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Turbulent mixing of stratified flows

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ABSTRACT

A mathematical model for turbulent mixing of stratified, sheared flows is developed and explored. Two applications are emphasized, one to the dynamics of the ocean well–mixed layer, and another to the analysis of the stability of equilibrium profiles away from boundaries.

1 Introduction

The ocean is a highly inhomogeneous medium, characterized by spatial and temporal contrasts in temperature and salinity, as well as in chemical composition and in the distribution of biological agents. The inhomogeneity of the ocean, however, is not static, but follows from a dynamical equilibrium, in which contrasts are permanently being created and attenuated. Local processes that generate contrasts include evaporation and rain, freezing and melting of sea–ice, river inflows, and volcanic activity. Attenuation is mainly due to mixing processes, such as turbulent diffusion, breaking waves, and stirring by surface winds, ocean currents, and planetary tides interacting

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with the ocean's bottom and lateral morphology. The dynamical equilibrium emerging from the balance of these processes is a determining factor to the Earth's climate: slight changes in the properties of the upper layers of the ocean, in particular, can lead to significant variations of its ice–coverage, of local patterns of convection and rain and, ultimately, to dramatic changes in the global patterns of surface temperature, humidity and prevailing winds.

Yet the quantification, and even the identification of some of the critical processes involved in this dynamical balance, remain to a large degree incomplete. The mixing side of the balance is particularly elusive, due to its vast distribution over whole basins, to the difficulties inherent to its observation and measurement, to its highly anisotropic nature, and to the incompletely understood physics of its underlying processes, such as turbulent diffusion, shear and convective instabilities and wave overturning. State-of-the-art computational ocean circulation models typically parameterize these processes, introducing empirical closures designed to fit as well as possible the [sparse] available experimental and observational data. Such an approach, driven by necessity, may yield large errors in the prediction of climatic changes, since the adjustment of parameters to match features of today's climate may fail to capture those of tomorrow's.

A mathematical criterion for the stability of a sheared, stratified flow, based on the Richardson number, was developed in [10, 5]. Physical experiments and dimensional analysis were used in [2, 12] to develop a closure for entrainment and mixing of ambient fluid into a plume of buoyant fluid. This closure was improved in [3] to better represent entrainment into oceanic dense overflows. In [9] and more recently [1], a closure for mixing was proposed based on turbulent diffusion, the approach that we explore in this article. Examples of observations, physical experiments, theory and parameterizations in general circulation models, of mixing of stratified flows, can be found in [7, 11, 4, 6, 8], and references therein.

A first, striking feature of oceanic mixing, is its highly anisotropic character. A tracer deposited in the water will diffuse much more rapidly along *isopycnals* –surfaces of constant density, typically close to horizontal– than across them. This is due, to a large extent, to the relative freedom that fluid parcels experiment to move along isopycnals, as contrasted to the relatively high rigidity that a stratified flow in a rotating environment opposes to vertical, *diapycnal* motion. This leads to the formation of horizontal eddies covering a wide range of scales, which work as very effective enhancers of mixing. Another way of understanding this disparity of mixing rates is through energetics: isopycnal mixing is energetically free, while diapycnal mixing is costly, since vertical mixing involves raising and sinking respectively heavy and light parcels of fluid, thus increasing the total amount of potential energy in the system. Our focus in this work is on vertical, diapycnal mixing, which in the long term determines the properties of the water at the sea–surface, by bringing up denser waters from beneath.

Mixing of waters of different properties occurs throughout the ocean and at all scales. However, there are highly localized areas of intense mixing, which play a fundamental role in establishing the main properties of large water masses. A locus of intense stirring and mixing, with a direct influence on the surface waters, is the upper mixed layer, occupying typically the first fifty to one hundred meters below the ocean's surface. This layer is characterized by its vertically nearly uniform properties, in high contrast with the ocean's interior relatively strong stratification. The main stirring agent for this layer is the wind which, through the generation of surface and internal waves and turbulence, leads to entrainment of water from the ocean's interior into the well–mixed layer, and subsequent mixing throughout it. Despite its obvious significance for the weather and climate –this is the only part of the ocean that communicates directly with the atmosphere–, the upper well–mixed layer is not properly resolved in current general circulation models. This is due not only to its small vertical scale, which makes it difficult to resolve in the relatively coarse grids of general circulations models, but also to many lacoons in our understanding of its basic dynamics and driving physical processes.

