CUBO *A Mathematical Journal*
Vol. 7, № 2, (139 - 169). August 2005.

Differential Forms and/or Multi-vector Functions

F. Brackx
Department of Mathematical Analysis
Ghent University. Galglaan 2, B-9000 Gent, Belgium
fb@cage.ugent.be

R. Delanghe
Department of Mathematical Analysis
Ghent University. Galglaan 2, B-9000 Gent, Belgium
rd@cage.ugent.be

F. Sommen
Department of Mathematical Analysis
Ghent University. Galglaan 2, B-9000 Gent, Belgium
fs@cage.ugent.be

"... on peut se poser la question: quel est le théorème mathématique le plus profond, le plus difficile, dont il existe une interprétation physique concrète et indubitable? (...) Pour moi, c'est le théorème de Stokes qui est le candidat numéro un. Et cela témoigne d'un fait: la différentielle extérieure est une notion très mystérieuse, dont la véritable nature, je crois, recèle encore bien des énigmes, et cela en dépit de la simplicité de sa définition formelle."

René Thom, La science malgré tout ...

ABSTRACT

Similarities are shown between the algebras of differential forms and of Clifford algebra-valued multi-vector functions in an open region of Euclidean space. The Poincaré Lemma and the Dual Poincaré Lemma are restated and proved in a refined version. In the case of real-analytic differential forms an alternative proof of the Poincaré Lemma is given using the Euler operator. A position is taken in the debate on the redundancy of either of the two algebras.

RESUMEN

Se muestran similitudes entre las álgebras de formas diferenciales y las de funciones multivectoriales valuadas de una álgebra de Clifford en una región abierta del espacio Euclidiano. El Lema de Poincaré y Lema de Poincaré dual son presentados y probados en una versión refinada. En el caso de formas diferenciales reales analíticas una prueba alternativa del Lema de Poincaré es dada usando el operador de Euler. Una posición es tomada en el debate en redundancia de cualquiera de las dos álgebras.

Key words and phrases: *differential forms, multi-vector functions, Poincaré Lemma*

Math. Subj. Class.: *58A10, 30G35*

1 Introduction

In this paper two mathematical languages are confronted with each other: the language of differential forms and the one of Clifford algebra-valued multi-vector functions.

The Cartan algebra $\bigwedge(\Omega)$ of smooth differential forms on an open subset Ω of Euclidean space $\mathbb{R}^{m+1}$, endowed with exterior multiplication, is of course well-known. A fundamental operator on $\bigwedge(\Omega)$ is the exterior derivative d with its important property that for any differential form ω, $d(d\omega) = 0$.

Introducing the Hodge co-derivative d^* leads to the differential operator $D = d + d^*$, by means of which the so-called "harmonic" r-forms $(0 < r < m+1)$ are characterized as smooth differential r-forms ω^r satisfying $D\omega^r = 0$.

The algebra $\mathcal{E}(\Omega)$ of smooth multi-vector functions is less well-known. Multi-vector functions arise in a natural way when considering functions defined in Ω and taking values in the universal real Clifford algebra $\mathbb{R}_{0,m+1}$ constructed over $\mathbb{R}^{0,m+1}$, i.e. $\mathbb{R}^{m+1}$ equipped with an anti-Euclidean metric. If $\mathbb{R}^r_{0,m+1}$ $(0 \leq r \leq m+1)$ denotes the space of r-vectors, then the Clifford algebra $\mathbb{R}_{0,m+1}$ is precisely the graded associative algebra $\mathbb{R}_{0,m+1} = \sum_{r=0}^{m+1} \oplus \mathbb{R}^r_{0,m+1}$, and an r-vector function F_r is a map $F_r : \Omega \to \mathbb{R}^r_{0,m+1}$. It was William Kingdon Clifford who introduced his so-called geometric algebra in the 1870s, building on earlier work of Hamilton and Grassmann. A fundamental operator on the space of smooth multi-vector functions, is the Dirac operator ∂, by means of which the so-called monogenic functions are characterized as the smooth functions f satisfying $\partial f = 0$. Note that the monogenic functions are at the core of so-called Clifford analysis, a function theory which developed extensively during the last decades, offering a direct and elegant generalization to higher dimension of the theory of holomorphic functions in the complex plane. Note also that the above mentioned Dirac equation may be expressed in the language of systems of partial differential equations by modelling Clifford algebra through its matrix representation.

The spaces of smooth differential forms on the one hand, and of smooth multi-vector functions on the other, are shown to be isomorphic in a natural way: a smooth r-form is identified with a smooth r-vector function, the action of the differential operator $D = d + d^*$ on the space $\bigwedge^r(\Omega)$ of smooth r-forms, is identified with the action of the Dirac operator ∂ on the space $\mathcal{E}_r(\Omega)$ of smooth r-vector functions, and the counterparts in the space of multi-vectors of the exterior derivative d and the co-derivative d^* are pinpointed. This isomorphism is moreover fully exploited in that proofs can be given in either of both languages and that the results obtained are mutually exchangeable (section 4).

In fact the paper also focusses on two well-known theorems on differential forms: the Poincaré Lemma and the Dual Poincaré Lemma. They are restated in a refined version which, to the authors' knowledge, rarely appears in the literature. Combining these two theorems, a structure theorem for monogenic multi-vector functions and its counterpart in the space of smooth differential forms is given (section 5). In proving these structure theorems, we heavily rely on the classical Poincaré Lemma and the classical Dual Poincaré Lemma. In section 6 an alternative proof of those lemmata are given in the special case of real-analytic differential forms in an open ball centred at the origin.

We wish to emphasize that the present paper may not be seen as a pleading to substitute one of the languages for the other, nor to prefer one language above the other. On the contrary, we are convinced that differential forms and multi-vector functions, despite the natural identification given, are quite different mathematical objects, the use of which is very much imposed by the mathematical context. This in-depth difference between and context-dependence of differential forms and multi-vector functions will be fully discussed in a forthcoming paper by one of the authors.

2 Multi-vector functions: preliminaries

In this section we recall some basic notions and results from Clifford algebra and Clifford analysis. For a detailed account we refer the reader to [10] and [2]; the recent book [3] gives a nice and broad overview of the intrinsic value and usefulness of Clifford algebra and Clifford analysis for mathematical physics.

The construction of the universal real Clifford algebra is well-known; we restrict ourselves to a schematic approach. Let $\mathbb{R}^{0,m+1}$ be the real vector space $\mathbb{R}^{m+1}$ $(m \geq 1)$ endowed with a non-degenerate symmetric bilinear form $\mathcal{B}$ of signature $(0, m+1)$, and let $(e_0, e_1, \cdots, e_m)$ be an associated orthonormal basis:

$$\mathcal{B}(e_i, e_j) = \begin{cases} -1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases} \quad (0 \leq i, j \leq m).$$

The anti-Euclidean metric on $\mathbb{R}^{0,m+1}$ is induced by the scalar product

$$< e_i, e_j >= -\mathcal{B}(e_i, e_j) = \delta_{ij}, \ 0 \le i, j \le m.$$

Introduce the anti-symmetric outer product by the rules:

$$\begin{aligned} e_i \wedge e_i &= 0, \quad 0 \le i \le m \\ e_i \wedge e_j + e_j \wedge e_i &= 0, \quad 0 \le i \ne j \le m. \end{aligned}$$

For each $A = \{i_1, i_2, \cdots, i_r\} \subset M = \{0, 1, \cdots, m\}$, ordered in the natural way: $0 \le i_1 < i_2 < \cdots < i_r \le m$, put

$$e_A = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_r}$$

and

$$e_\phi = 1.$$

Then for each $r = 0, 1, \cdots, m+1$, the set $\{e_A : A \subset M \text{ and } |A| = r\}$ is a basis for the space $\mathbb{R}^r_{0,m+1}$ of so-called *r-vectors*.

Introducing the inner product by

$$e_i \bullet e_j = - < e_i, e_j >, \quad 0 \le i, j \le m$$

leads to the so-called *geometric product* in the Clifford algebra, given by

$$e_i e_j = e_i \bullet e_j \ + \ e_i \wedge e_j, \quad 0 \le i, j \le m.$$

The respective definitions of the inner product, the outer product and the (geometric) product are then extended to r-vectors by the formulae:

$$e_j \bullet e_A = e_j \bullet (e_{i_1} \wedge \cdots \wedge e_{i_r}) = \sum_k (-1)^k \delta_{j i_k} e_{A \setminus \{i_k\}}$$

where

$$e_{A\setminus\{i_k\}} = e_{i_1} \wedge \cdots \wedge e_{i_{k-1}} \wedge [e_{i_k} \wedge] \, e_{i_{k+1}} \wedge \cdots \wedge e_{i_r}$$

and

$$\begin{cases} e_j \wedge e_A = e_j \wedge (e_{i_1} \wedge \cdots \wedge e_{i_r}) = e_j \wedge e_{i_1} \wedge \cdots \wedge e_{i_r}, & \text{if } j \notin A \\ e_j \wedge e_A = 0, \quad \text{if } j \in A \end{cases}$$

and finally

$$e_j e_A = e_j \bullet e_A + e_j \wedge e_A.$$

The inner and outer products are distributive over addition, and so is the (geometric) product.
The universal real Clifford algebra $\mathbb{R}_{0,m+1}$ is the graded associative algebra

$$\mathbb{R}_{0,m+1} = \sum_{r=0}^{m+1} \oplus \, \mathbb{R}^r_{0,m+1}.$$

If $[\ .\ ]_r : \mathbb{R}_{0,m+1} \to \mathbb{R}^r_{0,m+1}$ denotes the projection operator from $\mathbb{R}_{0,m+1}$ onto $\mathbb{R}^r_{0,m+1}$, then each Clifford number $a \in \mathbb{R}_{0,m+1}$ may be written as

$$a = \sum_{r=0}^{m+1} [a]_r .$$

Note that in particular for a 1-vector u and an r-vector v_r , one has

$$u\ v_r = u \bullet v_r + u\ \wedge\ v_r$$

with

$$u \bullet v_r \quad = \quad [u\ v_r]_{r-1} \quad = \quad \frac{1}{2}\Big(u\ v_r - (-1)^r v_r\ u\Big)$$

and

$$u\ \wedge\ v_r \quad = \quad [u\ v_r]_{r+1} \quad = \quad \frac{1}{2}\Big(u\ v_r + (-1)^r v_r\ u\Big).$$

Usually $\mathbb{R}$ and $\mathbb{R}^{m+1}$ are identified with $\mathbb{R}^0_{0,m+1}$ and $\mathbb{R}^1_{0,m+1}$ respectively. An element $x = (x_0, x_1, \cdots, x_m) \in \mathbb{R}^{m+1}$ is thus identified with the 1-vector $x = \sum_{j=0}^m\ x_j\ e_j$.

