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The ergodic measures related with nonautonomous hamiltonian systems and their homology structure. Part 1 [1]

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ABSTRACT

There is developed an approach to studying ergodic properties of time-dependent periodic Hamiltonian flows on symplectic metric manifolds having applications in mechanics and mathematical physics. Based both on J. Mather's [9] results about homology of probability invariant measures minimizing some Lagrangian

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functionals and on the symplectic field theory devised by A. Floer and others [3-8,12,15] for investigating symplectic actions and Lagrangian submanifold intersections, an analog of Mather's β-function is constructed subject to a Hamiltonian flow reduced invariantly upon some compact neighborhood of a Lagrangian submanifold. Some results on stable and unstable manifolds to hyperbolic periodic orbits having applications in the theory of adiabatic invariants of slowly perturbed integrable Hamiltonian systems are stated within the Gromov-Salamon-Zehnder [3,5,12] elliptic techniques in symplectic geometry.

RESUMEN

Un método para estudiar propiedades ergódicas de flujos Hamiltonianos que dependen del tiempo sobre variedades simplécticas es desarrollado. Basados tanto en un trabajo de J. Mather [9] sobre homología de medidas invariantes de probabilidad que minimizan algunos funcionales lagrangianos, como en la teoría de campos simplécticos, desarrollada por A. Floer y otros [3-8,12,15] para investigar acciones simplécticas e intersecciones de subvariedades lagrangianas, se construye un análogo de la función β de Mather sujeto a un flujo hamiltoniano reducido invariantemente sobre una vecindad compacta de una subvariedad Lagrangiana. Se plantean algunos resultados sobre variedades estables e intestables de órbitas hiperbólicas periódicas. Estas tienen aplicaciones en la teoría de sistemas hamiltonianos integrables con perturbaciones lentas, en el marco de las técnicas elípticas de Gromov-Salamon-Zehnder [3,5,12] en geometría simpléctica.

Key words: *Ergodic measures, Holonomy groups, Dynamical systems, Quasi-complex structures, Symplectic field theory*

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Introduction

The past years have given rise to several exciting developments in the field of symplectic geometry and dynamical systems [3-12], which introduced new mathematical tools and concepts suitable for solving many before too hard problems. When studying periodic solutions to non-autonomous Hamiltonian systems Salamon & Zehnder [3] developed a proper Morse theory for infinite dimensional loop manifolds based on previous results on symplectic geometry of Lagrangian submanifolds of Floer [4, 6]. Investigating at the same time ergodic measures related with Lagrangian dynamical systems on tangent spaces to configuration manifolds, Mather [9] devised a new approach to studying the correspondingly related invariant probabilistic measures based on a so called β-function. The latter made it possible to describe effectively the so called homology of these invariant probabilistic measures minimizing the corresponding Lagrangian action functional.

As one can easily see, the Mather approach doesn't allow any its direct application to the problem of describing the ergodic measures related naturally with a given periodic non-autonomous Hamiltonian system on a closed symplectic space. Thereby, to overcome constraints to this task we suggest in the present work some new way to imbedding the non-autonomous Hamiltonian case into the Mather β-function theory picture, making use of the mentioned above Salamon & Zehneder and Floer [3, 4, 6] loop space homology structures. Based further on the Gromov elliptic techniques in symplectic geometry, the latters make it possible to construct the invariant submanifolds of our Hamiltonian system, naturally related with corresponding compact Lagrangian submanifolds, and the related on them a β-function analog.

1 Symplectic and analytic problem setting

Let $(M^{2n}, \omega^{(2)})$ be a closed symplectic manifold of dimension $2n$ with a symplectic structure $\omega^{(2)} \in \Lambda(M^{2n})$ being weakly exact, that is $\omega^{(2)}(\pi_2(M^{2n})) = 0$. Every smooth enough time-dependent 2π-periodic function $H : M^{2n} \times \mathbb{S}^1 \to \mathbb{R}$ gives rise to the non-autonomous vector field $X_H : M^{2n} \times \mathbb{S}^1 \to T(M^{2n})$ defined by the equality

$$i_{X_H}\omega^{(2)} = -dH, \tag{1}$$

where as usually [1], the operation " i_{X_H} " denotes the intrinsic derivation of the Grassmann algebra $\Lambda(M^{2n})$ along the vector field X_H. The corresponding flow on $M^{2n} \times \mathbb{S}^1$ takes the form:

$$du/ds = X_H(u;t), \qquad dt/ds = 1, \tag{2}$$

where $u : \mathbb{R} \to M^{2n}$ is an orbit, $t \in \mathbb{R}/2\pi\mathbb{Z} \simeq \mathbb{S}^1$ and $s \in \mathbb{R}$ is an evolution parameter. We shall assume that solutions to (2) are complete and determine a one-parametric ψ-flow of diffeomorphisms $\psi^s : M^{2n} \times \mathbb{S}^1 \to M^{2n} \times \mathbb{S}^1$ for all $s \in \mathbb{R}$ which are due to (1) evidently symplectic, that is $\psi_{t_0}^{s*}\omega^{(2)} = \omega^{(2)}$ where $\psi_{t_0}^s := \psi^s|_{M^{2n}}$ at any fixed $t_0 \in \mathbb{R}/2\pi\mathbb{Z} \simeq \mathbb{S}^1$. Take now an $(n+1)$-dimensional submanifold $\mathcal{L}^{n+1} \subset M^{2n} \times \mathbb{R}$, such that for any closed contractible curve γ with $\gamma \subset \mathcal{L}^{n+1}$ the following integral equality

$$\oint_\gamma (\alpha^{(1)} - H(t)dt) = 0 \tag{3}$$

holds, where $\alpha^{(1)} \in \Lambda^1(M^{\ 2n})$ is such a 1-form on M^{2n} which satisfies the condition $\int_{D^2}(\omega^{(2)} - d\alpha^{(1)}) = 0$ for any compact two-dimensional disk $D^2 \subset M^{2n}$ due to the weak exactness of the symplectic structure $\omega^{(2)} \in \Lambda^2(M^{2n})$ and existing globally on $\mathcal{L}^{n+1}$ due to Floer results [4, 6]. Assume now also that for the flow of symplectomorphisms $\psi_{t_0}^s : M^{2n} \to M^{2n}$, $s \in \mathbb{R}$, the condition

$$\{(\psi_{t_0}^s \mathcal{L}_{t_0}^n, t_0 + s) : s \in \mathbb{R}\} \subset \mathcal{L}^{n+1} \tag{4}$$

holds for some compact Lagrangian submanifold $\mathcal{L}_{t_0}^n \subset M^{2n}$ upon which $\omega^{(2)}\big|_{\mathcal{L}_{t_0}^n} = 0$. The condition (4) in particular means [2] that the following expression

$$\alpha^{(1)} - H(t)dt = d\mathcal{A}(t), \tag{5}$$

$t = t_0 + s(\text{mod}2\pi) \in \mathbb{R}/2\pi\mathbb{Z}$, holds in some vicinity of the Lagrangian submanifold $\mathcal{L}_{t_0}^n \subset M^{2n}$, where a mapping $\mathcal{A} : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}$ is the so called [1, 2] generating function for the defined above continuous set of diffeomorphisms $\psi_{t_0}^s \in$Diff(M^{2n}), $s \in \mathbb{R}$. The expression (5) makes it possible to define naturally the following Poincare-Cartan type functional on a set of almost everywhere differentiable curves $\gamma : [0, \tau] \to M^{\ 2n} \times \mathbb{S}^1$

$$\mathcal{A}_{t_0}^{(\tau)}(\gamma) := \frac{1}{\tau}\int_\gamma (\alpha^{(1)} - H(t)dt), \tag{6}$$

with end points $\{\ \gamma(\tau) = \psi^\tau(\gamma(0))\ \}$, *supp* $\gamma \subset \mathcal{U}(\mathcal{L}_{t_0}^n) \times \mathbb{S}^1$ for all $\tau \in \mathbb{R}$ and $\mathcal{U}(\mathcal{L}_{t_0}^n)$ is some compact neighborhood of the Lagrangian submanifold $\mathcal{L}_{t_0}^n \subset M^{2n}$ satisfying the condition $\psi_{t_0}^s\mathcal{U}(\mathcal{L}_{t_0}^n) \subset \mathcal{U}(\mathcal{L}_{t_0}^n)$ for all $s \in \mathbb{R}$.

