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A Survey on the Oscillation of Solutions of First Order Delay Difference Equations

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ABSTRACT

In this paper, a survey of the most interesting results on the oscillation of all solutions of the first order delay difference equation of the form

$$x_{n+1} - x_n + p_n x_{n-k} = 0, \quad n = 0, 1, 2, ...,$$

where $\{p_n\}$ is a sequence of nonnegative real numbers and k is a positive integer is presented, especially in the case when neither of the well-known oscillation conditions

$$\limsup_{n\to\infty} \sum_{i=n-k}^{n} p_i > 1 \quad \text{and} \quad \liminf_{n\to\infty} \frac{1}{k} \sum_{i=n-k}^{n-1} p_i > \frac{k^k}{(k+1)^{k+1}}$$

is satisfied.

RESUMEN

En este artículo, hacemos una revisión de los resultados más interesantes sobre oscilaciones de las soluciones de la ecuación en diferencias de primer orden

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con retardo, de la forma

$$x_{n+1} - x_n + p_n x_{n-k} = 0, \quad n = 0, 1, 2, ...,$$

en donde $\{p_n\}$ es una sucesión de números reales no negativos, k es un entero positivo, en especial cuando ni siquiera se satisfacen las conocidas condiciones de oscilación

$$\limsup_{n\to\infty} \sum_{i=n-k}^{n} p_i > 1 \quad \text{y} \quad \liminf_{n\to\infty} \frac{1}{k} \sum_{i=n-k}^{n-1} p_i > \frac{k^k}{(k+1)^{k+1}}$$

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1 Introduction

In the last few decades the oscillation theory of delay differential equations has been extensively developed. The oscillation theory of discrete analogues of delay differential equations has also attracted growing attention in the recent few years. The reader is referred to [1-12,14-16, 21, 22, 24-26, 29-46] and the references cited therein. In particular, the problem of establishing sufficient conditions for the oscillation of all solutions of the delay difference equation

$$x_{n+1} - x_n + p_n x_{n-k} = 0, \quad n = 0, 1, 2, ..., \tag{1.1}$$

where $\{p_n\}$ is a sequence of nonnegative real numbers and k is a positive integer, has been the subject of many recent investigations. See, for example, [2-12, 14, 21, 22, 24-26, 29-39, 42-46] and the references cited therein. Strong interest in (1.1) is motivated by the fact that it represents a discrete analogue of the delay differential equation (see [13, 17-20, 23, 27, 28] and the references cited therein)

$$x'(t) + p(t)x(t-\tau) = 0, \quad p(t) \geq 0, \quad \tau > 0. \tag{1.2}$$

By a solution of (1.1) we mean a sequence $\{x_n\}$ which is defined for $n \geq -k$ and which satisfies (1.1) for $n \geq 0$. A solution $\{x_n\}$ of (1.1) is said to be *oscillatory* if the terms x_n of the solution are not eventually positive or eventually negative. Otherwise the solution is called *nonoscillatory*.

For convenience, we will assume that inequalities about values of sequences are satisfied eventually for all large n.

In this paper, our main purpose is to present the state of the art on the oscillation of solutions to (1.1) especially in the case that the oscillation conditions (see below)

$$\limsup_{n\to\infty} \sum_{i=n-k}^{n} p_i > 1 \quad \text{and} \quad \liminf_{n\to\infty} \frac{1}{k} \sum_{i=n-k}^{n-1} p_i > \frac{k^k}{(k+1)^{k+1}}$$

are not satisfied.

2 Oscillation criteria for Eq. (1.1)

In 1981, Domshlak [7] was the first who studied this problem in the case where $k = 1$. Then, in 1989, Erbe and Zhang [14] established the following oscillation criteria for (1.1).

