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The exact solution of the Potts models with external magnetic field on the Cayley tree

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ABSTRACT

The exact solution is found for the problem of phase transition in the Potts model and the Potts model with competing ternary and binary interactions with external magnetic field.

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RESUMEN

Se encuentran soluciones exactas para los problemas de transisión de fases en el modelo de Pott y también para el modelo de Pott con interacciones binarias y ternarias en un campo magnético externo.

Key words and phrases: *Cayley tree, Potts model, Competing interactions, External magnetic field.*

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1 Introduction

The Potts model was introduced as a generalization of the Ising model. The idea came from the representation of the Ising model as interacting spins which can be either parallel or antiparallel. An obvious generalization was to extend the number of directions of the spins. Such a model was proposed by C.Domb as a PhD thesis for his student R.Potts in 1952. At present the Potts model encompasses a number of problems in statistical physics and lattice theory. It has been a subject of incresing intense research interest in recent years. It includes the ice-rule vertex and bond percolation models as special cases.

We consider a semi-infinite Cayley tree J^k for order $k \geq 2$, i.e., a graph having no cycles, from each vertex of which, except on vertex x^0, emanates exactly $k+1$ edges and from vertex x^0, which is the root of the tree, emanates k edges.

The vertices x and y are called nearest neighbors, which is denoted by $< x, y >$, if there exists an edge connecting them. The vertices x,y and z are called a triple of neighbors ,which is denoted by $< x, y, z >$,if $< x, y >$ and $< y, z >$ are nearest neighbors and $x \neq z$

Let V be the set of vertices in J^k. We set

$$W_n = \{x \in V | d(x, x^0) = n\},$$

$$V_n = \cup_{m=0}^{n} W_m = \{x \in V | d(x, x^0) \leq n\}.$$

where the distance $d(x, y), x, y, \in V$ is given by the formula,

$$d(x, y) = min\{d | x = x_0, x_1, x_2, ..., x_{d-1}, x_d = y \in V$$

such that the pairs $< x_0, x_1 >, ..., < x_{d-1}, x_d >$ are nearest neighbors.
The set W_n is called n-th level of J^k and the set V_n is called n-storeyed home with root x^0.

We consider models where the spin takes values in the set $\Phi = \{0, 1, 2, ..., q\}$, $q \geq 2$ and assigned to the vertices of the tree.A configuration σ on V is then defined as a function $x \in V \to \sigma(x) \in \Phi$;the set of all configurations coincides with $\Omega = \Phi^V$. The Potts model on the Cayley tree is defined by the Hamiltonian

$$H(\sigma) = -J \sum_{<x,y>} \delta_{\sigma(x)\sigma(y)} - h \sum_{x \in V} \delta_{0\sigma(x)} \qquad (1)$$

where the first sum is taken over all nearest neighbors, δ in the first and second sums is the Kroneker's symbol,$J, h \in R$ are coupling constants and $\sigma \in \Omega$.

Along with this model, we will consider the Potts model with competing interactions on the Cayley tree which is defined by the Hamiltonian below

$$H(\sigma) = -J_1 \sum_{<x,y,z>} \delta_{\sigma(x)\sigma(y)\sigma(z)} - J_2 \sum_{<x,y>} \delta_{\sigma(x)\sigma(y)} - h \sum_{x \in V} \delta_{0\sigma(x)} \qquad (2)$$

where the first sum is taken over all neighbors tripples, and δ in this sum is the generalized Kroneker's symbol (see [1]-[4] for models with competing interactions). Such model was investigated in [3], where for the neighbors tripple $< x, y, z >$the generalized Kroneker's symbol δ had a form

$$\delta_{\sigma(x)\sigma(y)\sigma(z)} = \begin{cases} 1 & \text{if} \quad \sigma(x) = \sigma(y) = \sigma(z), \\ 0 & \text{else.} \end{cases}$$

For the neighbors tripple $< x, y, z >$, we assume

$$\delta_{\sigma(x)\sigma(y)\sigma(z)} = \begin{cases} 1 & \text{if} \quad \sigma(x) = \sigma(y) = \sigma(z), \\ \frac{1}{2} & \text{if} \quad \sigma(x) = \sigma(y) \neq \sigma(z) \quad \text{or} \quad \sigma(x) \neq \sigma(y) = \sigma(z); \\ 0 & \text{else.} \end{cases} \qquad (3)$$

where $x, z \in W_n$ for some n and $y \in W_{n-1}$. This definition is well coordinated with the theory of quadratic stochastic operators, where the quadratic stochastic operator corresponding to the generalized Kroneker's symbol (4) is the identity transformation [5].