Let us denote by $\Sigma_{t_0}(H)$ the subset of curves γ with support in $\mathcal{U}(\mathcal{L}_{t_0}^n) \times \mathbb{S}^1$ and fixed end-points as before minimizing the functional (6). If the infimum is realized, one easily shows that any such curve $\gamma \in \Sigma_{t_0}(H)$ solves the system (2). For the above set of curves $\Sigma_{t_0}(H)$ to be specified more suitably, choose, following Floer's ideas [3-8,12], an almost complex structure $J : M^{2n} \to End(T(M^{2n}))$ on the symplectic manifold M^{2n}, where by definition $J^2 = -I$, compatible with the symplectic structure $\omega^{(2)} \in \Lambda^2(M^{2n})$. Then the expexpression

$$<\xi, \eta> := \omega^{(2)}(\xi, J\eta), \tag{7}$$

where $\xi, \eta \in T(M^{2n})$, naturally defines a Riemannian metric on M^{2n}. Subject to the metric (7) our Hamiltonian vector field $X_H : M^{2n} \times \mathbb{S}^1 \to T(M^{2n})$ is now represented as $X_H = J\nabla H$, where $\nabla : \mathcal{D}(M^{2n}) \to T(M^{2n})$ denotes the usual gradient mapping with respect to this metric.

Consider now the space $\Omega := \Omega(M^{2n} \times \mathbb{S}^1)$ of all continuous curves in $M^{2n} \times \mathbb{S}^1$ with fixed end-points. Then one can similarly define the gradient mapping *grad* $\mathcal{A}_{t_0}^{(\tau)} : \Omega \to T(\Omega)$ as follows:

$$(grad\ \mathcal{A}_{t_0}^{(\tau)}(\gamma), \xi) := \frac{1}{\tau}\int_0^\tau ds < J(\gamma_{t_0})\dot{\gamma}_{t_0}(s) + \nabla H(\gamma_{t_0}; s + t_0), \xi >, \tag{8}$$

where $\gamma = \{(\gamma_{t_0}(s); t_0 + s(\mod 2\pi)) : s \in [0, \tau]\} \in \Omega$ as before, and $\xi \in T(\Omega)$. Since all critical curves $\gamma \in \Sigma_{t_0}(H)$ minimizing the functional (6) solve (2), this fact motivates a way of construction of an invariant subset $\Omega_H \subset \Omega$, such that $\Omega_H := \Omega(\mathcal{U}(\mathcal{L}_{t_0}^n) \times \mathbb{S}^1)$. Namely, define a curve $\gamma \in \Omega_H(\gamma^{(-)}) \subset \Omega_H$ as satisfying [3] the following gradient flow in $\mathcal{U}(\mathcal{L}_{t_0}^n) \times \mathbb{S}^1$:

$$\partial u_{t_0}/\partial z = -grad\ \mathcal{A}_{t_0}^{(\tau)}(u), \qquad \partial t/\partial z = 0 \tag{9}$$

for all $z \in \mathbb{R}$ and any $\tau \in \mathbb{R}$ under the asymptotic conditions

$$\lim_{z\to-\infty} u_{t_0}(s;z) = \gamma_{t_0}^{(-)}(s), \qquad \lim_{z\to\infty} \gamma_{t_0}(s;z) = \gamma_{t_0}(s) \tag{10}$$

with the corresponding curves $\gamma_{t_0}^{(-)}$, $\gamma_{t_0} : \mathbb{R} \to M^{2n}$ satisfying the system (2), and moreover, with the curve $\gamma_{t_0}^{(-)} : \mathbb{R} \to M^{2n}$ being taken to be hyperbolic [1, 2] with supp $\gamma_{t_0}^{(-)} \subset \mathcal{L}_{t_0}^n$. Now we can construct a so called [1] unstable manifold $W^u(\gamma_{t_0}^{(-)})$ to this hyperbolic curve $\gamma_{t_0}^{(-)}$ defined for all $\tau \in \mathbb{R}$. Thus due to the above construction, the functional manifold $W^u(\gamma_{t_0}^{(-)})$ when compact can be imbedded as a point submanifold into M^{2n} thereby interpreting supports of all curves solving (9) and (10) where supp $\gamma_{t_0} \subset \mathcal{L}_{t_0}^n$, as a compact neighborhood $\mathcal{L}_{t_0}^{(-)}(H) \subset \mathcal{U}(\mathcal{L}_{t_0}^n)$ of the compact Lagrangian submanifold $\mathcal{L}_{t_0}^n \subset M^{\,2n}$ looked for above.

The same construction can be done evidently for the case when the conditions (10) are changed either by

$$\lim_{z\to+\infty} \gamma_{t_0}(s;z) = \gamma_{t_0}^{(+)}(s), \qquad \lim_{z\to-\infty} \gamma_{t_0}(s;z) = \gamma_{t_0}(s), \tag{10a}$$

or by

$$\lim_{z\to-\infty} \gamma_{t_0}(s;z) = \gamma_{t_0}^{(-)}(s), \qquad \lim_{z\to\infty} \gamma_{t_0}(s;z) = \gamma_{t_0}^{(+)}(s), \tag{10b}$$

where $\gamma_{t_0}^{(-)} : \mathbb{R} \to M^{2n}$ and $\gamma_{t_0}^{(+)} : \mathbb{R} \to M^{2n}$ are some strictly different hyperbolic curves on M^{2n} with supp $\gamma_{t_0}^{(\pm)} \subset \mathcal{L}_{t_0}^n$ and solving (2). Based on (10a) one constructs similarly the stable manifold $W^s(\gamma_{t_0}^{(+)}(s))$ to a hyperbolic curve $\gamma_{t_0}^{(+)}$ and further the corresponding compact neighborhood $\mathcal{L}_{t_0}^{(+)}(H) \subset \mathcal{U}(\mathcal{L}_{t_0}^n)$ of the compact Lagrangian submanifold $\mathcal{L}_{t_0}^n \subset M^{2n}$ which is of crucial importance when studying intersection properties of stable $W^s(\gamma_{t_0}^{(+)})$ and unstable $W^u(\gamma_{t_0}^{(-)})$ manifolds. Based similarly on (10b), one constructs the neighborhood $\mathcal{L}_{t_0}(H) \subset \mathcal{U}(\mathcal{L}_{t_0}^n)$ of the compact Lagrangian submanifold $\mathcal{L}_{t_0}^n \subset M^{2n}$ being of interest when investigating so called adiabatic perturbations of integrable autonomous Hamiltonian flows on the symplectic manifold M^{2n}.

