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# Positive Operators and Maximum Principles for Ordinary Differential Equations

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**ABSTRACT**

We show an equivalence between a classical maximum principle in differential equations and positive operators on Banach Spaces. Then we shall exhibit many types of boundary value problems for which the maximum principle is valid. Finally, we shall present extended applications of the maximum principle that have arisen with the continued study of the qualitative properties of Green's functions.

**RESUMEN**

Mostramos una equivalencia entre el clásico principio del máximo en ecuaciones diferenciales y operadores positivos en espacios de Banach. Exhibiremos distintos tipos de problemas con valores en la frontera para los cuales el principio del máximo es válido. Finalmente, mostraremos aplicaciones generalizadas del principio del máximo que resultan del estudio de las propiedades cualitativas de las funciones de Green.

**Key words and phrases:** *Maximum principle, Positive operators, sign properties of Green's functions*
**Math. Subj. Class.:** *34B*

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# 0 Introduction

The purpose of this article is three-fold. First, in Section 1, we shall show a relationship between a classical maximum principle in differential equations and positive operators on Banach Spaces. In loose terms, we shall refer to the "equivalence" of the maximum principle and known sign properties of an associated Green's function. Second, in Section 2, we shall exhibit many types of boundary value problems (BVPs) for which the maximum principle is valid. Finally, in Section 3, we shall present some extended applications of the maximum principle that have arisen with the continued study of the qualitative properties of Green's functions.

To begin, we recall the monograph of Protter and Weinberger [63]. In this work, the authors exhibit the impact of the maximum principle in ordinary differential equations, and elliptic, parabolic and hyperbolic partial differential equations. They give applications of the maximum principle to unique solvability, approximation methods, Harnack inequalities, etc.

In Section 2, we shall exhibit families of BVPs, not discussed by Protter and Weinberger [63], which can be considered in the context of a maximum principle or a positive fixed point operator. These families include higher order ordinary differential equations with a wide variety of boundary conditions, finite difference equations, and hence, dynamic equations on time scales, impulsive equations, BVPs with nonlinear boundary conditions, and even some functional equations with delay.

In Section 3, we present extended applications of maximum principles that have emerged since the work of Protter and Weinberger [63]. Again, these applications are available through sign property analysis of appropriate Green's functions. Applications include elementary monotone methods coupled with upper and lower solution methods, rapid convergence methods, Krein-Rutman theory [53], new comparison theorems, generalizations of concavity through Harnack type inequalities, and limiting behavior on unbounded domains.

# 1 The Maximum Principle and Positive Operators on Banach Spaces

To introduce this section, we consider a specific boundary value problem of the form

$$x''(t) = f(t), \quad 0 < t < 1, \tag{1.1}$$

$$x(0) = x_1, \quad x(1) = x_2. \tag{1.2}$$

It is well-known and one can show directly, that the solution $x$ has the form

$$x(t) = l(t) + \int_0^1 G(t,s)f(s)ds, \tag{1.3}$$

where

$$G(t,s) = \begin{cases} t(s-1), & s \geq t, \\ s(t-1), & s \leq t, \end{cases} \tag{1.4}$$

and $l$ is the solution of the homogeneous differential equation, $x'' = 0$, that satisfies the boundary conditions, (1.2).

**Theorem 1.1** *The statement*

$$x''(t) \leq 0, \quad 0 < t < 1, \quad x(0) = 0, \quad x(1) = 0 \quad \Rightarrow x(t) \geq 0, \quad 0 \leq t \leq 1,$$

*is equivalent to the statement*

$$G(t,s) \leq 0, \quad (t,s) \in (0,1) \times (0,1).$$

**Remark 1.1** *The statement*

$$x''(t) \leq 0, \quad 0 < t < 1, \quad x(0) = 0, \quad x(1) = 0, \quad \Rightarrow x(t) \geq 0, \quad 0 \leq t \leq 1,$$

*is one form of the maximum principle [63], and more precisely it is a minimum principle. Many authors prefer that a Green's function be positive pointwise; hence many authors choose to work with the operator, $-d^2/dt^2$. In this setting, signs are reversed; in particular, a Green's function is pointwise positive and the equivalent principle is a maximum principle. We will be lax with the phrase, maximum principle, throughout the paper and we do not use the phrase minimum principle.*

**Proof.** One implication is immediately due to the representation, (1.3). The other implication employs a straightforward proof by contradiction. If $G(t_0, s_0) > 0$, one constructs a continuous nonpositive function $f$ such that

$$\int_0^1 G(t,s)f(s)ds < 0. \qquad \blacksquare$$

The above theorem is trivial; but the above example carries over immediately to abstract boundary value problems of the form

$$Lx = f, \quad Bx = c, \tag{1.5}$$

where $L$ denotes a linear operator on a Banach space and $B$ denotes linear boundary conditions. Assume that (1.5) can be inverted; that is assume there exists $G(t,s)$ such that the solution of (1.5) has the form

$$x(t) = l(t) + \int_\Omega G(t,s)f(s)ds,$$

where $l$ satisfies, $Lx = 0, Bx = c$. Then the equivalence theorem applies in this setting as well.

# 2 Classes of Boundary Value Problems

In this section, we intend to show that the simple equivalence that is exposed in the preceding section applies to a wide variety of problems. For a given operator, there will be applications to various examples depending on the boundary conditions. Moreover, the equivalence applies to a broad variety of operators as well. We shall exhibit as examples ordinary differential operators, discrete and time scale operators, delay operators, and impulse operators.

## 2.1 Ordinary Differential Equations

We will present six types of BVPs in this subsection. These are conjugate, right focal, Lidstone, periodic, nonlocal, and Sturm-Liouville type BVPS. This list is not exhaustive.

The obvious place to begin is with the $n$th order disconjugate ordinary differential operator and the conjugate type boundary conditions. Let $I$ denote a bounded interval of the reals and let $v_i \in C^{n-i}(I)$, $i = 0, \ldots, n$, be positive. For $x \in C^n(I)$, define the $n$th order ordinary differential operator, $L_n$, by

$$L_n x(t) = v_n(v_{n-1}(\ldots(v_0x)'\ldots)')'(t), \quad t \in I. \tag{2.6}$$

This particular factored operator has a long history of study and we refer the reader to the monograph of Coppel [15]. Let it suffice to comment that the primary motivation for the study of the operator, $L_n$, is that the operator is homotopic to the operator, $d^n/dt^n$ ($v_i \equiv 1$), and solutions of $L_n x = 0$ exhibit the same qualitative behavior as the family of $n$th order polynomials. Hence, the study of BVPs for disconjugate operators reduces to a study in interpolation theory.

Let $a, b \in I$, $a < b$. Let $k \in \{2, \ldots, n\}$ and let $a = t_1 < t_2 < \cdots < t_k = b$ be given. Assume $n_1, \ldots, n_k$ are positive integers such that

$$\alpha = \sum_{i=1}^{k} n_i = n.$$

For each $j = 1, \dots k-1$, set

$$\alpha_j = \sum_{l=j+1}^{k} n_l.$$

The *conjugate* BVP associated with (2.6) is the problem, solve

$$L_n x(t) = f, \quad t \in [a, b], \tag{2.7}$$

$$x^{(j-1)}(t_i) = 0, \quad j = 1, \dots, n_i, \quad i = 1, \dots, k. \tag{2.8}$$

The BVP, (2.7), (2.8), is invertible in the following sense: there is a function $G(t,s) : [a,b]^2 \to \mathbb{R}$ such that the solution $x$ of the BVP, (2.7), (2.8), has the representation,

$$x(t) = \int_a^b G(t,s) f(s) ds, \quad t \in [a, b]. \tag{2.9}$$

Before we give the sign properties of $G$, we make two observations, the first with respect to the construction of $G$ and the second with respect to nonlinear mathematics.