Let

$$S^{m-1} = \{x = (x_1, \cdots, x_m) \in \mathbb{R}^m : \sum_{i=1}^{m} x_i = 1\} \ x_i \geq 0 \ \forall i = 1, \cdots, m\}$$

be the $(m-1)$-dimensional simplex in $\mathbb{R}^m$. The transformation $V : S^{m-1} \to S^{m-1}$ is called quadratic stochastic operator , if

$$(Vx)_k = \sum_{i,j=1}^{m} p_{ij,k} x_i x_j$$

where $p_{ij,k} \geq 0$, $p_{ij,k} = p_{ji,k}$ and $\sum_{k=1}^{m} p_{ij,k} = 1$ for arbitrary $i, j, k \in \{1, \cdots, m\}$. Such operator have applications in mathematical biology, namely theory of heredity, where the coefficients $p_{ij,k}$ are interpreted as coefficients of heredity. Assume $p_{ij,k} = \delta_{ijk}$,where the generalized Kroneker's symbol δ has a form

$$\delta_{ijk} = \begin{cases} 1 & \text{if} \quad i = j = k, \\ \frac{1}{2} & \text{if} \quad i = k \neq j \quad \text{or} \quad i \neq j = k; \\ 0 & \text{else.} \end{cases} \tag{4}$$

Then it is easy to show that the corresponding quadratic stochastic operator is the identity transformation .

2 Recurrent Equations for partition function

There are several approaches to derive the equation or a system of equations describing the limiting Gibbs measures for lattice models on a Cayley tree. One approach is based on the properties of the Markov random fields on a Cayley tree [6, 7]. Another approach is based on recurrent equations for partition functions(see for example [8]). Naturally both approaches lead to the same equation(for example [9]). The second approach, however, is more suitable for models with competing interactions.

Let Λ be a finite subset of V. Assume $\sigma(\Lambda)$ and $\sigma(V \setminus \Lambda)$ are the restriction of σ to Λ and $V \setminus \Lambda$ respectively. Let $\overline{\sigma}(V \setminus \Lambda)$ be a fixed boundary configuration. The total energy of configuration $\sigma(\Lambda)$ under condition $\overline{\sigma}(V \setminus \Lambda)$ is defined as

$$H(\sigma(\Lambda)|\overline{\sigma}(V \setminus \Lambda)) = -J \sum_{\substack{<x,y> \\ x,y \in \Lambda}} \delta_{\sigma(x)\sigma(y)} - J \sum_{\substack{<x,y> \\ x \in \Lambda, y \notin \Lambda}} \delta_{\sigma(x)\sigma(y)} - h \sum_{x \in \Lambda} \delta_{0\sigma(x)}.$$

in the first case and

$$\begin{aligned} H(\sigma(\Lambda)|\overline{\sigma}(V \setminus \Lambda)) &= -J_1 \sum_{\substack{<x,y,z> \\ x,y,z \in \Lambda}} \delta_{\sigma(x)\sigma(y)\sigma(z)} - J_2 \sum_{\substack{<x,y> \\ x,y \in \Lambda}} \delta_{\sigma(x)\sigma(y)} - h \sum_{x \in \Lambda} \delta_{0\sigma(x)} \\ & \quad -J_1 \sum_{\substack{<x,y,z> \\ x \notin \Lambda, y \in \Lambda, z \notin \Lambda}} \delta_{\sigma(x)\sigma(y)\sigma(z)} - J_2 \sum_{\substack{<x,y> \\ x \in \Lambda, y \notin \Lambda}} \delta_{\sigma(x)\sigma(y)} . \end{aligned}$$

for the second Hamiltonian respectively.

The partition function $Z_\Lambda(\overline{\sigma}(V \setminus \Lambda))$ in volume Λ under boundary condition $\overline{\sigma}(V \setminus \Lambda)$ is defined as

$$Z_\Lambda = \sum_{\sigma(\Lambda) \in \Omega(\Lambda)} \exp(-\beta H(\sigma(\Lambda))|\overline{\sigma}(V \setminus \Lambda)), \tag{5}$$

where $\Omega(\Lambda)$ is the set of all configuration on Λ, and $\beta = \dfrac{1}{T}$ is the inverse temperature. We consider the configurations $\sigma(V_n)$, the partition functions Z_{V_n} in the volume V_n and for brevity we denote it as σ_n, $Z^{(n)}$ respectively.

