi(x)$ is continuous and bounded. Moreover,

$$\left| \partial^\beta \varphi(x) \right| \leq C_m p_{m+n+1,0} \circ \mathcal{F}(\varphi).$$

for every multi-index β with $|\beta| \leq m$. So, according to Lemma 2.3 and Theorem 3.1, $\varphi \in \mathfrak{S}$ and the identity map from $(\mathfrak{B}, B)$ to $\left(\mathfrak{S}, \overline{\overline{S}}\right)$ is continuous.

This completes the proof of Theorem 3.4. ∎

Remark 3.5 *This short proof of Theorem 3.4 relies on Theorem 3.1. It is worth noticing that in the first proof of Theorem 3.1 we could obtain explicit linear estimates. Moreover, we are actually using in* $\mathfrak{S}$ *two equivalent sets of norms,* $\overline{\overline{S}}$ *and* A*, for which the equivalence can be formulated in terms of explicit linear estimates.*

Remark 3.6 *The characterization of the space* $\mathfrak{S}$ *given by Theorem 3.4 impose separate conditions on the function and on its Fourier transform. This characterization could be interpreted as treating time and frequency independently. Karlheinz Gröchenig and Georg Zimmermann have obtained in [7] and [8] joint time-frequency characterizations of* $\mathfrak{S}$ *and* $\mathfrak{S}_w$ *using the short-time Fourier transform.*

4 A characterization of the space $\mathfrak{S}_w$

We consider in this section the spaces $\mathcal{M}_c$ and $\mathfrak{S}_w$, as defined in Section 2, and we present the following characterization of $\mathfrak{S}_w$.

Theorem 4.1 ([4]) *Given* $w \in \mathcal{M}_c$*, the space* $\mathfrak{S}_w$ *can be described as a set as well as topologically by*

$$\mathfrak{S}_w = \left\{ \begin{array}{c} \varphi : \mathbb{R}^n \to \mathbb{C} : \varphi \text{ is continuous and for all} \\ k = 0, 1, 2, ..., \; q_{k,0}(\varphi) < \infty, q_{k,0} \circ \mathcal{F}(\varphi) < \infty \end{array} \right\} \tag{21}$$

Proof. Let us indicate $\mathfrak{B}_w$ the space defined in (21). The condition $q_{k,0}(\varphi) < \infty$ for each $k = 0, 1, 2, ...$ implies that $\varphi \in L^1$, so the formulation of the second condition $q_{k,0} \circ \mathcal{F}(\varphi) < \infty$ makes sense for all $k = 0, 1, 2, ...$ Moreover, (21) implies that φ and $\widehat{\varphi}$ are C^∞ functions. The space $\mathfrak{B}_w$ becomes a Fréchet space with respect to the family of norms

$$B_w = \{q_{k,0}, q_{k,0} \circ \mathcal{F}\}_{k=0}^{\infty}.$$

From the definitions, it is clear that $\mathfrak{S}_w \subseteq \mathfrak{B}_w$ and that the inclusion is continuous. We will prove the converse by using induction on m and the general idea of Landau's inequality as given in Lemma 2.9.

We fix $\varphi \in \mathfrak{B}_w$ not identically zero. We want to show that $\left\|e^{kw(x)}\partial^\beta \varphi\right\|_{L^\infty}$ and $\left\|e^{kw(\xi)}\partial^\beta \widehat{\varphi}\right\|_{L^\infty}$ are finite, for every $k = 0, 1, 2, ...$ and every multi-index β. It is true for all k, when $\beta = 0$. We assume that it is true for all k, when $|\beta| \leq m$, and we want to prove it for all k and for $|\beta| = m + 1$.

We start with $\left\|e^{kw(x)}\partial^\beta \varphi\right\|_{L^\infty}$. As in the first proof of Theorem 3.1, we assume for simplicity that $\beta = (\beta_1 + 1, \beta_2, ..., \beta_n)$ with $\beta_1 + \beta_2 + ... + \beta_n = m$, $m = 0, 1, 2, ...$We also indicate $\beta' = (\beta_1, \beta_2, ..., \beta_n)$, $\partial^\beta \varphi = \partial_{x_1} \partial^{\beta'} \varphi$, $f_{x'}(x_1) = \partial^{\beta'} \varphi(x_1, x')$ for $x' = (x_2, ..., x_n)$ fixed, $\partial^\beta \varphi(x) = f'_{x'}(x_1)$. Moreover, if $h \neq 0$, we have

