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Convergence rates in regularization for ill-posed variational inequalities

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ABSTRACT

In this paper the convergence rates for ill-posed inverse-strongly monotone variational inequalities in Banach spaces are obtained on the base of choosing the regularization parameter by the generalized discrepancy principle.

RESUMEN

En este artículo se obtienen tasas de convergencia para desigualdades variacionales en problemas inversos mal puestos fuertemente monótonos en espacios de Banach, sobre la base de la elección del parámetro de regularización por medio del principio de discrepancia generalizada.

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1 Introduction.

Let X be a real reflexive Banach space having the E-property and X^*, the dual space of X, be strictly convex. For the sake of simplicity, the norms of X and X^* will be denoted by the symbol $\|.\|$. We write $\langle x^*, x\rangle$ instead of $x^*(x)$ for $x^* \in X^*$ and $x \in X$. Let A be a hemi-continuous and monotone operator from X into X^*, and K be a closed convex subset of X.

For a given $f \in X^*$, consider the variational inequality: find an element $x_0 \in K$ such that

$$\langle A(x_0) - f, x - x_0\rangle \geq 0, \quad \forall x \in K. \tag{1.1}$$

Variational inequalities and their approximations have been extensively studied in the last two decates. Existence and approximations of solutions of variational inequalities for various classes of operators in Hilbert and Banach spaces have been considered in [1]-[5], [7], [8], [10], [11] and [13]. We mention, in particular, the paper [3], [11], where the operator method or iterative method of regularization are considered. Further, in [7] the convergence rates of the operator method of regularization is investigated under the inverse-strongly monotone A in Hilbert space when the parameter of regularization α is chosen a priory.

In the Banach space X, the operator method of regularization is the following variational inequality

$$\langle A_h(x^\tau_\alpha) + \alpha U(x^\tau_\alpha - x^0) - f_\delta, x - x^\tau_\alpha\rangle \geq 0, \quad x^\tau_\alpha \in K, \quad \forall x \in K, \tag{1.2}$$

where A_h are also monotone operators from X into X^* and approximate A in the sense

$$\|A_h(x) - A(x)\| \leq hg(\|x\|) \tag{1.3}$$

with a nonegative continuous and bounded (image of bounded set is bounded) function $g(t)$, U is the normalized duality mapping of X, i.e., U is the mapping from X onto X^* satisfying the condition (see [14])

$$\langle U(x), x\rangle = \|x\|^2, \quad \|U(x)\| = \|x\|,$$

f_δ are the approximations of f : $\|f_\delta - f\| \leq \delta$, $\tau = (h, \delta)$, and x^0 is some element in X playing the role of a criterion selection. By the choice of x^0, we can influence which solution we want to approximate.

In [11], it is showed the existence and uniqueness of the solution x^τ_α for every $\alpha > 0$ and for arbitrary A_h, f_δ. And, the regularized solution x^τ_α converges to $x_0 \in S_0$, the set of solutions of (1.1) which is assumed to be nonempty, with

$$\|x_0 - x^0\| = \min_{x\in S_0} \|x - x^0\|,$$

if $(h+\delta)/\alpha, \alpha \to 0$. Moreover, for each fixed $\tau = (\delta, h)$ the papameter of regularization α can be chosen by the discrepancy principle

$$\rho(\overline{\alpha}) = (k-1)(\delta + h)^p + \delta^p + g(\|x^\tau_{\overline{\alpha}}\|)h^p, \quad 0 < p < 1, k > 1,$$

where $\rho(\alpha) = \alpha\|x_\alpha^\tau - x^0\|$, under the conditions: $x^0 \in \text{int } K$ and

$$\|A_h(x^0) - f_\delta\| > (k-1)(\delta+h)^p + \delta^p + g(\|x^0\|)h^p$$

for $0 < \delta < \overline{\delta} < 1, \quad 0 < h < \overline{h} < 1$. The case $x^0 \in \partial K$ also is considered when $x_\alpha^\tau \in \text{int } K$.

In this paper, under the condition $x^0 \in K\backslash S_0$ without the restriction $x_\alpha^\tau \in \text{int } K$ we shall show that the parameter of regularization $\overline{\alpha} = \alpha(\delta, h)$ can be chosen by the generalized discrepancy principle

$$\rho(\alpha) = (\delta+h)^p\alpha^{-q}, \quad p, q > 0, \tag{1.4}$$

for arbitrary monotone operator A, and on the base of the result we can estimate the convergence rates when A is an inverse-strongly monotone operator, i.e., A possesses the property

$$\langle A(x) - A(y), x - y\rangle \geq \frac{1}{\beta}\|A(x) - A(y)\|^2, \quad \forall x, y \in X, \tag{1.5}$$

where β is some positive constant. In facts, variational inequalities with inverse-strongly monotone operator belong to a class of nonlinear ill-posed problems (see [7]).