Let us first consider the model (1).We decompose the partition function $Z^{(n)}$ into the following summands

$$Z^{(n)} = \sum_{i=1}^{q} Z_i^{(n)},$$

where

$$Z_i^{(n)} = \sum_{\sigma_n \in \Omega(V_n): \sigma_n(x^0)=i} \exp(-\beta H_n(\sigma_n)). \tag{6}$$

Let $\theta = \exp(\beta J)$, $\theta_3 = \exp(\beta h)$. From (5) and (6), the following system of recurrent equations can be easily derived

$$\begin{aligned} Z_0^{(n)} &= \theta_3 \left[\theta Z_0^{(n-1)} + Z_1^{(n-1)} + Z_2^{(n-1)} + ... + Z_q^{(n-1)}\right]^k \\ Z_i^{(n)} &= \left[Z_0^{(n-1)} + ... + Z_{i-1}^{(n-1)} + \theta Z_i^{(n-1)} + Z_{i+1}^{(n-1)} ... Z_q^{(n-1)}\right]^k \end{aligned} \tag{7}$$

for i=1,2,...,q , where $Z_i^{(n-1)}$ is a partition function in $(n-1)$-storeyed home with root located a vertex $x \in W_1$ for which $\sigma(x) = i$.

After replacing $u_i^{(n)} = \frac{Z_i^{(n)}}{Z_0^{(n)}}$, we have the following system of recurrent equations

$$u_i^{(n)} = \frac{1}{\theta_3}\left(\frac{1 + (\theta-1)u_i^{(n-1)} + \sum_{j=1}^{q} u_j^{(n-1)}}{\theta + \sum_{j=1}^{q} u_j^{(n-1)}}\right)^k; \tag{8}$$

for i=1,2,...,q and n=2,3,...

We describe the fixed points of this system recurrent equation (5). For this, it suffices to solve the system of equations

$$u_i = \frac{1}{\theta_3}\left(\frac{1 + (\theta-1)u_i + \sum_{j=1}^{q} u_j}{\theta + \sum_{j=1}^{q} u_j}\right)^k; \quad i = 1, 2, ..., q. \tag{9}$$

Before we begin to solve this system of equations, we turn to the model (2). Here we consider a slight modification of the Hamiltonian (2)

Definition 1 *A triple of neighbours $< x, y, z >$ is said to be two-level and is denoted by $< x, \bar{y}, z >$ if the vertices x and z belong to W_n for some n, i.e. they are located on the same level and $y \in W_{n-1}$.*

We consider the Hamiltonian

$$H(\sigma) = -J_1 \sum_{<x,\bar{y},z>} \delta_{\sigma(x)\sigma(y)\sigma(z)} - J_2 \sum_{<x,y>} \delta_{\sigma(x)\sigma(y)} - h \sum_{x \in V} \delta_{0\sigma(x)} \tag{10}$$

where $J_1 \neq 0$ and in contrast to (2), the first sum inlcudes only the two-level triples of neighbours. Such a model is called a two-level model (see [3], [4] and the references there for the physical motivation underlying the study of these model).

It is not hard to derive, in this case, the system of recurrent equations is as the following

$$Z_0^{(n)} = \theta_3 \left[\theta_1\theta_2 Z_0^{(n-1)} + Z_1^{(n-1)} + Z_2^{(n-1)} + ... + Z_q^{(n-1)}\right]^k$$
$$Z_i^{(n)} = \left[Z_0^{(n-1)} + ... + Z_{i-1}^{(n-1)} + \theta_1\theta_2 Z_i^{(n-1)} + Z_{i+1}^{(n-1)} ... Z_q^{(n-1)}\right]^k \quad \text{for} \quad i = 1, 2, ..., q$$

where $\theta_1 = \exp(\beta J_1)$, $\theta_2 = \exp(\beta J_2)$ and $\theta_3 = \exp(\beta h)$.

Thus both models (1) and (10) are described by the same system of recurrent equations.

3 The proof of existence of phase transitions for zero external field

In this section, we let $J > 0$ for model (1), that is we consider model (1) as a ferromagnetic Potts model and $J_1 + J_2 > 0$ for model (10). Then $\theta > 1$ in the first case and $\theta_1\theta_2 > 1$ for second case. We consider the system of equations (9).Assume $u_i = \exp h_i, i = 1, 2, ..., q$ Then

$$h_i' = k \ln \frac{1}{\theta_3}\left(\frac{1 + (\theta - 1)h_j + \sum_{j=1}^{q} \exp(h_j)}{\theta + \sum_{j=1}^{q} \exp h_j}\right); \quad i = 1, 2, ..., q \tag{11}$$

is the transformation R^q into R^q. Evidently the line $l_0 : h_1 = h_2 = ... = h_q$ in R^q is invarinat with respect to transformation (11) and the restriction of (11) on the line l_0 has the following form

$$h' = k \ln \frac{1}{\theta_3}\left(\frac{(\theta + q - 1)\exp(h) + 1}{q \exp(h) + \theta}\right)$$

where $h \in R$. Again, after renaming $u = \exp(h)$, we have

$$u = \frac{1}{\theta_3}\left(\frac{(\theta + q - 1)u + 1}{qu + \theta}\right)^k.$$

The following Lemma is a generalization of the Proposition 10.7 from [8].