Now let Ω be an open region in $\mathbb{R}^{m+1}$. A smooth *r-vector function* F_r is a map

$$F_r : \Omega \to \mathbb{R}^r_{0,m+1},\ x \mapsto \sum_{|A|=r} F_{r,A}(x)\ e_A$$

where for each A, $F_{r,A}$ is a smooth real-valued function in Ω.
We denote by $\mathcal{E}_r(\Omega)$ the space of smooth r-vector functions in Ω, and we put

$$\mathcal{E}(\Omega) = \sum_{r=0}^{m+1}\ \oplus\ \mathcal{E}_r(\Omega).$$

The projection operator from $\mathcal{E}(\Omega)$ onto $\mathcal{E}_r(\Omega)$ is denoted by $[\ .\ ]_r$. For the linear operator $T : \mathcal{E}_r(\Omega) \to \mathcal{E}(\Omega)$ we denote by $\overset{r}{\ker}\ T$ the kernel of T in $\mathcal{E}_r(\Omega)$, while $\overset{r}{\mathrm{im}}\ T$ stands for the image of $\mathcal{E}_r(\Omega)$ under T.

A fundamental operator in Clifford analysis is the so-called *Dirac operator*, a vector differential operator given by

$$\partial = \sum_{j=0}^{m} e_j\ \partial_{x_j} .$$

Due to the non-commutativity of the multiplication in the Clifford algebra, it can act from the left or from the right on a function. For $F = \sum_A e_A F_A \in \mathcal{E}(\Omega)$ these actions are given by

$$\partial F = \sum_j \sum_A e_j e_A\ \partial_{x_j} F_A \quad \text{and} \quad F\partial = \sum_J \sum_A e_A e_j\ \partial_{x_j} F_A .$$

A function $F \in \mathcal{E}(\Omega)$ is called *left* (resp. *right*) *monogenic* in Ω iff it satisfies $\partial F = 0$ (resp. $F\partial = 0$) in Ω.

Restricting the Dirac operator ∂ to the space $\mathcal{E}_r(\Omega)$ we find for an r-vector function F_r, that ∂F_r and $F_r\partial$ split up into an $(r-1)$-vector and an $(r+1)$-vector function:

$$\partial F_r \quad = \quad \sum_{j=0}^{m} e_j \; \partial_{x_j} F_r \quad = \quad \sum_j e_j \bullet \partial_{x_j} F_r \quad + \quad \sum_j e_j \;\wedge\; \partial_{x_j} F_r$$

and

$$F_r \partial \quad = \quad \sum_{j=0}^{m} \partial_{x_j} F_r \; e_j \quad = \quad \sum_j \partial_{x_j} F_r \bullet e_j \quad + \quad \sum_j \partial_{x_j} F_r \;\wedge\; e_j.$$

It readily follows that

$$[\partial F_r]_{r-1} \quad = \quad \sum_j e_j \bullet \partial_{x_j} F_r \quad = \quad (-1)^{r+1} \sum_j \partial_{x_j} F_r \bullet e_j \quad = \quad (-1)^{r+1} [F_r \partial]_{r-1}$$

$$[\partial F_r]_{r+1} \quad = \quad \sum_j e_j \;\wedge\; \partial_{x_j} F_r \quad = \quad (-1)^{r} \sum_j \partial_{x_j} F_r \;\wedge\; e_j \quad = \quad (-1)^{r} [F_r \partial]_{r+1}.$$

Consequently, for an r-vector function F_r, the notions of left monogenicity and right monogenicity coincide.
Moreover, if for $F \in \mathcal{E}(\Omega)$ we put $F_E = \sum_{|A|=even} e_A \; F_A$ and $F_O = \sum_{|A|=odd} e_A \; F_A$, then F is monogenic in Ω iff both F_E and F_O are monogenic in Ω.

Commonly one introduces the notations:

$$\begin{array}{rclcrcl} \partial \bullet F_r & = & [\partial F_r]_{r-1} & , & \partial \;\wedge\; F_r & = & [\partial F_r]_{r+1} \\ F_r \bullet \partial & = & [F_r \partial]_{r-1} & , & F_r \;\wedge\; \partial & = & [F_r \partial]_{r+1}. \end{array}$$

The action of the Dirac operator ∂ on $\mathcal{E}_r(\Omega)$ thus gives rise to two auxiliary differential operators:

$$\partial^- \;:\; \mathcal{E}_r(\Omega) \;\rightarrow\; \mathcal{E}_{r-1}(\Omega) \;:\; F_r \;\mapsto\; \partial^- F_r \;=\; \partial \bullet F_r \;=\; [\partial F_r]_{r-1}$$

and

$$\partial^+ \;:\; \mathcal{E}_r(\Omega) \;\rightarrow\; \mathcal{E}_{r+1}(\Omega) \;:\; F_r \;\mapsto\; \partial^+ F_r \;=\; \partial \;\wedge\; F_r \;=\; [\partial F_r]_{r+1}.$$

Symbolically these operators may be written as:

$$\partial^- \quad = \quad (\partial \bullet \;) \quad = \quad \sum_j (e_j \bullet \;) \partial_{x_j}$$

and

$$\partial^+ \quad = \quad (\partial \;\wedge) \quad = \quad \sum_j (e_j \;\wedge) \partial_{x_j}.$$

Their action on $\mathcal{E}_r(\Omega)$ is two-fold in the sense that they act on the multi-vector by means of the inner and outer product with basis vectors, and at the same time on the function coefficients by partial differentiation.

As on $\mathcal{E}_r(\Omega)$ holds:

$$\partial \;=\; \partial^- + \partial^+$$

we obtain that a smooth r-vector function F_r is left monogenic (as well as right monogenic) in Ω iff in Ω

$$\partial F_r = 0 \iff F_r \partial = 0 \iff \left\{ \begin{array}{lcl} \partial^- F_r & = & 0 \\ \partial^+ F_r & = & 0 \end{array} \right. . \tag{I}$$

As the Dirac operator ∂ splits the Laplace operator:

$$\partial^2 \;=\; \partial \bullet \partial + \partial \wedge \partial \;=\; \partial \bullet \partial \;=\; -<\partial, \partial> = \; -\triangle$$

a monogenic function in Ω is also harmonic in Ω, but the converse clearly is not true.

As moreover

$$(\partial^-)^2 \;=\; (\partial^+)^2 \;=\; 0$$

we have

$$-\triangle \;=\; (\partial^- + \partial^+)^2 \;=\; \partial^-\partial^+ + \partial^+\partial^- .$$

The second order differential operators $\partial^-\partial^+$ and $\partial^+\partial^-$ are scalar operators in the sense that they keep the order of the multi-vector function, but the function coefficients, while being differentiated, are interchanged w.r.t. the basis multi-vectors.

Now observe that the system (I), expressing the monogenicity of an r-vector function, is also equivalent to

$$\widetilde{\partial} F_r \;=\; (\partial^+ - \partial^-) F_r \;=\; 0$$

or

$$F_r \widetilde{\partial} \;=\; F_r (\partial^+ - \partial^-) \;=\; 0$$

where we have introduced the *modified Dirac operator*

$$\widetilde{\partial} \;=\; \partial^+ - \partial^- .$$

We directly have the basic formulae:

$$\begin{array}{rcl} \partial\widetilde{\partial} & = & \partial^-\partial^+ - \partial^+\partial^- \\ \widetilde{\partial}\partial & = & \partial^+\partial^- - \partial^-\partial^+ \\ \widetilde{\partial}\widetilde{\partial} & = & -\partial^+\partial^- - \partial^-\partial^+ \;=\; \triangle \end{array}$$

which leads to the *modified Laplace operator*

$$\widetilde{\triangle} = \partial^- \partial^+ - \partial^+ \partial^-$$

which clearly is a scalar operator in the sense that it keeps the order of the multi-vector function on which it acts.

Taking into account the *main involution*, also called *inversion*, of the Clifford algebra, for which

$$(e_{i_1} \cdots e_{i_r})^* = (e_{i_1} \wedge \cdots \wedge e_{i_r})^* = (-1)^r \, e_{i_1} \wedge \cdots \wedge e_{i_r}$$

we get the formulae:

$$\begin{array}{rclcccl}
\partial F_r & = & F_r^* \widetilde{\partial} & \text{and} & \widetilde{\partial} F_r & = & F_r^* \partial \\
-\triangle F_r & = & \partial\partial F_r & = & \partial F_r^* \widetilde{\partial} & = & (-1)^r \, \partial F_r \widetilde{\partial} \\
\triangle F_r & = & \widetilde{\partial}\widetilde{\partial} F_r & = & \widetilde{\partial} F_r^* \partial & = & (-1)^r \, \widetilde{\partial} F_r \partial \\
\widetilde{\triangle} F_r & = & \partial\widetilde{\partial} F_r & = & \partial F_r^* \partial & = & (-1)^r \, \partial F_r \partial \\
-\widetilde{\triangle} F_r & = & \widetilde{\partial}\partial F_r & = & \widetilde{\partial} F_r^* \widetilde{\partial} & = & (-1)^r \, \widetilde{\partial} F_r \widetilde{\partial}.
\end{array}$$

3 Differential forms: preliminaries

This section is also introductory; there is a vast literature on differential forms; we may refer to e.g. [8], [15].
Let $\mathbb{R}^{m+1}$ be endowed with the standard Euclidean metric.
Denoting by $\bigwedge^r \mathbb{R}^{m+1}$ the space of alternating (or skew-multilinear) real-valued r-forms $(0 \leq r \leq m+1)$, the Grassmann algebra or exterior algebra over $\mathbb{R}^{m+1}$ is the graded associative algebra

$$\bigwedge \mathbb{R}^{m+1} = \sum_{r=0}^{m+1} \oplus \bigwedge^r \mathbb{R}^{m+1}$$

endowed with the exterior multiplication.
A basis for $\bigwedge^r \mathbb{R}^{m+1}$ is obtained as follows. Let $\{dx^0, dx^1, \cdots, dx^m\}$ be a basis for the dual space $(\mathbb{R}^{m+1})^*$ of $\mathbb{R}^{m+1}$. If again the set $A = \{i_1, \ldots, i_r\} \subset M = \{0, 1, \cdots, m\}$ is ordered in the natural way, put

$$dx^A = dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_r}$$

and

$$dx^\phi = 1.$$

Then for each $r = 0, 1, \cdots, m+1$, the set $\{dx^A : A \subset M \text{ and } |A| = r\}$ is a basis for $\bigwedge^r \mathbb{R}^{m+1}$.
Note that in particular

$$dx^i \wedge dx^i = 0, \qquad i = 0, 1, \cdots, m+1$$

and

$$dx^i \wedge dx^j + dx^j \wedge dx^i = 0, \qquad 0 \leq i \neq j \leq m.$$

A smooth r-form in an open region Ω of $\mathbb{R}^{m+1}$ is a map

$$\omega^r : \Omega \to \bigwedge^r \mathbb{R}^{m+1}, \quad x \mapsto \sum_{|A|=r} \omega^r_A(x)\, dx^A$$

where for each A, ω^r_A is a smooth real-valued function in Ω.

