CUBO *A Mathematical Journal*
Vol. 7, Nº 2, (57 - 67). August 2005.

Quaternionic analysis and Maxwell's equations

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ABSTRACT

Methods of quaternionic analysis are used to obtain solutions of Maxwell' s equations. By the help of time-discretisation Maxwell's equations are reduced to an equation of Yukawa type. Initial value and boundary value conditions are realized by a representation formula in each time step. Approximation and stability is proved.

RESUMEN

Se usan los métodos de análisis quaternionico para obtener soluciones de las ecuaciones de Maxwell. Con la ayuda de la discretización del tiempo, las ecuaciones de Maxwell son reducidas a una ecuación del tipo Yukawa. Valores iniciales y condiciones de valores en la frontera son realizadas por una fórmula de representación en cada paso de tiempo. Se demuestra la aproximación y estabilidad.

Key words and phrases: *Maxwell equations, quaternionic analysis, operator calculus.*
Math. Subj. Class.: *35F10, 30G35*

1 Introduction

In 1873 J. C. Maxwell' s fundamental paper *A Treatise on Electricity and Magnetism* was published. Since this time generations of physicists and mathematicians felt facsinated from the deepness and beauty of these equations. From the very beginning scientists tried to give Maxwell's equations a more simply algebraical structure maybe in the form

$$Du + Au = F$$

with a suitable derivation operator D and a potential operator A. In this connection we should mention people like L. Silberstein (*The theory of relativity*, 1914), H. Weyl (*Raum -Zeit-Materie*, 1921) and M. Mercier *Expression des équations de electromagnetique au moyen des nombres de Clifford*, 1946). New algebraical notions were introduced and used for the description of Maxwell's equations (for instance: C. Lanczos (1929): spinors, A. Proca (1930): Clifford numbers, A. Einstein/A. Mayer (1932): semi-vectors, F. Bolinder (1957): 4-d forms, G. Kron (1959): skew-symmetric tensors and D. Hestenes (1968): multivector calculus).

We will use in our conception real and complex quaternionic analysis, more exactly a quaternionic operator calculus, which also contains a corresponding quaternionic numerical analysis.

We intent to apply a time-discretisation method (Rothe' s method) in order to reduce Maxwell's equations to a disturbed Yukawa equation. The latter one is considered under realization of initial and boundary values by means of a suited quaternionic calculus. This paper belongs to a series of papers where we use Rothe's method to involve time-dependent problems in our quaternionic calculus. This paper can also be seen as an alternative supplement to latest papers by V.V. Kravchenko et al (cf. [3], [8] and [9].

2 Maxwell' s equations in a chiral medium

Let $G \subset \mathbb{R}^3$ a bounded domain with a sufficiently smooth boundary Γ. In MKS Maxwell's equations read as follows:

$$\operatorname{div} \mathcal{D} = \frac{\rho}{\varepsilon_0}, \tag{1}$$

$$\operatorname{rot} E = -\partial_t \mathcal{B}, \tag{2}$$

$$\operatorname{div} \mathcal{B} = 0, \tag{3}$$

$$\operatorname{rot} H = \mu_0 J + \varepsilon_0 \mu_0 \partial_t \mathcal{D}, \tag{4}$$

where μ_0 is the permeability of the free space, ε_0 the permittivity of the free space, E the imposed electric field, $\mathcal{B}$ the magnetic field, ρ the (free) charge density, H the effective magnetic field inside the dielectric medium, $\mathcal{D}$ the effective electric field inside the dielectric and J the charge density. In a homogeneous chiral medium G the operators $\mathcal{B}$ and $\mathcal{D}$ have to fulfil the Drude-Born-Feodorov constitutive relations:

$$\mathcal{B} : \; = \; \mu H + \mu\beta \mathrm{rot}\, H, \tag{5}$$
$$\mathcal{D} : \; = \; \varepsilon E + \varepsilon\beta \mathrm{rot}\, E, \tag{6}$$

where β is the chirality measure of the medium. Further, we assume:
Initial value conditions:

$$E(x,0) = E_0(x) \quad \text{and} \quad H(x,0) = H_0(x). \tag{7}$$

Boundary condition:

$$E(x,t) = g(x,t) \qquad (x \in \Gamma). \tag{8}$$

Good references for Maxwell's equations in chiral media are the book by A. Lakhtakia [11] in 1994 as well as the article [1].