Note that the generalized discrepancy principle for parameter choice is presented first in [6] for the ill-posed operator equation

$$A(x) = f \tag{1.6}$$

when A is a linear and bounded operator in Hilbert space. Recently, it is considered and applied in estimating convergence rates of the regularized solution for equation (1.6) involving an m-accretive (in general nonlinear) operator (see [9]).

Later, the symbols $\rightharpoonup$ and $\to$ denote weak convergence and convergence in norm, respectively, and the notation $a \sim b$ is meant that $a = O(b)$ and $b = O(a)$.

2. Main result

To obtain the result on the convergence rate for $\{x_{\alpha(\delta,h)}^\tau\}$ as in [6] we need the following lemmas.

Lemma 1. *For each $p, q, \delta, h > 0$, there exists at least a value α such that (1.4) holds.*

Proof. It follows from [11] that $\rho(\alpha)$ is a continuous and nondecreasing function on $[\alpha_0, +\infty), \alpha_0 > 0$. Moreover, $\rho(\alpha) > 0 \quad \forall \quad \alpha \neq 0$. Indeed, if $\alpha_1 \neq 0$ with $\rho(\alpha_1^\tau) = 0$, then $x_{\alpha_1}^\tau = x^0$ and from (1.2) it follows

$$\langle A_h(x^0) - f_\delta, x - x^0\rangle \geq 0, \quad \forall x \in K.$$

After passing δ and h to zero in this inequality we see $x^0 \in S_0$. This contradicts the assumption $x^0 \in K\backslash S_0$. Therefore, $\alpha^q\rho(\alpha) \to +\infty$, as $\alpha \to +\infty$. On the other hand, since

$$\begin{aligned} 0 \leq \rho(\alpha) &= \alpha\|x_\alpha^\tau - x^0\| \\ &\leq \delta + hg(\|x_0\|\|) + 2\alpha\|x_0 - x^0\| \end{aligned}$$

(see also [11]), we have $\alpha^q\rho(\alpha) \to 0$, as $\alpha \to +0$. Hence, there exists a value α such that (1.4) holds.

Lemma 2. $\lim_{\delta,h\to 0} \alpha(\delta, h) = 0$.

Proof. Let $\delta_n, h_n \to 0$, and $\alpha_n = \alpha(\delta_n, h_n) \to \infty$ as $n \to \infty$. From (1.3),

$$\langle A_{h_n}(x^{\tau_n}_{\alpha_n}) + \alpha_n U(x^{\tau_n}_{\alpha_n} - x^0) - f_{\delta_n}, x - x^{\tau_n}_{\alpha_n}\rangle \geq 0, \quad \forall x \in K, \tag{2.1}$$

the monotone property of A_{h_n} and $x^0 \in K$ it follows

$$\|x^{\tau_n}_{\alpha_n} - x^0\| \leq \|A_{h_n}(x^0) - f_{\delta_n}\|/\alpha_n \to 0,$$

as $n \to \infty$. Therefore, $x^{\tau_n}_{\alpha_n} \to x^0$, as $n \to \infty$. On the other hand, by using the monotone property of A_{h_n} and the property of U we can write (2.1) in the form

$$\begin{aligned}\langle A_{h_n}(x) - f_{\delta_n}, x - x^{\tau_n}_{\alpha_n}\rangle &\geq -\alpha_n\langle U(x^{\tau_n}_{\alpha_n} - x^0), x - x^{\tau_n}_{\alpha_n}\rangle\\ &\geq -\alpha_n\|x^{\tau_n}_{\alpha_n} - x^0\|\|x - x^{\tau_n}_{\alpha_n}\|\\ &\geq -\rho(\alpha_n)\|x - x^{\tau_n}_{\alpha_n}\|\\ &\geq -(\delta_n + h_n)^p\alpha_n^{-q}\|x - x^{\tau_n}_{\alpha_n}\| \to 0,\end{aligned}$$

as $n \to \infty$. It means that

$$\langle A(x^0) - f, x - x^0\rangle \geq 0, \quad \forall x \in K,$$

i.e., x^0 is a solution of (1.1). It contradicts $x^0 \notin S_0$.