Theorem 2.1.([14]) *Assume that*

$$\beta := \liminf_{n\to\infty} p_n > 0 \quad \textit{and} \quad \limsup_{n\to\infty} p_n > 1 - \beta \tag{C_1}$$

Then all solutions of (1.1) *oscillate.*

Theorem 2.2.([14]) *Assume that*

$$\liminf_{n\to\infty} p_n > \frac{k^k}{(k+1)^{k+1}}. \tag{C_2}$$

Then all solutions of (1.1) *oscillate.*

Theorem 2.3.([14]) *Assume that*

$$A := \limsup_{n\to\infty} \sum_{i=n-k}^{n} p_i > 1. \tag{C_3}$$

Then all solutions of (1.1) *oscillate.*

In the same year 1989 Ladas, Philos and Sficas [22] proved the following theorem.

Theorem 2.4.([22]) *Assume that*

$$\liminf_{n\to\infty} \frac{1}{k} \sum_{i=n-k}^{n-1} p_i > \frac{k^k}{(k+1)^{k+1}}. \tag{C_4}$$

Then all solutions of (1.1) *oscillate.*

Therefore they improved the condition (C_2) by replacing the p_n of (C_2) by the arithmetic mean of the terms $p_{n-k}, ..., p_{n-1}$ in (C_4).

Concerning the constant $\frac{k^k}{(k+1)^{k+1}}$ in (C_2) and (C_4) it should be empasized that, as it is shown in [14], if

$$\sup p_n < \frac{k^k}{(k+1)^{k+1}}, \tag{N_1}$$

then (1.1) has a nonoscillatory solution.

In 1990, Ladas [21] conjectured that Eq. (1.1) has a nonoscillatory solution if

$$\frac{1}{k} \sum_{i=n-k}^{n-1} p_i \le \frac{k^k}{(k+1)^{k+1}}$$

holds eventually. However this conjecture is false and a counterexample was given in 1994 by Yu, Zhang and Wang [43].

It is interesting to establish sufficient conditions for the oscillation of all solutions of (1.1) when (C_3) and (C_4) are not satisfied. (For the equation (1.2) this question has been investigated by many authors, see, for example, [13, 17-20, 23, 27, 28] and the references cited therein.)

In 1993, Yu, Zhang and Qian [42] and Lalli and Zhang [24], trying to improve (C_3), established the following (false) sufficient oscillation conditions for (1.1)

$$0 < \alpha := \liminf_{n\to\infty} \sum_{i=n-k}^{n-1} p_i \le \left(\frac{k}{k+1}\right)^{k+1} \quad \text{and} \quad A > 1 - \frac{\alpha^2}{4} \tag{F_1}$$

and

$$\sum_{i=n-k}^{n} p_i \ge d > 0 \text{ for large } n \text{ and } A > 1 - \frac{d^4}{8}\left(1 - \frac{d^3}{4} + \sqrt{1 - \frac{d^3}{2}}\right)^{-1} \tag{F_2}$$

respectively.

Unfortunately, the above conditions (F_1) and (F_2) are not correct. This is due to the fact that they are based on the following (false) discrete version of Koplatadze-Chanturia Lemma. (See [6] and [10]).

Lemma A (False). *Assume that* $\{x_n\}$ *is an eventually positive solution of* (1.1) *and that*

$$\sum_{i=n-k}^{n} p_i \ge M > 0 \quad \textit{for large } n. \tag{1.3}$$

Then

$$x_n > \frac{M^2}{4} x_{n-k} \quad \textit{for large } n.$$

As one can see, the erroneous proof of Lemma A is based on the following (false) statement. (See [6] and [10]).

Statement A (False). *If* (1.3) *holds, then for any large* N*, there exists a positive integer* n *such that* $n-k \le N \le n$ *and*

$$\sum_{i=n-k}^{N} p_i \ge \frac{M}{2}\ , \quad \sum_{i=N}^{n} p_i \ge \frac{M}{2}.$$

It is obvious that all the oscillation results which have made use of the above Lemma A or Statement A are incorrect. For details on this problem see the paper by Cheng and Zhang [6].

Here it should be pointed out that the following statement (see [22], [31]) is correct and it should not be confused with the Statement A.