3 Preliminaries

Let us now introduce notations and operators needed in our approach. Let $\mathbb{H}$ be the set of real quaternions. Each quaternion permits the representation

$$a = \sum_{k=0}^{3} a_k e_k \qquad (a_k \in \mathbb{R};\; k = 0,1,2,3),$$

where $e_0 = 1$ and e_1, e_2, e_3 are the so-called imaginary units. By definition these units e_k obey the following arithmetic rules:

$$e_0^2 = 1, e_1e_2 = -e_2e_1 = e_3, e_2e_3 = -e_3e_2 = e_1 \text{ and } e_3e_1 = -e_1e_3 = e_2.$$

Addition and multiplication in $\mathbb{H}$ turn it into a non-commutative number field. The main-involution in $\mathbb{H}$ is called quaternionic conjugation and defined by

$$\overline{e_0} = e_0, \quad \overline{e_k} = -e_k \quad (k = 1,2,3),$$

which can be extented onto $\mathbb{H}$ by $\mathbb{R}$-linearity. Therefore, we have

$$\overline{a} = a_0 - \sum_{k=1}^{3} a_k e_k.$$

Note that

$$a\overline{a} = \overline{a}a = \sum_{k=0}^{3} a_k^2 =: |a|_{\mathbb{H}}^2.$$

If $a \in \mathbb{H} \setminus \{0\}$ then the quaternion

$$a^{-1} := \frac{\overline{a}}{|a|^2}$$

is the inverse to a. For $a, b \in \mathbb{H}$ we have $\overline{ab} = \overline{b}\,\overline{a}$. The set of complex quaternions, which we also need, is denoted by $\mathbb{H}(\mathbb{C})$ and consist of all elements of the form

$$a = \sum_{k=0}^{3} a_k e_k \qquad (a_k \in \mathbb{C};\ k = 0, 1, 2, 3).$$

By definition we state: $\mathbf{i}e_k = e_k\mathbf{i},\ k = 0, 1, 2, 3$. Here $\mathbf{i}$ denotes the usual imaginary unit in $\mathbb{C}$. Elements of $\mathbb{H}(\mathbb{C})$ can also be represented in the form

$$a = a^1 + \mathbf{i}a^2 \qquad (a^k \in \mathbb{H};\ k = 1, 2).$$

Notice that the quaternionic conjugation acts only on the quaternionic units and not on $\mathbf{i}$.

Let now $G \subset \mathbb{R}^3$ be a bounded domain with sufficient smooth boundary Γ. Assume that all function spaces $\mathcal{B}(G, \mathbb{H}(\mathbb{C})) =: \mathcal{B}(G)$ have the usual componentwise meaning. Let $u \in C^1(G)$. The Dirac-Operator D is defined by

$$Du = \sum_{k=1}^{3} e_k \partial_k u.$$

The operator D is right-linear with respect to complex numbers. On $C^2(G))$ the 3-dimensional Laplacian permits the factorization

$$\Delta = -D^2.$$

We consider the disturbed Laplacian

$$\Delta - \nu^2, \qquad (\nu \in \mathbb{R}).$$

which is called Yukawa operator and acts on $C^2(G)$. This operator has the factorization property

$$\Delta - \nu^2 = (\mathbf{i}\nu + D)(\mathbf{i}\nu - D).$$

The factors $i\nu + D$ and $i\nu - D$ are called mutually generalized Dirac type operators. Functions $u \in \ker(D + \mathbf{i}\nu)$ are called left-($\mathbf{i}\nu$)-hyperholomorphic. The fundamental solution of the Yukawa operator in $\mathbb{R}^3$ is given by

$$\Theta_\nu := -\frac{1}{4\pi}\frac{1}{|x|}e^{-\nu|x|}.$$

Then the corresponding fundamental solution of the operator $\mathbf{i}\nu + D$ is given by

$$e_{\mathbf{i}\nu}(x) := (\mathbf{i}\nu - D)\Theta_\nu(x) = \left(\mathbf{i}\nu + \frac{x}{|x|^2} + \nu\frac{x}{|x|}\right)\Theta_\nu(x)$$