There are various constructions of $G$. $G$ is uniquely characterized by four properties [13, page 192]; this characterization leads to solving a linear system of $2n$ equations for $2n$ unknowns. We will refer to this type of characterization throughout this paper. For a second construction, $G$, as a function of $t$, satisfies $L_n x = 0$ on triangles $t < s$, $t > s$; as a function of $s$, $G$ satisfies the adjoint equation, $L_n^* x = 0$. This observation leads to an independent construction of $G$, one we do not employ in this paper. A third construction, and one we shall use is as follows: let $\chi(t,s)$ denote the Cauchy function associated with $L_n$; that is, let $\chi$ denote the solution of the initial value problem,

$$L_n x(t) = 0, \quad t \in [a, b], \quad x^{(i-1)}(s) = \delta_{i,n}, \quad i = 1, \dots, n.$$

$\chi$ is essentially the impulse function for this $n$th order ordinary problem. Assume $t_l < s < t_{l+1}$. Construct $G$ as

$$G(t,s) = \begin{cases} u(t,s), & s > t, \\ u(t,s) + \chi(t,s), & s \le t, \end{cases}$$

where is $u$ is the solution of the BVP,

$$L_n x(t) = 0, \quad t \in [a, b],$$

$$x^{(j-1)}(t_i) = 0, \quad j = 1, \dots, n_i, \quad i = 1, \dots, l,$$
$$x^{(j-1)}(t_i) = -\chi^{(j-1)}(t_i), \quad j = 1, \dots, n_i, \quad i = l+1, \dots, k.$$

The factored form of $L_n$ implies that $u$ is uniquely determined; if one thinks in terms of qualitative properties of polynomials, this is a well-posed interpolation problem. In the case, $L_n = d^n/dt^n$, it is easy to see from this construction that as a function of $t$, $G$ has precisely the $n$ roots, counting multiplicities, given by the boundary conditions

(2.8), and no more. $G$ is a $C^{n-2}$ function and if $G$ has an additional root, repeated applications of Rolle's theorem imply $G^{(n-2)}$ has three roots in $(a,b)$. Thus, $G^{(n-1)}$ vanishes for $t < s$ or $t > s$. The zeros of each $G^{(j)}$ have been located by Rolle's theorem and so one can show inductively that each $G^{(j)} \equiv 0$ for $t < s$ or $t > s$, $j = 0, \ldots, n-1$. For $t < s$, $u \equiv 0$ implies $\chi \equiv 0$ by disconjugacy. This contradicts the construction of $\chi$. A similar contradiction is obtained for $s \le t$.

The representation (1.4) implies the following observation with respect to nonlinear problems.

**Theorem 2.1** *Assume $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ is continuous. Then $x$ is a solution of the nonlinear BVP,*

$$L_n x(t) = f(t, x(t)), \quad t \in (a,b),$$

*with boundary conditions, (2.8), if, and only if, $x \in C[a,b]$ and $x$ satisfies the integral equation,*

$$x(t) = \int_a^b G(t,s) f(s, x(s)) ds, \quad t \in [a,b]. \tag{2.10}$$

The following sign condition of $G$ is well-known [15].

**Theorem 2.2** *Assume $L_n$ has the factored form (2.6). Then for $(t,s) \in (t_i, t_{i+1}) \times (a,b)$,*

$$(-1)^{\alpha_i} G(t,s) > 0.$$

Thus, the parity of the sign of $G$ agrees with the number of boundary conditions specified to the right of $t$. Couple the sign conditions given in Theorem 2.2 with (2.10) and it is easy to see that cone theoretic fixed point theorems on Banach spaces are very useful in the study of boundary value problems for nonlinear $n$th order ordinary differential equations.

We also point out here that the nonlinear term $f$ in Theorem 2.1 is carefully constructed. Only the sign of $G$ is known; thus, the nonlinear term depends only on $x$ and not on higher order derivatives of $x$. If one considers nonlinear effects from higher order derivatives of $x$, then one appeals to Nagumo conditions [51].

Although the theory of disconjugacy is summarized in the authoritative account due to Coppel [15], we also refer to Hartman [48], and Levin [55], as these authors played key roles in the development of the theory.

The second BVP we consider is the related right focal problem. For simplicity of exposition, we now consider the special case of (2.6), $L_n = d^n/dt^n$. Let $k \in \{2, \ldots, n-1\}$. Two-point $(k, n-k)$ right focal boundary conditions are of the form

$$x^{(i-1)}(a) = 0, \quad i = 1, \ldots, k, \quad x^{(i-1)}(b) = 0, \quad i = k+1, \ldots, n. \tag{2.11}$$

If one employs the more general factored operator, $L_n$, then one needs stronger hypotheses than just the sign conditions on the $v_i s$ [1]; if one chooses to employ two term differential operators [58], [21], then commonly, one employs quasi-derivatives in the boundary conditions, (2.11). Hence, we only consider $L_n = d^n/dt^n$.

**Theorem 2.3** *A unique Green's function, $G(t,s)$, exists for the BVP,*

$$x^{(n)}(t) = f, \quad t \in (a,b),$$

*with two-point right focal boundary conditions, (2.11). Let $G_i(t,s) = \partial^i(G(t,s))/\partial t^i$. Then*

$$(-1)^{n-k}G_{i-1}(t,s) > 0, (t,s) \in (a,b] \times (a,b) \quad i = 1, \dots k,$$
$$(-1)^{n-i+1}G_{i-1}(t,s) \geq 0, (t,s) \in [a,b) \times (a,b) \quad i = k+1, \dots n.$$

This theorem is very believable; beginning with the $n-1$ order derivative, one can sign $G_{n-1}$ as the kernel of a first order initial value problem. One then signs lower order derivatives inductively via definite integration.

Note that each derivative of $G$ satisfies a known sign condition; in particular, under suitable hypotheses, each derivative of a solution satisfies a maximum principle. So, cone theoretic methods apply immediately to nonlinear problems of the form,

$$x^{(n)}(t) = f(t, x(t), \dots, x^{(n-1)}(t)), \quad t \in (a,b),$$

with boundary conditions (2.11). No Nagumo conditions [51] are required in an analysis here.

A third and related BVP is the Lidstone BVP [56], [2], [37], for example. Again we restrict the discussion to a simple even order operator, $L_{2n} = d^{2n}/dt^{2n}$, and we consider two-point boundary conditions of the form,

$$x^{(2(i-1))}(a) = 0, \quad x^{(2(i-1))}(b) = 0, \quad i = 1, \dots, n. \tag{2.12}$$

**Theorem 2.4** *A unique Green's function, $G(t,s)$, exists for the BVP,*

$$x^{(2n)}(t) = f, \quad t \in (a,b),$$

*with two-point Lidstone boundary conditions, (2.12) and*

$$(-1)^{n-i+1}G_{2(i-1)}(t,s) > 0, \quad (t,s) \in (a,b)^2, \quad i = 1, \dots n.$$

As with Theorem 2.3, Theorem 2.4 is believable. The function, $G_{2(n-1)}$ is the Green's function for a second order conjugate BVP, $x''(t) = 0$, $a < t < b$, $x(a) = 0$, $x(b) = 0$. Hence, $G_{2(n-1)}$ has the representation (1.4) (with $a = 0$ and $b = 1$) and $G_{2(n-1)}(t,s) < 0$, $(t,s) \in (a,b)^2$.