Statement 2.1.([22], [31]) *If*

$$\sum_{i=n-k}^{n-1} p_i \ge M > 0 \quad \textit{for large } n, \tag{1.4}$$

then for any large n, there exists a positive integer n^* *with* $n-k \le n^* \le n$ *such that*

$$\sum_{i=n-k}^{n^*} p_i \ge \frac{M}{2}\ , \quad \sum_{i=n^*}^{n} p_i \ge \frac{M}{2}.$$

In 1995, Stavroulakis [31], based on this correct Statement 2.1, proved the following theorem.

Theorem 2.5.([31]) *Assume that*

$$0 < \alpha \le \left(\frac{k}{k+1}\right)^{k+1}$$

and

$$\limsup_{n\to\infty} p_n > 1 - \frac{\alpha^2}{4}. \tag{C_5}$$

Then all solutions of (1.1) *oscillate.*

In 1999, Domshlak [10] and in 2000, Cheng and Zhang [6] established the following lemmas, respectively, which may be looked upon as (exact) discrete versions of Koplatadze-Chanturia Lemma.

Lemma 2.1.([10]) *Assume that* $\{x_n\}$ *is an eventually positive solution of* (1.1) *and that the condition* (1.4) *holds. Then*

$$x_n > \frac{M^2}{4} x_{n-k} \quad \text{for large} \ \ n. \tag{1.5}$$

Lemma 2.2.([6]) *Assume that* $\{x_n\}$ *is an eventually positive solution of* (1.1) *and that the condition* (1.4) *holds. Then*

$$x_n > M^k x_{n-k} \ \ \text{for large} \ \ n. \tag{1.6}$$

Based on these lemmas the following theorem was established in [32].

Theorem 2.6.([32]) *Assume that*

$$0 < \alpha \le \left(\frac{k}{k+1}\right)^{k+1}.$$

Then either one of the conditions

$$\limsup_{n\to\infty} \sum_{i=n-k}^{n-1} p_i > 1 - \frac{\alpha^2}{4} \tag{C_6}$$

or

$$\limsup_{n\to\infty} \sum_{i=n-k}^{n-1} p_i > 1 - \alpha^k \tag{C_7}$$

implies that all solutions of (1.1) *oscillate.*

Remark 2.1.([32]) From the above theorem it is now clear that

$$0 < \alpha := \liminf_{n\to\infty} \sum_{i=n-k}^{n-1} p_i \le \left(\frac{k}{k+1}\right)^{k+1} \quad \text{and} \quad \limsup_{n\to\infty} \sum_{i=n-k}^{n-1} p_i > 1 - \frac{\alpha^2}{4}$$

is the correct oscillation condition by which the (false) condition (F_1) should be replaced.

Remark 2.2.([32]) Observe the following:

(i) When $k = 1, 2$,

$$\alpha^k > \frac{\alpha^2}{4},$$

(since, from the above mentioned conditions, it makes sense to investigate the case when $\alpha < \left(\frac{k}{k+1}\right)^{k+1}$) and therefore condition (C_6) implies (C_7).

(ii) When $k = 3$,

$$\alpha^3 > \frac{\alpha^2}{4} \text{ when } \alpha > \frac{1}{4}$$

while

$$\alpha^3 < \frac{\alpha^2}{4} \text{ when } \alpha < \frac{1}{4}.$$

So in this case the conditions (C_6) and (C_7) are independent.

(iii) When $k \ge 4$,

$$\alpha^k < \frac{\alpha^2}{4},$$

and therefore condition (C_7) implies (C_6).

(iv) When $k < 12$ condition (C_6) or (C_7) implies (C_3).

(v) When $k \ge 12$ condition (C_6) may hold but condition (C_3) may not hold.

We illustrate these by the following examples.