4 A time discretisation method

For simplicity we introduce the following abbreviations:

$$f := \frac{\rho}{\varepsilon_0 \varepsilon},\ a := -\mu,\ b := -\mu\beta,\ c := \varepsilon_0\mu_0\varepsilon,\ d := \varepsilon_0\mu_0\varepsilon\beta.$$

A simply calculation shows that Maxwell's equations in a homogeneous chiral medium transform into

$$\begin{aligned} DE &= -f + \partial_t H + b\partial_t \mathrm{rot}\, H, \\ DH &= \mu_0 J + c\partial_t E + d\partial_t \mathrm{rot}\, E. \end{aligned} \tag{9}$$

Let $T > 0$. The equations (9) are considered in the time-intervall $[0, T]$. A decomposition of $[0, T)$into n equal parts yields $T = n\tau$, where τ is called the meshwidth of the decomposition. We briefly write for $k = 0, 1, ..., n$:

$$E_k := E(k\tau, x), \quad H_k := H(k\tau, x), \quad f_k := f_k(k\tau, x) \quad \text{and} \quad J_k := J(k\tau, x).$$

We want to approximate the time derivatives $\partial_t E$ and $\partial_t H$ by the finite forward differences:

$$\frac{E_{k+1} - E_k}{\tau} \quad \text{and} \quad \frac{H_{k+1} - H_k}{\tau}.$$

More detailed we will consider the case $\beta = 0$. From (9) we obtain for $k = 0, 1, ..., n-1$:

$$DE_{k+1} = -f_k + \frac{a}{\tau}(H_{k+1} - H_k), \tag{10}$$

$$DH_{k+1} = \mu_0 J_k + \frac{c}{\tau}(E_{k+1} - E_k). \tag{11}$$

Setting now

$$\nu^2 = -\frac{ca}{\tau^2} \quad \text{and} \quad L := -\sqrt{-\frac{a}{c}},$$

then we have

$$DE_{k+1} = -f_k + L\nu(H_{k+1} - H_k), \tag{12}$$

$$DH_{k+1} = \mu_0 J_k - \frac{1}{L}\nu(E_{k+1} - E_k). \tag{13}$$

Applying the Dirac operator D from the left we get

$$\begin{aligned} DDE_{k+1} &= -Df_k + L\nu DH_{k+1} - L\nu DH_, \\ &= -Df_k - L\nu DH_k + L\nu\mu_0 J_k - \nu^2 E_{k+1} + \nu^2 E_k \end{aligned}$$

and so

$$(\Delta - \nu^2)E_{k+1} = Df_k + L\nu DH_k - L\nu\mu_0 J_k - \nu^2 E_k. \tag{14}$$

With the same actions we obtain the dual relation

$$(\Delta - \nu^2)H_{k+1} = -\mu_0 DJ_k - \frac{1}{L}\nu f_k - \frac{1}{L}\nu DE_k - \nu^2 H_k. \tag{15}$$

We intent to consider equations (14) and (15) in the hypercomplex setting in (cf.[4], [12]). Main idea is to factorize the Yukawa operator on the left hand side. This leads to

$$(D + i\nu)(D - i\nu)E_{k+1} = -Df_k + L\nu\mu_0 J_k - L\nu DH_k + \nu^2 E_k. \tag{16}$$

5 Borel-Pompeiu's formula

Let G be a bounded domain in $\mathbb{R}^3$ with the Liapunov boundary Γ and let $n = (n_1, n_2, n_3)$ be the unit vector of the outward pointing normal at the point $y \in \Gamma$. The kernel $e_{\mathbf{i}\mu}(x)$ function generates two important integrals: Teodorescu transforms, which are defined by:

$$(T_{\pm\mathbf{i}\nu}u)(x) = \int_G e_{\pm\mathbf{i}\nu}(y - x)u(y)dy$$

as well as the Cauchy-Bizadse operators:

$$(F_{\pm\mathbf{i}\nu}u)(x) = \int_\Gamma e_{\pm\mathbf{i}\mu}(x - y)n(y)u(y)d\Gamma_y.$$