Now we make use of some statements [3, 5, 12] about the properties of the set Ω_H constructed above. For a generic choice of the Hamiltonian function $H : M^{2n} \times \mathbb{S}^1 \to \mathbb{R}$ the functional space of curves Ω_H is proved to be finite-dimensional what gives rise right away to hereditary finite-dimensionality of the neighborhood $\mathcal{L}_{t_0}^{(-)}(H)$ with the compact manifold structure. To see this linearize equation (9) in the direction of a vector field $\xi \in T(\Omega_H)$. This leads to the linearized first-order differential operator:

$$F_{t_0}(u)\xi := \nabla_z \xi + J(u)\nabla_s \xi + \nabla_\xi J(u)\partial u/\partial s + \nabla_\xi \nabla H(u; t_0 + s), \tag{11}$$

where $u \in \Omega_H$ satisfies the following equation stemming from (9) :

$$\partial u/\partial z + J(u)\partial u/\partial s + \nabla H(u; s + t_0) = 0 \tag{12}$$

and ∇_z, ∇_s and ∇_ξ denote here the corresponding covariant derivatives with respect to the metric (7) on M^{2n}. If $u \in \Omega_H$ satisfies (12), the curve γ_{t_0} in M^{2n} has supp $\gamma_{t_0} \subset \mathcal{L}^n_{t_0}$ and a curve $\gamma^{(-)}_{t_0}$ in $\mathcal{L}^n_{t_0}$ is hyperbolic and nondegenerate [3], then the operator $F_{t_0}(u) : T(\Omega_H) \to T(\Omega_H)$ defined by (11) is a Fredholm operator [12] between appropriate Sobolev spaces. The corresponding pair (H, J) with $J : M^{2n} \to End(T(M^{2n}))$ satisfying (7) is called regular [3] if every hyperbolic solution to (2) is nondegenerate [1, 3] and the operator $F_{t_0}(u)$ is onto for $u \in \Omega_H$. In general one can prove that the space $(\mathcal{H}, \mathcal{J})_{reg} \subset (\mathcal{H}, \mathcal{J})$ of regular pairs $(H, J) \in (\mathcal{H}, \mathcal{J})$ is dense with respect to the C^∞-topology. Thus, for the regular pairs it follows from an implicit function theorem [1] that the space $\Omega_H(\gamma^{(-)}_{t_0})$ is indeed for any curve γ_{t_0} with supp $\gamma_{t_0} \subset \mathcal{L}^n_{t_0}$ a finite-dimensional compact functional submanifold whose local dimension near $u \in \Omega_H(\gamma^{(-)}_{t_0})$ is exactly the Fredholm index of the operator $F_{t_0}(u)$. As a simple inference from the finite-dimensionality of the set $\Omega_H(\gamma^{(-)}_{t_0})$ and its compactness one gets that the corresponding point set $\mathcal{L}^{(-)}_{t_0}(H)$ is finite-dimensional and compact submanifold smoothly imbedded into M^{2n}. The same is evidently true for the point manifolds $\mathcal{L}^{(+)}_{t_0}(H)$ and $\mathcal{L}_{t_0}(H)$ supplying us with compact neighborhoods of the compact Lagrangian submanifold $\mathcal{L}^n_{t_0} \subset M^{2n}$. Let us specify the structure of the manifold $\mathcal{L}^{(-)}_{t_0}(H)$ more exactly making use of the Floer type analytical results [3, 8, 12] about the space of solutions to the problem (9) and (10). One has that for any two curves $\gamma^{(-)}$, $\gamma : [0, \tau] \to \mathcal{L}^n_{t_0} \times \mathbb{S}^1$ satisfying the system (2), the following functional

$$\Phi^{(\tau)}_{t_0}(u) := \frac{1}{\tau}\int_0^\tau ds \int_{\mathbb{R}} dz(|\partial u/\partial z|^2 + |\partial u/\partial s - X_H(u; s + t_0)|^2) \tag{13}$$

if bounded satisfies the characteristic equality

$$\Phi^{(\tau)}_{t_0}(u) = \mathcal{A}^{(\tau)}_{t_0}(\gamma^{(-)}) - \mathcal{A}^{(\tau)}_{t_0}(\gamma) \tag{14}$$

for any $\tau \in \mathbb{R}$. Thereby, in the case when the right hand side of (14) doesn't vanish, the functional space $\Omega_H(\gamma^{(-)})$ will be *a priori* nontrivial. Similarly, for any $u \in \mathcal{L}^{(+)}_{t_0}(H)$ one finds that

$$\Phi^{(\tau)}_{t_0}(u) = \mathcal{A}^{(\tau)}_{t_0}(\gamma) - \mathcal{A}^{(\tau)}_{t_0}(\gamma^{(+)}), \tag{14a}$$

where the corresponding curve $\gamma^{(+)}_{t_0} : [0, \tau] \to M^{2n}$ satisfies the system (2), is hyperbolic having supp $\gamma^{(+)}_{t_0} \subset \mathcal{L}^n_{t_0}$, and the curve $\gamma_{t_0} : [0, \tau] \to M^{2n}$ also satisfies the system (2) having supp $\gamma_{t_0} \subset \mathcal{L}^n_{t_0}$, and at last, for $u \in \mathcal{L}_{t_0}(H)$

$$\Phi^{(\tau)}_{t_0}(u) = \mathcal{A}^{(\tau)}_{t_0}(\gamma^{(-)}) - \mathcal{A}^{(\tau)}_{t_0}(\gamma^{(+)}), \tag{14b}$$

where $\gamma^{(\pm)} : [0, \tau] \to M^{2n} \times \mathbb{S}^1$, $\tau \in \mathbb{R}$, are taken to be strictly different, hyperbolic and having supp $\gamma^{(\pm)} \subset \mathcal{L}^n_{t_0}$. The case when $\gamma^{(+)}_{t_0} = \gamma^{(-)}_{t_0}$ needs some modification of the construction presented above on which we shall not dwell here. Thus we have constructed the corresponding neighborhoods $\mathcal{L}^{(\pm)}_{t_0}(H)$ and $\mathcal{L}_{t_0}(H)$ of the compact Lagrangian submanifold $\mathcal{L}^n_{t_0} \subset M^{2n}$ consisting of all bounded solutions to the corresponding equations (9), (10) and (10a,b). Based now on this fact and the analytical expressions (14) and (14a,b) one derives the following important lemma.

Lemma 1.1. *All neighborhoods* $\mathcal{L}_{t_0}^{(\pm)}(H)$ and $\mathcal{L}_{t_0}(H)$ *constructed via the scheme presented above are compact and invariant with respect to the Hamiltonian flow of diffeomorphisms* $\psi^s \in \mathrm{Diff}(M^{2n}) \times \mathbb{S}^1$, $s \in \mathbb{R}$.

Let us consider below the case of the neighborhood $\mathcal{L}_{t_0}(H) \subset M^{2n}$. The preceding characterization of the space of curves Ω_H leads us following Mather's approach [9] to another important for applications description of the compact neighborhood $\mathcal{L}_{t_0}(H)$ by means of the space of normalized probability measures $\mathcal{M}_{t_0}(H) := \mathcal{M}(T(\mathcal{L}_{t_0}(H)) \times \mathbb{S})$ with compact support and invariant with respect to our Hamiltonian ψ-flow of diffeomorphisms $\psi^s \in \mathrm{Diff}(M^{2n}) \times \mathbb{S}^1$, $s \in \mathbb{R}$, naturally extended on $T(\mathcal{L}_{t_0}(H)) \times \mathbb{S}$. The Hamiltonian ψ-flow due to Lemma 1.1 can be reduced invariantly upon the compact submanifold $\mathcal{L}_{t_0}(H) \times \mathbb{S} \subset M^{2n} \times \mathbb{S}$. For the behavior of this reduced ψ-flow upon $\mathcal{L}_{t_0}(H) \times \mathbb{S}$ to be studied in more detail let us assume that our extended Hamiltonian ψ_*-flow on $T(\mathcal{L}_{t_0}(H)) \times \mathbb{S}$ is ergodic, that is the $\lim_{\tau\to\infty} \mathcal{A}_{t_0}^{(\tau)}(\gamma)$ doesn't depend on initial points $(u_0, \dot{u}_0; t_0) \in T(\mathcal{L}_{t_0}(H)) \times \mathbb{S}$.