In this paper, we describe some mathematical and physical tools useful for the description of mixing in the ocean. We focus on conceptual, idealized models that, though lacking the richness, complexity and diversity of the real ocean, may help shed light on some of its underlying processes. In particular, we explore some of the properties of a model of mixing based on *turbulent diffusion*, i.e. a diffusive process supported not in the microscopic scale of Brownian motion, but in the intermediate, imprecise scales of turbulence.

The plan of the paper is the following: after this introduction, in section 2, we describe a mathematical model of mixing basin on nonlinear, turbulent diffusion. In section 3, we apply this model to the dynamics of the well–mixed layer. In section 4, we show how the model sheds light on subtle issues arising in the shear instability of stratified flows. Finally, in 5, we make some closing remarks.

2 A mathematical model for turbulent diffusion

How do ocean waters with different properties mix? There is not a simple answer to this question: a plurality of mixing scenarios exist, each shedding a distinct light on the mixing process. When a dense mass of water is placed above a ligther mass – as when the ocean surface layers are cooled by very cold winds –, a convective instability occurs. When localized currents give rise to a marked shear, either horizontal or vertical, this shear may go unstable and shed mixing eddies. Pronounced internal waves may nonlinearly deform and break, leading to intense, localized mixing bursts. Dense overflows descending into the ocean may generate internal hydraulic jumps, with high rates of localized entrainment of lighter ambient waters.

Diverse as these scenarios are, they all share a common feature: flow instabilities or hydraulic constraints give rise to highly turbulent bursts, which rapidly homogenize the fluid properties. It is therefore attractive to treat them all under the common umbrella of turbulence driven diffusion. Models of this kind are currently used in general circulation models [9].

In this section, we describe in some detail a candidate model of mixing based on turbulent diffusion. For simplicity, we shall restrict our attention to flows which are horizontally uniform, so that the problem becomes one–dimensional. This is not to say, however, that we are modeling a vertical slab of fluid, since horizontal velocities will be allowed, though depending only on the vertical coordinate z.

Since density variations in the ocean are very small, typically ranging bellow 3%, we will adopt in our model the Boussinnesq approximation, whereby only the buoyancy effect of density variations is retained, while their effect on the fluid's inertia is neglected.

Our variables are the buoyancy $b = g\frac{\rho-\rho_0}{\rho_0}$, where ρ is the fluid's density and ρ_0 some reference value, the horizontal velocity u, and the turbulent kinetic energy per unit of mass e. We assume that both the buoyancy and the momentum are turbulently diffused, so that the equations of mass and momentum conservation are

$$b_t = (K_b b_z)_z, \tag{1}$$

$$u_t = (K_u u_z)_z, \tag{2}$$

where K_b and K_u are the turbulent buoyancy and shear diffusivities, which we model bellow.

Since we anticipate that the diffusivities K_b and K_u will depend on the local amount of turbulence in the system, characterized for instance by a typical value of the turbulent velocity field $w = \sqrt{2e}$, still another equation is needed in order to close the system. Our choice is an equation for the diffusion of the turbulent energy itself, which reads

$$e_t = (K_e e_z)_z + K_b b_z + K_u (u_z)^2. \tag{3}$$

The first term on the right-hand side represents diffusion; the other two are required so that the total energy

$$\int \left(b\,z + \frac{u^2}{2} + e \right) dz$$

is preserved by the flow. The physical interpretation of these two terms is straightforward. The first, a sink, represents the energetic cost of mixing a stratified fluid, raising heavy and bringing down light parcels of fluid. The second, a source, accounts for the energy surplus provided by mixing a shear flow; it follows from the mathematical fact that the mean of a square is always bigger than the square of the mean.

Finally, we must determine the turbulent diffusivities. One reasonable assumption is that each diffusivity must be proportional to the mean turbulent velocity. In order to simplify our approach, we will assume this simplest scenario, and set

$$K_j = l e^{\frac{1}{2}} S_j \tag{4}$$

for $j \in \{b, u, e\}$, where l is some fixed lenghtscale –the typical size of a mixing eddy– and the S_j's are dimensionless parameters accounting for the possibly different rates of mixing for the various physical quantities of the system.