We denote by $\bigwedge^r(\Omega)$ the space of smooth r-forms in Ω and we put

$$\bigwedge(\Omega) = \sum_{r=0}^{m+1} \oplus \bigwedge^r(\Omega).$$

The projection operator from $\bigwedge(\Omega)$ onto $\bigwedge^r(\Omega)$ is denoted by $[\ .\ ]^r$, and the notations of the foregoing section are kept for the kernel and the image of a linear operator $T : \bigwedge^r(\Omega) \longrightarrow \bigwedge(\Omega)$.

A fundamental linear operator on the space of smooth forms is the *exterior derivative* d. It is first defined on $\bigwedge^r(\Omega)$ $(r < m+1)$ by

$$d : \bigwedge^r(\Omega) \longrightarrow \bigwedge^{r+1}(\Omega)$$

$$\omega^r = \sum_{|A|=r} \omega^r_A\, dx^A \quad \longmapsto \quad d\omega^r = \sum_A \sum_j \partial_{x_j} \omega^r_A\, dx^j \wedge dx^A$$

and this definition is then extended to $\bigwedge(\Omega)$ by linearity.
The kernel of the exterior derivative d ,

$$\overset{r}{\ker}\, d = \{\omega^r \in \bigwedge^r(\Omega) : d\omega^r = 0\}$$

consists of the so-called *closed* r-forms in Ω, while its image of $\bigwedge^{r-1}(\Omega)$ in $\bigwedge^r(\Omega)$

$$\overset{r-1}{\operatorname{im}}\, d = \{d\omega^{r-1} : \omega^{r-1} \in \bigwedge^{r-1}(\Omega)\}$$

consists of the so-called *exact* r-forms in Ω.

The quotient space

$$H^r(\Omega) = \overset{r}{\ker}\, d \,/\, \overset{r-1}{\operatorname{im}}\, d$$

is the so-called *de Rham r-th cohomology space.*

The well-known Poincaré Lemma (see also section 5) asserts that if Ω is contractible to a point, then for each $r > 0$, $H^r(\Omega) = 0$, in other words: if Ω is contractible to a point and $\omega^r \in \bigwedge^r(\Omega)$ is closed, then ω^r is exact. The converse, i.e. that any exact r-form in an open region of $\mathbb{R}^{m+1}$ is also closed, follows at once from the observation that $d(d\omega) = 0$.

A second fundamental linear operator on the space of smooth forms is the Hodge co-derivative d^*. For $A = \{i_i, \cdots, i_r\} \subset M$ we denote

$$dx^{A\setminus\{i_j\}} \ = \ dx^{i_1} \ \wedge \ \cdots \ \wedge \ dx^{i_{j-1}} \ \wedge \ [dx^{i_j} \ \wedge] \ dx^{i_{j+1}} \ \wedge \ \cdots \ \wedge \ dx^{i_r}$$

and in a first step we put:

$$d^*(\omega_A dx^A) \ = \ \sum_{j=1}^{r}(-1)^j \ \partial_{x_j}\omega_A \ dx^{A\setminus\{i_j\}}.$$

Then d^* is defined on $\bigwedge^r(\Omega)$ $(r > 0)$ by

$$d^* : \ \bigwedge^r(\Omega) \longrightarrow \bigwedge^{r-1}(\Omega)$$

$$\omega^r \ = \ \sum_{|A|=r} \omega^r_A \ dx^A \quad \longmapsto \quad d^*(\omega^r) \ = \ \sum_{|A|=r} d^*(\omega^r_A \ dx^A)$$

and this definition is extended to $\bigwedge(\Omega)$ by linearity.
The kernel of the co-derivative d^* acting on $\bigwedge(\Omega)$:

$$\overset{r}{\ker} \ d^* \ = \ \{\omega^r \in \ \bigwedge^r(\Omega) : \ d^*\omega_r \ = \ 0\}$$

consists of the so-called *co-closed* r-forms in Ω, while its image of $\bigwedge^{r+1}(\Omega)$ in $\bigwedge^r(\Omega)$

$$\overset{r+1}{\mathrm{im}} \ d^* \ = \ \{d^*\omega^{r+1} : \ \omega^{r+1} \ \in \ \bigwedge^{r+1}(\Omega)\}$$

consists of the so-called *co-exact* r-forms in Ω.
By observing that for any smooth form in Ω, $d^*(d^*\omega) = 0$, it follows that each co-exact r-form in Ω is also co-closed. The quotient space

$$H_r(\Omega) \ = \overset{r}{\ker} \ d^* \ / \ \overset{r+1}{\mathrm{im}} \ d^*$$

is the so-called *de Rham r-th homology space.*
It could be confusing to use the term "homology" here, since it usually refers to the complex associated with the algebra of chains subject to the action of the boundary-operator; in the space of currents however there is a connection (see [8], p.313).

By virtue of the Weyl duality we have for a region Ω which is contractible to a point, and for each $r < m+1$, that $H_r(\Omega) = 0$, in other words: if Ω is contractible to a point, then each co-closed r-form in Ω is also co-exact; this is dealt with in the so-called Dual Poincaré Lemma (see section 5).

A smooth r-form in Ω which is at the same time closed and co-closed is called *harmonic* in Ω (in the sense of Hodge). Introducing the operator $D = d + d^*$, a necessary and sufficient condition for a smooth r-form ω^r in Ω to be harmonic in Ω thus reads:

$$D\omega^r \;=\; (d+d^*)\omega^r \;=\; 0 \iff \left\{ \begin{array}{lll} d\omega^r & = & 0 \\ d^*\omega^r & = & 0 \end{array} \right. . \tag{II}$$

The system (II) is called the *Hodge-de Rham system*.

Note that if ω^r is harmonic in an open region Ω of $\mathbb{R}^{m+1}$ then automatically ω^r satisfies $\triangle\omega^r = 0$ in Ω, since

$$D^2 \;=\; (d+d^*)^2 \;=\; d\; d^* + d^*\; d \;=\; -\triangle\, .$$

The converse is however not true.

The action of the operators d and d^* on differential forms is two-fold in the sense that they act on the form itself as well as on the function coefficients by partial differentiation. In order to explicit this double action we introduce the following symbolic notations for the operators d and d^*. For d the following notation is rather obvious:

$$d \;=\; \sum_{j=0}^{m} (dx^j\wedge)\; \partial_{x_j}.$$

We then indeed have

$$\begin{aligned} d\omega^r \;&=\; \left(\sum_j (dx^j\wedge)\partial_{x_j}\right)\left(\sum_{|A|=r} \omega^r_A\; dx^A\right) \\ &=\; \sum_j \sum_A \partial_{x_j}\omega^r_A\; dx^j \wedge dx^A \end{aligned}$$

illustrating the above mentioned double action and the fact that d acts in an "exterior" way.
But this raises the question whether there exists a differential operator on forms acting in an "inner" way, to which end an "inner product" in the Grassmann algebra should be defined. Inspired by the inner product in the Clifford algebra, we put by definition:

$$dx^i \bullet dx^j \;=\; - < dx^i, dx^j > = \; -\delta_{ij}, \qquad 0 \le i, j \le m.$$

In fact this scalar product in the Grassmann algebra already tacitly exists. Indeed, as $\mathbb{R}^{m+1}$ is endowed with the standard Euclidean metric, there is a canonical isomorphism between the tangent space $T_x\Omega \;\cong\; \mathbb{R}^{m+1}$ and its dual $T^*_x\Omega \;\cong\; (\mathbb{R}^{m+1})^*$, given by

$$e_j \quad \overset{iso}{\longleftrightarrow} \quad < e_j, \bullet > = \ e_j^* \ = \ dx^j$$

and hence

$$< dx^i, dx^j > = < e_i^*, e_j^* > = < e_i, e_j > = \ \delta_{ij}, \qquad 0 \leq i, j \leq m.$$

So we introduce the operator

$$\sum_{j=0}^{m} (dx^j \bullet \ ) \ \partial_{x_j}$$

clearly an operator with a double action.

In a next step we put

$$dx^j \bullet dx^A \ = \ dx^j \bullet (dx^{i_1} \ \wedge \ \cdots \ \wedge \ \ dx^{i_r}) \ = \ \sum_{k=1}^{r} (-1)^k \ \delta_{j i_k} dx^{A \setminus \{i_k\}}.$$

We then get, by linearity, for a smooth r-form ω^r:

$$\left(\sum_{j=0}^{m} (dx^j \bullet \ ) \partial_{x_j} \right) \left(\sum_{|A|=r} \omega_A^r dx^A \right) \ = \ \sum_{k=1}^{r} \sum_{|A|=r} (-1)^k \ (\partial_{x_{i_k}} \omega_A^r) \ dx^{A \setminus \{i_k\}}$$

in which we recognize the action of the co-derivative d^* on ω_r.
Consequently this co-derivative may be written as:

$$d^* \ = \ \sum_{j=0}^{m} (dx^j \bullet \ ) \partial_{x_j}$$

also nicely illustrating the double action of d^*. From this point of view the co-derivative d^* might as well have been called "interior derivative".

Finally for the operator $D \ = \ d + d^*$ we obtain the expressions

$$\begin{aligned} D \quad = \quad d + d^* \quad &= \quad \sum_{j=0}^{m} (dx^j \wedge \ ) \partial_{x_j} \ + \ \sum_{j=0}^{m} (dx^j \bullet \ ) \partial_{x_j} \\ &= \quad \sum_{j=0}^{m} (dx^j \wedge \ + dx^j \bullet \ ) \partial_{x_j} \\ &= \quad \sum_{j=0}^{m} (Dx^j \vee \ ) \partial_{x_j} \end{aligned}$$

where

$$Dx^j \vee \ = \ dx^j \bullet \ + dx^j \wedge$$

is the so-called "vee-product"-operator, which was introduced in e.g. [7] and [12] in the more general context of a metric with (p,q)-signature on $\mathbb{R}^{m+1}$.