Recall now that the basic result [13] in functional analysis (the Riesz representation theorem) states that the set of Borel probability measures on a compact metric space X is a subset of the dual space $C(X)^*$ of the Banach space $C(X)$ of continuous functions on X. It is obviously a convex set and it is well known [13] to be metrizable and compact with respect to the weak topology on $C(X)^*$ defined by $C(X)$, also called the weak $(*)$-topology. The restriction of this topology to the set of Borel measures is frequently called the vague topology on measures [9]. Since the space $\mathcal{P}_{t_0} := T(\mathcal{L}_{t_0}(H)) \times \mathbb{S}$ is metrizable and can be as well compactified, it follows that the set of Borel probability measures on $\mathcal{P}_{t_0}$ is a metrizable, compact and convex subset of the dual to the Banach space of continuous functions on $\mathcal{P}_{t_0}$. The corresponding set $\mathcal{M}_{t_0}(H)$ is then evidently a compact, convex subset of this set. The well known result of the Kryloff and Bogoliuboff [14] states that any ψ-flow on a compact metric space X has an invariant probability measure. This result one can suitably adapt [9] to our metric compactified space $\mathcal{P}_{t_0} := T(\mathcal{L}_{t_0}(H)) \times \mathbb{S}$ as follows. Take a trajectory $\gamma \in \Omega_H$ of the extended ψ_*-flow on $\mathcal{P}_{t_0}$ with supp $\gamma \subset \mathcal{L}_{t_0}(H) \times \mathbb{S}$ defined on a time interval $[0, \tau] \subset \mathbb{R}$ and let a measure μ_τ on $T(\mathcal{L}_{t_0}(H)) \times \mathbb{S}$ be evenly distributed along the orbit γ. Then evidently $||\psi_*^s \mu_\tau - \mu_\tau|| \leq 2s/\tau$ for $s \in [0, \tau]$. Denote by μ a point of accumulation of the set $\{\mu_\tau : \tau \in \mathbb{R}_+\}$ as $\tau \to \infty$ with respect to the before mentioned vague topology. For any continuous function $f \in C(\mathcal{P}_{t_0})$, any $s \in \mathbb{R}$ and any $\tau_0, \varepsilon > 0$ there exists $\tau > \tau_0$ such that $|\int_{\mathcal{P}_{t_0}} f \circ \psi_*^{\bar{s}} d\mu - \int_{\mathcal{P}_{t_0}} f \circ \psi_*^{\bar{s}} d\mu_\tau| < \varepsilon$ for $\bar{s} \in \{0, s\}$. Then it follows from the above estimations

$$
\begin{aligned}
&|\int_{\mathcal{P}_{t_0}} f \circ \psi_*^s d\mu - \int_{\mathcal{P}_{t_0}} f d\mu| \leq |\int_{\mathcal{P}_{t_0}} f \circ \psi_*^s d\mu - \\
&\int_{\mathcal{P}_{t_0}} f \circ \psi_*^s d\mu_\tau| + |\int_{\mathcal{P}_{t_0}} f \circ \psi_*^s d\mu_\tau - \int_{\mathcal{P}_{t_0}} f d\mu_\tau| + |\int_{\mathcal{P}_{t_0}} f d\mu_\tau - \\
&\int_{\mathcal{P}_{t_0}} f d\mu| \leq 2\varepsilon + ||f||\, ||\psi_*^s \mu_\tau - \mu_\tau|| \leq 2\varepsilon + 2s||f||/\tau,
\end{aligned}
$$

that is $|\int_{\mathcal{P}_{t_0}} f\circ\psi_*^s d\mu - \int_{\mathcal{P}_{t_0}} f d\mu| = 0$ since $\varepsilon > 0$ can be taken arbitrarily small and $\tau_0 > 0$ arbitrarily large. Thereby one sees that the constructed measure $\mu \in \mathcal{M}_{t_0}(H)$, that is it is normalized and invariant with respect to the extended Hamiltonian ψ_*-flow on $\mathcal{P}_{t_0}$.

Thus, in the case of ergodicity of the ψ_*-flow on $T(\mathcal{L}_{t_0}(H)) \times \mathbb{S}$ the mentioned above limit

$$\lim_{\tau\to\infty} \mathcal{A}_{t_0}^{(\tau)}(\gamma) = \int_{\mathcal{P}_{t_0}} (\alpha^{(1)} - H)d\mu, \tag{15}$$

with 1-form $\alpha^{(1)} \in \Lambda^1(M^{2n})$ being considered above as a function $\alpha^{(1)} : \mathcal{P}_{t_0} \to \mathbb{R}$, since the submanifold $\mathcal{L}_{t_0}(H)$ by construction is compact and invariantly imbedded into M^{2n} due to Lemma 1.1. So, it is natural to study properties of the functional

$$\mathcal{A}_{t_0}(\mu) := \int_{\mathcal{P}_{t_0}} (\alpha^{(1)} - H)d\mu \tag{16}$$

on the space $\mathcal{M}_{t_0}(H)$, where we omitted for brevity the natural pullback of the 1-form $\alpha^{(1)} \in \Lambda^1(M^{2n})$ upon the invariant compact submanifold $\mathcal{L}_{t_0}(H) \subset M^{2n}$. Being interested namely in ergodic properties of ψ_*-orbits on $T(\mathcal{L}_{t_0}(H)) \times \mathbb{S})$, we shall develop below an analog of the J. Mather Lagrangian measure homology technique [9, 10] to a more general and complicated case of the reduced Hamiltonian ψ-flow on the invariant compact submanifold $\mathcal{L}_{t_0}(H) \subset M^{2n}$. In particular, we shall construct an analog of the so called Mather β-function [9] on the homology group $H_1(\mathcal{L}_{t_0}(H);\mathbb{R})$ whose linear domains generate exactly ergodic components of a measure $\mu \in \mathcal{M}_{t_0}(H)$ minimizing the functional (16), being of great importance for studying regularity properties of ψ_*-orbits on $T(\mathcal{L}_{t_0}(H)) \times \mathbb{S}$. The results can be extended further to adiabatically perturbed integrable Hamiltonian systems depending on a small parameter $\varepsilon \downarrow 0$ via the continuous dependence $H(t) := \tilde{H}(\varepsilon t)$, where $\tilde{H}(\tau + 2\pi) = \tilde{H}(\tau)$ for all $\tau \in [0, 2\pi]$. It makes also possible to state the existence of so called adiabatic invariants with compact supports in $\mathcal{L}_{t_0}(H)$ having many applications in mathematical physics and mechanics. Some of the results can be also applied to investigating the problem of transversal intersections of corresponding stable and unstable manifolds to hyperbolic curves or singular points, related closely with existence of highly irregular motions in a periodic time-dependent Hamiltonian dynamical system under regard.