These operators are well studied in several papers, see e.g. [6],[10] and [12]. In [7] was obtained the following Borel-Pompeiu formula:

$$u = (D \pm i\nu)T_{\pm\mathbf{i}\nu}u = T_{\pm i\nu}(D \pm \mathbf{i}\nu)u + F_{\pm\mathbf{i}\nu}u \qquad \text{in } G, \tag{17}$$

where $u \in C^1(G) \cup C(\overline{G})$. Notice that Borel-Pompeiu' s formula is also valid for $u \in W_2^1(G)$. On the boundary Γ we have $tr_\Gamma \mathbf{u} \in W_2^{1/2}(\Gamma)$.

6 Representations

Applying Teodorescu transforms $T_{\pm\mathbf{i}\nu}$ to formula (16) we get the iteration procedures:

$$E_{k+1} = T_{-i\nu}T_{\mathbf{i}\nu}\left[-Df_k + L\nu\mu_0 J_k - L\nu DH_k + \nu^2 E_k\right] + T_{-i\nu}\Phi_+ + \Phi_- \tag{18}$$

and

$$H_{k+1} = T_{-i\nu}T_{\mathbf{i}\nu}\left[\mu_0 DJ_k + \frac{1}{L}\nu\mu_0 f_k + \frac{1}{L}\nu DE_k - \nu^2 H_k\right] + T_{-\mathbf{i}\nu}\Psi_+ + \Psi_-,$$

where $\Phi_\pm$ and $\Psi_\pm$ belong to the kernel of the operators $D \pm \mathbf{i}\nu$. Notice also that holds

$$H_{k+1} = H_k + \frac{1}{L}\nu \mathrm{rot}\, E_k.$$

The unknown functions Φ_+ and Φ_- have to be determined now. In [12] is shown that holds $(D \pm i\mu)T_{\pm \mathbf{i}\mu}u = 0$ and therefore also $F_{\pm \mathbf{i}\mu}T_{\pm \mathbf{i}\mu}u = 0$ for any function $u \in W_2^1$.This leads to

$$\Phi_- = F_{-\mathbf{i}\nu}E_{k+1} = F_{-\mathbf{i}\nu}tr_\Gamma E_{k+1} = F_{-\mathbf{i}\nu}g_{k+1}$$

where $g_k := g(k\tau, x)$. The determination of Φ_+ is more complicated. Using $F_{\mathbf{i}\nu}\Phi_+ = \Phi_+$, which is a consequence of Borel-Pompeiu's formula and $\mathrm{tr}_\Gamma u_{k+1} = g_{k+1}$ we have

$$Q_{\Gamma,-\mathbf{i}\nu}g_{k+1} = \mathrm{tr}_\Gamma T_{-\mathbf{i}\nu}T_{\mathbf{i}\nu}\left[-Df_k + L\nu\mu_0 J_k - L\nu DH_k + \nu^2 E_k\right] + T_{-\mathbf{i}\nu}F_{\mathbf{i}\nu}\Phi_+,$$

where $Q_{\Gamma,-\mathbf{i}\nu}$ is one of the so-called Plemelj projections, which are defined by

$$n.-t.-\lim_{\substack{x'\in\Omega_\pm \\ x'\to x\in\Gamma}} F_{\pm\mathbf{i}\nu}u(x') =: \begin{cases} (P_{\Gamma,\pm\mathbf{i}\nu}u)(x), & x' \in \Omega_+ = G \\ (Q_{\Gamma,\pm\mathbf{i}\nu}u)(x), & x' \in \Omega_- = \mathbb{R}^3 \setminus \overline{G} \end{cases}.$$