Lemma 1 *The equation*

$$\theta_3 u = \left(\frac{(\theta + q - 1)u + 1}{qu + \theta}\right)^k \tag{12}$$

(with $u > 0$, $k \geq 2$, $q \geq 2$) has a single solution if

$$1 < \theta < \theta_{cr} = \frac{-(k-1)(q-1) + \sqrt{(k-1)^2(q+1)^2 + 8q(k-1)}}{2(k+1)}$$

If $\theta > \theta_{cr}$ then there are numbers $\eta_1(\theta, q, k)$, $\eta_2(\theta, q, k)$ with $0 < \eta_1(\theta, q, k) < \eta_2(\theta, q, k)$ such that equation (12) has three roots, when $0 < \eta_1(\theta, q, k) < \theta_3 < \eta_2(\theta, q, k)$ and it has two roots if either $\theta_3 = \eta_1(\theta, q, k)$ or $\theta_3 = \eta_2(\theta, q, k)$ or $\theta = \theta_{cr}$. The numbers $\eta_i, i = 1, 2$ are defined from the formula

$$\eta_i(\theta, q, k) = \frac{1}{u_i}\left(\frac{(\theta + q - 1)u_i + 1}{qu_i + \theta}\right)^k \tag{13}$$

where u_1 and u_2 are the solution of the equation

$$(\theta + q - 1)qu^2 - [k(\theta - 1)(\theta + q) - \theta(\theta + q - 1) - q]u + \theta = 0 \tag{14}$$

Proof. Assume $f(u) = (\frac{(\theta+q-1)u+1}{qu+\theta})^k$. It is easy to check that equation (12) has more than one root if and only if the equation $uf' = f(u)$ has more than one solution. The equation $uf' = f(u)$ is no other than just equation (14).

Although there are three solutions for the system of equations (9) for $\theta > \theta_{cr}$, one cannot claim that there is a phase transition. Among these solutions, only one of them is a stable solution. It is necessary to find other stable solutions. This problem is rather complete for arbitrary k and q when $\theta \neq 1$. The case with $\theta \neq 1$ will be considerd separately when $k = 2$ and $q = 2$. We shall now solve this problem for $\theta_3 = 1$, that is, $h = 0$. Then, the system of equation (9) has the following form

$$u_i = \left(\frac{1 + (\theta - 1)u_i + \sum\limits_{j=1}^{q} u_j}{\theta + \sum\limits_{j=1}^{q} u_j}\right)^k \quad i = 1, 2, ..., q \tag{15}$$

and the transformation R^q into R^q (11) has the following form

$$h_i' = k\ln\left(\frac{1 + (\theta - 1)\exp h_j + \sum\limits_{j=1}^{q} \exp(h_j)}{\theta + \sum\limits_{j=1}^{q} \exp h_j}\right); \quad i = 1, 2, ..., q. \tag{16}$$

Then, apart from the invariant line l_0 we can find other q invariant lines, namely the line $l_j : h_1 = ... = h_{j-1} = h_{j+1} = ... = h_q = 0$, $j = 1, 2, ..., q$. The transformation (11) reduces to the following transformation of R:

$$h' = k\ln\left(\frac{\theta \exp h + q}{\exp h + \theta + q - 1}\right)$$

on each invariant line $l_j, j = 1, 2, ..., q$.

Now we will solve this simpler equation

$$u = (\frac{\theta u + q}{u + \theta + q - 1})^k \tag{17}$$

Let us consider the function $\phi(u) = (\frac{\theta u + q}{u + \theta + q - 1})^k$. With the help of the Lemma, it is not hard to show that the equation (17) has three solutions when $\theta > \theta^*_{cr} = \frac{k + 2q - 1}{k - 1}$. In this case only one of these roots is stable, namely, largest of them. For equation (12), when $\theta_3 = 1$, we showed above that it has three solutions when $\theta > \theta^*_{cr}$ (see Lemma) and only one of them is stable. It is easy to check that $\theta^*_{cr} > \theta_{cr}$.