2 Invariant measures and mather's type β-function

Before studying the average functional (16) on the measure space $\mathcal{M}_{t_0}(H)$, let us first analyze properties of the functional

$$\oint_\sigma a^{(1)} := \prec a^{(1)}, \sigma \succ \tag{17}$$

on $H^1(\mathcal{L}_{t_0}(H);\mathbb{R})$ at a fixed $\sigma \in H_1(\mathcal{L}_{t_0}(H);\mathbb{R})$. Since the 1-form $a^{(1)} \in H^1(\mathcal{L}_{t_0}(H);\mathbb{R})$ in (17) can be considered as a function $a^{(1)} : \mathcal{P}_{t_0} \to \mathbb{R}$, in virtue of the Riesz theorem

[13] there exists a Borel measure $\mu : \mathcal{P}_{t_0} \rightarrow \mathbb{R}_+$ (still not necessary ψ-invariant), such that

$$\prec a^{(1)}, \sigma \succ = \int_{\mathcal{P}_{t_0}} a^{(1)} d\mu. \tag{18}$$

The following lemma characterizing the right hand side of (18) holds.
Lemma 2.1. *Let a 1-form $a^{(1)} = d\lambda^{(0)} \in \Lambda^1(\mathcal{L}_{t_0}(H))$ be exact, that is the cohomology class $[d\lambda^{(0)}] = 0 \in H^1(\mathcal{L}_{t_0}(H); R)$. Then for any $\mu \in \mathcal{M}_{t_0}(H)$*

$$\oint_\sigma a^{(1)} = 0. \tag{19}$$

$\lhd$ Really, for $a^{(1)} = d\lambda^{(0)}$, where $\lambda^{(0)} : \mathcal{L}_{t_0}(H) \rightarrow \mathbb{R}$ is an absolutely continuous mapping, the following holds due to The Fubini theorem for any $\tau \in \mathbb{R}_+$:

$$\begin{array}{c} |\int_{\mathcal{P}_{t_0}} d\lambda^{(0)} d\mu.| = |\frac{1}{\tau}\int_0^\tau ds \int_{\mathcal{P}_{t_0}} d\lambda^{(0)}(\psi_*^s d\mu)| = \\ |\frac{1}{\tau}\int_{\mathcal{P}_{t_0}} d\mu \int_0^\tau ds d(\lambda^{(0)} \circ \psi_*^s)/ds| \\ = |\frac{1}{\tau}\int_{\mathcal{P}_{t_0}} d\mu [\lambda^{(0)} \circ \psi_*^\tau - \lambda^{(0)} \circ \psi_*^0]| \leq 2||\lambda^{(0)}||/\tau. \end{array} \tag{20}$$

The latter inequality as $\tau \rightarrow \infty$ gives rise to the wanted equality (19), that proves the lemma.$\rhd$

Thus, the right hand side of (18) defines a true functional

$$H^1(\mathcal{L}_{t_0}(H); \mathbb{R}) \ni a^{(1)} \rightarrow \int_{\mathcal{P}_{t_0}} a^{(1)} d\mu \in \mathbb{R} \tag{21}$$

on the cohomology space $H^1(\mathcal{L}_{t_0}(H); \mathbb{R})$. All the above can be formulated as the following theorem.

Theorem 2.2. *Let an element $\sigma \in H_1(\mathcal{L}_{t_0}(H); \mathbb{R})$ be fixed. Then there exists a ψ-invariant probability measure (not unique) $\mu \in \mathcal{M}_{t_0}(H)$, such that the representation (18) holds and vice versa, for any measure $\mu \in \mathcal{M}_{t_0}(H)$ there exists the homology class $\sigma := \rho_{t_0}(\mu) \in H_1(\mathcal{L}_{t_0}(H); \mathbb{R})$, such that*

$$\prec a^{(1)}, \rho_{t_0}(\mu) \succ = \int_{\mathcal{P}_{t_0}} a^{(1)} d\mu \tag{22}$$

for all $a^{(1)} \in H^1(\mathcal{L}_{t_0}(H); \mathbb{R})$.

Definition 2.3. ([10]) *For any measure $\mu \in \mathcal{M}_{t_0}(H)$ the homology class $\rho_{t_0}(\mu) \in H_1(L_{t_0}(H); \mathbb{R})$ is called its homology.*

Corollary 2.4. *The homology mapping $\rho_{t_0} : \mathcal{M}_{t_0}(H) \rightarrow H_1(L_{t_0}(H); \mathbb{R})$ defined within Theorem 2.2 is surjective.*

$\lhd$ Sketch of a proof of Theorem 2.2. The fact that for each $\mu \in \mathcal{M}_{t_0}(H)$ there exists the unique homology class $\sigma := \rho_{t_0}(\mu) \in H_1(\mathcal{L}_{t_0}(H); \mathbb{R})$ is based on the well known Poincare duality theorem [1]. The inverse statement is about the surjectivity of the mapping $\rho_{t_0} : \mathcal{M}_{t_0}(H) \rightarrow H_1(\mathcal{L}_{t_0}(H); \mathbb{R})$. For it to be stated, consider following [8-10] a covering space $\overline{\mathcal{L}}_{t_0}(H)$ over $\mathcal{L}_{t_0}(H)$ defined by the condition that $\pi_1(\overline{\mathcal{L}}_{t_0}(H)) = \ker h_{t_0}$, where $h_{t_0} : \pi_1(\mathcal{L}_{t_0}(H)) \to H_1(\mathcal{L}_{t_0}(H);\mathbb{R})$ denotes the Hurewicz homomorphism [10]. Since in reality the functional (22) is defined on the covering space $\overline{\mathcal{L}}_{t_0}(H)$, it is necessary to lift all curves $\gamma \in \Omega_H$ on $\mathcal{L}_{t_0}(H)\times\mathbb{S}$ to curves $\tilde{\gamma} \in\in \tilde{\Omega}_H$ on $\overline{\mathcal{L}}_{t_0}(H)\times\mathbb{S}$. In the case when the homotopy group $\pi_1(\mathcal{L}_{t_0}(H))$ is abelian, the covering space $\tilde{\mathcal{L}}_{t_0}(H)$ becomes universal, but in general it is obtained as some universal covering of $\tilde{\mathcal{L}}_{t_0}(H)$ quotioned further with respect to the action of the kernel of the corresponding Hurewicz homomorphism $h_{t_0} : \pi_1(\mathcal{L}_{t_0}(H)) \to H_1(\mathcal{L}_{t_0}(H);\mathbb{R})$.

Take now any element $\sigma \in H_1(\mathcal{L}_{t_0}(H);\mathbb{R})$ and construct a set of approximating it so called Deck transformations $\tau^{-1}\sigma_\tau \in im\ h_{t_0} \subset H_1(\mathcal{L}_{t_0}(H);\mathbb{R})$, $\tau \in \mathbb{R}_+$, such that weakly $\lim_{\tau\to\infty}\tau^{-1}\sigma_\tau = \sigma$ holds. Put further $\tilde{x}_\tau := \sigma_\tau \circ \tilde{x}_0 \in \overline{\mathcal{L}}_{t_0}(H)\times\mathbb{S}$, $\tau \in \mathbb{R}_+$, where $\tilde{x}_0 \in \overline{\mathcal{L}}_{t_0}(H)\times\mathbb{S}$ is taken arbitrary and consider such a curve $\tilde{\gamma} : [0,\tau] \to \overline{\mathcal{L}}_{t_0}(H)\times\mathbb{S}$ with end-points $\tilde{\gamma}(0) = \tilde{x}_0$, $\tilde{\gamma}(\tau) = \tilde{x}_\tau$ whose projection on $\mathcal{L}_{t_0}(H)\times\mathbb{S}$ is the curve $\gamma \in \Sigma_{t_0}(H)$, minimizing the functional (6). Consider also a set $\{\mu_\tau : \tau \in \mathbb{R}_+\}$ of probability measures on $\mathcal{P}_{t_0}$ evenly distributed along corresponding curves $\gamma \in \Sigma_{t_0}(H)$ for each $\tau \in \mathbb{R}_+$ and denote by μ a point of its accumulation as $\tau \to \infty$. Due to the uniform distribution of measures μ_τ, $\tau \in \mathbb{R}_+$, along curves $\gamma \in \Sigma_{t_0}(H)$ having the end-points agreed with chosen above Deck transformations $\sigma_\tau \in H_1(\mathcal{L}_{t_0}(H);\mathbb{R})$, $\tau \in \mathbb{R}_+$, one gets right away from the Birkhoff-Khinchin ergodic theorem [1, 2] that