Other models may be produced by treating l not as a constant, but as a function of the dependent variables. Dimensional analysis suggests the possibilities $l = \sqrt{\frac{e}{b_z}}$,

$l = \sqrt{\frac{u^2}{b_z}}$, and $l = \sqrt{\frac{e}{(u_z)^2}}$. Still another possibility is to let l evolve dynamically, following a dynamic equation modeling the cascade of turbulent energy across scales.

In this note, however, we concentrate on the simple choice that has l fixed at some externally provided value. The arbitrariness of this value should work as a reminder that treating turbulent mixing as a diffusive process is not a first–principled approach, but a convenient, often deceptively convincing closure.

3 An application to the dynamics of the well–mixed layer

One of the most relevant feature of the oceans is the existence of a turbulent and well–mixed top layer, typically occupying the top 50–100 meters. In this layer the temperature and salt content of the water are almost independent of depth. At the bottom of the layer there is a shallow region where both salinity and temperature change very rapidly, appearing at times to be nearly discontinuous.

The mixed layer plays a fundamental role in regulating our climate, since it is through it that all heat and momentum exchanges between the ocean and the atmosphere take place. Its depth is largely regulated by the atmosphere, thus generating a nontrivial feedback mechanism. We describe three ways by which this regulation may happen:

1. Storms over the ocean may stir the waters directly bellow it, adding to the total turbulent energy at the top of the mixed layer. This extra turbulence is nonlinearly diffused through the layer and enhances mixing at its bottom, thus increasing its depth.

2. The daily variations of atmospheric temperature causes a periodic warming and cooling of the ocean's surface, reflected in a periodic change of buoyancy. When the top water becomes colder than its surroundings, it also becomes heavier, an unstable situation that causes convection. This, in turn, releases potential energy that is transformed into turbulence, inducing further mixing.

3. Winds in the atmosphere transmit their momentum to the ocean. The surface then acquires a velocity distinct from its surroundings, developing a marked shear. This velocity shear may develop instabilities and mix, thus releasing kinetic energy that also becomes turbulent.

As an application of our turbulent mixing model, we show numerically how these three scenarios may generate a mixed layer. In our numerical runs, the atmospheric influence will be represented by boundary fluxes. Our numerical scheme uses finite differencing in conservation form. The conserved quantities b, u and e are represented by their averages over numerical cells, and sit at their centers, while the fluxes are computed at the interfaces between cells. These fluxes are further limited so as to proscribe negative turbulent energies, that numerical inaccuracies might otherwise produce. In all runs, the eddy mixing length l is set to $1/4$, the diffusivities S_b and S_e are set to one, as is S_u, except for the results displayed in Figure 9, where $S_u = -0.3$.

The initial vertical profile for the buoyancy is linear, representing a background stratification, and the velocity profile is initially depth independent. Finally, the turbulent energy is initialized as zero everywhere except for some small initial turbulence close to the surface, necessary in our model to start the boundary fluxes.

In the first run (see Figure 1), a boundary flux of turbulent energy represents the input of turbulence into the ocean by the storm, while the fluxes of buoyancy and momentum are set to zero. Without an initial shear or a mechanism to generate it, the momentum equation 2 is satisfied trivially, so we have run our model without it.

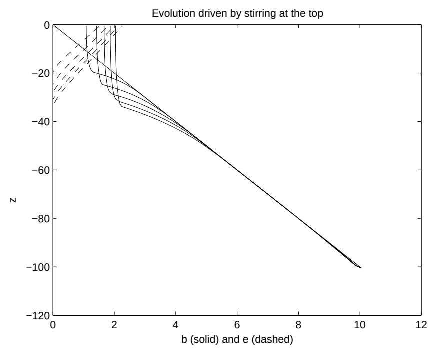

Figure 1: Formation of a well–mixed layer in a stratified fluid by a constant flux of turbulent energy from the top. The solid line represents the buoyancy profile, and the dashed line the turbulent energy. The snapshots plotted are 100 time units apart. The energy flux through the top surface is given by $K_e e_z = 0.025$.