In the sequel we will deal with the operators d and d^* on the same footing and systematically mention the properties of d^* next to those of d, for the sake of aesthetical symmetry. However, from the mathematical point of view this is superfluous; considering the operator d^* only leads to new results when it appears in connection with the operator d. Note in this context the interesting operators dd^* and d^*d, which are the "components" of the Laplace operator $(-\triangle)$.

4 Differential forms and multi-vector functions: an identification

In becomes clear from sections 2 and 3 that the world of differential forms in an open region Ω of $\mathbb{R}^{m+1}$ and the world of multi-vector functions in Ω, may be identified in a natural way. If for each $A \subset M$, f_A is a smooth real-valued function in Ω, then the following correspondence table may already be drawn (see next page).

This identification is now further developed. First one may wonder what the counterpart is of the Hodge $*$ (star) operator. On the one hand one has

$$* \left(dx^{j_1} \wedge \cdots \wedge dx^{j_r}\right) = \sigma\, dx^{j_{r+1}} \wedge \cdots \wedge dx^{j_{m+1}}$$

where $j_1 < \cdots < j_r$, $j_{r+1} < \cdots < j_{m+1}$, $\{j_1, \cdots, j_r\} \cup \{j_{r+1}, \cdots, j_{m+1}\} = M = \{0, 1, \cdots, m\}$ and σ is the signature of the permutation $(j_{r+1}, \cdots, j_{m+1}, j_1, \cdots, j_r)$. This corresponds, for $A = \{j_1, \cdots, j_r\} \subset M$ to

$$*e_A = (-1)^r\, e_M e_A^\dagger$$

where $e_M = e_0 \wedge e_1 \wedge \cdots \wedge e_m$ is the so-called *pseudoscalar* and $\dagger$ stands for the *main anti-involution* of the Clifford algebra, also called *reversion*, given by

$$e_A^\dagger = (e_{j_1} \wedge \cdots \wedge e_{j_r})^\dagger = e_{j_r} \wedge \cdots \wedge e_{j_1} = (-1)^{\frac{r(r-1)}{2}} e_A.$$

Next we identify some differential operators and establish similar formulae in both worlds.

To start with, the Euler operator

$$E = \sum_{j=0}^{m} x_j\, \partial_{x_j}$$

dx^j	e_j
$dx^i \wedge dx^j$	$e_i \wedge e_j$
$dx^i \bullet dx^j$	$e_i \bullet e_j$
$\omega^r = \sum_{\lvert A\rvert=r} f_A\ dx^A$	$F_r = \sum_{\lvert A\rvert=r} f_A\ e_A$
$d = \sum_{j=0}^{m} (dx^j \wedge)\partial_{x_j}$	$\partial^+ = \sum_{j=0}^{m} (e_j \wedge)\partial_{x_j}$
$d^* = \sum_{j=0}^{m} (dx^j\ \bullet)\partial_{x_j}$	$\partial^- = \sum_{j=0}^{m} (e_j\ \bullet)\partial_{x_j}$
$D = d + d^* = \sum_{j=0}^{m} (Dx^j \vee)\partial_{x_j}$	$\partial = \partial^+ + \partial^- = \sum_{j=0}^{m} e_j\ \partial_{x_j}$
ω^r harmonic in $\Omega \subset \mathbb{R}^{m+1}$	F_r monogenic in $\Omega \subset \mathbb{R}^{m+1}$
$d^2 = dd = 0$	$\partial^{+2} = \partial^+\partial^+ = 0$
$d^{*2} = d^*d^* = 0$	$\partial^{-2} = \partial^-\partial^- = 0$
dd^*	$\partial^+\partial^-$
d^*d	$\partial^-\partial^+$
$D^2 = (d + d^*)^2 = dd^* + d^*d = -\triangle$	$\partial^2 = (\partial^+ + \partial^-)^2 = \partial^+\partial^- + \partial^-\partial^+ = -\triangle$
$\widetilde{D} = d - d^*$	$\widetilde{\partial} = \partial^+ - \partial^-$
$\widetilde{D}^2 = (d - d^*)^2 = -dd^* - d^*d = \triangle$	$\widetilde{\partial}^2 = -\partial^+\partial^- - \partial^-\partial^+ = \triangle$
$D\widetilde{D} = -\widetilde{D}D = d^*d - dd^* = \widetilde{\triangle}$	$\partial\widetilde{\partial} = -\widetilde{\partial}\partial = \partial^-\partial^+ - \partial^+\partial^- = \widetilde{\triangle}$

defined by

$$E\omega^r \;=\; \sum_{j=0}^{m} x_j\, \partial_{x_j}\omega^r \;=\; \sum_{|A|=r} dx^A \sum_{j=0}^{m} x_j\, \partial_{x_j}\omega^r_A$$

and

$$EF_r \;=\; \sum_{j=0}^{m} x_j\, \partial_{x_j}\, F_r \;=\; \sum_{|A|=r} e_A \sum_{j=0}^{m} x_j\, \partial_{x_j}\, F_{r,A}$$

is a scalar operator, measuring the degree of homogenicity of a function, and not affecting the order of a differential form or a multi-vector function. The Euler operator thus has the same defining expression in both worlds.

From the world of differential forms we now focus on the contraction operators $\partial_{x_j}\rfloor$, $j = 0, 1, \cdots m$, acting only on the basis elements of the differential form, but not on the function coefficients, and given by

$$\partial_{x_j}\rfloor dx^A \;=\; \partial_{x_j}\rfloor \left(dx^{i_1} \;\wedge\; \cdots \;\wedge\; dx^{i_r}\right) \;=\; \sum_{k=1}^{r} (-1)^{k-1}\, \delta_{j i_k}\; dx^{A\setminus\{i_k\}} \quad .$$

Apparently the contraction operator $\partial_{x_j}\rfloor$ is, up to a minus sign, nothing else but the "inner product"-operator $(dx^j \bullet\ )$:

$$\partial_{x_j}\rfloor \;=\; (-dx^j \bullet\ )\ , \quad j = 0, 1, \cdots, m.$$

However bear in mind that contractions are more fundamental than dot products. Indeed, they can be introduced independently of a scalar product, and their behaviour is invariant under diffeomorphisms, which is not the case for the dot product.

For a first order operator $v = \sum_{j=0}^{m} v_j(x)\ \partial_{x_j}$, v_j being a scalar-valued smooth function, mostly called a *vector field*, one may consider the associated contraction operator

$$v\rfloor \;=\; \sum_{j=0}^{m} v_j(x)\ \partial_{x_j}\rfloor$$

which also takes the form

$$v\rfloor \;=\; \sum_{j=0}^{m} v_j(x)(-dx^j \bullet\ ).$$

This inspires an associated "inflation" operator

$$v\rceil \;=\; \sum_{j=0}^{m} v_j(x)\ \partial_{x_j}\rceil = \sum_{j=0} v_j(x)\left(-dx^j\wedge\right)$$

where the action of $\partial_{x_j}\rceil = (-dx^j\wedge)$ is given by

$$\partial_{x_j}\rceil\ dx^A \;=\; -dx^j \;\wedge\; dx^A.$$

So from the Euler operator E we deduce the operators

$$E\rfloor \;=\; \sum_j x_j\ \partial_{x_j}\rfloor \;=\; \sum_j x_j(-dx^j \bullet\ )$$

and

$$E\rceil \;=\; \sum_j x_j(-dx^j \wedge\ )$$

which are in a sense complementary to the operators d and d^* — think of replacing x_j by dx^j and dx^j by x_j. So the operators $E\rfloor$ and $E\rceil$ must show properties similar to those of the operators d en d^*, which they indeed do, as shown in the next lemma.

Lemma 4.1

The operators $E\rfloor$ and $E\rceil$ enjoy the following fundamental properties:

(i) $(E\rfloor)^2 \;=\; 0$

(ii) $(E\rceil)^2 \;=\; 0$

(iii) $E\rfloor + E\rceil \;=\; -\sum_{j=0}^{m} x_j\ (Dx_j\ \vee)$

(iv) $(E\rfloor + E\rceil)^2 \;=\; E\rfloor E\rceil + E\rceil E\rfloor \;=\; -|x|^2$

The counterparts in the Clifford setting of the operators $(-dx^j\ \bullet)$ and $(-dx^j\wedge)$ clearly are $(-e_j\ \bullet)$ and $(-e_j\wedge)$. The properties of the operators

$$\sum_{j=0}^{m} x_j(-e_j\ \bullet) = (-x\ \bullet) \text{ and } \sum_{j=0}^{m} x_j(-e_j\wedge) = (-x\wedge)$$

corresponding to the ones in Lemma 4.1, are then straightforward:

$$\begin{array}{rlcl}
(i) & (-x\bullet\ )(-x\bullet\ ) & = & 0 \\
(ii) & (-x\wedge\ )(-x\wedge\ ) & = & 0 \\
(iii) & (-x\bullet\ ) + (-x\wedge\ ) & = & -x \quad \text{(Clifford product understood)} \\
(iv) & ((-x\bullet\ ) + (-x\wedge\ ))^2 & = & (-x\bullet\ )(-x\wedge\ ) + (-x\wedge\ )(-x\bullet\ ) \;=\; -|x|^2.
\end{array}$$

Note that the operators $(e_j\ \bullet)$ and $(e_j\wedge)$, j=0,1, ..., m, coincide with the so-called de Witt basis of the algebra of endomorphisms on the Clifford algebra $\mathbb{R}_{0,m+1}$. Indeed, if e_j and ε_j, $j = 0, 1, \ldots, m$ denote the endomorphisms, given for an arbitrary Clifford number a, by

$$\begin{array}{rcl}
e_j : a & \longmapsto & e_j a \\
\varepsilon_j : a & \longmapsto & \varepsilon_j a = \widetilde{a} e_j
\end{array}$$

then the Witt basis is formed by

$$\mathfrak{F}_j = \frac{1}{2}(e_j - \varepsilon_j)\ ,\ \mathfrak{F}'_j = \frac{1}{2}(e_j + \varepsilon_j), \quad j = 0, 1, \ldots, m$$

and apparently $\mathfrak{F}_j = (e_j \bullet)$ and $\mathfrak{F}'_j = (e_j \wedge)$.