$$\int_{\mathcal{P}_{t_0}} a^{(1)} d\mu_\tau = \prec a^{(1)}, \tau^{-1}\sigma_\tau) \succ \tag{23}$$

for any $a^{(1)} \in H^1(\mathcal{L}_{t_0}(H);\mathbb{R})$. Passing now to the limit in (23) as $\tau \to \infty$ and taking into account that weakly $\lim_{\tau\to\infty}\tau^{-1}\sigma_\tau = \sigma$, one gets right away that the equality (22) holds for some measure $\mu \in \mathcal{M}_{t_0}(H)$, such that $\rho_{t_0}(\mu) = \sigma \in H_1(\mathcal{L}_{t_0}(H);\mathbb{R})$, thereby giving rise to the surjectivity of the mapping $\rho_{t_0} : \mathcal{M}_{t_0}(H) \to H_1(\mathcal{L}_{t_0}(H);\mathbb{R})$ and proving the theorem. $\triangleright$

Return now to treating the average functional (16) subject to the space of all invariant measures $\mathcal{M}_{t_0}(H)$. Namely, consider the following β-function $\beta_{t_0} : H_1(\mathcal{L}_{t_0}(H);\mathbb{R}) \to \mathbb{R}$ defined as

$$\beta_{t_0}(\sigma) := \inf_\mu\{\mathcal{A}_{t_0}(\mu) : \rho_{t_0}(\mu) = \sigma \in H_1(\mathcal{L}_{t_0}(H);\mathbb{R})\} \tag{24}$$

It will be further called a Mather type β-function due to its analogy to the definition given in [9,10]. The following lemma holds.

Lemma 2.5. *Let a 1-form $a^{(1)} \in H^1(\mathcal{L}_{t_0}(H);\mathbb{R})$ be taken arbitrary. Then the Mather type β-function*

$$\beta^{(a)}_{t_0}(\sigma) := \inf_\mu\{\mathcal{A}^{(a)}_{t_0}(\mu) : \rho_{t_0}(\mu) = \sigma \in H_1(\mathcal{L}_{t_0}(H);\mathbb{R})\}, \tag{25}$$

where by definition

$$\mathcal{A}^{(a)}_{t_0}(\mu) := \int_{\mathcal{P}_{t_0}} (\alpha^{(1)} + a^{(1)} - H)d\mu, \tag{26}$$

satisfies the following equation:

$$\beta^{(a)}_{t_0}(\sigma) = \beta_{t_0}(\sigma) + \prec a^{(1)}, \sigma) \succ . \tag{27}$$

$\triangleleft$ The proof easily stems from the definition (25) and the equality (22). $\triangleright$

Assume now that the infimum in (24) is attained at a measure $\mu(\sigma) \in \mathcal{M}_{t_0}(H)$. Then evidently, $\rho_{t_0}(\mu(\sigma)) = \sigma$ for any homology class $\sigma \in H_1(\mathcal{L}_{t_0}(H);\mathbb{R})$. Denote by $\mathcal{M}_{t_0}^{(\sigma)}(H)$ the set of all minimizing the functional (24) measures of $\mathcal{M}_{t_0}(H)$. In the next chapter we shall proceed on study its ergodic and homology properties.

3 Ergodic measures and their homologies

Consider the introduced above Mather type β-function $\beta_{t_0}^{(a)} : H_1(\mathcal{L}_{t_0}(H);\mathbb{R}) \to \mathbb{R}$ for any $a^{(1)} \in H^1(\mathcal{L}_{t_0}(H);\mathbb{R})$. It is evidently a convex function on $H_1(\mathcal{L}_{t_0}(H);\mathbb{R})$, that is for any $\lambda_1, \lambda_2 \in [0,1]$, $\lambda_1 + \lambda_2 = 1$, and $\sigma_1, \sigma_2 \in H_1(\mathcal{L}_{t_0}(H);\mathbb{R})$ there holds the inequality

$$\beta_{t_0}^{(a)}(\lambda_1\sigma_1 + \lambda_2\sigma_2) \le \lambda_1\beta_{t_0}^{(a)}(\sigma_1) + \lambda_2\beta_{t_0}^{(a)}(\sigma_2). \tag{28}$$

As usually dealing with convex functions, one says that an element $\sigma \in H_1(\mathcal{L}_{t_0}(H);\mathbb{R})$ is extremal point [13] if $\beta_{t_0}^{(a)}(\lambda_1\sigma_1 + \lambda_2\sigma_2) < \lambda_1\beta_{t_0}^{(a)}(\sigma_1) + \lambda_2\beta_{t_0}^{(a)}(\sigma_2)$ for all $\lambda_1, \lambda_2 \in (0,1)$, $\lambda_1 + \lambda_2 = 1$, and $\sigma = \lambda_1\sigma_1 + \lambda_2\sigma_2$. Correspondingly, we shall call a convex set $Z_{t_0}(H) \subset H_1(\mathcal{L}_{t_0}(H);\mathbb{R})$ by a linear domain of the Mather type function (25) if

$$\beta_{t_0}^{(a)}(\lambda_1\sigma_1 + \lambda_2\sigma_2) = \lambda_1\beta_{t_0}^{(a)}(\sigma_1) + \lambda_2\beta_{t_0}^{(a)}(\sigma_2) \tag{29}$$

for any $\sigma_1, \sigma_2 \in Z_{t_0}(H)$ and $\lambda_1, \lambda_2 \in \mathbb{R}$. It is easy to see now that if $\sigma \in H_1(\mathcal{L}_{t_0}(H);\mathbb{R})$ is extremal, then the set $\mathcal{M}_{t_0}^{(\sigma)}(H)$ contains [15] ergodic minimizing measure components. Namely, following [9, 10] one states that if $Z_{t_0}(H)$ is a linear domain and $\mathcal{P}_{t_0}^{(\sigma)} \subset \mathcal{P}_{t_0}$ is the closure of the union of the supports of measures $\mu(\sigma) \in \mathcal{M}_{t_0}^{(\sigma)}(H)$ with $\sigma \in Z_{t_0}(H)$, then the set $\mathcal{P}_{t_0}^{(\sigma)}$ is compact and the inverse mapping $(p_{t_0}|_{\mathcal{P}_{t_0}^{(\sigma)}})^{-1} : p_{t_0}(\mathcal{P}_{t_0}^{(\sigma)}) \to \mathcal{P}_{t_0}^{(\sigma)}$ is Lipschitzian, where $p_{t_0} : \mathcal{P}_{t_0} \to \mathcal{L}_{t_0}(H) \times \mathbb{S}$ is the standard projection, being injective upon $\mathcal{P}_{t_0}^{(\sigma)}$. Moreover, one can show [9] that if a measure $\mu \in \mathcal{M}_{t_0}^{(\sigma)}(H)$ is minimizing the functional (26), then its support supp $\mu \subset \mathcal{P}_{t_0}^{(\sigma)}$ and all its ergodic components $\{\bar{\mu}\}$ are minimizing this functional too, and the convex hull of the corresponding homologies $conv\{\rho_{t_0}(\bar{\mu})\}$ is a linear domain $Z_{t_0}^{(\sigma)}(H)$ of the Mather type β-function (25). These results are of very interest concerning many applications in dynamics. Especially, the ergodic measures, as is well known, possess the crucial property that every invariant Borel set has measure either 0 or 1, giving rise to the following important equality:

$$\lim_{\tau\to\infty} \mathcal{A}_{t_0}^{(\tau)}(\gamma) = \mathcal{A}_{t_0}(\bar{\mu})) \tag{30}$$

uniformly on $(\gamma_{t_0},(0), \dot{\gamma}_{t_0}(0); t_0) \in \mathcal{P}_{t_0} \cap$ supp $\bar{\mu}$, where $\gamma \in \Sigma_{t_0}(H)$. All of the properties formulated above are inferred from the following theorem modeling the similar one in [10].