Thus, $\alpha(\delta, h)$ remains bounded as $\delta, h \to 0$. Let $\delta_n, h_n \to 0$ as $n \to \infty$, and meantime $\alpha_n \to c > 0$. Since $\alpha_n^{1+q}\|x^{\tau_n}_{\alpha_n} - x^0\| = (\delta_n + h_n)^p$, we have $\|x^{\tau_n}_{\alpha_n} - x^0\| \to 0$, as $n \to \infty$. Again, $x^0 \in S_0$. Hence, $\lim_{\delta,h\to 0}\alpha(\delta, h) = 0$.

Lemma 3. *If* $0 < p < q$, *then* $\lim_{\delta,h\to 0}(\delta + h)/\alpha(\delta, h) = 0$.

Proof. It is easy to see that

$$\begin{aligned}&\left[\frac{\delta + h}{\alpha(\delta, h)}\right]^p [(\delta + h)^p\alpha(\delta, h)^{-q}]\alpha(\delta, h)^{q-p}\\ &\qquad= \rho(\alpha(\delta, h))\alpha(\delta, h)^{q-p} = \alpha(\delta, h)\|x^{\tau}_{\alpha(\delta,h)} - x^0\|\alpha(\delta, h)^{q-p}\\ &\qquad\leq \left[\delta + hg(\|x^0\|) + 2\alpha(\delta, h)\|x_0 - x^0\|\right]\alpha(\delta, h)^{q-p} \to 0\end{aligned}$$

as $\delta, h \to 0$. Therefore,

$$\lim_{\delta,h\to 0}\left[\frac{\delta + h}{\alpha(\delta, h)}\right]^p = 0.$$

The lemma is proved.

Lemma 4. *Let $0 < p < q$. Then, there exist constants $C_1, C_2 > 0$ such that, for sufficiently small $\delta, h > 0$, the relation*

$$C_1 \le (\delta + h)^p \alpha(\delta, h)^{-1-q} \le C_2$$

holds.

Proof. From

$$\begin{aligned}(\delta + h)^p \alpha(\delta, h)^{-1-q} &= \alpha(\delta, h)^{-1}\rho(\alpha(\delta, h)) = \|x^\tau_{\alpha(\delta,h)} - x^0\| \\ &\le \frac{\delta}{\alpha(\delta, h)} + \frac{h}{\alpha(\delta, h)} g(\|x_0\|) + 2\|x_0 - x^0\|\end{aligned}$$

and lemma 3, it implies the existence of a positive constant C_2 in the lemma.

On the other hand, as X is reflexive and $\{x^\tau_{\alpha(\delta,h)}\}$ is bounded, there exists a subsequence of the sequence $\{x^\tau_{\alpha(\delta,h)}\}$ that converges weakly to some element $\tilde{x}_0$ in K such that

$$\|\tilde{x}_0 - x^0\| \le \liminf \|x^\tau_{\alpha(\delta,h)} - x^0\|.$$

We can conclude that $\tilde{x}^0 \ne x^0$. Indeed, if $\tilde{x}^0 = x^0$, then from the monotone hemi-continuous property of A_h and (1.2) it follows

$$\langle A_h(x) + \alpha U(x - x^0) - f_\delta, x - x^\tau_\alpha \rangle \ge 0, \quad \forall x \in K.$$

After passing δ and h in the last inequality to zero we obtain

$$\langle A(x) - f, x - \tilde{x}^0 \rangle \ge 0, \quad \forall x \in K$$

which is equivalent to (1.1). It is meant that $\tilde{x}^0 \in S_0$. It contradicts $x^0 \notin S_0$. Therefore, there exists a constant C_1 in the lemma.

To estimate the convergence rates for $\{x^\tau_{\alpha(\delta,h)}\}$ we assume that

$$\langle U(x) - U(y), x - y \rangle \ge m_U \|x - y\|^s, \quad m_U > 0, s \ge 2, \quad \forall x, y \in X. \tag{2.2}$$

It is well-known that when $X \equiv H$, the Hilbert space, $m_U = 1$, $s = 2$, and when $X = L_p$ or W_p, $m_U = p - 1, s = 2$ for the case $1 < p < 2$. In the case $p > 2$ we have to use the duality mapping U^s satisfying the condition

$$\langle U^s(x), x \rangle = \|x\|^s, \quad \|U^s(x)\| = \|x\|^{s-1}, \quad s \ge 2$$

instead of U. Then, $m_{U^s} = 2^{2-p}/p$ and $s = p$ in (2.2) (see [12]).