Next, note that $G_{2(n-2)}$ is the solution of a nonhomogeneous BVP,

$$x''(t) = G_{2(n-1)}(t,s), \quad a < t < b,$$

$$x(a) = 0, \quad x(b) = 0.$$

Thus, $G_{2(n-2)}$ is the convolution

$$G_{2(n-2)}(t,s) = \int_a^b G_{2(n-1)}(t,r)G_{2(n-1)}(r,s)dr.$$

Proceed inductively and calculate the convolution

$$G_{2(i-2)}(t,s) = \int_a^b G_{2(n-1)}(t,r)G_{2(i-1)}(r,s)dr$$

at each step.

The illustration in the preceding paragraph indicates that the Lidstone problem can be considered as a nested collection of second order conjugate problems. Upon further reflection, and in the same context, note that right focal BVPs (and focal BVPs as well) are nested initial value problems.

With the maximum principle given in Theorem 2.4, the natural nonlinear problem to consider has the form,

$$x^{(2n)}(t) = f(t, x(t), x''(t), \dots, x^{(2j)}(t), \dots, x^{(2(n-1))}(t)), \quad t \in (a,b).$$

If one wishes to address nonlinear dependence on odd order derivatives of $x$, one must again appeal to Nagumo conditions [18].

Considerable work has been done recently in relation to the maximum principle for periodic BVPs of the form,

$$x^{(n)}(t) = f, \quad t \in (a,b),$$

$$x^{(i-1)}(a) = x^{(i-1)}(b), \quad i = 1, \dots, n.$$

See [12] for example.

There is a current flurry to study nonlocal boundary value problems. Key works include Lomtatidze and co-authors [57], and Gupta [46]. For example, consider the BVP, (1.1), with boundary conditions given by

$$x(0) = 0, \quad x(1) = x(1/2).$$

A Green's function can be constructed directly and has the form

$$G(t,s) = \begin{cases} G_1(t,s), & 0 < s \le 1/2, \\ G_2(t,s), & 1/2 \le s \le 1, \end{cases}$$

where

$$G_1(t,s) = \begin{cases} -t, & t < s, \\ -s, & s < t, \end{cases}$$

and

$$G_2(t,s) = \begin{cases} 2(s-1)t, & t < s, \\ -s + (2s-1)t, & s < t. \end{cases}$$

Hence, the maximum principle is valid for these nonlocal BVPs as well.

Finally, for second order problems, conjugate, right focal and periodic boundary conditions are all special cases of the general Sturm-Liouville boundary conditions [45].

## 2.2 Discrete, Time Scale Problems

The disconjugacy theory for forward difference equations was developed by Philip Hartman [47] in a landmark paper which has generated so much activity in the study of difference equations. Sturm theory for a second order finite difference equation goes back to Fort [43]. Hartman considers the $n$th order linear finite difference equation,

$$Pu(m) = \sum_{j=0}^{n} \alpha_j(m)u(m+j) = 0,$$

$\alpha_n\alpha_0 \neq 0$, $m \in I = \{a, a+1, a+2, \dots\}$, with conjugate boundary conditions,

$$u(m_i) = 0, \quad i = 1, \dots, n,$$

where $a \leq m_1 < m_2 < \cdots < m_n$.

Completely analogous to the development of disconjugacy for ordinary differential equations, he shows the existence of a Green's function, $G(m,s)$, for this problem and

$$(-1)^{\sigma(m)}G(m,s) > 0,$$

$m \in I$, $m \neq m_j$, $s \in I$, $\sigma(m) =$card $\{j : m < m_j\}$.

Much of the theory for ordinary differential equations carries over to the discrete problems and we refer the reader to a comprehensive bibliography in [3].

In an effort to unify the continuous and discrete calculus, **Hilger** [50] invented the calculus on time scales. Rather than discuss the material here, we refer the reader to the authoritative account in [10]. A Green's function for the second order scalar function is developed there. In Chapter 8 of [11], a Green's function for an $n$th order disconjugate equation with conjugate boundary conditions is signed.

Many of the applications discussed in Section 3 for ordinary differential equations have extensions to difference equations. Most of the extensions to time scales have not been developed. See Chapter 8 of [11].

## 2.3 Systems of Ordinary Equations

Werner [66] wrote an interesting paper in which he developed an abstract maximum principle for systems of ordinary differential equations. Let $I = [a,b]$ be a subset of the reals and assume that $f : I \to R^n$. Let $M, N$ denote $n \times n$ matrices with real entries and let $c \in R^n$. Consider a two-point BVP of the form,

$$x'(t) = f(t), \quad t \in (a,b), \tag{2.13}$$

$$Mx(a) + Nx(b) = c. \tag{2.14}$$

The BVP, (2.13), (2.14), is not invertible. So, one considers an equivalent equation,

$$x'(t) - D(t)x(t) = f(t) - D(t)x(t), \quad t \in (a,b), \tag{2.15}$$

where $D$ is an $n \times n$ matrix with entries in $C(I)$. One constructs $D$ so that the equivalent BVP, (2.13), (2.14), is invertible, and so that a maximum principle is valid on a partial order induced by $D$. The Green's function has the form

$$G(t,s) = \begin{cases} U(t)AMU(a)U^{-1}(s), & a \le s < t \le b, \\ U(t)(AMU(a)-E)U^{-1}(s), & a \le t < s \le b, \end{cases}$$

where $U$ denotes a fundamental matrix for the system, $x' - Dx = 0$, $E$ denotes the $n \times n$ identity matrix and

$$A = (MU(a) + NU(b))^{-1}.$$

Let $B$ denote the Banach space $C_n(I)$ with $||x|| = \max ||x_k||$ and $||x_k||$ denotes the usual supremum norm on the $k$th component of $x$. Define a partial order on $B$ by $x \le z$ if, and only if, $x_k(t) \le z_k(t)$, $t \in I$, $k = 1, \ldots, n$. If $H : B \to B$ is an invertible linear operator, define a relation, $\le_H$, by $x \le_H z$ if, and only if, $Hx \le Hz$. Then $\le_H$ denotes a partial order on $B$ and $B$ is a partially ordered Banach space with respect to $\le_H$. Werner [66] defines a partial order $\le_{HU^{-1}}$ and a partial order $\le_{JU^{-1}}$ where $H$ and $J$ are chosen so that

$$HAMU(a)J^{-1} \ge 0, \quad H(AMU(a)-E)J^{-1} \ge 0$$

elementwise. Consider a modification of $G$:

$$\hat{G}(t,s) = \begin{cases} U(t)AMU(a)J^{-1}, & a \le s < t \le b, \\ U(t)(AMU(a)-E)J^{-1}, & a \le t < s \le b. \end{cases}$$

Then $\hat{G} \ge_{HU^{-1}} 0$ and the maximum principle applies. Werner makes a very nice application to a second order scalar problem with periodic boundary conditions. This development in systems has been extended to multipoint problems [27] and problems with impulse [34].

### 2.4 Impulsive Problems with Nonlinear Boundary Conditions

In this subsection, we will briefly discuss a second order impulsive BVP with nonlinear boundary conditions.