Theorem 3.1. *Let a measure $\mu \in \mathcal{M}_{t_0}(H)$ be minimizing the functional (26) satisfying the condition $\beta_{t_0}^{(a)}(\rho_{t_0}(\mu)) = \mathcal{A}_{t_0}(\mu)$. Then supp $\mu \subset \Sigma_{t_0}(H)$ and the convex* *hull of the set of homologies $\rho_{t_0}(\bar{\mu}) \in H_1(\mathcal{L}_{t_0}(H);\mathbb{R})$, where $\{\bar{\mu}\} \subset \mathcal{M}_{t_0}(H)$ are the corresponding ergodic components of the measure $\mu \in \mathcal{M}_{t_0}(H)$, is a linear domain $Z_{t_0}(H)$ of the Mather type β-function (25).*

$\lhd$ Sketch of a proof. Let $h_{t_0} : \pi_1(\mathcal{L}_{t_0}(H)) \to H_1(\mathcal{L}_{t_0}(H);\mathbb{R})$ be the corresponding Hurewicz homomorphism and take some basis $\sigma_k \in im\ h_{t_0} \subset H_1(\mathcal{L}_{t_0}(H);\mathbb{R})$, $k = \overline{1,r}$, where $r = \dim im\ h_{t_0}$, being its dual basis $a_j^{(1)} \in H^1(\mathcal{L}_{t_0}(H);\mathbb{R})$, $j = \overline{1,r}$. Then for any points $\tilde{x}, \tilde{y} \in \overline{\mathcal{L}}_{t_0}(H) \times \mathbb{S}$ one can define an element $\xi^{(\tau)}(\tilde{x},\tilde{y}|\tilde{\gamma}) \in H_1(\mathcal{L}_{t_0}(H);\mathbb{R})$ as the sum

$$\xi^{(\tau)}(\tilde{x},\tilde{y}|\tilde{\gamma}) := \frac{1}{\tau}\sum_{j=1}^{r}\sigma_j \int_0^\tau \tilde{a}_j^{(1)}(\tilde{\gamma}), \tag{31}$$

where $\gamma : [0,\tau] \to \mathcal{L}_{t_0}(H) \times \mathbb{S}$ is any continuous arc joining these two chosen points $\tilde{x}, \tilde{y} \in \overline{\mathcal{L}}_{t_0}(H) \times \mathbb{S}$, and $\tilde{a}_j^{(1)} \in H^1(\overline{\mathcal{L}}_{t_0}(H);\mathbb{R})$ are the corresponding liftings to $\overline{\mathcal{L}}_{t_0}(H)$ of 1-forms $a_j^{(1)} \in H^1(\mathcal{L}_{t_0}(H);\mathbb{R})$, $j = \overline{1,r}$. One can show then that if $\mu \in \mathcal{M}_{t_0}(H)$ is ergodic and supp $\mu \subset \Sigma_{t_0}(H)$, then the measure μ is minimizing the functional (26). Put $\sigma := \rho_{t_0}(\mu)$ and let a set $Z_{t_0}(H) \subset H_1(\mathcal{L}_{t_0}(H);\mathbb{R})$ be a supporting domain containing this homology class $\sigma \in H_1(\mathcal{L}_{t_0}(H);\mathbb{R})$. Thus, one can see that the extremal points of the convex set $Z_{t_0}(H)$ are extremal points also of the Mather type β-function (25). Next expand the homology class $\sigma = \rho_{t_0}(\mu)$ as a convex combination of extremal points $\bar{\sigma}_j \in Z_{t_0}(H)$, $j = \overline{1,m}$, for some $m \in \mathbb{Z}_+$. Then, since elements $\bar{\sigma}_j \in Z_{t_0}(H)$, $j = \overline{1,m}$, are extremal, there exist ergodic measures $\bar{\mu}_j \in \mathcal{M}_{t_0}^{(\sigma_j)}(H)$, $j = \overline{1,m}$, such that $\rho_{t_0}(\bar{\mu}_j) = \bar{\sigma}_j$, $j = \overline{1,m}$. Moreover, since $Z_{t_0}^{(\sigma)}(H)$ is a linear domain, one easily brings about that

$$\beta_{t_0}^{(a)}(\sigma) = \sum_{j=1}^{m} c_j \beta_{t_0}^{(a)}(\bar{\sigma}_j) = \sum_{j=1}^{m} c_j \mathcal{A}_{t_0}^{(a)}(\bar{\mu}_j), \tag{32}$$

where $\sigma = \sum_{j=1}^m c_j\bar{\sigma}_j$ with some real coefficients $c_j \in \mathbb{R}$, $j = \overline{1,m}$. Due to the ergodicity of the measure $\mu \in \mathcal{M}_{t_0}(H)$ from the Birkhoff-Khinchin ergodic theorem [1] one derives that there exists an orbit $\tilde{\gamma} : [0,\tau\} \to \overline{\mathcal{L}}_{t_0}(H) \times \mathbb{S}$ with the supp $\gamma \subset$ supp μ, such that the property (30) together with the equality

$$\sigma := \rho_{t_0}(\mu) = \lim_{\tau\to\infty} \xi^{(\tau)}(\tilde{x},\tilde{y}|\tilde{\gamma}) \tag{33}$$

hold. Further, there exist curves $\tilde{\gamma}_j \in \Sigma_{t_0}(H)$, supp $\gamma_j \subset$ supp $\bar{\mu}_j$, $j = \overline{1,m}$, such the expressions

$$\bar{\sigma}_j := \rho_{t_0}(\bar{\mu}_j) = \lim_{\tau\to\infty} \xi^{(\tau)}(\tilde{x},\tilde{y}|\tilde{\gamma}_j) \tag{34}$$

as well as $\beta_{t_0}^{(a)}(\bar{\sigma}_j) = \mathcal{A}_{t_0}^{(a)}(\bar{\mu}_j) = \lim_{\tau\to\infty} \mathcal{A}_{t_0}^{(\tau)}(\tilde{\gamma}_j)$ hold for every $j = \overline{1,m}$. Under the conditions (14b) involved on the invariant neighborhood $\mathcal{L}_{t_0}(H)$ one shows that for any measure $\mu \in \mathcal{M}_{t_0}(H)$ such that $\rho_{t_0}(\mu) = \sigma$, the inequality $\mathcal{A}_{t_0}^{(a)}(\mu) \leq \beta_{t_0}^{(a)}(\rho_{t_0}(\mu))$ holds thereby proving its minimality. Suppose now that the measure $\mu \in \mathcal{M}_{t_0}(H)$ has all its ergodic components with supports contained in $\Sigma_{t_0}(H)$ and the convex hull of its homologies is a linear domain of the Mather type function (25). One can approximate (in the weak topology) a measure $\mu \in \mathcal{M}_{t_0}(H)$ by means of a convex combination $\hat{\mu} := \sum_{j=1}^{m} \hat{c}_j \bar{\mu}_j$, where $\hat{c}_j \in \mathbb{R}$ and $\bar{\mu}_j \in \mathcal{M}_{t_0}(H)$, $j = \overline{1,m}$, are ergodic components of the measure $\mu \in \mathcal{M}_{t_0}(H)$. Then supp $\bar{\mu}_j \subset \Sigma_{t_0}(H)$ implying that all $\bar{\mu}_j \in \mathcal{M}_{t_0}(H)$, $j = \overline{1,m}$, are minimizing (26), that is are minimal. Therefore, since the convex hull of homologies $\{\rho_{t_0}(\bar{\mu}_j) \in H_1(\mathcal{L}_{t_0}(H);\mathbb{R}) : j = \overline{1,m}\}$ is a linear domain due to its minimality, one gets that