$$f_{x'}(x_1 + h) = f_{x'}(x_1) + f'_{x'}(x_1)h + \frac{1}{2} f''_{x'}(y)h^2,$$

where y is some number between x_1 and $x_1 + h$. Thus,

$$f'_{x'}(x_1)h = f_{x'}(x_1 + h) - f_{x'}(x_1) - \frac{1}{2} f''_{x'}(y)h^2,$$

or,

$$\left|f'_{x'}(x_1)\right| \leq \frac{\left|f_{x'}(x_1+h)\right| + \left|f_{x'}(x_1)\right|}{|h|} + \frac{|h|}{2}\left|f''_{x'}(y)\right|.$$

We can write

$$\left|e^{kw\left(x_1+h,x'\right)} f_{x'}(x_1+h)\right| \leq q_{k,m}(\varphi),$$
$$\left|e^{kw(x)} f_{x'}(x_1)\right| \leq q_{k,m}(\varphi).$$

If we pick h with the same sign as x_1, we have

$$\left|(x_1+h, x')\right|^2 = |x|^2 + 2|x_1||h| + h^2 \geq |x|^2.$$

Moreover, the assumptions on w imply that $w(x) \leq w(x_1+h, x')$. So,

$$\left|f_{x'}(x_1+h)\right| + \left|f_{x'}(x_1)\right| \leq C_m q_{k,m}(\varphi) e^{-kw(x)}.$$

We need to estimate $f''_{x'}(y) = \partial_{x_1}\partial^{\beta}\varphi(y)$. For $r = 0, 1, 2, \ldots$ to be fixed later, we have

$$\int_{\mathbb{R}^n} \left|2\pi i\xi_1 (2\pi i\xi)^{\beta} \widehat{\varphi}(\xi)\right| d\xi \leq C_{\beta,m} \int_{\mathbb{R}^n} (1+|\xi|)^{m+2} e^{-rw(\xi)} e^{rw(\xi)} \left|\widehat{\varphi}(\xi)\right| d\xi. \qquad (22)$$

Using Definition 2.6, we can write

$$e^{-rw(\xi)} \leq e^{-r(a+b\log(1+|\xi|))} = e^{-ar}(1+|\xi|)^{-br}.$$

Thus, we can estimate the integral on the right side of (22) as

$$\leq C_{\beta,m} q_{r,0} \circ \mathcal{F}(\varphi) \int_{\mathbb{R}^n} (1+|\xi|)^{m+2-br} d\xi,$$

which is finite when $m+2-br < -n$. Thus, we have

$$\left|\partial_{x_1}\partial^{\beta}\varphi(y)\right| \leq C_m q_{\left[\frac{m+n+2}{b}\right]+1,0} \circ \mathcal{F}(\varphi).$$

So

$$\left|\partial^{\beta}\varphi(x)\right| \leq C_m \left[\frac{1}{t} q_{k,m}(\varphi) e^{-kw(x)} + t q_{\left[\frac{m+n+2}{b}\right]+1,0} \circ \mathcal{F}(\varphi)\right] \qquad (23)$$

for every $t > 0$. As a function of t, the right side of (23) has a global minimum at

$$t = \left(q_{k,m}(\varphi) e^{-kw(x)}\right)^{\frac{1}{2}} \left(q_{\left[\frac{m+n+2}{b}\right]+1,0} \circ \mathcal{F}(\varphi)\right)^{-\frac{1}{2}}.$$

Thus, we obtain the inequality

$$\left|\partial^{\beta}\varphi(x)\right| \leq C_m \left(q_{k,m}(\varphi)\right)^{\frac{1}{2}} \left(q_{\left[\frac{m+n+2}{b}\right]+1,0} \circ \mathcal{F}(\varphi)\right)^{\frac{1}{2}} e^{-\frac{k}{2}w(x)},$$

or

$$\left|e^{kw(x)}\partial^{\beta}\varphi\left(x\right)\right|\leq C_{m}\left(q_{2k,m}\left(\varphi\right)\right)^{\frac{1}{2}}\left(q_{\left[\frac{m+n+2}{b}\right]+1,0}\circ\mathcal{F}\left(\varphi\right)\right)^{\frac{1}{2}}. \tag{24}$$

The way to estimate $e^{kw(\xi)}\partial^{\beta}\widehat{\varphi}\left(\xi\right)$ is similar to what we have just done, so we write the final estimate without going into details.