In the same order of ideas and starting from the operators d and d^*, we introduce the contraction and "inflation" operators

$$d\rfloor = \sum_{j=0}^{m} (dx^j \wedge)\ \partial_{x_j}\rfloor = \sum_{j=0}^{m} (dx^j \wedge)(-dx^j \bullet)$$

$$d^*\rceil = \sum_{j=0}^{m} (dx^j \bullet)\ \partial_{x_j}\rceil = \sum_{j=0}^{m} (dx^j \bullet)(-dx^j \wedge)$$

The operators $d\rfloor$ and $d^*\rceil$ have $\mathcal{E}^r(\Omega)$ as an eigenspace since

$$d\rfloor\omega^r = r\ \omega^r \qquad \text{and} \qquad d^*\rceil\omega^r = (m+1-r)\ \omega^r.$$

In other words: they measure the order of a differential form. They are sometimes called *fermionic Euler operators*.
In the Clifford analysis setting we get

$$\partial^+\rfloor = \sum_{j=0}^{m} (e_j \wedge)(-e_j \bullet) \qquad \text{and} \qquad \partial^-\rceil = \sum_{j=0}^{m} (e_j \bullet)(-e_j \wedge)$$

for which indeed:

$$\partial^+\rfloor\ F_r = r\ F_r \qquad \text{and} \qquad \partial^-\rceil\ F_r = (m+1-r)\ F_r.$$

Now we turn our attention, still in the world of differential forms, to a so-called *Lie-derivative* of differential forms. For a given scalar vector field $v = \sum_j v_j \partial_{x_j}$ we define

$$\mathcal{L}_v \omega = d\ v\rfloor\ \omega + v\rfloor d\ \omega.$$

It is clear that the operators $\mathcal{L}_v$ and d, as well as $\mathcal{L}_v$ and $v\rfloor$, commute, since

$$d\ \mathcal{L}_v = d\ v\rfloor\ d = \mathcal{L}_v\ d$$

and

$$v\rfloor \mathcal{L}_v = v\rfloor\ d\ v\rfloor = \mathcal{L}_v\ v\rfloor.$$

This implies that closedness and exactness of differential forms are preserved under "Lie-derivation".
We now prove a fundamental formula about the Lie-derivative of the Euler operator.

Lemma 4.2

For any smooth differential form $\omega \in \bigwedge(\Omega)$ one has

$$\mathcal{L}_E\ \omega = (E\rfloor d + d\ E\rfloor)\ \omega = (E + d\rfloor)\ \omega.$$

Proof.

First we have

$$E\rfloor\ d\ \omega = \sum_j x_j(-dx^j \bullet)\Big(\sum_k (dx^k\wedge)\partial_{x_k}\omega\Big)$$

$$= \sum_j\sum_k\ x_j\ \delta_{jk}\ \partial_{x_k}\ \omega + \sum_j\sum_k x_j\ dx^k\ \wedge\ \partial_{x_k}\ dx^j \bullet\ \omega$$

$$= \sum_j x_j\ \partial_{x_j} + \sum_j\sum_k\ x_j\ dx^k\ \wedge\ \partial_{x_k}(dx^j \bullet\ \omega)$$

while

$$\begin{aligned} d\ E\rfloor\ \omega &= \sum_k (dx^k\ \wedge)\ \partial_{x_k}\ \sum_j x_j\ (-dx_j \bullet \omega) \\ &= \sum_j\sum_k\ dx^k\ \wedge\ (-dx^j \bullet \omega)\ \delta_{jk} - \sum_j\sum_k\ dx^k\ \wedge\ x_j\ (dx^j \bullet \partial_{x_k}\ \omega) \\ &= -\sum_j\ dx^j\ \wedge\ (dx^j \bullet \omega) - \sum_j\sum_k x_j\ dx^k\ \wedge\ \partial_{x_k}\ (dx^j \bullet\ \omega). \end{aligned}$$

Hence

$$E\rfloor\ d\ \omega = E\ \omega + \sum_j\sum_k\ x_j\ dx^k\ \wedge\ \partial_{x_k}\ (dx^j \bullet\ \omega)$$

while

$$d\ E\rfloor\ \omega = d\rfloor\ \omega - \sum_j\sum_k\ x_j\ dx^k\ \wedge\ \partial_{x_k}\ (dx^j \bullet\ \omega)$$

and the desired result follows. ■

By transposing the identity of Lemma 4.1 into Clifford analysis language we get

Corollary 4.3.

For any smooth multi-vector function $F \in \mathcal{E}(\Omega)$ one has

$$\Big((-x \bullet)\ \partial^+ + \partial^+(-x \bullet)\Big)F = (E + \partial^+\rfloor)\ F.$$

Corollary 4.4.

(i) For $\omega^r \in \bigwedge^r(\Omega)$ one has

$$\mathcal{L}_E\ \omega^r = (E\rfloor\ d + d\ E\rfloor)\ \omega^r = (E + r)\ \omega^r.$$

(ii) For $F_r \in \mathcal{E}_r(\Omega)$ one has

$$\Big((-x \bullet)\ \partial^+ + \partial^+(-x \bullet)\Big)F_r = (E + r)\ F_r.$$

Corollary 4.5.

(i) If $\omega_k^r \in \bigwedge^r(\Omega)$ *is homogeneous of degree* k*, then*

$$\mathcal{L}_E\ \omega_k^r = (E\rfloor\ d + d\ E\rfloor)\ \omega_k^r = (k+r)\ \omega^r.$$

(ii) If $F_{r,k} \in \mathcal{E}_r\ (\Omega)$ *is homogeneous of degree* k*, then*

$$\Big((-x\ \bullet)\ \partial^+ + \partial^+(-x\ \bullet)\Big)F_{r,k} = (k+r)\ F_{r,k}.$$

The similar fundamental identity involving the operators $E\rceil$ and d^* is now proven in the language of multi-vector functions.

Corollary 4.6.

For any smooth multi-vector function $F \in \mathcal{E}(\Omega)$ *one has*

$$\Big((-x\ \wedge)\ \partial^- + \partial^-(-x\ \wedge)\Big)F = (E + \partial^-\rceil)\ F.$$

Proof.

On the one hand we have

$$x\ \wedge\ (\partial^- F) = \sum_j\ x_j\ (e_j\ \wedge)\ \sum_k\ (e_k\ \bullet)\ \partial_{x_k}F$$

$$= \sum_j\ x_j\ (e_j\wedge)(e_j\ \bullet)\ \partial_{x_j}F + \sum_{j\neq k}\ x_j\ (e_j\wedge)(e_k\ \bullet)\ \partial_{x_k}F$$

$$= -\sum_j\ x_j\ \partial_{x_j}\ \overset{(j)}{F} + \sum_{j\neq k}\ x_j\ (e_j\wedge)(e_k\ \bullet)\ \partial_{x_k}F$$

where $\overset{(j)}{F}$ denotes that part of F containing the basis vector e_j.
On the other hand we have

$$\partial^-(x\wedge F) = \sum_k\ (e_k\ \bullet)\ \partial_{x_k}\ \sum_j\ x_j\ e_j\wedge F$$

$$= \sum_j\ (e_j\ \bullet)(e_j\wedge)F + \sum_j\sum_k\ x_j\ (e_k\ \bullet)(e_j\wedge)\ \partial_{x_k}F$$

$$= -\partial^-\rceil\ F + \sum_j\ x_j\ (e_j\ \bullet)(e_j\wedge)\ \partial_{x_j}F + \sum_{j\neq k}\ x_j\ (e_k\ \bullet)(e_j\wedge)\ \partial_{x_k}F$$

$$= -\partial^-\rceil\ F - \sum_j\ x_j\ \partial_{x_j}\ \overset{co(j)}{F} + \sum_{j\neq k}\ x_j\ (e_k\ \bullet)(e_j\wedge)\ \partial_{x_k}F$$

where $\overset{co(j)}{F}$ denotes that part of F not containing the basis vector e_j.
Adding both expressions yields the desired result. ■

Corollary 4.7.

For any smooth differential form $\omega \in \bigwedge(\Omega)$ *one has*

$$(E\rceil\ d^* + d^*\ E\rceil)\ \omega = (E + d^*\rceil)\ \omega.$$

Corollary 4.8.

(i) For $\omega^r \in \bigwedge^r(\Omega)$ *one has*

$$(E\rceil\ d^* + d^* E\rceil)\ \omega^r = (E + m + 1 - r)\ \omega^r.$$

(ii) For $F_r \in \mathcal{E}_r(\Omega)$ *one has*

$$\Big((-x\ \wedge)\ \partial^- + \partial^-\ (-x\wedge)\Big)\ F_r = (E + m + 1 - r)\ F_r.$$

Corollary 4.9.

(i) If $\omega^r_k \in \bigwedge\ (\Omega)$ *is homogeneous of degree* k*, then*

$$(E\rceil\ d^* + d^* E\rceil)\ \omega^r_k = (k + m + 1 - r)\ \omega^r_k.$$

(ii) If $F_{r,k} \in \mathcal{E}_r\ (\Omega)$ *is homogeneous of degree* k*, then*

$$\Big((-x\ \wedge)\ \partial^- + \partial^-\ (-x\ \wedge)\Big)\ F_{r,k} = (k + m + 1 - r)\ F_{r,k}.$$

The above considerations lead to the completion of our identification table set up at the beginning of this section.

$E = \sum_j x_j\ \partial_{x_j}$	$E = \sum_j x_j\ \partial_{x_j}$
$\partial_{x_j}\rfloor\ = -dx^j\ \bullet$	$-e_j\ \bullet$
$\partial_{x_j}\rceil\ = -dx^j\ \bullet$	$-e_j \wedge$
$E\rfloor = \sum_j x_j(-dx_j\ \bullet)$	$\sum_j x_j(-e_j\ \bullet) = -x\ \bullet$
$E\rceil = \sum_j x_j(-dx^j \wedge)$	$\sum_j x_j(-e_j \wedge) = -x \wedge$

$E\rfloor + E\rceil = \sum_j x_j\ (dx^j \vee)$	$(-x\ \bullet) + (-x\wedge) = -x$ Clifford product understood
$d\rfloor = \sum_j (dx^j\wedge)(-dx^j\ \bullet)$	$\partial^+\rfloor = \sum_j (e_j\wedge)(-e_j\ \bullet)$
$d^*\rfloor = \sum_j (dx^j\ \bullet)(-dx^j\wedge)$	$\partial^-\rceil = \sum_j (e_j\ \bullet)(-e_j\wedge)$
$\mathcal{L}_E\ = d\ E\rfloor + E\rfloor\ d = E + d\rfloor$	$\partial^+\ (-x\ \bullet) + (-x\ \bullet)\ \partial^+ = E + \partial^+\rfloor$
$\mathcal{L}^*_E\ = d^*\ E\rceil + E\rceil\ d^* = E + d^*\rceil$	$\partial^-\ (-x\wedge) + (-x\wedge)\ \partial^- = E + \partial^-\rceil$

5 The Poincaré and the dual Poincaré Lemmata revisited

In this section we formulate refinements of the classical Poincaré Lemma and its dual, both in the language of differential forms and in the one of multi-vector functions, exploiting the identification established in the previous section.
As it appears to us that these refinements are rarely cited in the literature, we add their proofs.
We start with a classical result, which in the language of three dimensional vector fields is usually called the *Helmholtz decomposition.*

Proposition 5.1.