As $u_j = \frac{P(x^0=j)}{P(x^0=0)}$ for some limiting Gibbs measure P with $\theta > \theta^*_{cr} = \frac{k+2q-1}{k-1}$, we have $q+1$ differences translated invariant limiting Gibbs measures. The same way as in [9], it is possible to prove that all of them are extremal.

Theorem 1 *For Potts model (1) with null external field, a phase transition occurs when,*

$$\theta > \frac{k + 2q - 1}{k - 1}.$$

Similar assertation is also valid for the two-level Potts model with competing ternary and binary potentials with null external field.

Theorem 2 *For the two-level Potts model (10) with competing ternaty and binary potentials with null external magnetic field, a phase transition occurs when $\theta_1\theta_2 > \frac{k+2q-1}{k-1}$.*

4 The case of non-zero external magnetic field when $k = q = 2$

Here we consider Potts models both (1) and (10) with external magnetic field $h \neq 0$, when $k = q = 2$ and $\theta > 1$ for model (1) and $\theta_1\theta_2 > 1$ for model (10) respectively (The case $h = 0$ was considered in [9]for model (1) and for model (10) in [10]). Then the system of equations (9) reduces to the following

$$\begin{aligned} x &= \frac{1}{\theta_3}(\frac{\theta x + y + 1}{x + y + \theta})^2 \\ y &= \frac{1}{\theta_3}(\frac{x + \theta y + 1}{x + y + \theta})^2 \end{aligned} \tag{18}$$

where $x = u_1$, $y = u_2$ for brevity. As

$$x - y = \frac{1}{\theta_3}\frac{(\theta_1)(x - y)[(\theta + 1)(x + y) + 1]}{(x + y + \theta^2)},$$

then some solutions of (18) can be found from equation

$$u = \frac{1}{\theta_3}(\frac{(\theta+1)u+1}{2u+\theta})^2 \tag{19}$$

where $x = y = u$ and other solutions can be found from equation

$$\theta_3 z^2 - (\theta^2 - 2\theta_3\theta - 1)z + \theta_3\theta^2 - 2\theta + 2 = 0 \tag{20}$$

where $z = x+y$. First of all, let us consider equation (19).Then the equation (13)(see Lemma)has the following form

$$2(\theta+1)u^2 - (\theta^2 + \theta - 6)u + \theta = 0. \tag{21}$$

This equation has two roots u_1, u_2 if $\theta > \frac{\sqrt{73}-1}{2}$. Then by Lemma, equation (19)have three roots if $\theta > \frac{\sqrt{73}-1}{2}$ and $\eta_1(\theta) < \theta_3 < \eta_2(\theta)$, where $\eta_i = \frac{1}{u_i}(\frac{(\theta+1)u_i+2}{3u_i+\theta})^2$, $i = 1, 2$.

Now we consider the equation (20). Again with the help of elementary analysis it is not hard to show that the equation has two solutions for $\theta_3 > \frac{1}{2}$ with $\theta > 2\theta_3 - 1 + 2\sqrt{\theta_3(\theta_3+1)}$ and for $0 < \theta_3 < \frac{1}{2}$, with $\theta > \frac{1+\sqrt{1-2\theta_3}}{\theta_3}$. By virtue of symmetry of equations (18) we have two stable solutions.

Assume $A = \{(\theta_3, \theta) : \eta_1(\theta) < \theta_3 < \eta_2(\theta) \ ; \theta > \dfrac{\sqrt{73}-1}{2}\}$ where $\eta_1(\theta)$ and $\eta_2(\theta)$ as above and $B = \{(\theta_3, \theta)\text{: } 0 < \theta_3 < \frac{1}{2} \ ; \ \theta > \frac{1+\sqrt{1-2\theta_3}}{\theta_3}\} \cup \{(\theta_3, \theta)\text{: } \theta_3 > \frac{1}{2} \ ; \ \theta > 2\theta_3 - 1 + 2\sqrt{\theta_3(\theta_3+1)}\ \}$. Then for arbitrary $(\theta_3, \theta) \in A \cap B$ there are three stable solutions of the equations (18). We have thus proved the following theorems.

Theorem 3 *For Potts model (1) with $q = k = 2$ and non-zero external magnetic field, a phase transition occurs when $(\theta_3, \theta) \in A \cap B$*

A similar result is valid for model (10).

Theorem 4 *For the two-level Potts model (10) with competing ternary and binary potentials $q = k = 2$ and non-zero external magnetic field a phase transition occurs when $(\theta_3, \theta_1\theta_2) \in A \cap B$.*

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