One remarkable feature is the large gradients at the bottom of the mixed layer. Our model is diffusive, and diffusion is usually associated with attenuation of disparities, but in this case the strongly nonlinear nature of this diffusion yields the inverse phenomenon. In fact, if the storm stops, setting the energy flux at the surface to zero, the base of the well mixed layer rapidly becomes discontinuous. This counterintuitive and mathematically appealing feature is in good agreement with physical reality.

It is also worth noticing that, as time evolves, the rate of growth of the mixed layer slows down considerably. This can be understood by a simple energetic argument, where the mixed layer is taken to be completely homogeneous, with the same buoyancy throughout its depth. If $b = g'z$ is the linear background stratification of the ocean, and the mixed layer thickness is H, mass conservation implies that the buoyancy in the layer is $\bar{b} = \frac{g'H}{2}$. Then, if the mixed layer is deepened to $H + \Delta H$, the potential energy increase is:

$$\begin{aligned}\Delta P.E. &= \int_{-H}^{0}\left(\frac{g'\,(H+\Delta H)}{2}-\frac{g'(H)}{2}\right)z\,dz+\\ &+\int_{H+\Delta H}^{H}\left(\frac{g'H+\Delta H}{2}-g'z\right)z\,dz=\\ &=\frac{\Delta H H^2}{4}+O((\Delta H^2)).\end{aligned}$$

So we see that the work per unit time required for the growth of the mixed layer increases as the square of the depth, explaining the reduced velocity observed in the numerical runs.

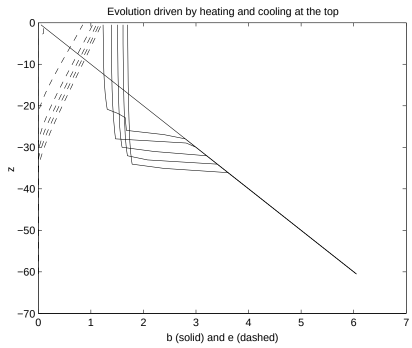

Figure 2: Formation of a well–mixed layer in a stratified fluid by a periodic cycle of heating and cooling at the surface. The solid line represents the buoyancy profile, and the dashed line the turbulent energy. The times of the snapshots plotted correspond to the beginning of cooling periods. The buoyancy flux at the top is given by $K_b b_z = 0.0015\,\sin\left(\frac{2\pi t}{100}\right)$. In order to trigger convection, there is a small constant flux of turbulent energy through the top as well, given by $K_e e_z = 0.0003$.

In the second run (see Figure 2), a daily periodic, sinusoidal boundary flux of buoyancy mimics the cooling and heating of the ocean's surface. The times shown correspond to the beginning of cooling periods. Again, the momentum equation has not been used, since there is no external source of momentum in this experiment. Two features are particularly noticeable: the sharp discontinuity at the base of the mixed layer, and the presence, in the first day plotted, of an inversion layer of warm water next to the surface. In later days, this layer diffuses rapidly throughout the mixed layer, due to the persistence of significant amounts of turbulence from previous cooling events.

In the third run (figures 3 and 4), a boundary flux of momentum accounts for the action of the wind. We see a substantial horizontal velocity developing near the surface and diffusing rapidly, due to the turbulence generated by shear instability, throughout the mixed layer. Hence the mixed layer decouples from the bulk of the ocean, developing a mean velocity of its own. The base of the mixed layer is smoother here than in the previous runs, since turbulence is more effectively generated precisely at this interface, which has the maximum shear. Hence diffusion is locally enhanced, and the potential discontinuity at the base is smoothed away.

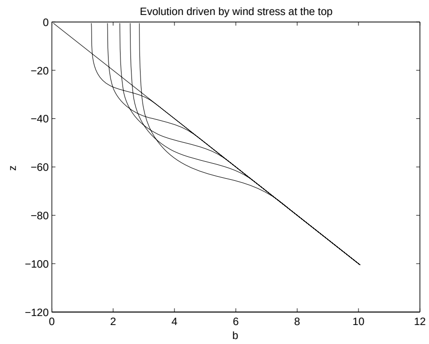

Figure 3: Buoyancy profile corresponding to the evolution of a well–mixed layer driven by wind stress, represented by a constant flux of horizontal momentum through the surface: $K_u u_z = 0.15$. As in figure 2, there is a turbulent energy flux as well, given by $K_e e_z = 0.0003$. The snapshots are displayed 700 time units apart.