For each r-form $\omega^r \in \bigwedge^r(\Omega)$ $(0 < r < m+1)$ *there exist* $a^{r+1} \in \bigwedge^{r+1}(\Omega)$ *and* $b^{r-1} \in \bigwedge^{r-1}(\Omega)$ *such that*

(i) $d\ a^{r+1} = 0$;

(ii) $d^*\ b^{r-1} = 0$;

(iii) $\omega^r = d^*\ a^{r+1} + d\ b^{r-1}$.

Proposition 5.2.

For each r-vector function $F_r \in \mathcal{E}_r(\Omega)$ $(0 < r < m+1)$ *there exist* $A_{r+1} \in \mathcal{E}_{r+1}(\Omega)$ *and* $B_{r-1} \in \mathcal{E}_{r-1}(\Omega)$ *such that*

(i) $\partial^+\ A_{r+1} = 0$;

(ii) $\partial^-\ B_{r-1} = 0$;

(iii) $F_r = \partial^- \ A_{r+1} + \partial^+ \ B_{r-1}$.

Proof.

As the Laplace operator $\triangle : \ \mathcal{E}_r(\Omega) \longrightarrow \mathcal{E}_r(\Omega)$ is surjective (see e.g. [14], there ought to exist $G_r \in \mathcal{E}_r(\Omega)$ such that $(-\triangle) \ G_r = F_r$ or $(\partial^- \ \partial^+ + \partial^+\partial^-) \ G_r = F_r$. Put $A_{r+1} = \partial^+ \ G_r$ and $B_{r-1} = \partial^- \ G_r$ to obtain the desired result. ■

Note that $d\omega^r = 0$ iff the $(r+1)$-form a^{r+1} in the above Helmholtz decomposition is harmonic (in the sense of Hodge), while $d^*\omega^r = 0$ iff b^{r-1} is harmonic. Similarly, we have that $\partial^+F_r = 0$ iff A_{r+1} is monogenic, while $\partial^-F_r = 0$ iff B_{r-1} is monogenic. But there is more. The Poincaré Lemma and the Dual Poincaré Lemma will assert that one of those harmonic forms a^{r+1} and b^{r-1}, respectively one of those monogenic multi-vector functions A_{r+1} and B_{r-1}, is absorbed in the other remaining term.

Lemma 5.3. (Poincaré)

Let $r \geq 1$ and let Ω be an open region contractible to a point. Then

$$\overset{r}{\ker} \ d = d \ (\ \overset{r-1}{\ker} \ d^* \ )$$

i.e. the following are equivalent:

(i) $d\omega^r = 0$

(ii) there exists $\omega^{r-1} \in \bigwedge^{r-1}(\Omega)$ *such that* $d^*\omega^{r-1} = 0$ *and* $\omega^r = d\omega^{r-1}$.

Lemma 5.4. (Poincaré)

Let $r \geq 1$ and let Ω be an open region contractible to a point. Then

$$\overset{r}{\ker} \ \partial^+ = \partial^+ \ (\ \overset{r-1}{\ker} \ \partial^- \ )$$

i.e. the following are equivalent:

(i) $\partial^+F_r = \partial \wedge \ F_r = 0$

(ii) there exists $F_{r-1} \in \mathcal{E}_{r-1}(\Omega)$ *such that* $\partial^-F_{r-1} = \partial \bullet F_{r-1} = 0$
and $F_r = \partial^+F_{r-1} = \partial \wedge F_{r-1}$.

Proof.

We prove Lemma 5.3.
$(i) \Longrightarrow (ii)$
From the classical Poincaré Lemma follows the existence of $\alpha^{r-1} \in \bigwedge^{r-1}(\Omega)$ such that $\omega^r = d\alpha^{r-1}$.
As $\triangle : \bigwedge^{r-1}(\Omega) \longrightarrow \bigwedge^{r-1}(\Omega)$ is surjective, there ought to exist $\beta^{r-1} \in \bigwedge^{r-1}(\Omega)$

such that $\triangle \beta^{r-1} = \alpha^{r-1}$.
Put

$$\omega^{r-1} = \alpha^{r-1} + dd^* \beta^{r-1}.$$

Then clearly $d\omega^{r-1} = d\alpha^{r-1} = \omega^r$. Moreover

$$\begin{aligned} d^*\omega^{r-1} &= d^*\alpha^{r-1} + d^*dd^*\beta^{r-1} &&= d^*\alpha^{r-1} + d^*\ (dd^* + d^*d)\ \beta^{r-1} \\ &= d^*\ \alpha^{r-1} - d^*\triangle\beta^{r-1} &&= 0. \end{aligned}$$

$(ii) \Longrightarrow (i)$
Trivial. ■

Lemma 5.5. (Dual Poincaré Lemma)

Let $r < m+1$ and let Ω be an open region contractible to a point. Then

$$\overset{r}{\ker}\ d^* = d^*\ (\ \overset{r+1}{\ker}\ d\ )$$

i.e. the following are equivalent:

(i) $d^*\omega^r = 0$

(ii) there exists $\omega^{r+1} \in \bigwedge^{r+1}(\Omega)$ *such that* $d\omega^{r+1} = 0$ *and* $\omega^r = d^*\omega^{r+1}$.

Lemma 5.6. (Dual Poincaré Lemma)

Let $r < m+1$ and let Ω be an open region contractible to a point. Then

$$\overset{r}{\ker}\ \partial^- = \partial^-\ (\ \overset{r+1}{\ker}\ \partial^+\ )$$

i.e. the following are equivalent:

(i) $\partial^- F_r = \partial \bullet\ F_r = 0$

(ii) there exists $F_{r+1} \in \mathcal{E}_{r+1}(\Omega)$ *such that* $\partial^+ F_{r+1} = 0$ *and* $F_r = \partial^-\ F_{r+1}$.

Proof.

We prove Lemma 5.6.
$(i) \Longrightarrow (ii)$
For each $F_r \in \mathcal{E}_r(\Omega)$, $F_r\, e_M = F_r\ e_o e_1 \ldots e_m = G_{m+1-r}$ belongs to $\mathcal{E}_{m+1-r}(\Omega)$. As $\partial\ G_{m+1-r} = (\partial\ F_r)\ e_M$, we get:

$$\partial^-\ G_{m+1-r} = \partial \bullet\ G_{m+1-r} = [\partial\ G_{m+1-r}]_{m-r} = [\partial\ F_r]_{r+1}\ e_M = (\partial^+\ F_r)\ e_M$$

and also

$$\partial^+\ G_{m+1-r} = \partial \wedge\ G_{m+1-r} = [\partial\ G_{m+1-r}]_{m+2-r} = [\partial\ F_r]_{r-1}\ e_M = (\partial^-\ F_r)\ e_M.$$

Hence F_r will satisfy $\partial^- F_r = 0$ iff $\partial^+ G_{m+1-r} = 0$. Lemma 5.4 then asserts the existence of $G_{m-r} \in \mathcal{E}_{m-r}(\Omega)$ such that $\partial^- G_{m-r} = 0$ and $G_{m+1-r} = \partial^+ G_{m-r}$. As $e_M^2 = \varepsilon_M$, $\varepsilon_M = \pm 1$, we get, putting $G_{m-r}\, e_M\, \varepsilon_M = F_{r+1}$:

$$\begin{aligned} F_r &= G_{m+1-r}\, e_M\, \varepsilon_M = (\partial^+ G_{m-r})\, e_M\, \varepsilon_M = [\partial\, G_{m-r}]_{m+1-r}\, e_M\, \varepsilon_M \\ &= [\partial\, F_{r+1}]_r = \partial^- F_{r+1} \end{aligned}$$

while

$$\partial^+ F_{r+1} = \partial^- G_{m-r} = 0.$$

$(ii) \Longrightarrow (i)$
Trivial. ■

Corollary 5.7.

If the open region Ω is contractible to a point, then the differential operators:

(i) $\partial^-\partial^+ : \overset{r}{\ker}\, \partial^- \longrightarrow \overset{r}{\ker}\, \partial^-$

(ii) $\partial^+\partial^- : \overset{r}{\ker}\, \partial^+ \longrightarrow \overset{r}{\ker}\, \partial^+$

(iii) $\tilde{\triangle} : \mathcal{E}_r(\Omega) \longrightarrow \mathcal{E}_r(\Omega)$

are surjective.

Proof.

(i) Take $F_r \in \overset{r}{\ker}\, \partial^-$. By Lemma 5.6 there exists $F_{r+1} \in \mathcal{E}_{r+1}\,\Omega$ such that $\partial^+ F_{r+1} = 0$ and $\partial^- F_{r+1} = F_r$. So by Lemma 5.4 there exists $G_r \in \mathcal{E}_r(\Omega)$ such that $\partial^- G_r = 0$ and $\partial^+ G_r = F_{r+1}$.
It follows that $\partial^-\partial^+ G_r = \partial^- F_{r+1} = F_r$ with $G_r \in \overset{r}{\ker}\, \partial^-$.

(ii) Similar to the proof of (i).