$$\left|e^{kw(\xi)}\partial^{\beta}\widehat{\varphi}\left(\xi\right)\right|\leq C_{m}\left(q_{2k,m}\circ\mathcal{F}\left(\varphi\right)\right)^{\frac{1}{2}}\left(q_{\left[\frac{m+n+2}{b}\right]+1,0}\left(\varphi\right)\right)^{\frac{1}{2}}. \tag{25}$$

Using (24) and (25), the inductive hypothesis implies that $\varphi\in\mathfrak{S}_{w}$. The Open Mapping Theorem can provide once again the continuity of the inclusion. However, solving the recursive inequalities (24) and (25), we can obtain an explicit proof, although non-linear in nature. Indeed, reiterating the two recursive inequalities m times and recalling that the norms $q_{k,m}$ and $q_{k,m}\circ\mathcal{F}$ are increasing functions of k and m separately, we finally have

$$\left|e^{kw(x)}\partial^{\beta}\varphi\left(x\right)\right|\leq C_{m}\left(q_{2^{m+1}k,0}\left(\varphi\right)\right)^{2^{-m-1}}\left(q_{\left[\frac{m+n+2}{b}\right]+1,0}\circ\mathcal{F}\left(\varphi\right)\right)^{1-2^{-m-1}},$$
$$\left|e^{kw(\xi)}\partial^{\beta}\widehat{\varphi}\left(\xi\right)\right|\leq C_{m}\left(q_{2^{m+1}k,0}\circ\mathcal{F}\left(\varphi\right)\right)^{2^{-m-1}}\left(q_{\left[\frac{m+n+2}{b}\right]+1,0}\left(\varphi\right)\right)^{1-2^{-m-1}}.$$

This completes the proof of Theorem 4.1. ∎

Remark 4.2 *Estimating in (24) and (25) the geometric mean with the arithmetic mean, we obtain the linear recursive inequalities*

$$q_{k,m+1}\left(\varphi\right)\leq C_{m}\left(q_{2k,m}\left(\varphi\right)+q_{\left[\frac{m+n+2}{b}\right]+1,0}\circ\mathcal{F}\left(\varphi\right)\right),$$
$$q_{k,m+1}\circ\mathcal{F}\left(\varphi\right)\leq C_{m}\left(q_{2k,m}\circ\mathcal{F}\left(\varphi\right)+q_{\left[\frac{m+n+2}{b}\right]+1,0}\left(\varphi\right)\right).$$

If we write these recursive inequalities in closed form we have the linear estimates

$$q_{k,m+1}\left(\varphi\right)\leq C_{m}\left(q_{2^{m+1}k,0}\left(\varphi\right)+\sum_{j=0}^{m}q_{\left[\frac{m+1-j+n+1}{b}\right]+1,0}\circ\mathcal{F}\left(\varphi\right)\right),$$
$$q_{k,m+1}\circ\mathcal{F}\left(\varphi\right)\leq C_{m}\left(q_{2^{m+1}k,0}\circ\mathcal{F}\left(\varphi\right)+\sum_{j=0}^{m}q_{\left[\frac{m+1-j+n+1}{b}\right]+1,0}\left(\varphi\right)\right),$$

or,

$$q_{k,m+1}\left(\varphi\right)\leq C_{m}\left(q_{2^{m+1}k,0}\left(\varphi\right)+\left(m+1\right)q_{\left[\frac{m+n+2}{b}\right]+1,0}\circ\mathcal{F}\left(\varphi\right)\right),$$
$$q_{k,m+1}\circ\mathcal{F}\left(\varphi\right)\leq C_{m}\left(q_{2^{m+1}k,0}\circ\mathcal{F}\left(\varphi\right)+\left(m+1\right)q_{\left[\frac{m+n+2}{b}\right]+1,0}\left(\varphi\right)\right).$$

When $w\left(x\right)=\log\left(1+|x|\right)$, the characterization of $\mathfrak{S}_{w}$ given by Theorem 4.1 reduces to the second characterization of $\mathfrak{S}$ given by Theorem 3.4. So it is obvious *that* $\mathfrak{S}_w$ *becomes* $\mathfrak{S}$ *for this choice of* w*. For a proof of this fact using Definitions 2.1 and 2.7, see [2] p. 375.*

If we try to adapt the second proof of Theorem 3.1 to the space $\mathfrak{S}_w$*, it becomes clear very quickly that we need to impose too strong smoothness and size conditions on the weight function* w*.*

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