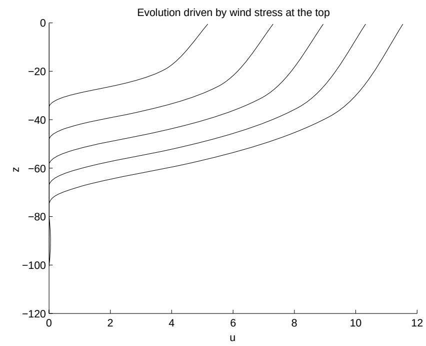

Figure 4: Horizontal velocity profile for the same wind stress driven evolution of figure 3.

4 Flow stability and the Richardson number

In this section we analyze what is the qualitative behavior of the solutions to the equations far from boundaries and close to equilibrium. The relevant parameter here is the Richardson number, defined as $Ri = -\frac{b_z}{(u_z)^2}$, which measures the relative stabilizing influence of the stratification versus the unstabilizing influence of the shear. In this section we shall consider scenarios with no or little initial turbulent energy.

There are two critical values for Ri. The first one arises from considerations involving the total energy of the system, while the second follows from the details of the dynamics.

If one would replace a stably stratified and sheared fluid at a given layer $z_0-\Delta z \le z \le z_0+\Delta z$ by a homogeneous fluid with the same mass and momentum, there would be an increase in the potential energy of the layer, but a decrease in the kinetic energy of the flow. The difference in the potential and kinetic energy would be, to leading order in Δz,

$$\begin{aligned}
\Delta P.E. &= \int_{-\Delta z}^{\Delta z} b(z_0)(z_0+s)ds - \int_{-\Delta z}^{\Delta z} (b(z_0)+b_z(z_0)s)(s+z_0)ds \\
&= \int_{-\Delta z}^{\Delta z} -b_z(z_0)s^2 ds \\
\Delta K.E. &= \int_{-\Delta z}^{\Delta z} \frac{(u(z_0))^2}{2}ds - \int_{-\Delta z}^{\Delta z} \frac{(u(z_0)+u_z(z_0)s)^2}{2}ds \\
&= \int_{-\Delta z}^{\Delta z} \frac{(u_z(z_0)^2 s^2)}{2}ds.
\end{aligned}$$

So we see that locally, if the Richardson number is smaller than $\frac{1}{2}$, then the kinetic energy of the shear is larger than the potential energy necessary to completely mix the stratification, and one could expect a final state where the fluid is neither stratified nor sheared, and all of the extra energy has been converted into turbulence.

The other critical value follows from the turbulent energy equation. If $Ri < \frac{S_u}{S_b}$, where the S_j's are the coefficients defining the diffusivities in 4, then the input of kinetic energy, $K_b b_z + K_u(u_z)^2 = l\, e^{\frac{1}{2}}(S_u - Ri\, S_b){u_z}^{\frac{1}{2}}$, is positive; i.e., the potential energy sink due to mixing is smaller than the kinetic energy gain produced by the suppression of shear. This gives rise to instability. Interestingly, this instability is related in this case to a problem of non–uniqueness: Neglecting diffusion, the energy equation 3 can be written as

$$e_t = l\, e^{\frac{1}{2}}(S_u - Ri\, S_b){u_z}^{\frac{1}{2}}\,. \tag{5}$$

If e is initially zero but the Richardson number Ri is smaller than $\frac{S_u}{S_b}$, this equation has a similar nature to the classical textbook example for non–uniqueness,

$$X_t = cX^{\frac{1}{2}}\,.$$

In other words, a profile with no initial turbulence gives rise to a large family of solutions, included but not limited to the trivial equilibrium. We shall prove below that, consequently, the equilibrium is indeed unstable.

This leaves us with two situations to consider, depending on whether the critical value $\frac{S_u}{S_b}$ is larger or smaller than $\frac{1}{2}$. In the numerical examples that illustrate the discussion that follows, we have always used as initial profile linear backgrounds of buoyancy and horizontal velocity, and a profile of turbulent energy that is zero everywhere except for a small bump included to trigger potential instabilities (see Figure 5).

a) $\frac{S_u}{S_b} > \frac{1}{2}$

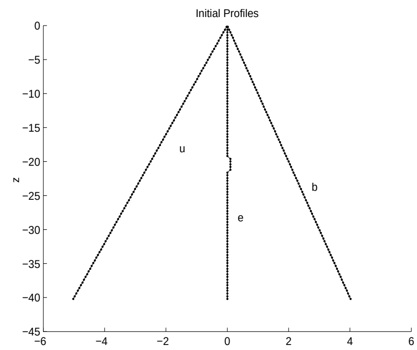

Figure 5: Qualitative initial buoyancy, velocity and turbulent energy profile for all numerical experiments on shear instability.