Theorem 1. *Assume that the following conditions hold:*

(i) A is an inverse-strongly-monotone operator in X with

$$\|A(x) - A(x_0) - A'(x_0)(x - x_0)\| \le \tilde{\tau} \|A(x) - A(x_0)\|, \quad \forall x \in X,$$

where $\tilde{\tau}$ is some positive constant;

(ii) There exists an element $z \in X$ such that $A'(x_0)^* z = U(x_0 - x^0)$;
(iii) The parameter α is chosen by (1.4) with $p < q$.
Then, we have

$$\|x^\tau_{\alpha(\delta,h)} - x_0\| = O((\delta + h)^\theta), \quad \theta = \frac{p}{(1+q)(2s-1)}.$$

Proof. From (1.1) - (1.3) it follows

$$\begin{aligned}\langle A(x^\tau_{\alpha(\delta,h)}) - A(x_0), x^\tau_{\alpha(\delta,h)} - x_0\rangle &+ \alpha(\delta, h)\\ &\times \langle U(x^\tau_{\alpha(\delta,h)} - x^0) - U(x_0 - x^0), x^\tau_{\alpha(\delta,h)} - x_0\rangle\\ &\le (\delta + hg(\|x_{\alpha(\delta,h)}\|))\|x_{\alpha(\delta,h)} - x_0\|\\ &+\alpha(\delta, h)\langle U(x_0 - x^0), x_0 - x^\tau_{\alpha(\delta,h)}\rangle. \end{aligned} \tag{2.3}$$

Thus, by using (1.5) and the monotone property of U we obtain

$$\|A(x^\tau_{\alpha(\delta,h)}) - A(x_0)\| \le O(\sqrt{\delta + h + \alpha(\delta, h)})\|x^\tau_{\alpha(\delta,h)} - x_0\|^{1/2}.$$

On the other hand, from (2.2), (2.3) and the monotone property of A which is followed from (1.5) we have

$$\begin{aligned}m_U\|x^\tau_{\alpha(\delta,h)} - x_0\|^s &\le \langle U(x^\tau_{\alpha(\delta,h)} - x^0) - U(x_0 - x^0), x^\tau_{\alpha(\delta,h)} - x_0\rangle\\ &\le \frac{\delta + \tilde{C}_0 h}{\alpha(\delta, h)}\|x^\tau_{\alpha(\delta,h)} - x^0)\| + \langle z, A'(x_0)(x_0 - x^\tau_{\alpha(\delta,h)})\rangle\end{aligned}$$

where $\tilde{C}_0$ is some positive constant, and

$$\begin{aligned}\left|\langle z, A'(x_0)(x_0 - x^\tau_{\alpha(\delta,h)})\rangle\right| &\le \|z\|(\tilde{\tau} + 1)\|A(x^\tau_{\alpha(\delta,h)}) - A(x_0)\|\\ &\le \|z\|(\tilde{\tau} + 1)O(\sqrt{\delta + h + \alpha(\delta, h)})\|x^\tau_{\alpha(\delta,h)}) - x_0\|^{1/2}.\end{aligned}$$

Now, from lemma 4 it implies that

$$\alpha(\delta, h) \le C_1^{-1/(1+q)}(\delta + h)^{p/(1+q)}.$$

and

$$\begin{aligned}\frac{\delta + h}{\alpha(\delta, h)} &\le C_2(\delta + h)^{1-p}\alpha(\delta, h)^q\\ &\le C_2C_1^{-q/(1+q)}(\delta + h)^{1-p}(\delta + h)^{pq/(1+q)}\\ &\le C_2C_1^{-q/(1+q)}(\delta + h)^{1-p/(1+q)}.\end{aligned}$$

In final, we have

$$\begin{aligned}m_U\|x^\tau_{\alpha(\delta,h)} - x_0\|^{s-1/2} &\le \max\{1, \tilde{C}_0\}C_2C_1^{-q/(1+q)}(\delta + h)^{1-p/(1+q)}\\ &\times \|x^\tau_{\alpha(\delta,h)} - x_0\|^{1/2} + O(\sqrt{\delta + h + \alpha(\delta, h)})\\ &\le O((\delta + h)^{1-p/(1+q)}\|x^\tau_{\alpha(\delta,h)} - x_0\|^{1/2} + O((\delta + h)^{p/2(1+q)}).\end{aligned}$$

Using the implication

$$a, b, c \geq 0, s > t, a^s \leq ba^t + c \Longrightarrow a^s = O(b^{s/(s-t)} + c)$$

we obtain

$$\|x^{\tau}_{\alpha(\delta,h)} - x_0)\| = O((\delta + h)^{\theta}).$$

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