Example 2.1.([32]) Consider the equation

$$x_{n+1} - x_n + p_n x_{n-3} = 0, \quad n = 0, 1, 2, ...,$$

where

$$p_{2n} = \frac{1}{10}, \quad p_{2n+1} = \frac{1}{10} + \frac{64}{95}\sin^2\frac{n\pi}{2}, \qquad n = 0, 1, 2, ...$$

Here $k = 3$ and it is easy to see that

$$\alpha = \liminf_{n\to\infty} \sum_{i=n-3}^{n-1} p_i = \frac{3}{10} < \left(\frac{3}{4}\right)^4$$

and

$$\limsup_{n\to\infty} \sum_{i=n-3}^{n-1} p_i = \frac{3}{10} + \frac{64}{95} > 1 - \alpha^3.$$

Thus condition (C_7) is satisfied and therefore all solutions oscillate. Observe, however, that condition (C_6) is not satisfied.

If, on the other hand, in the above equation

$$p_{2n} = \frac{8}{100}, \quad p_{2n+1} = \frac{8}{100} + \frac{746}{1000}\sin^2\frac{n\pi}{2}, \qquad n = 0, 1, 2, ...,$$

then it is easy to see that

$$\alpha = \liminf_{n\to\infty} \sum_{i=n-3}^{n-1} p_i = \frac{24}{100} < \left(\frac{3}{4}\right)^4$$

and

$$\limsup_{n\to\infty} \sum_{i=n-3}^{n-1} p_i = \frac{24}{100} + \frac{746}{1000} > 1 - \frac{\alpha^2}{4}.$$

In this case condition (C_6) is satisfied and therefore all solutions oscillate. Observe, however, that condition (C_7) is not satisfied.

Example 2.2.([32]) Consider the equation

$$x_{n+1} - x_n + p_n x_{n-16} = 0, \quad n = 0, 1, 2, ...,$$

where

$$p_{17n} = p_{17n+1} = \quad = p_{17n+15} = \frac{2}{100}, \; p_{17n+16} = \frac{2}{100} + \frac{655}{1000}, \quad n = 0, 1, 2, ...$$

Here $k = 16$ and it is easy to see that

$$\alpha = \liminf_{n\to\infty} \sum_{i=n-16}^{n-1} p_i = \frac{32}{100} < \left(\frac{16}{17}\right)^{17}$$

and

$$\limsup_{n\to\infty} \sum_{i=n-16}^{n-1} p_i = \frac{32}{100} + \frac{655}{1000} = 0.975 > 1 - \frac{\alpha^2}{4}.$$

We see that condition (C_6) is satisfied and therefore all solutions oscillate. Observe, however, that

$$A = \limsup_{n\to\infty} \sum_{i=n-16}^{n} p_i = \frac{34}{100} + \frac{655}{1000} = 0.995 < 1;$$

that is, condition (C_3) is not satisfied.

In 1995, Chen and Yu [2], following the above mentioned direction, derived a condition which formulated in terms of α and A says that all solutions of (1.1) oscillate if $0 < \alpha \leq \frac{k^{k+1}}{(k+1)^{k+1}}$ and

$$A > 1 - \frac{1 - \alpha - \sqrt{1 - 2\alpha - \alpha^2}}{2}. \qquad (C_8)$$

In 1998, Domshlak [9], studied the oscillation of all solutions and the existence of nonoscillatory solution of (1.1) with r -periodic positive coefficients $\{p_n\}, p_{n+r} = p_n$. It is very important that in the following cases where $\{r = k\}, \{r = k+1\}$, $\{r = 2\}, \{k = 1, r = 3\}$ and $\{k = 1, r = 4\}$ the results obtained are stated in terms of necessary and sufficient conditions and it is very easy to check them.

In 2000, Tang and Yu [38] improved condition (C_8) to the condition

$$A > \lambda_2^k(1 - k\ln\lambda_2) - \frac{1-\alpha-\sqrt{1-\alpha-\alpha^2}}{2}, \qquad (C_9)$$

where λ_2 is the greater root of the algebraic equation

$$k\lambda^k(1-\lambda) = \alpha.$$

In 2000, Shen and Stavroulakis [30], using new techniques, improved the previous results.