(iii) Take $F_r \in \mathcal{E}_r(\Omega)$. By Proposition 5.2 there exist $A_{r+1} \in \mathcal{E}_{r+1}(\Omega)$ and $B_{r-1} \in \mathcal{E}_{r-1}(\Omega)$ such that $\partial^+ A_{r+1} = 0$, $\partial^- B_{r-1} = 0$ and $F_r = \partial^- A_{r+1} + \partial^+ B_{r-1}$. By (i) and (ii) there exist $G_r \in \overset{r}{\ker}\, \partial^-$ and $H_r \in \overset{r}{\ker}\, \partial^+$ such that $\partial^-\partial^+ G_r = \partial^- A_{r+1} \in \overset{r}{\ker}\, \partial^-$ and $\partial^+\partial^- H_r = -\partial^+ B_{r-1} \in \overset{r}{\ker}\, \partial^+$. Hence $\partial^-\partial^+(G_r + H_r) = \partial^- A_{r+1}$ and $\partial^+\partial^-(G_r + H_r) = -\partial^+ B_{r-1}$, and thus also $\tilde{\triangle}(G_r + H_r) = (\partial^-\partial^+ - \partial^+\partial^-)(G_r + H_r) = \partial^- A_{r+1} + \partial^+ B_{r-1} = F_r$. ■

Now combining the Poincaré Lemma and the Dual Poincaré Lemma, we obtain the following structure theorem on monogenic multi-vector functions and its counterpart on harmonic differential forms.

Theorem 5.8.

If the open region Ω *is contractible to a point, then for each* $\omega^r \in \bigwedge^r(\Omega)$ $(0 < r < m+1)$ *the following are equivalent:*

(i) ω^r *is harmonic in* Ω*, i.e.* $D\omega^r = (d + d^*)\,\omega^r = 0$ *in* Ω

(ii) *there exists* $\omega^{r-1} \in \bigwedge^{r-1}(\Omega)$ *such that* $d^*\,\omega^{r-1} = 0$, $\triangle\,\omega^{r-1} = 0$ *and* $\omega^r = d\,\omega^{r-1}$

(ii') *there exists* $\omega^{r-1} \in \bigwedge^{r-1}(\Omega)$ *such that* $d^*\,\omega^{r-1} = 0$, $\tilde{\triangle}\,\omega^{r-1} = 0$ *and* $\omega^r = d\,\omega^{r-1}$

(iii) *there exists* $\omega^{r+1} \in \bigwedge^{r+1}(\Omega)$ *such that* $d\,\omega^{r+1} = 0$, $\triangle\,\omega^{r+1} = 0$ *and* $\omega^r = d^*\omega^{r+1}$

(iii') *there exists* $\omega^{r+1} \in \bigwedge^{r+1}(\Omega)$ *such that* $d\,\omega^{r+1} = 0$, $\tilde{\triangle}\,\omega^{r+1} = 0$ *and* $\omega^r = d^*\omega^{r+1}$.

Theorem 5.9.

If the open region Ω *is contractible to a point, then for each* $F_r \in \mathcal{E}_r(\Omega)$ $(0 < r < m+1)$ *the following are equivalent:*

(i) F_r *is monogenic in* Ω*, i.e.* $\partial\,F_r = (\partial^+ + \partial^-)\,F_r = 0$ *in* Ω

(ii) *there exists* $F_{r-1} \in \mathcal{E}_{r-1}(\Omega)$ *such that* $\partial^- F_{r-1} = 0$, $\triangle\,F_{r-1} = 0$ *and* $F_r = \partial^+ F_{r-1}$

(ii') *there exists* $F_{r-1} \in \mathcal{E}_{r-1}(\Omega)$ *such that* $\partial^- F_{r-1} = 0$, $\tilde{\triangle}\,F_{r-1} = 0$ *and* $F_r = \partial^+ F_{r-1}$

(iii) *there exists* $F_{r+1} \in \mathcal{E}_{r+1}(\Omega)$ *such that* $\partial^+ F_{r+1} = 0$, $\triangle\,F_{r+1} = 0$ *and* $F_r = \partial^- F_{r+1}$

(iii') *there exists* $F_{r+1} \in \mathcal{E}_{r+1}(\Omega)$ *such that* $\partial^+ F_{r+1} = 0$, $\tilde{\triangle}\,F_{r+1} = 0$ *and* $F_r = \partial^- F_{r+1}$.

Proof.

$(ii) \Rightarrow (i)$ and $(ii') \Rightarrow (i)$: trivial
$(iii) \Rightarrow (i)$ and $(iii') \Rightarrow (i)$: trivial

$(i) \Rightarrow (ii)$ and $(i) \Rightarrow (ii')$
If F_r is monogenic in Ω then $\partial^+ F_r = 0$ and $\partial^- F_r = 0$ in Ω. By Lemma 5.4 there exists $F_{r-1} \in \mathcal{E}_{r-1}(\Omega)$ such that $\partial^-\,F_{r-1} = 0$ and $\partial^+\,F_{r-1} = F_r$. It follows that in Ω:

$$\partial\,F_{r-1} = \partial^+\,F_{r+1} = F_r$$

and

$$(-\triangle)\; F_{r-1} = \partial\,(\partial\, F_{r-1}) = \partial\, F_r = 0.$$

It also follows that in Ω

$$\tilde{\partial}\; F_{r-1} = (\partial^+ - \partial^-)\; F_{r-1} = F_r$$

and

$$\tilde{\triangle}\; F_{r-1} = \partial\,(\tilde{\partial}\, F_{r-1}) = \partial\, F_r = 0.$$

$(i)\ \Rightarrow (iii)$ and $(i)\ \Rightarrow (iii')$

By Lemma 5.6 there exists $F_{r+1} \in \mathcal{E}_{r+1}(\Omega)$ such that $\partial^+ F_{r+1} = 0$ and $\partial^- F_{r+1} = F_r$. It follows that in Ω :

$$\partial\, F_{r+1} = \partial^-\; F_{r+1} = F_r$$

and

$$(-\triangle)\; F_{r+1} = \partial\,(\partial\, F_{r+1}) = \partial\, F_r = 0.$$

It also follows that in Ω

$$\tilde{\partial}\; F_{r+1} = (\partial^+ - \partial^-)\; F_{r+1} = -\partial^-\; F_{r+1} = -F_r$$

and

$$\tilde{\triangle}\; F_{r+1} = \partial\,(\tilde{\partial}\, F_{r+1}) = -\partial\, F_r = 0.$$

■

Remarks 5.10.

(i) The above Theorems 5.8. and 5.9 may be rephrased as follows.
If the open region Ω is contractible to a point and $0 < r < m+1$, then

$$\begin{aligned}
\overset{r}{\ker}\; D = d\left(\overset{r-1}{\ker}\; d^* \;\cap\; \overset{r-1}{\ker}\,(d^* d)\right) \quad &= \quad d\;(\overset{r-1}{\ker}\; \triangle \;\cap\; \overset{r-1}{\ker}\; d^*)\\
&= \quad d\;(\overset{r-1}{\ker}\; \tilde{\triangle} \;\cap\; \overset{r-1}{\ker}\; d^*)\\
\overset{r}{\ker}\; D = d^*\left(\overset{r+1}{\ker}\; d \;\cap\; \overset{r+1}{\ker}\,(d d^*)\right) \quad &= \quad d^*\;(\overset{r+1}{\ker}\; \triangle \;\cap\; \overset{r+1}{\ker}\; d)\\
&= \quad d^*\;(\overset{r+1}{\ker}\; \tilde{\triangle} \;\cap\; \overset{r+1}{\ker}\; d)\\
\overset{r}{\ker}\; \partial = \partial^+\left(\overset{r-1}{\ker}\; \partial^- \;\cap\; \overset{r-1}{\ker}\,(\partial^- \partial^+)\right) \quad &= \quad \partial^+\;(\overset{r-1}{\ker}\; \triangle \;\cap\; \overset{r-1}{\ker}\; \partial^-)\\
&= \quad \partial^+\;(\overset{r-1}{\ker}\; \tilde{\triangle} \;\cap\; \overset{r-1}{\ker}\; \partial^-)\\
\overset{r}{\ker}\; \partial = \partial^-\left(\overset{r+1}{\ker}\; \partial^+ \;\cap\; \overset{r+1}{\ker}\,(\partial^+ \partial^-)\right) \quad &= \quad \partial^-\;(\overset{r+1}{\ker}\; \triangle \;\cap\; \overset{r+1}{\ker}\; \partial^+)\\
&= \quad \partial^-\;(\overset{r+1}{\ker}\; \tilde{\triangle} \;\cap\; \overset{r+1}{\ker}\; \partial^+).
\end{aligned}$$

(*ii*) For the equivalence (*i*) $\Longleftrightarrow$ (*ii*) of Theorem 5.8 we also refer to [4].

6 From the Euler operator to the Poincaré Lemma

The proof of Lemma 5.3. heavily relies on the classical Poincaré Lemma. In this section we reflect upon the proof of this classical Poincaré Lemma and we present an alternative proof, however restricted to *real-analytic* differential forms in an open ball.

The essence of the proof of the Poincaré Lemma for one-forms is easily grasped.

Indeed, one-forms may be integrated along curves and the integral of a closed one-form from a fixed point to a variable endpoint, in a homologically trivial domain such as a ball, only depends on this endpoint; in other words: for closed one-forms there is a natural notion of primitive.

For higher-order forms the integral operators in the proof of the Poincaré Lemma, are still one-dimensional. How is it possible that such a kind of method is still successful? The answer to this question, at least for the case of a ball, lies in considering the Euler operator E (see also section 4).

Let $\mathcal{P}$ be the algebra of polynomials generated by $\{x_0, x_1, \ldots, x_m\}$ and let $\mathcal{P}_k$ be the subspace of homogeneous polynomials of degree k, $k \in \mathbb{N}$. Then it is clear that

$$\mathcal{P} = \sum_{k=0}^{+\infty} \oplus \, \mathcal{P}_k$$

is the eigenspace decomposition of $\mathcal{P}$ associated with the Euler operator E.

Next consider the algebra Φ of polynomial differential forms, i.e. the free associative algebra generated by $\{x_0, x_1, \ldots, x_m, dx^0, dx^1, \ldots, dx^m\}$. If Φ_k^r denotes the subspace of r-forms with function coefficients in $\mathcal{P}_k$, then one has the decomposition

$$\Phi = \sum_{r=1}^{m+1} \sum_{k=0}^{+\infty} \oplus \, \Phi_k^r$$

and the question arises with which operator this decomposition is associated. The answer to this question is given by Corollary 4.5.(i): for each $\varphi_k^r \in \Phi_k^r$ we indeed have

$$\mathcal{L}_E \; \varphi_k^r = (E \rfloor d + d\, E \rfloor) \; \varphi_k^r = \; (k+r) \; \varphi_k^r \quad ,$$

showing that Φ_k^r is an eigenspace of the operator $\mathcal{L}_E$, which, for $r \geq 1$, has only positive eigenvalues.