In [6] is shown that

$$\mathrm{tr}_\Gamma T_{-\mathbf{i}\nu}F_{\mathbf{i}\nu} : \mathrm{im}\, P_{\Gamma,\mathbf{i}\nu} \cap W_2^{k-1/2}(\Gamma) \to \mathrm{im}Q_{\Gamma,-\mathbf{i}\nu} \cap W_2^{k+1/2}(\Gamma) \quad (k > 1)$$

is an isomorphism. Notice that the pair of Plemelj projections act within corresponding Hardy spaces (cf. [12],[6]). Further, we obtain

$$\begin{aligned} \Phi_+ &= F_{\mathbf{i}\nu}(\mathrm{tr}_\Gamma T_{-\mathbf{i}\nu}F_{\mathbf{i}\nu})^{-1}\left[\mathrm{tr}_\Gamma T_{-\mathbf{i}\nu}T_{\mathbf{i}\nu}(Df_k - L\nu\mu_0 J_k + L\nu DH_k - \nu^2 E_k)\right] + \\ &\quad + F_{\mathbf{i}\nu}(\mathrm{tr}_\Gamma T_{-\mathbf{i}\nu}F_{\mathbf{i}\nu})^{-1}Q_{\Gamma,-\mathbf{i}\nu}g_{k+1}. \end{aligned}$$

Replacing Φ_+ in (18) we get

$$\begin{aligned} &E_{k+1} = T_{-\mathbf{i}\nu}[T_{\mathbf{i}\nu}\left(-Df_k + L\nu\mu_0 J_k - L\nu DH_k + \nu^2 E_k\right) \\ &+F_{\mathbf{i}\nu}(\mathrm{tr}_\Gamma T_{-\mathbf{i}\nu}F_{\mathbf{i}\nu})^{-1}\mathrm{tr}_\Gamma T_{-i\nu}T_{i\nu}(Df_k - L\nu\mu_0 J_k + \nu DH_k - \nu^2 E_k) \\ &+F_{\mathbf{i}\nu}(\mathrm{tr}_\Gamma T_{-\mathbf{i}\nu}F_{\mathbf{i}\nu})^{-1}Q_{\Gamma,-\mathbf{i}\nu}g_{k+1}] + F_{\mathbf{i}\nu}g_{k+1} \\ &= T_{-\mathbf{i}\nu}\left[I - F_{\mathbf{i}\nu}(\mathrm{tr}_\Gamma T_{-\mathbf{i}\nu}F_{\mathbf{i}\nu})^{-1}\mathrm{tr}_\Gamma T_{-\mathbf{i}\nu}\right]T_{\mathbf{i}\nu}(-Df_k + L\nu\mu_0 J_k - L\nu DH_k + \nu^2 E_k) \\ &+T_{-\mathbf{i}\nu}F_{\mathbf{i}\nu}(\mathrm{tr}_\Gamma T_{-\mathbf{i}\nu}F_{\mathbf{i}\nu})^{-1}\mathrm{tr}_\Gamma T_{-\mathbf{i}\nu}(D - \mathbf{i}\nu)\tilde{g}_{k+1} + F_{\mathbf{i}\nu}g_{k+1}, \end{aligned}$$

where $\tilde{g}_{k+1}$ is a smooth extension of g_{k+1} into the domain G. We now introduce the orthoprojections $\mathcal{Q}_{\mathbf{i}\nu} := I - \mathbb{P}_{\mathbf{i}\nu}$, where $\mathbb{P}_{\mathbf{i}\nu} := F_{\mathbf{i}\nu}(\mathrm{tr}_\Gamma T_{-i\nu}F_{i\nu})^{-1}T_{-i\nu}$ is a modified Bergman projection onto the subspace $\ker(D + \mathbf{i}\nu)$ and $\mathcal{Q}_{\mathbf{i}\nu}$ a projection onto the subspace $(D - \mathbf{i}\nu)\dot{W}_2^1$. Finally, we find

$$\begin{aligned} E_{k+1} &= T_{-\mathbf{i}\nu}\mathcal{Q}_{\mathbf{i}\nu}T_{\mathbf{i}\nu}\left[-Df_k + L\nu\mu_0 J_k - L\nu DH_k + \nu^2 E_k\right] && (19) \\ &+ T_{-\mathbf{i}\nu}\mathcal{P}(D - \mathbf{i}\nu)\tilde{g}_{k+1} + F_{-\mathbf{i}\nu}g_{k+1}. && (20) \end{aligned}$$

7 Realization of boundary conditions

The realization of boundary conditions for the imposed electric field makes use of the follwing proposition:

Proposition 7.1 *Let $u \in L_2(G, \mathbb{H}(\mathbb{C}))$. The condition*

$$\mathrm{tr}_\Gamma T_{-\mathbf{i}\nu} u = 0$$

is sufficient and necessary that $u \in \mathrm{im}\mathcal{Q}_{i\nu} \cap L_2(G, \mathbb{H}(\mathbb{C}))$.