We consider a problem with impulses at $0 < t_1 < \cdots < t_m < 1$; define an impulse $\Delta x(t) = x(t^+) - x(t^-)$. First, consider an impulsive problem with conjugate boundary conditions and consider the problem as linear, nonhomogeneous.

$$x''(t) = f(t), \quad t \in (0,1) \setminus \{t_1, \ldots, t_m\}, \tag{2.16}$$

$$\Delta x(t_i) = u_i, \quad \Delta x'(t_i) = v_i, \quad i = 1, \ldots, m, \tag{2.17}$$

$$x(0) = x_1, \quad x(1) = x_2. \tag{2.18}$$

Then, see [35],

$$x(t) = p(t) + \sum_{i=1}^{m} I_i(t) + \int_a^b G(t,s)f(s)ds, \tag{2.19}$$

where

$$p(t) = x_1(t-1) + x_2 t,$$

and

$$I(t_i) = \begin{cases} t(-u_i - (1 - t_i v_1)), & t \le t_i, \\ (1-t)(u_i - t_i v_1), & t_i \le t, \end{cases}$$

$i = 1, \ldots, m$. $G$ has the representation given by (1.4). If one analyzes the $G$ in terms of the maximum principle, it is clear that the crux of the matter is that the terms $t, (1-t), s$, and $(1-s)$ are of fixed sign on $(0,1)$. Note that the analogous terms in $p$ and $I$ are precisely $t$ and $(1-t)$. As one readily imposes a sign condition on $f$ to invoke a maximum principle, one can readily impose conditions on $u_i, v_i$ or $x_1, x_2$ to invoke a maximum principle. We shall briefly return to this development below when we discuss monotone methods.

If the problem is nonlinear in any term, differential equation, impulse, or boundary condition, then the integral expression (2.19) readily becomes a fixed point equation as in Theorem 2.1.

## 2.5 Delay Equations

Azbelev, and co-authors in the Perm group, have studied functional differential equations (fdes) for many years [5], [6]. Domoshnitsky [17] and Domoshnitsky and Bainov [7] have studied the sign properties of associated Green's functions for linear fdes.

Eloe and Henderson [31] studied a second order linear operator of the form

$$Lx(t) = x''(t) + q(t)x'(0) + \sum_{i=1}^{m} p_i(t)x(h_i(t)), \quad 0 < t, \tag{2.20}$$

where $q, p_i, h_i \in C[0,\infty)$, $i = 1, \ldots, m$. Eloe and Henderson make the additional assumption, not assumed by the Azbelev and co-authors; $0 \le h_i(t) \le t$, $i = 1, \ldots, m$. For each $b > 0$, consider the conjugate boundary conditions

$$x(0) = 0, \quad x(b) = 0.$$

One can employ the construction of a Green's function as outlined by Coddington and Levinson [13, page 192] to show that a Green's function, $G(b; t, s)$, exists for the conjugate BVP associated with (2.20). In this setting Eloe and Henderson proved that if $q(t) \ge 0$ and $0 < b_1 < b_2$, then

$$0 > G(b_1; t, s) > G(b_2; t, s), \quad (t, s) \in (0, b_1)^2,$$

and

$$0 > G_t(b_1; 0, s) > G_t(b_2; 0, s), \quad s \in (0, b_1).$$

### 2.6 Singular Equations

Bebernes and Jackson [9] studied a singular BVP on an unbounded domain of the form,

$$x''(t) = f, \quad 0 < t,$$

$$x(0) = 0, \quad x \in BC[0, \infty),$$

where $BC[0, \infty)$ denotes the bounded continuous functions on $[0, \infty)$ with a usual supremum norm. A Green's function for this problem exists in the following sense: if $f$ satisfies suitable asymptotic properties, then $x$ is a solution of the the singular BVP, if, and only if,

$$x(t) = \int_0^\infty G(t, s) f ds.$$

The Green's function has the form

$$G(t, s) = \begin{cases} -s, & 0 \le s < t, \\ -t, & 0 \le t < s. \end{cases}$$

If one calculates the second order conjugate Green's function on an interval $[0, b]$ and formally lets $b$ diverge to $\infty$, the characterization of a conjugate Green's functions "converges" to the form given above. Likewise, if one calculates a second order right focal Green's function on an interval $[0, b]$ and formally lets $b$ diverge to $\infty$, the characterization of a right focal Green's functions "converges" to the form given above. This interesting behavior is addressed in the book by Coddington and Levinson [13] when they discuss limit circle and limit point cases. This would illustrate an example of the limit point case.

Elias [19] studied an $n$th order ordinary differential operator with a family of related two-point BVPs and obtained a unique limiting Green's function satisfying known sign conditions. Eloe and Kaufmann [39] were motivated by Elias, changed many of Elias' assumed sign conditions, and obtained a unique limiting Green's function satisfying known sign conditions. Eloe and Kaufmann [38] have shown similar results to be valid for discrete problems. Eloe and Henderson [30] have constructed unique limiting Green's functions with known sign conditions for singular problems on bounded domains as well.

## 3 Applications

### 3.1 Monotone Methods

Collatz [14] provides an elegant discussion of monotone methods for positive operators on partially ordered Banach spaces. Let $B$ be a partially ordered Banach space with partial order, $\le$. Suppose $T : B \to B$ is an isotone operator; i.e.,

$$x_1 \le x_2 \Rightarrow Tx_1 \le Tx_2.$$

Suppose there exist $\alpha_0, \beta_0 \in B$ such that

$$\alpha_0 \leq \beta_0, \tag{3.21}$$

$$\alpha_0 \leq T\alpha_0, \quad T\beta_0 \leq \beta_0. \tag{3.22}$$

Then if $T$ is a completely continuous map, $T$ has a fixed point $x_0 \in B$ such that

$$\alpha_0 \leq x_0 \leq \beta_0.$$

This follows immediately by the Schauder fixed point theorem. Define the convex set

$$D := \{x \in B : \alpha_0 \leq x_0 \leq \beta_0\}.$$

The monotonicity assumptions on $T$ coupled with the assumptions (3.21), (3.22) immediately imply that $T : D \to D$.

In our setting, $Tx(t) = \int_\Omega G(t,s)f(s,x(s))ds$. For the simple problem, (1.1), (1.2), it is the case that $G(t,s) < 0$ on $(0,1) \times (0,1)$. Hence, if $f$ is decreasing in $x$, $T$ is an isotone operator. There are methods of forced monotonicity as well (see [26], [45]). The goal then would be to construct a forcing term that essentially forces the nonlinear term to be decreasing in $x$. Applications abound; we only list a few citations.

If $\alpha_0, \beta_0$ satisfy (3.21), (3.22) then $\alpha_0$ and $\beta_0$ are typically called *lower solution* and *upper solution*, respectively. (3.21) is a straight forward assumption. To obtain (3.22), let us assume that $G(t,s) < 0$ on $(0,1) \times (0,1)$, and consider the equation (1.1). Then one assumes

$$\alpha_0''(t) \geq f(t, \alpha_0(t)), \quad \beta_0''(t) \leq f(t, \beta_0(t)), \quad t \in (0,1).$$

The inequalities make sense if one considers mental images due to concavity. They are precisely the correct inequalities when one appeals to the representation in (2.10) and notes the representation,

$$\alpha_0(t) = \int_\Omega G(t,s)\alpha_0''(s)ds.$$

The analogous representation is valid for any $x$ including $\beta_0$. Thus, couple the differential inequalities with the sign of $G$ and obtain (3.22).

Šeda [65] has written a very interesting paper developing and exploiting properties of upper and lower solutions.

A different type of monotone method coupled with the method of upper and lower solutions is a method of rapid convergence. An authoritative account has recently been published by Lakshmikantham and Vatsala [54]. The methods, and related rapid convergence methods are simply numerical methods related to Taylor series expansions. How far one expands a Taylor series dictates the order of convergence of the monotone iterates. Regardless, it is an interesting and very delicate balance of the monotone methods and the method of upper and lower solutions. In addition to being applications of the maximum principle, the methods require uniqueness of solutions as well.