In this situation we can expect three different behaviors.

First, if the value of Ri is larger then the critical value $\frac{S_u}{S_b}$, then small disturbances to the main flow will have little effect. Any sufficiently small initial turbulent energy added to the flow will be consumed and transferred mainly to potential energy. If the initial turbulent energy is confined to a portion of the domain, say some layer between the depths a and b, then the mixing will take place only in a somewhat broader layer, but it will still be localized. A numerical run of this situation can be seen in Figure 6.

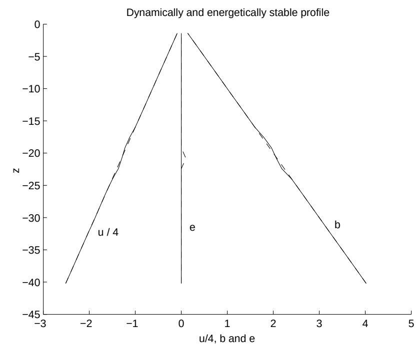

Figure 6: Evolution of a profile that is stable, both on dynamic and energetic grounds. A small patch of turbulence added to the flow yields a localized and moderate amount of mixing. In this run, $S_b = S_u = S_e = 1$, and initially $B_z = -0.1$ and $U_z = 0.25$. In this and in all remaining figures, the dashed and solid lines correspond to the initial and final profiles respectively.

Second, if $\frac{1}{2} < Ri < \frac{S_u}{S_b}$, we expect any small initial turbulent energy to grow and to produce more mixing. On the other hand, since the total energy is not sufficient to completely mix the fluid, this process must end at some point, which can only happen if Ri grows beyond $\frac{S_u}{S_b}$. So, at the final state, we expect the value of the Richardson number to be everywhere larger than the dynamical critical value. Even though the initial turbulent kinetic energy is confined to a small layer, the mixing will spread through the full depth of the fluid, and the final state will still be stratified and sheared, but the mean absolute values of b_z and u_z will decrease. This is indeed the case, as the numerical experiment displayed in Figure 7 shows. This scenario corresponds to the double diffusive instability, where a seemingly (energetically) stable profile can grow unstable due to disparities in the diffusivities of two quantities involved.

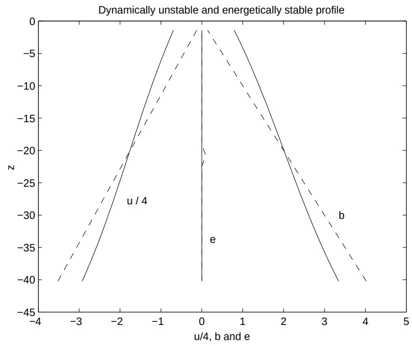

Figure 7: Evolution of a profile that is dynamically unstable, yet lacks enough kinetic energy to fully mix. Part of the energy in the shear is used for mixing, but the final state has both shear and stratification. In this run, $S_b = S_u = S_e = 1$, and initially $B_z = -0.1$ and $U_z = 0.35$.

Finally, if $Ri < \frac{1}{2}$, then the mixing will be able to completely overcome the stratification, and the final state will be homogeneous with uniform buoyancy and velocity, with all the excess energy converted into turbulence. The results of a numerical experiment confirming this scenario are plotted in Figure 8.

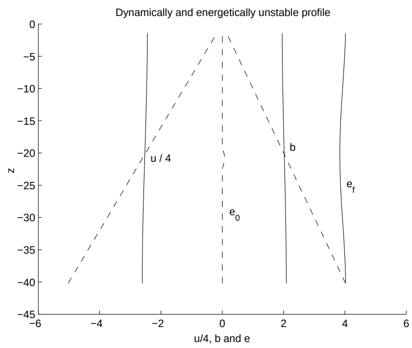

Figure 8: Evolution of a profile that is dynamically and energetically unstable. The final profile is fully homogeneous, with neither shear nor stratification, and all extra energy converted into turbulence. In this run, $S_b = S_u = S_e = 1$, and initially $B_z = -0.1$ and $U_z = 0.5$.

b) $\frac{S_u}{S_b} < \frac{1}{2}$

In this situation there are only two relevant cases.