Proof. At first let $u \in \mathrm{im}\mathcal{Q}_{i\nu} \cap L_2(G, \mathbb{H}(\mathbb{C}))$. Then u permits the representation

$$u = (D - \mathbf{i}\nu)w \quad \text{with} \quad w \in \dot{W}_2^1(G, \mathbb{H}(\mathbb{C}))$$

and therefore $\mathrm{tr}_\Gamma T_{-\mathbf{i}\nu} u = 0$. On the other hand, if $\mathrm{tr}_\Gamma T_{-\mathbf{i}\nu} u = 0$ then immediately we obtain from Hodge's decomposition of the Hilbert space $L_2(G, \mathbb{H}(\mathbb{C}))$and the representation of the generalized Bergman projection:

$$u = F_{\Gamma,\mathbf{i}\nu}(\mathrm{tr}_\Gamma T_{-\mathbf{i}\nu} F_{\Gamma,\mathbf{i}\nu})^{-1} \mathrm{tr}_\Gamma T_{-\mathbf{i}\nu} u + \mathcal{Q}_{\mathbf{i}\nu} u.$$

The first term vanishes and we have $u = \mathcal{Q}_{\mathbf{i}\nu} u$. #

Let

$$\tilde{E}_{k+1} := T_{-\mathbf{i}\nu} \mathcal{P}(D - \mathbf{i}\nu)\tilde{g}_{k+1} + F_{-\mathbf{i}\nu} g_{k+1},$$

Borel-Pompeiu' s formula yields

$$(-\Delta + \nu^2)\tilde{E}_{k+1} = (D + \mathbf{i}\nu)(D - \mathbf{i}\nu)\tilde{E}_{k+1} = 0.$$

Furthermore, we get from proposition (7.1)

$$\begin{aligned}
\mathrm{tr}_\Gamma \tilde{E}_{k+1} &= \mathrm{tr}_\Gamma T_{-\mathbf{i}\nu} \mathcal{P}_{\mathbf{i}\nu}(D - \mathbf{i}\nu)\tilde{g}_{k+1} + P_{\Gamma,-\mathbf{i}\nu} g_{k+1} \\
&= \mathrm{tr}_\Gamma T_{-\mathbf{i}\nu}(D - \mathbf{i}\nu)\tilde{g}_{k+1} - \mathrm{tr}_\Gamma T_{-\mathbf{i}\nu} \mathcal{Q}_{\mathbf{i}\nu}(D - \mathbf{i}\nu)\tilde{g}_{k+1} + P_{\Gamma,-\mathbf{i}\nu} g_{k+1} \\
&= g_{k+1} - P_{\Gamma,-\mathbf{i}\nu} g_{k+1} + P_{\Gamma,-\mathbf{i}\nu} g_{k+1} - \mathrm{tr}_\Gamma T_{-\mathbf{i}\nu} \mathcal{Q}_{\mathbf{i}\nu}(D - \mathbf{i}\nu)\tilde{g}_{k+1} = g_{k+1}.
\end{aligned}$$

8 Approximation and Stability

From (15) we obtain by setting $F_k := E_k + iLH_k$ the representation

$$F_{k+1} = T_{\mathbf{i}\nu T_{-\mathbf{i}\nu}(\mathbf{i}\nu - D)(f_k - \mathbf{i}\mu_0 L J_k) - \mathbf{i}\nu T_{\mathbf{i}\nu} T_{-i}\nu}(\mathbf{i}\nu - D)F_k + T_{\mathbf{i}\nu}\chi_- + \chi_+,$$

where $\chi_\pm$ belongs to the sets ker $(D \pm \mathbf{i}\nu)$. These functions can be defined by boundary conditions. Further, we abbreviate $M_k := f_k - \mathbf{i}\mu_0 J_k$. Using Borel-Pompeiu' s formula

$$T_{-\mathbf{i}\nu}(\mathbf{i}\nu - D)u = -T_{-\mathbf{i}\nu}(D - \mathbf{i}\nu)u = F_{\Gamma,-\mathbf{i}\nu} u - u$$

we get

$$F_{k+1} = -T_{-\mathbf{i}\nu} M_k + T_{-\mathbf{i}\nu} F_{\Gamma,-\mathbf{i}\nu} M_k + \mathbf{i}\nu T_{\mathbf{i}\nu} F_k - \mathbf{i}\nu T_{-\mathbf{i}\nu} F_{\Gamma,-\mathbf{i}\nu} F_k + T_{\mathbf{i}\nu}\chi_- + \chi_+.$$