One requirement of the method is that if $\alpha$ and $\beta$ satisfy (3.22), then $\alpha \leq \beta$; that is, (3.22) implies (3.21). This property implies the uniqueness of solutions since solutions are simultaneously lower solution and upper solutions.

Because of the requirement, (3.22) implies (3.21), the rapid convergence methods were initially applied to initial value problems. The methods have proved suitable to second order problems of the form

$$x''(t) = f(t, x(t), x'(t)), \quad t \in I,$$

with conjugate conditions, Sturm Liouville conditions, nonlocal conditions, or singular problems on unbounded domains, for example. In this setting, if $f$ is increasing with respect to the second component, then one has uniqueness of solutions for various families of boundary conditions.

One does not find many applications of the rapid convergence methods to higher order scalar problems. The methods have worked very nicely for $n$th order problems with periodic boundary conditions [12]; the methods should work as well on Lidstone problems or other nested type problems based on the initial or boundary value problems discussed in Section 2. The methods have yet to be successfully applied to the $n$th order conjugate type problems addressed in [15], for example. Reasonable conditions for uniqueness of solutions have not been formulated for such problems.

## 3.2 Krein-Rutman Theory

A theorem due to Perron states that if $T$ is a square matrix with only positive entries, then the eigenvalue of largest magnitude is positive and simple, and there exists an associated eigenfunction that has all positive entries. The Krein-Rutman theory carries this idea over to partially ordered Banach spaces. Applications of the maximum principle carry this idea over to many of the problems illustrated in Section 2.

We shall begin with some definitions and results from the theory of cones. Please refer to Krasnosel'skiĭ [52], Amann [4], Deimling [16], Krein and Rutman [53], Schmitt and Smith [64], and Zeidler [67] for accounts of the material stated here. We shall assume the reader has some familiarity with cones in Banach spaces.

Let $B$ denote a real Banach space, and let $P$ denote a reproducing cone in $B$. Recall that the cone is reproducing if for each $x \in B$, there exist $u, v \in P$ such that $x = u - v$. Let $\leq$ be the partial order on $B$ induced by $P$; that is, $x \leq y$ if, and only if, $x - y \in P$. Let $N_1, N_2 : B \to B$ be bounded, linear operators. We will say $N_1 \leq N_2$ if $N_1 u \leq N_2 u$ for each $u \in P$ and we will say $N$ is positive with respect to $P$ if $N : P \to P$. Let $r(N)$ denote the spectral radius of $N$.

Nussbaum [59] is responsible for the following result.

**Theorem 3.1** *Let $N_b$, $\alpha \leq b \leq \beta$, be a family of compact, linear operators on a Banach space such that the mapping $b \to N_b$ is continuous in the uniform operator topology. Then the mapping $b \to r(N_b)$ is continuous.*

Refer to [4] or [52] for proofs of the following three theorems. Assume the maps $N, N_1, N_2$ are compact, linear, and positive with respect to a reproducing cone, $P$.

**Theorem 3.2** *Assume $r(N) > 0$. Then $r(N)$ is an eigenvalue of $N$, and there is a corresponding eigenvector in $P$.*

**Theorem 3.3** $N_1 \leq N_2$ *implies* $r(N_1) \leq r(N_2)$.

**Theorem 3.4** *Suppose there exists $\mu > 0$, $u \in B$, $-u \notin P$ such that $Nu \geq \mu u$. Then $N$ has an eigenvector in $P$ which corresponds to an eigenvalue, $\lambda \geq \mu$.*

For the sake of exposition we will consider the second order problem with Dirichlet boundary conditions on an arbitrary interval $(a, b)$. That is, we consider

$$x''(t) = p(t)x(t), \quad a < t < b, \tag{3.23}$$

$$x(a) = 0, \quad x(b) = 0. \tag{3.24}$$

The Green's function has the form

$$G(b; t, s) = \begin{cases} (t-a)(s-b)/(b-a), & a \leq t < s \leq b, \\ (s-a)(t-b)/(b-a), & a \leq s < t \leq b. \end{cases}$$

Assume that $p$ is a nonpositive continuous function defined on $[a, \infty)$ and assume $p$ does not vanish identically on each compact subinterval of $[a, \infty)$. Of course, if $p = -1$ the eigenfunctions are sine functions.

Set

$$B = \{x \in BC[a, \infty) : x(a) = 0\},$$

where $B$ is equipped with the the usual supremum norm. Let $P = \{x \in B : x(t) \leq 0, \quad a \leq t\}$. Define $N_b : B \to B$ by

$$N_b x(t) = \begin{cases} \int_a^b G(b; t, s)p(s)x(s)ds, & a \leq t \leq b, \\ 0, & b < t. \end{cases}$$

**Theorem 3.5** *The following are equivalent:*

1. $b_0 = \inf\{b > a : (3.23), (3.24)$ *has a nontrivial solution*$\}$;
2. *there exists a nontrivial solution $x_0$ of the BVP (for $b = b_0$) (3.23), (3.24) such that $x \in P_{b_0}$;*
3. $r(N_{b_0}) = 1$.

It is common to call $b_0$ the principal eigenvalue with principal eigenvector $x_0$. The principal eigenvector can be useful in nonlinear problems as an upper solution.

We don't prove Theorem 3.5; details can be found in [25]. We do discuss how Theorems 3.2, 3.3, and 3.4 can be applied due to the maximum principle. First, if one defines $B_b = \{x \in C[a, b] : x(a) = 0\}$ and one defines the obvious cone, $P_b$, on a compact domain, then the restriction of $N_b$ to $B_b$ maps $P_b \setminus \{0\}$ into the interior of $P_b$. This observation is useful in the application of the theory of $\mu_0$ positive operators, a closely related theory [52].

That the restriction of $N_b$ to $B_b$ maps $P_b\{0\}$ into the interior of $P_b$ follows because

$$(\partial/\partial t)G(b;a,s) < 0, \quad (\partial/\partial t)G(b;b,s) > 0.$$

The above observation generalizes to all Green's functions discussed in Section 2.

More pertinent to the Krein-Rutman theory, one will compare $(N_{b_2} - N_{b_1})x(t)$. If $b_2 > b_1$ and $t \in [a, b_1]$, one considers

$$\int_a^{b_1} (G(b_2;t,s) - G(b_1;t,s))p(s)x(s)ds.$$

Hence, one becomes interested in sign analysis of

$$H(b;t,s) = (\partial/\partial b)G(b;t,s).$$

The authors that come to mind that analyze $H$ are Bates and Gustafson [8] and Henderson [49]. $H$ is a $C_n$ solution of a BVP and it's sign can be analyzed using methods that will be discussed in the next section. For our simple second order example in this section, $H$ satisfies the BVP,

$$x''(t) = 0, \quad a < t < b, \quad x(a) = 0, \quad x(b) = -(\partial/\partial t)G(b;b,s) < 0.$$

The Krein-Rutman theory has been applied to $n$th order BVPs with conjugate conditions [25], right focal and in between two point problems [28], impulse problems [41] and [37], and functional differential equations [31]. It is interesting to note that in the impulse citations, the corresponding impulse function $(\partial/\partial b)I$ satisfies the same sign properties as $H$. Schmitt and Smith [64] have shown the Krein-Rutman theory applies to elliptic BVPs with Dirichlet boundary conditions as well. The shape of the domain is a carefully constructed rectangle so that a concept of disconjugacy can be defined.