The injective linear operator $\mathcal{L}_E : \Phi \longrightarrow \Phi$ thus has a left inverse $\mathcal{L}_E^{-1}$, given by

$$\mathcal{L}_E^{-1} \; \varphi = \sum_{r=1}^{m+1} \; \sum_{k} \; \mathcal{L}_E^{-1}(\varphi_k^r) = \; \sum_{r=1}^{m+1} \; \sum_{k} \frac{1}{k+r} \; \varphi_k^r \quad ,$$

which is also a right inverse:

$$\mathcal{L}_E^{-1}\ \mathcal{L}_E\ \varphi = \mathcal{L}_E\ \mathcal{L}_E^{-1}\ \varphi = \varphi, \quad \text{for all } \varphi \in \Phi.$$

Moreover, as in the case for $\mathcal{L}_E$, the operator $\mathcal{L}_E^{-1}$ commutes with the operators d and $E\rfloor$:

$$\mathcal{L}_E^{-1} d = d\,\mathcal{L}_E^{-1} \quad \text{and} \quad \mathcal{L}_E^{-1} E\rfloor = E\rfloor\ \mathcal{L}_E^{-1}.$$

For any polynomial differential form φ, not containing a scalar part, we thus have

$$\begin{aligned} \varphi &= \mathcal{L}_E^{-1}\ E\rfloor\, d\ \varphi + \mathcal{L}_E^{-1}\ d\ E\rfloor\,\varphi \\ &= \mathcal{L}_E^{-1}\ E\rfloor\, d\ \varphi + d\ \mathcal{L}_E^{-1}\ E\rfloor\,\varphi \end{aligned}$$

and, in particular, for any closed polynomial differential form φ_{closed} we find

$$\varphi_{closed} = d\,(\mathcal{L}_E^{-1} E\rfloor\ \varphi_{closed}) = d\,(E\rfloor \mathcal{L}_E^{-1}\ \varphi_{closed}).$$

This proves the Poincaré Lemma for closed polynomial differential forms in any open region of $\mathbb{R}^{m+1}$.

Finally, let ω^r be a closed *real-analytic* r-form in a ball centred at the origin, say $\overset{\circ}{B}(0,R)$. Then the series

$$\omega^r(x) = \sum_{k=0}^{\infty} \omega_k^r\,(x) \quad , \quad \omega_k^r \in \Phi_k^r$$

together with all its derived series, converges uniformly on the compact subsets of $\overset{\circ}{B}(0,R)$. As for each k,

$$\mathcal{L}_E^{-1}\,\omega_k^r = \frac{1}{k+r}\,\omega_k^r$$

and as the series

$$\sum_{k=0}^{\infty} \frac{1}{k+r}\ \omega_k^r\,(x)$$

together with all its derived series, also converges uniformly on the compact subsets of $\overset{\circ}{B}(0,R)$, we may define

$$\mathcal{L}_E^{-1}\,\omega^r = \sum_{k=0}^{\infty}\ \mathcal{L}_E^{-1}\,\omega_k^r.$$

Hence

$$\begin{aligned} \omega^r &= \sum_{k=0}^{\infty}\ \omega_k^r \;=\; \sum_{k=0}^{\infty}\ d\,(E\rfloor\ \mathcal{L}_E^{-1}\,\omega_k^r) \;=\; d\,(E\rfloor \sum_{k=0}^{\infty}\ \mathcal{L}_E^{-1}\,\omega_k^r) \\ &= d\,(E\rfloor\ \mathcal{L}_E^{-1}\,\omega^r) \end{aligned}$$

which concludes the proof of the Poincaré Lemma for closed real-analytic r-forms in an open ball centred at the origin.

Remark 6.1.

In a similar way the Dual Poincaré Lemma for co-closed real-analytic differential forms in an open ball may be proved. The key steps in the proof are

(i) Corollary 4.8.(i) stating that for each $\varphi_k^r \in \Phi_k^r$:

$$\mathcal{L}_E^* \ \varphi_k^r = (E\rceil d^* + d^* E\rceil)\ \varphi_k^r = (k+m+1-r)\ \mathcal{E}_k^r \quad ;$$

(ii) the commutation rules:

$$\begin{aligned} d^*\mathcal{L}_E^* &= d^* E\rceil\, d^* &= \mathcal{L}_E^*\, d^* \\ E\rceil\,\mathcal{L}_E^* &= E\rceil\, d^* E\rceil &= \mathcal{L}_E^*\, E\rceil \quad ; \end{aligned}$$

(iii) the inversion formula for a polynomial differential form φ :

$$\varphi = (\mathcal{L}_E^{*-1} E\rceil\, d^* + \mathcal{L}_E^{*-1}\, d^* E\rceil)\ \varphi \quad ;$$

(iv) and in particular for a co-closed polynomial differential form $\varphi_{co-closed}$:

$$\varphi_{co-closed} = \mathcal{L}_E^{*-1}\, d^* E\rceil\, \varphi_{co-closed} = d^*\, (\mathcal{L}_E^{*-1} E\rceil)\ \varphi_{co-closed}.$$

7 Differential forms versus multivector functions

In the previous sections we established and illustrated a "natural" isomorphism between on the one hand the Cartan algebra of differential forms (extended with the Hodge star operator and the inner product or dot product), with the underlying structure of the Grassmann algebra, and on the other hand the algebra of multi-vector functions in Clifford analysis with the underlying structure of Clifford algebra. This could easily lead to the conclusion that either one of both is redundant. Indeed it is true that the equations of Clifford analysis may often be rewritten using vector calculus or more generally differential forms. This is nicely illustrated by the correspondence table of section 4 and in particular by the correspondence between the action of the Dirac operator ∂ on multi-vector functions and the action of the operator $D = d + d^*$ on differential forms. Historically this redundancy issue has led to a long and repeated discussion between those who advocate the use of differential forms and those who consider differential forms as an intermediate concept that can be fully replaced by Clifford algebra. Examples of papers where Clifford algebra is realized by means of Grassmann algebra are [7], [9] and [12].
A typical construct in these is the so-called "vee-product" or Clifford product of differential forms (see e.g. [2]). The Dirac operator D on the Cartan algebra $\bigwedge(\Omega)$ may then be defined by

$$D\omega = D \vee \omega \quad , \quad \omega \in \bigwedge(\Omega)\ .$$

It turns out that $D = d + d^*$.
On the other hand, in their book [6] Hestenes and Sobczyk recover most of the theory and calculus of differential forms by interpreting them as alternating tensors which may be represented by means of linear functions on the subspaces of r-vectors in a Clifford algebra, an approach which was made more explicit in [5].
Strictly speaking both points of view are mathematically correct. What we do not agree with is the conclusion that either the use of an extra Clifford basis $(e_0, e_1, \ldots, e_m)$ next to $(dx^0, dx^1, \ldots, dx^m)$ or the use of the differential forms $dx^0, dx^1, \ldots, dx^m$ as basic elements of calculus, is redundant. Despite the similarities as depicted in this paper, the dx^j and e_j are different calculus objects with a different calculus behaviour, which will be fully demonstrated and illustrated in the forthcoming paper [13].
Many examples illustrating the falsity of the "redundancy idea" could be given, but the main counter-argument relies in the success and the richness of the results obtained by considering both the basic differential forms dx^j and the Clifford algebra generators e_j as independent calculus elements. This is nicely demonstrated in e.g. [11] where Chapter 9 focusses on the interplay between *complex* differential forms and *complex* Clifford algebras and its usefulness for classical several complex variables theory is shown.

Received: June 2004. Revised: August 2004.

References

[1] F. BRACKX, R. DELANGHE AND F. SOMMEN, *Clifford Analysis*, Pitman Advanced Publishing Program, Boston - London - Melbourne, 1982

[2] R. DELANGHE, F. SOMMEN AND V. SOUČEK, *Clifford Algebra and Spinor-Valued Functions*, Kluwer Academic Publishers, Dordrecht - Boston - London, 1992

[3] CH. DORAN AND A. LASENBY, *Geometric Algebra for Physicists*, Cambridge University Press, Cambridge, 2003

[4] J. E. GILBERT, J. A. HOGAN AND J. D. LAKEY, *Frame Decompositions of Form-Valued Hardy Spaces*, In: *Clifford Algebras in Analysis and Related Topics*, J. Ryan (ed.), CRC Press, Boca Raton, New York, 1996, 239-259

[5] D. HESTENES, *Differential Forms in Geometric Calculus*, In: *Clifford Algebras and their Applications in Mathematical Physics*, F. Brackx, R. Delanghe

and H. Serras (eds.), Kluwer Academic Publishers, Dordrecht - Boston - London, 1993, 269-285

[6] D. HESTENES AND G. SOBCZYK, *Clifford Algebra to Geometric Calculus*, D. Reidel Publishing Company, Dordrecht, 1984

[7] E. KÄHLER, *Der innere Differentialkalkül*, Rend. Mat.(Roma), 21, 1962, 425-523

[8] K. MAURIN, *Analysis, part II*, D. Reidel Publishing Company, Dordrecht - Boston - London, PWN-Polish Scientific Publishers, Warszawa, 1980

[9] Z. OZIEWICZ, *From Grassmann to Clifford*, In: *Clifford Algebras and their Applications in Mathematical Physics*, J.S.R. Chisholm and A.K. Common (eds.), Nato ASI Series, D. Reidel Publishing Company, Dordrecht - Boston - Lancaster - Tokyo, 1986, 245-255

[10] I. R. PORTEOUS, *Topological Geometry*, Van Nostrand Reinhold Company, London - New York - Toronto - Melbourne, 1969

[11] R. ROCHA-CHÁVEZ, M. SHAPIRO AND F. SOMMEN, *Integral theorems for functions and differential forms in* $\mathbb{C}^m$, Research Notes in Mathematics 428, Chapman & Hall / CRC, Boca Raton - London - New York - Washington, D.C., 2002

[12] N. A. SALINGAROS AND G. P. WENE, *The Clifford Algebra of Differential Forms*, Acta Appl. Math., 4, 1985, 271-292

[13] F. SOMMEN, *Differential Forms in Clifford Analysis*, to appear

[14] F. TRÈVES, *Linear Partial Differential Equations with Constant Coefficients*, Gordon and Breach, New York, 1966

[15] C. VON WESTENHOLZ, *Differential Forms in Mathematical Physics*, Stud. Math. Appl., vol 3, North-Holland, Amsterdam, 1978