$$\begin{aligned} \mathcal{A}_{t_0}^{(a)}(\hat{\mu}) = \textstyle\sum_{j=1}^{m} \hat{c}_j \mathcal{A}_{t_0}^{(a)}(\bar{\mu}_j) = \sum_{j=1}^{m} \hat{c}_j \beta_{t_0}^{(a)}(\rho_{t_0}(\bar{\mu}_j)) \\ = \beta_{t_0}^{(a)}(\rho_{t_0}(\textstyle\sum_{j=1}^{m} \hat{c}_j \bar{\mu}_j)) = \beta_{t_0}^{(a)}(\rho_{t_0}(\mu), \end{aligned} \tag{35}$$

meaning evidently that the measure $\hat{\mu} \in \mathcal{M}_{t_0}(H)$ is minimal too. Making use now of the fact that limits of minimizing measures are minimizing too, one obtains finally that the measure $\mu \in \mathcal{M}_{t_0}(H)$ is minimizing the functional (26), thereby proving the theorem. $\triangleright$

Consider some properties of a so called [10] supporting domain

$$Z_{t_0}^{(a)}(H) := \{\sigma \in H_1(\mathcal{L}_{t_0}(H);\mathbb{R}) : \beta_{t_0}^{(a)}(\sigma) = \prec a^{(1)}, \sigma \succ + c_{t_0}^{(a)}\} \tag{36}$$

for the Mather type β-function (25) at some fixed $a^{(1)} \in H^1(\mathcal{L}_{t_0}(H);\mathbb{R})$ with $c_{t_0}^{(a)} \in \mathbb{R}$ properly defined by (27). Define also by $\mathcal{P}_{t_0}^{(a)} := \cup_{\sigma \in Z_{t_0}^{(a)}(H)} \mathrm{supp}\ \mu(\sigma)$, where $\mu(\sigma) \in \mathcal{M}_{t_0}(H)$ and $\rho_{t_0}(\mu(\sigma)) = \sigma \in Z_{t_0}^{(a)}(H)$. Present now a supporting domain $Z_{t_0}^{(a)}(H) \subset H_1(\mathcal{L}_{t_0}(H);\mathbb{R})$ due to the expression (27) as follows:

$$Z_{t_0}^{(a)}(H) = \{\sigma \in H_1(\mathcal{L}_{t_0}(H);\mathbb{R}) : \beta_{t_0}^{(0)}(\sigma) = c_{t_0}^{(a)}\}, \tag{37}$$

where the function $\beta_{t_0}^{(0)} : H_1(\mathcal{L}_{t_0}(H);\mathbb{R})$ being bounded from below is chosen in such a way that $\beta_{t_0}^{(0)}(\sigma) \geq c_{t_0}^{(a)}$ for all $\sigma \in H_1(\mathcal{L}_{t_0}(H);\mathbb{R})$. Take now a measure $\mu \in \mathcal{M}_{t_0}(H)$ and suppose that supp $\mu \subset \Sigma_{t_0}(H)$. Since $\beta_{t_0}^{(0)}(\sigma) \geq c_{t_0}^{(a)}$ for all $\sigma \in H_1(\mathcal{L}_{t_0}(H);\mathbb{R})$ and due to (37) $Z_{t_0}^{(a)}(H) = (\beta_{t_0}^{(0)})^{-1}\{c_{t_0}^{(a)}\}$ at some fixed $a^{(1)} \in H^1(\mathcal{L}_{t_0}(H);\mathbb{R})$, this evidently implies that the measure $\mu \in \mathcal{M}_{t_0}(H)$ is minimizing the functional (26) and $\rho_{t_0}(\mu) \in Z_{t_0}^{(a)}(H)$. Thereby the following theorem is stated.

Theorem 3.2. *Suppose that $Z_{t_0}^{(a)}(H) \subset H_1(\mathcal{L}_{t_0}(H);\mathbb{R})$ is a supporting domain of the Mather type function (27) and a measure $\mu \in \mathcal{M}_{t_0}(H)$ satisfies the condition supp $\mu \subset \Sigma_{t_0}(H)$. Then this measure $\mu \in \mathcal{M}_{t_0}(H)$ is minimizing and $\rho_{t_0}(\mu) \in Z_{t_0}^{(a)}(H)$.*

The following corollaries from the Theorem 3.2 as in [10] hold.

Corollary 3.3. *The minimizing measure $\mu \subset \mathcal{M}_{t_0}(H)$ with* supp $\mu \subset \Sigma_{t_0}(H)$ *satisfies the condition $\mathcal{A}_{t_0}^{(0)}(\mu) = c_{t_0}^{(a)}$. By means of choosing the element $a^{(1)} \in H^1(\mathcal{L}_{t_0}(H);\mathbb{R})$ one can make the value $c_{t_0}^{(a)}$ be zero, that is one can put $c_{t_0}^{(a)} = 0$.*

Corollary 3.4. *For any strictly extremal closed curve $\sigma \in H_1(\mathcal{L}_{t_0}(H);\mathbb{R})$ the following properties take place*:

i) there exists an ergodic measure $\bar{\mu}(\sigma) \in \mathcal{M}_{t_0}(H)$ whose support is a minimal set and $\rho_{t_0}(\bar{\mu}(\sigma)) = \sigma$;

ii) for every closed 1-form $a^{(1)} \in H^1(\mathcal{L}_{t_0}(H);\mathbb{R})$ *the equality* $\prec a^{(1)},\sigma \succ = \lim_{\tau\to\infty}\frac{1}{\tau}\int_{t_0}^{t_0+\tau} a^{(1)}(\dot{\gamma})ds$ *holds uniformly for all* $(\gamma_{t_0(0)},\dot{\gamma}_{t_0(0)};t_0) \in \mathcal{P}_{t_0} \cap \text{supp}\ \bar{\mu}(\sigma)$, $\rho_{t_0}(\bar{\mu}(\sigma)) = \sigma$ *and* $\gamma \in \Sigma_{t_0}(H)$;

iii) if $(\gamma_{t_0(0)},\dot{\gamma}_{t_0(0)};t_0) \in \mathcal{P}_{t_0} \cap \text{supp}\ \bar{\mu}(\sigma)$, $\rho_{t_0}(\bar{\mu}(\sigma)) = \sigma$ *and* $\gamma \in \Sigma_{t_0}(H)$ *is the corresponding orbit in* $\mathcal{L}_{t_0}(H)\times\mathbb{S}$, *then* $\beta_{t_0}^{(a)}(\sigma) = \lim_{\tau\to\infty}\mathcal{A}_{t_0}^{(\tau)}(\gamma)$ *uniformly.*

The statements formulated above can be effectively used for studying dynamics of many perturbed integrable Hamiltonian flows and their regularity properties. As it is well known, they are strongly based on the intersection theory of stable and unstable manifolds related with hyperbolic either closed orbits or singular points of a Hamiltonian system under regard. These aspects of our study of ergodic measure and homology properties of such Hamiltonian flows are supposed to be treated in a proceeding article under preparation.

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