### 3.3 Comparison Theorems and a Hierarchy of Boundary Value Problems

For equations of one independent variable, the difference of two Green's functions is, in fact, sufficiently smooth. The jump in the appropriate derivative to give the delta impulse effect adds out and so the difference is sufficiently smooth. This observation applies above to intuitively see that $H$ is sufficiently smooth as well. This observation also applies in the case of Laplace's elliptic equation. Here the delta impulse is generated with a logarithm term that adds out. In particular, the theory developed by Schmitt and Smith [64] can carry over to more irregularly shaped domains.

Before discussing a general problem, consider, for the sake of motivation, the second order equation,

$$x''(t) = 0, \quad 0 < t < 1.$$

Let $G(1;t,s)$ denote the Green's function associated with the conjugate conditions,

$$x(0) = 0, \quad x(1) = 0,$$

and let $G(2;t,s)$ denote the Green's function associated with the right focal conditions,

$$x(0) = 0, \quad x'(1) = 0.$$

Then, for $s \in (0,1)$, $h(t) = G(2;t,s) - G(1;t,s)$ is the solution of BVP,

$$x''(t) = 0, \quad 0 < t < 1, \quad x(0) = 0, \quad x(1) = G(2;1,s) < 0.$$

Note that

$$x''(t) = 0, \quad 0 < t < 1, \quad x(0) = 0, \quad x(1) < 0,$$

is a form of the maximum principle and so $x(t) < 0$, $0 < t < 1$.

To discuss the application of the maximum principle in this section, we shall appeal to an $n$th order ordinary differential operator

$$Lx = x^{(n)}(t) + \sum_{i=1}^{n} a_i(t)x^{(n-i)}(t), \quad a < t < b. \tag{3.25}$$

where each $a_i \in C[a,b]$. Let $k \in \{1,\dots,n-1\}$ be fixed; let $W$ denote the set of nonnegative integers and define $\Omega_{n-k} \subset W^{n-k}$ by

$$\Omega_{n-k} = \{\alpha = (\alpha_1,\dots,\alpha_{n-k}) : 0 \le \alpha_1 < \cdots < \alpha_{n-k} \le n-1\}. \tag{3.26}$$

Consider two-point boundary conditions of the form

$$x^{(l)}(a) = 0, \quad l = 0,\dots,k-1, \quad x^{(l)}(b) = 0, \quad l = \alpha_1,\dots,\alpha_{n-k}. \tag{3.27}$$

Note the boundary conditions (3.27) depend on $\alpha$ and on $b$. Also note that there is a natural partial order on $\Omega_{n-k}$:

$$\alpha \le \beta \Leftrightarrow \alpha_i \le \beta_i, \quad i = 1,\dots,n-k.$$

We have need to assume that the operator $L$ in (3.25) is right disfocal on an interval $[a,b_0]$ where $a < b \le b_0$; that is, the only solution of $Lx = 0$ satisfying $x^{(l)}(t_l) = 0$, $l = 0,\dots,n-1$, where $a \le t_0 \le \cdots \le t_{n-1} \le b_0$, is $x \equiv 0$. For a given $k,\alpha,b$, let $G(k,\alpha,b;t,s)$ denote the Green's function corresponding to the BVP, (3.25), (3.27). Eloe and Ridenhour [40] proved the following theorems.

**Theorem 3.6** *Let $k \in \{1,\dots,n-1\}$, $\alpha,\beta \in \Omega_{n-k}$, $\alpha < \beta$, $a < b \le b_0$. Then, for $l = 0,\dots,\alpha_1$,*

$$(-1)^{n-k}G_l(k,\beta,b;t,s) > (-1)^{n-k}G_l(k,\alpha,b;t,s) > 0, \tag{3.28}$$

*$(t,s) \in (0,b)^2$, and*

$$(-1)^{n-k}G_k(k,\beta,b;a,s) > (-1)^{n-k}G_k(k,\alpha,b;a,s) > 0, \tag{3.29}$$

*$s \in (0,b)$.*

**Theorem 3.7** *Let $k \in \{1, \ldots, n-1\}$, $\alpha, \beta \in \Omega_{n-k}$, $\alpha_{n-k} < n-1$. Assume $\alpha \le \beta$, $a < b_1 \le b_2 \le b_0$ and that one of the inequalities ($\alpha \le \beta$ or $b_1 \le b_2$) is strict. Then, for $l = 0, \ldots, \alpha_1$,*

$$(-1)^{n-k} G_l(k, \beta, b_2; t, s) > (-1)^{n-k} G_l(k, \alpha, b_1; t, s) > 0, \tag{3.30}$$

*$(t, s) \in (0, b_1)^2$, and*

$$(-1)^{n-k} G_k(k, \beta, b_2; a, s) > (-1)^{n-k} G_k(k, \alpha, b_1; a, s) > 0, \tag{3.31}$$

*$s \in (0, b_1)$.*

In particular, $G$ is monotone with respect to $b$; $G$ is monotone with respect to $\alpha$.

Peterson [60], [61], Elias [19], and Peterson and Ridenhour [62] have developed related inequalities for the two term operator $x^{(n)}(t) + p(t)x(t)$. Peterson considers the cases $p(t) < 0$ and $p(t) > 0$ independently and employs an adjoint equation argument. Elias considers the case $(-1)^{n-k}p(t) < 0$ and obtains a wealth of inequalities that contain Theorems 3.6 and 3.7. Elias appeals to special features of the two-term operator [21]. Peterson and Ridenhour address the case when $p$ is independent of sign. Again they appeal to an adjoint equation argument. To obtain Theorems 3.6 and 3.7 for the general operator $L$, one first observes the difference of two Green's functions to be $n$ times continuously differentiable. One then replaces the adjoint equation argument with a double induction on $k$ and $\sum_{i=1}^{n-k} \alpha_i$.

## 3.4 A Generalization of Concavity

In recent applications of cone theoretic fixed point theorems to boundary value problems (BVPs), Hadamard type inequalities that provide lower bounds for positive functions as a function of the supremum norm have been applied. A particular inequality to which we refer is as follows: if $y''(t) \le 0, 0 \le t \le 1$ and $y(t) \ge 0, 0 \le t \le 1$, then for $1/4 \le t \le 3/4$,

$$y(t) \ge ||y||/4, \tag{3.32}$$

where $||\cdot|| = \sup_{0 \le t \le 1} |y(t)|$. Analogous inequalities are valid for functions that satisfy the differential inequality piecewise. In particular, analogous inequalities are valid for associated Green's functions.

To obtain the above inequality, assume $y''(t) < 0, 0 \le t \le 1$, $y(0) \ge 0$, $y(1) \ge 0$. Let $t_0 \in (0, 1)$ be such that $||y|| = y(t_0)$. Construct the tent function

$$= \begin{cases} (||y||/t_0)t, & 0 \le t \le t_0, \\ (||y||/(t_0 - 1))(t - 1), & t_0 \le t \le 1. \end{cases}$$

$y$ lies above $p$ because of concavity and and one evaluates $p$ to obtain the estimate given in (3.32). The estimate in (3.32) was employed in [44] to obtain existence of solutions for BVPs for singular ordinary differential equations. Erbe and Wang [42] employed the estimate in (3.32) in conjunction with Krasnosel'skiĭ-Guo cone theoretic fixed point theorem and obtained existence of solutions in a cone when the nonlinear term satisfies superlinear or sublinear asymptotic behavior. This landmark paper, [42], has stimulated considerable research in the applications of fixed point methods. See an authoritative account of the recent activity in the area of time scales in Chapter 7 of [11].