If the initial Richardson number is smaller than the critical dynamical value, than mixing will spread throughout the depth of the fluid and the final state will again be one of a homogeneous fluid, much as in the last scenario discussed above.

If, on the other hand, $Ri > \frac{S_u}{S_b}$, then the initial turbulent energy will be insufficient to trigger a large mixing process. The most interest case is when $Ri < \frac{1}{2}$. Even though the kinetic energy of the shear is sufficient in principle to overcome the stratification completely, the dynamics do not allow this to happen, at least for small perturbations. The existing turbulent energy will be transformed into potential energy faster than it can collect kinetic energy from the shear, and the turbulence will eventually disappear, not allowing for any further mixing to occur. Figure 9 displays this interesting behavior. This scenario, where the state of maximal entropy is not dynamically reachable, is reminiscent of other geophysical situations, such as the high potential energy states in geostrophic balance with zonal winds prevailing in the atmosphere, that can only acquire entropy by eliminating some potential energy through violent nonlinear instabilities, yielding mid–latitude storms.

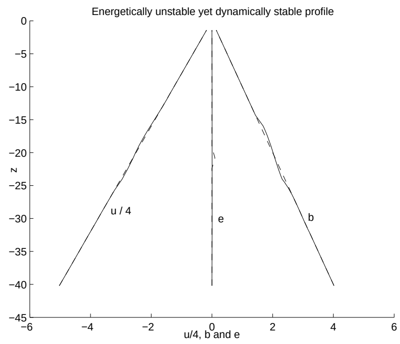

Figure 9: Evolution of a profile that is dynamically stable, even though the kinetic energy in the shear is sufficient to fully homogenize the buoyancy. Small perturbations are not enough to trigger nonlinear instabilities, and so yield only small, localized mixing. In this run, $S_b = S_e = 1$, $S_u = 0.3$, and initially $B_z = -0.1$ and $U_z = 0.5$.

4.1 Stability Analysis

We analyze now the equilibrium states for the system of equations 1 to 3.

Of course, if no–flux boundary conditions are applied, the only equilibrium states are those in which $e = 0$. On the other hand, in the ocean interior, fluxes of buoyancy and horizontal momentum coming from the upper and lower boundaries do exist: the buoyancy flux arises mainly from heating and cooling of the ocean surface, while the momentum flux has a more diverse source: wind stress, bottom roughness, and inhomogeneous buoyancy effects coupled with rotation, though the thermal wind effect. If we allow boundary fluxes to exist, there are other relevant equilibrium states, in which $Ri = \frac{S_u}{S_b}$, the buoyancy and velocity gradients are constant, and e may assume any constant positive value.

We will first analyze this case, where $\bar{b}_z = B$, $\bar{u}_z = U$, $S_b B = -S_u (U)^2$ and $\bar{e} = E$. Let $b = \bar{b} + b'$, $u = \bar{u} + u'$ and $e = E(1 + e')$. The linearization of the equations read:

$$\begin{aligned} b_t &= (lE^{\frac{1}{2}} S_b)(b_{zz} + \frac{e_z B}{2}), \\ u_t &= (lE^{\frac{1}{2}} S_u)(u_{zz} + \frac{e_z U}{2}), \\ e_t &= (lE^{\frac{1}{2}} S_e) e_{zz} + (lE^{-\frac{1}{2}} S_b)(b_z) + (lE^{-\frac{1}{2}} S_u U)(2u_z), \end{aligned} \tag{6}$$

where we have dropped the primes in the new variables, and made use of the fact that $S_b B = -S_u U^2$.