Lemma 2.3.([30]) *Let the number $M \geq 0$ be such that*

$$\sum_{i=1}^{k} p_{n-i} \geq M \quad \text{for large } n.$$

Assume that (1.1) *has an eventually positive solution $\{x_n\}$. Then $M \leq k^{k+1}/(k+1)^{k+1}$ and*

$$\limsup_{n\to\infty} \frac{x_{n-k}}{x_n} \prod_{i=1}^{k}\sum_{j=1}^{k} p_{n-i+j} \leq [\overline{d}(M)]^k,$$

where $\overline{d}(M)$ is the greater real root of the algebraic equation

$$d^{k+1} - d^k + M^k = 0, \quad \text{on } [0,1].$$

Note that from this lemma we obtain a better and perhaps optimal bound which essentially improves (1.6).

Theorem 2.7.([30]) *Assume that $0 \leq \alpha \leq k^{k+1}/(k+1)^{k+1}$ and that there exists an integer $l \geq 1$ such that*

$$\limsup_{n\to\infty}\left\{\sum_{i=1}^{k} p_{n-i} + [\overline{d}(\alpha)]^{-k}\prod_{i=1}^{k}\sum_{j=1}^{k} p_{n-i+j} + \sum_{m=0}^{l-1}[d(\alpha/k)]^{-(m+1)k}\sum_{i=1}^{k}\prod_{j=0}^{m+1} p_{n-jk-i}\right\} > 1, \qquad (C_{10})$$

where $\overline{d}(\alpha)$ and $d(\alpha/k)$ are the greater real roots of the equations

$$d^{k+1} - d^k + \alpha^k = 0$$

and

$$d^{k+1} - d^k + \alpha/k = 0,$$

respectively. Then all solutions of (1.1) *oscillate.*

Notice that when $k = 1$, $d(\alpha) = \overline{d}(\alpha) = (1+\sqrt{1-4\alpha})/2$ (see [30]), and so condition (C_{10}) reduces to

$$\limsup_{n\to\infty}\left\{Cp_n + p_{n-1} + \sum_{m=0}^{l-1} C^{m+1}\prod_{j=0}^{m+1} p_{n-j-1}\right\} > 1, \qquad (C_{11})$$

where $C = 2/(1+\sqrt{1-4\alpha})$, $\alpha = \liminf_{n\to\infty} p_n$. Therefore, from Theorem 2.7, we have the following corollary.

Corollary 2.1.([30]) *Assume that* $0 \le \alpha \le 1/4$ *and that* (C_{11}) *holds. Then all solutions of the equation*

$$x_{n+1} - x_n + p_n x_{n-1} = 0 \qquad (1.7)$$

oscillate.

A condition derived from (C_{11}) and which can be easier verified, is given in the next corollary.

Corollary 2.2.([30]) *Assume that* $0 \le \alpha \le 1/4$ *and that*

$$\limsup_{n\to\infty} p_n > \left(\frac{1+\sqrt{1-4\alpha}}{2}\right)^2. \qquad (C_{12})$$

Then all solutions of (1.7) *oscillate.*

Remark 2.2.([30]) Observe that when $\alpha = 1/4$, condition (C_{12}) reduces to

$$\limsup_{n\to\infty} p_n > 1/4$$

which can not be improved in the sense that the lower bound $1/4$ can not be replaced by a smaller number. Indeed, by condition (N_1) (Theorem 2.3 in [14]), we see that (1.7) has a nonoscillatory solution if

$$\sup p_n < 1/4.$$

Note, however, that even in the critical state where $\lim_{n\to\infty} p_n = 1/4$, (1.7) can be either oscillatory or nonoscillatory. For example, if $p_n = \frac{1}{4} + \frac{c}{n^2}$ then (1.7) will be oscillatory in case $c > 1/4$ and nonoscillatory in case $c < 1/4$ (the Kneser-like theorem, [8]).