Upon inspection, $p$ satisfies the differential equation piecewise. So, (3.32) has been very conducive to generalizations. The first such generalization was produced in [32]. Let $n \geq 2$ be an integer, and assume $k \in \{1, \ldots, n-1\}$. Assume

$$(-1)^{(n-k)} y^{(n)} \geq 0, \quad 0 \leq t \leq 1, \tag{3.33}$$

$$y^{(j)}(0) = 0, \quad j = 0, \ldots, k-1, \quad y^{(j)}(1) = 0, \quad j = 0, \ldots, n-k-1. \tag{3.34}$$

Then for $1/4 \leq t \leq 3/4$,

$$y(t) \geq ||y||/4^m, \tag{3.35}$$

where $m = \max\{k, n-k\}$. With the development of (3.35), the work of Erbe and Wang [42] with superlinear or sublinear growth readily carried over to the two-point conjugate type boundary value, (3.33), (3.34) [33]. The method to obtain (3.35) is merely a generalization of the method to obtain (3.32). First, assume strict differential inequality in (3.33). Let $t_0 \in (0,1)$ be such that $||y|| = y(t_0)$. Construct the tent function

$$p(t) = \begin{cases} (||y||/(t_0^k))t^k, & 0 \leq t \leq t_0, \\ (||y||/(t_0-1)^{n-k})(t-1)^{n-k}, & t_0 \leq t \leq 1. \end{cases}$$

One cannot appeal to generalized concavity to argue that $y$ lies above $p$; using contradiction and repeated applications of Rolle's theorem, one proves that $y$ lies above $p$ and then one evaluates $p$ to obtain the estimate given in (3.35).

Following [32], (3.32) has been generalized in various ways with applications to multipoint BVPs [36], discrete problems, [22], and Kiguradze type inequalities [20].

## 3.5 Unbounded Domains and Unique Green's Functions

We will close the paper with a discussion on a singular BVP,

$$x''(t) + q(t)x(t) = f(t, x(t)), \quad t \in \mathbb{R}^+, \tag{3.36}$$

$$x(0) = x_0, \quad x(t) \text{ bounded on } \mathbb{R}^+, \tag{3.37}$$

where $x_0$ is real, $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is continuous, $q : \mathbb{R}^+ \to \mathbb{R}^-$ is continuous, and $q(t) \leq -c^2 < 0$, $t \in \mathbb{R}^+$, for some $c^2 > 0$. We model the singular BVP based on the work of Bebernes and Jackson [9]. Assume that

$$x_1 \leq x_2 \Rightarrow f(t, x_1) \leq f(t, x_2), \quad t \in R^+. \tag{3.38}$$

A unique limiting $L_1$ Green's function for this singular BVP exists. A method of upper and lower solutions can be applied to obtain a fundamental existence of solutions theorem for the BVP, (3.36), (3.37). The condition, (3.38), gives uniqueness of solutions and in fact yields that (3.22) implies (3.21). So, the quasilinearization method has been developed for the BVP, (3.36), (3.37). The details of this development can be found in [23].

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## References


[1] R.P. AGARWAL, *Focal Boundary Value Problems for Differential and Difference Equations*, Mathematics and its Applications 436 Kluwer Academic Publishers, 1998.

[2] R.P. AGARWAL AND P.J.Y. WONG, *Lidstone polynomials and boundary value problems*, Comput. Math. Appl., **17** (1989), 1397–1421.

[3] R.P. AGARWAL, *Difference Equations and Inequalities*, Marcel Dekker, Inc. New York, 1992.

[4] H. AMANN, *Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces*, SIAM Rev., **18** (1976), 620–709.

[5] N. AZBELEV, V. MAKSIMOV, AND L. RAKHMATULLINA, *Introduction to the Theory of Functional Differential Equations*, "Nauka", Moscow, 1991. (Russian)

[6] N. AZBELEV, V. MAKSIMOV, L. RAKHMATULLINA, *Introduction to the Theory of Linear Functional-Differential Equations*, Advanced Series in Mathematical Science and Engineering, World Federation Publishers Company, Atlanta, GA, 1995.

[7] D. BAINOV AND A. DOMOSHNITSKY, *Theorems on differential inequalities for second order functional differential equations*, Glas. Mat. Ser. III, **29** (**49**) (1994), 275–289.

[8] P.W. BATES AND G.B. GUSTAFSON, *Maximization of Green's functions over classes of multipoint boundary value problems*, SIAM J. Math. Anal., **7** (1976), 858–871.

[9] J. BEBERNES AND L. JACKSON, *Infinite interval boundary value problems for $y'' = f(t,y)$*, Duke Math. J., **34** (1967), 39–47.

[10] M. BOHNER AND A. PETERSON, *Dynamic Equations on Time Scales*, Birkhäuser, Boston, 2001.

[11] M. BOHNER AND A. PETERSON, *Advances in Dynamic Equations on Time Scales*, Birkhäuser, Boston, 2003.

[12] A. CABADA, E. LIZ AND S. LOIS, *Green's function and maximum principle for higher order ordinary differential equation with impulse*, Rocky Mountain J. Math., **30** (2000), 435–446.

[13] E. CODDINGTON AND N. LEVINSON, *Theory of Ordinary Differential Equations*, McGraw-Hill, New York, 1955.

[14] L. COLLATZ, *Functional Analysis and Numerical Mathematics*, Academic Press, New York, 1966.

[15] W. COPPEL, *Disconjugacy*, Lecture Notes in Mathematics, Vol. 220, Springer-Verlag, New York/Berlin, 1971.

[16] K. DEIMLING, *Nonlinear Functional Analysis*, Springer-Verlag, New York, 1985.

[17] A DOMOSHNITSKY, *Preserving the sign of the Green function of a two-point boundary value problem for an nth order functional differential equation* (Russian), Differ. Uravn. **25** (1989), no. 6, 934–937, 1097; translation in Differential Equations **25** (1989), no. 6, 666–669.

[18] J. EHME, P. ELOE AMD J. HENDERSON, *Upper and lower solution methods for fully nonlinear boundary value problems*, J. Differential Equations, **180** (2002), 51–64.

[19] U. ELIAS, *Green's functions for a nondisconjugate differential operator*, J. Differential Equations, **37** (1980), 318–350.

[20] U. ELIAS, *Generalizations of an inequality of Kiguradze*, J. Math. Anal. Appl., **97** (1983), 277–290.

[21] U. ELIAS, *Oscillation Theory of Two-Term Differential Equations*, Mathematics and its Applications **396**, Kluwer Academic Publishers, Dordrecht, 1997.

[22] P. ELOE, *A generalization of concavity for finite differences*, Comput. Math. Appl., **36** (1998), 109–113.

[23] P. ELOE, *The quasilinearization method on an unbounded domain*, Proc. Amer. Math. Soc., **131** (2003), 1481–1488.

[24] P. ELOE AND Y. GAO, *The method of quasilinearization and a three-point boundary value problem*, J. Korean Math. Soc., **39** (2002), 319–330.

[25] P. ELOE, D. HANKERSON, AND J. HENDERSON, *Positive solutions and conjugate points for multipoint boundary value problems*, J. Differential Equations, **95** (1992), 20–32.

[26] P. ELOE AND L.J. GRIMM, *Monotone iteration and Green's functions for boundary value problems*, Proc. Amer. Math. Soc., **78** (1980), 533–538.

[27] P. ELOE AND L.J. GRIMM, *Differential systems and multipoint boundary value problems*, ZAMM, **62** (1982), 630–632.