Because the image of the Cauchy-Bizadse-type operators $F_{\Gamma,\pm\mathbf{i}\nu}$ belongs to the kernels ker $(D \pm \mathbf{i}\nu)$ eventually we achieve the formula:

$$F_{k+1} = \mathbf{i}\nu T_{-\mathbf{i}\nu} F_k - T_{\mathbf{i}\nu} M_k + T_{\mathbf{i}\nu}\chi_-^* + \chi_+^*.$$

The operator $\mathbf{i}\nu T_{\mathbf{i}\nu}$ is bounded in $L_2(G, \mathbb{H}(\mathbb{C}))$ which follows from ([2] Corollary 2.5). There it is deduced the estimation for the generalized Teodorescu transform

$$\|T_{\pm\mathbf{i}\nu}\|_{[L_2,L_2]} \leq \frac{d}{\nu},$$

where d depends on the diameter of the domain G and tends to zero for diam $G \to 0$. Now it remains to analyze the approximation property of our semi-discretisation procedure for convergence.

Put

$$L^1 := c\partial_t E(t,x) - DH(t,x) \quad \text{and} \quad L^2 := a\partial_t H(t,x) - DE(t,x).$$

Furthermore, we introduce the operators:

$$\begin{aligned} L_\tau^1 &= c\frac{E(t+\tau,x) - E(t,x)}{\tau} + DH(t+\tau,x), \\ L_\tau^2 &= a\frac{H(t+\tau,x) - H(t,x)}{\tau} + DE(t+\tau,x). \end{aligned}$$

We have to estimate the differences $L^j - L_\tau^j$ for $j = 1,2$. With $t = k\tau$ it follows

$$\begin{aligned} |(L^1 - L_\tau^1)| &\leq \frac{c}{\tau}|(E_{k+1} - E_k - \tau\partial_t E_k) + D(H_{k+1} - H_k)|, \\ |(L^2 - L_\tau^2)| &\leq \frac{a}{\tau}|(H_{k+1} - H_k - \tau\partial_t H_k) + DE_{k+1} - E_k|. \end{aligned}$$

We intent to continue with the first estimate. It is easy to show that

$$\begin{aligned} |L^1 - L_\tau^1| &\leq c\tau|\partial_{tt}E(k\tau + \theta\tau, x)| + \mu_0\tau|\partial_t J_k(k\tau + \theta'\tau, x)| + c\tau|\partial_{tt}E(k\tau + \theta''\tau, x)| \\ &\leq \tau C_k^1(E, J) \end{aligned}$$

with $\theta, \theta', \theta'' \in (k\tau, (k+1)\tau)$. Analogously, we obtain for the second estimation

$$|L^2 - L_\tau^2| \leq \tau C_k^2(H, f).$$

On this way the truncation error is estimated for sufficient smooth E, H, J and ρ

Remark 8.1 With the same principle also the chiral case can be considered. One obtains with similar calculations the following iteration procedure:

$$\left(D+\frac{\mathbf{i}\nu}{1+\beta\mathbf{i}\nu}\right)F_{k+1}=\frac{M_k}{1+\beta\mathbf{i}\nu}+\left(D+\frac{\mathbf{i}\nu}{1+\beta\mathbf{i}\nu}\right)\left(\frac{\beta\mathbf{i}\nu}{1+\beta\mathbf{i}\nu}\right)F_k-\left(\frac{\mathbf{i}\nu}{1+\beta\mathbf{i}\nu}\right)\left(\frac{\beta\mathbf{i}\nu}{1+\beta\mathbf{i}\nu}\right)F_k.$$

Received: June 2004. Revised: November 2004.

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