Example 2.2.([30]) Consider the equation

$$x_{n+1} - x_n + \left(\frac{1}{4} + a\sin^4\frac{n\pi}{8}\right)x_{n-1} = 0,$$

where $a > 0$ is a constant. It is easy to see that

$$\liminf_{n\to\infty} p_n = \liminf_{n\to\infty}\left(\frac{1}{4} + a\sin^4\frac{n\pi}{8}\right) = \frac{1}{4},$$

$$\limsup_{n\to\infty} p_n = \limsup_{n\to\infty}\left(\frac{1}{4}+a\sin^4\frac{n\pi}{8}\right)=\frac{1}{4}+a.$$

Therefore, by Corollary 2.2, all solutions oscillate. However, none of the conditions $(C_1)-(C_9)$ is satisfied.

The following corollary concerns the case when $k>1$.

Corollary 2.3.([30]) *Assume that* $0\le \alpha \le k^{k+1}/(k+1)^{k+1}$ *and that*

$$\limsup_{n\to\infty}\sum_{i=n-k}^{n-1} p_i > 1-[\overline{d}(\alpha)]^{-k}\alpha^k-\frac{k[d(\alpha/k)]^{-k}\beta^2}{1-[d(\alpha/k)]^{-k}\beta}, \qquad (C_{13})$$

where $\overline{d}(\alpha), d(\alpha/k)$ *are as in Theorem* 2.7. *Then all solutions of* (1.1) *oscillate.*

In 2000, Shen and Luo [29] proved the following theorems.

Theorem 2.8.([29]) *Assume that there exists some positive integer* l *such that*

$$\limsup_{n\to\infty}\left\{\sum_{i=0}^{k}p_{n-i}+\prod_{i=0}^{k}\sum_{j=1}^{k}p_{n-i+j}+\sum_{m=0}^{l-1}\sum_{i=1}^{k}\prod_{j=0}^{m+1}p_{n-jk-i}\right\}>1. \qquad (C_{14})$$

Then all solutions of (1.1) *oscillate.*

Theorem 2.9.([29]) *Assume that there exists some positive integer* l *such that*

$$\limsup_{n\to\infty}\left\{\sum_{i=1}^{k}p_{n-i}+\prod_{i=1}^{k}\sum_{j=1}^{k}p_{n-i+j}+\sum_{m=0}^{l-1}\sum_{i=1}^{k}\prod_{j=0}^{m+1}p_{n-jk-i}\right\}>1. \qquad (C_{15})$$

Then all solutions of (1.1) *oscillate.*

From Theorem 2.8 and Theorem 2.9 the following corollaries are derived.

Corollary 2.4. ([29]) *Assume that*

$$A>1-\alpha^{k+1}-\frac{k\beta^2}{1-\beta}. \qquad (C_{16})$$

Then all solutions of (1.1) *oscillate.*

Corollary 2.5. ([29]) *Assume that*

$$\limsup_{n\to\infty}\sum_{i=n-k}^{n-1}p_i>1-\alpha^k-\frac{k\beta^2}{1-\beta}. \qquad (C_{17})$$

Then all solutions of (1.1) *oscillate.*

Following this historical (and chronological) review we also mention that in the case where

$$\frac{1}{k}\sum_{i=n-k}^{n-1}p_i\ge\frac{k^k}{(k+1)^{k+1}} \quad \text{and} \quad \lim_{n\to\infty}\frac{1}{k}\sum_{i=n-k}^{n-1}p_i=\frac{k^k}{(k+1)^{k+1}},$$

the oscillation of (1.1) has been studied in 1994 by Domshlak [8] and in 1998 by Tang [33] (see also Tang and Yu [35]). In a case when p_n is asymptotically close to one of the periodic critical states, unimprovable results about oscillation preperties of the equation

$$x_{n+1} - x_n + p_n x_{n-1} = 0$$

were obtained by Domshlak in 1999 [11] and in 2000 [12].

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