[28] P. ELOE, D. HANKERSON, AND J. HENDERSON, *Positive solutions and j-focal points for two point boundary value problems*, Rocky Mountain J. Math., **22** (1992), 1283–1293.

[29] P. ELOE AND J. HENDERSON, *Focal points and comparison theorems for a class of two point boundary value problems*, J. Differential Equations, **102** (1993), 375–386.

[30] P. ELOE AND J. HENDERSON, *Differential inequalities for a singular boundary value problem*, World Sci. Ser. Appl. Anal **3** World Scientific Publishing Company (1994), 197–205.

[31] P. ELOE AND J. HENDERSON, *Positive solutions and conjugate points for a class of linear functional differential equations*, in Boundary Value Problems for Functional Differential Equations (ed. J. Henderson), World Scientific, Singapore, (1995), 131–142.

[32] P. ELOE AND J. HENDERSON, *Inequalities based on a generalization of concavity*, Proc. Amer. Math. Soc., **125** (1997), 2103–2108.

[33] P. ELOE AND J. HENDERSON, *Singular nonlinear $(k, n-k)$ conjugate boundary value problems*, J. Differential Equations, **133** (1997), 136–151.

[34] P. ELOE AND J. HENDERSON, *A boundary value problem for a system of ordinary differential equations with impulse effects*, Rocky Mountain J. Math., **27** (1997), 785–799.

[35] P. ELOE AND J. HENDERSON, *Positive solutions of boundary value problems for ordinary differential equations with impulse*, Dynam. Contin. Discrete Impuls.ive Systems, **4** (1998), 285–294.

[36] P. ELOE AND J. HENDERSON, *Inequalities for solutions of multipoint boundary value problems*, Rocky Mountain J. Math., **29** (1999), 821–829.

[37] P. ELOE, J. HENDERSON AND B. THOMPSON, *Extremal points for impulsive Lidstone boundary value problems*, Math. Comput. Modelling, **32** (2000), 687–698.

[38] P. ELOE AND E. KAUFMANN, *A unique limiting Green's function for a class of singular boundary value problems*, Comput. Math. Appl., **28** (1994), 93–99.

[39] P. ELOE AND E. KAUFMANN, *A singular boundary value problem for a right disfocal differential operator*, Dynam. Systems Appl., **5** (1996), 174–182.

[40] P. ELOE AND J. RIDENHOUR, *Sign properties of Green's functions for a family of two point boundary value problems*, Proc. Amer. Math. Soc., **120** No. 2 (1994), 443–452.

[41] P. ELOE AND M. SOKOL, *Positive solutions and conjugate points for a boundary value problem with impulse*, Dynam. Systems Appl., **7** (1998), 441–450.

[42] L. ERBE AND H. WANG, *On the existence of positive solutions of ordinary differential equations*, Proc. Amer. Math. Soc., **120** (1994), 743–748.

[43] T. FORT, *Finite Differences*, Oxford Univ. Press, Oxford, 1948.

[44] J. GATICA, V. OLIKER, AND P. WALTMAN, *Singular nonlinear boundary value problems for second-order ordinary differential equations*, J. Differential Equations, **79** (1989), 62–78.

[45] D. GUO AND V. LAKSHMIKANTHAM, *Nonlinear Problems in Abstract Cones*, Academic Press, Boston, 1988.

[46] C. GUPTA, *Solvability of a three-point nonlinear boundary value problem for a second order ordinary differential equation*, J. Math. Anal. Appl., **168** (1992), 540–551.

[47] P. HARTMAN, *Difference equations: disconjugay, principle solutions, Green's functions, complete monotonicity*, Trans. Amer. Math. Soc., **246** (1978), 1–30.

[48] P. HARTMAN, *Principal solutions of disconjugate nth order linear differential equations*, Amer. J. Math., **91** (1969), 306–362.

[49] J. HENDERSON, *Disconjugacy, disfocality, and differentiation with respect to boundary conditions*, J. Math. Anal. Appl., **121** (1987), 1–9.

[50] S. HILGER, *Ein Maßkettenkalkül mit Anwendung auf Zentrumsmannigfaltigkeiten*, Ph.D. thesis, Universität Würzburg, Germany, 1988.

[51] L. JACKSON, *Boundary value problems for ordinary differential equations* in "Studies in Ordinary Differential Equations," MAA Studies in Mathematics (J.K. Hale, Ed.), Vol. 14, Mathematical Association of America, Washington, D.C., 1977.

[52] M.A. KRASNOSEL'SKIĬ, *Positive Solutions of Operator equations*, Fizmatgiz, Moscow, 1962; English Translation: Noordhoff, Groningen, The Netherlands, 1964.

[53] M.G. KREIN AND M.A. RUTMAN, *Linear operators leaving a cone invariant in a Banach space* in American Mathematical Society Translations, Series 1, Vol. 10, pp. 199–325, Amer. Math. Soc., Providence, R.I. 1962.

[54] V. LAKSHMIKANTHAM AND A. VATSALA, *Generalized Quasilinearization for Nonlinear Problems*, Mathematics and its Applications 4**40**, Kluwer Academic Publishers, Dordrecht, 1998.

[55] A. JU. LEVIN, *Non-oscillation of solutions of the equations* $x^{(n)} + p_1(t)x^{(n-1)} + \cdots + p_n(t)x = 0$, Russian Math. Surveys, 24, 1969, pp. 43–99.

[56] G.J. LIDSTONE, *Notes on the extension of Aitken's theorem (for polynomial interpolation) to the Everett types*, Proc. Edinburgh Math. Soc., **2** (1929), 16–19.

[57] A. LOMTATIDZE, *On a nonlocal boundary value problem for second order linear ordinary differential equations*, J. Math. Anal. Appl., **193** (1995), 889–908.

[58] Z. NEHARI, *Disconjugate linear differential operators*, Trans. Amer. Math. Soc., **129** (1967), 500–516.

[59] R. D. NUSSBAUM, *Periodic solutions of some nonlinear integral equations* in Proceedings Internatl. Conf. on Differential Equations, Gainesville, FL, 1976.

[60] A. PETERSON, *Green's functions for focal type boundary value problems*, Rocky Mountain J. Math, **9** (1979), 721–732.

[61] A. PETERSON, *Focal Green's functions for fourth-order differential equations*, J. Math. Anal. Appl., **75** (1980), 602–610.

[62] A. PETERSON AND J. RIDENHOUR, *Comparison theorems for Green's functions for focal boundary value problems*, Recent Trends in Ordinary Differential Equations, World Sci. Ser. Appl. Anal. Vol. 1, World Sci. Publ., Teaneck, NJ, 1992, pp. 493–506.

[63] M.H. PROTTER AND H.F. WEINBERGER, *Maximum Principles in Differential Equations*, Prentice-Hall Inc., Englewood Cliffs, N.J., 1967.

[64] K. SCHMITT AND H. L. SMITH, *Positive solutions and conjugate points for systems of differential equations*, Nonlinear Anal., **2** (1978), 93–105.

[65] V. ŠEDA, *Two remarks on boundary value problems for ordinary differential equations*, J. Differential Equations, **26** (1977), 278–290.

[66] J. WERNER, *Einschließungssätze bei nichtlinearen gewöhnlichen Randwertaufgaben und erzwungenen Schwingungen*, Numer. Math., **13** (1969), 24–38.

[67] E. ZEIDLER, *Nonlinear Functional Analysis and its Applications*, Vol. I, Springer-Verlag, New York, 1985.