Let us now propose solutions of the form $(b, u, e) = (b_0, u_0, e_0)\, e^{i(kz - wt)}$, where (b_0, u_0, e_0) is a constant. It is easy to see that, in this case, the system 6 adopts the form

$$A\, v = 0,$$

where $A = lE^{\frac{1}{2}} \begin{bmatrix} \frac{iw}{lE^{\frac{1}{2}}} - k^2 S_b & 0 & ik\frac{S_b B}{2} \\ 0 & \frac{iw}{lE^{\frac{1}{2}}} - k^2 S_u & ik\frac{S_u U}{2} \\ \frac{ikS_b}{E} & \frac{2ikS_u U}{E} & \frac{iw}{lE^{\frac{1}{2}}} - k^2 S_e \end{bmatrix}$ and $v = \begin{bmatrix} b_0 \\ u_0 \\ e_0 \end{bmatrix}$.

Clearly, this system has nontrivial solutions only when $det(A) = 0$. If we set $x = \frac{iw}{lE^{\frac{1}{2}}}, \alpha = -k^2 S_b, \beta = -k^2 S_u, \gamma = -k^2 S_e$, then

$$det(A)(x) = (lE^{\frac{1}{2}})^3((x-\alpha)(x-\beta)(x-\gamma) + (x-\alpha)\frac{\beta S_u U^2}{E} + (x-\beta)\frac{\alpha S_b B}{2E}).$$

At $x = 0$ we see that $det(A)(x) = (lE^{\frac{1}{2}})^3(-\alpha\beta\gamma - \alpha\beta\frac{S_u U^2}{2E}) < 0$. Since $\lim_{x\to\infty} det(A) = +\infty$, there is a positive real $\bar{x}$ such that the determinant is null. This corresponds to a frequency $\bar{w}$ with positive imaginary part and so the linear system is unstable. In fact, since the frequency $\bar{w}$ is purely imaginary, the disturbances will grow in fixed locations, without moving. This is in fact verified by numerical experiments in this regime.

The analysis of the system when $e = 0$ is somewhat different. Here b and u can be any regular functions in the domain. If we perturb an equilibrium state $b = b(\bar{z}), u = u(\bar{z}), e = 0$, then the equations, to leading order, are

$$\begin{aligned} b'_t &= (l(e')^{\frac{1}{2}} S_b \bar{b}_z)_z \\ u'_t &= (l(e')^{\frac{1}{2}} S_u \bar{u}_z)_z \\ e'_t &= l(e')^{\frac{1}{2}}(S_b \bar{b}_z + S_u(\bar{u}_z)^2). \end{aligned} \tag{7}$$

We first note that the right–hand side of the system is independent of b' and u'. One can see that, if the Richardson number of the equilibrium state is everywhere larger than the critical dynamical value, then $S_b\bar{b}_z + S_u(\bar{u}_z)^2 < 0$ and we may find a positive C such that $e'_t < -C(e')^{\frac{1}{2}}$ everywhere in the domain. This implies that $e'_t(z,t) \leq (e'(z,0)^2 - \frac{C}{2}t)^2$, at any time t and any depth z, so the solutions to the system 7 will reach $e' = 0$ in a time smaller than $\max_{-H\leq z\leq 0} \frac{2(e'(z,0))^2}{C}$. This shows that this equilibrium is stable.

Finally, if the Richardson number is smaller than $\frac{S_u}{S_b}$ at some point in the domain, e' will grow at this depth until the terms of larger order begin to matter, and so the equilibrium is unstable.

5 Conclusions

Fluid mixing is a problem of high scientific and practical significance, and full of mathematical and physical challenges. Here we have concentrated on one possible mathematical description of mixing, probably the simplest, based on the assumption that turbulent mixing can be conceptualized as a nonlinear diffusive process. Independently of its range of validity, this model has a number of appealing features:

- It yields a phenomenology very much in agreement with physical reality, such as the formation of well–mixed layers with sharp interfaces.
- It provides a description of the mixing process easy to grasp intuitively, helping clarify complex concepts, such as the stability of sheared and stratified flows, and the distinction between dynamic (local) and energetic (global) stability.
- It is mathematically treatable, though far from trivial.

It should be remembered, however, that this class of models is phenomenological, and not based on a first–principled approach. A reminder of this is the free parameter l, the "eddy mixing length", which needs to be provided externally.

In this paper, we have concentrated on one–dimensional scenarios, where the flow is assumed to be horizontally homogeneous. Straightforward extensions of the model can be applied to two and three dimensional situations, shedding light on phenomena such as the anisotropy of mixing rates in the ocean, and the effects of breaking waves. Such extensions will be explored in further work.

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