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Some special classes of neutral functional differential equations

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ABSTRACT

This paper is dedicated to the investigation of existence, mainly local, of solutions of two classes of neutral functional differential equations. A reduction method and fixed point methods are emphasized.

RESUMEN

Este artículo está dedicado al estudio de existencia, principalmente local, de soluciones de dos clases de ecuaciones diferenciales funcionales neutrales. Un método de reducción y de punto fijo son puestos con algún énfasis.

Key words and phrases: *functional equation, neutral equation, existence of solution*
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1 Introduction

In several recent papers of the author [3], [4], [5], as well as in some joint papers with M. Mahdavi [7], [8], certain types of neutral functional equations (including functional–differential ones) have been investigated in regard to the existence of solutions. Such equation, in a rather general form, can be written as

$$(Ux)(t) = (Vx)(t), \quad t \in [0, T], \tag{1}$$

or, in functional differential form,

$$\frac{d}{dt}(Ux)(t) = (Vx)(t), \quad t \in [0, T]. \tag{2}$$

In (1) and (2), U and V stand for some operators acting on convenient function spaces, whose elements are defined on $[0, T]$. The case $[0, T)$, $T \leq \infty$, has been also discussed, while the solution has been sought in various function spaces (usually, C, L^p, $1 \leq p < \infty$).

Let us notice that equation (2), by integrating both sides on $[0, t]$, $t \leq T$, takes the form (1):

$$(Ux)(t) = C + \int_0^t (Vx)(s)ds, \; t \in [0, T]. \tag{3}$$

Moreover, when V is a causal operator, the right hand side of (3) is also causal. The case of causal operators has been dealt with in the author's recent book [6].

The basic method of investigating the existence of solutions to equations(1) and (2) consisted in reducing such equations to the simpler form

$$x(t) = (Wx)(t), \quad t \in [0, T], \tag{4}$$

and then applying existing results available for (4). In our book [2] we provided existence results for (4), based on the existing literature, while in [6] we have illustrated how this method works for neutral equations like (1) or (2).

The aim of this paper is to investigate some classes of neutral equations encountered in the existing literature, by using mainly the above described method. In other words, to reduce such equations to the form (4), and then to apply known results. We are not necessarily intended to reobtain results already known in the literature, but to see what kind of results one obtains by means of the above described method.

We shall particularly refer to the papers of T.A. Burton [1] and Loris Faina [9], in which various classes of functional differntial equations, of neutral type, are investigated.

2 Reduction of neutral equations to the form (4)

Let us start with the neutral equation

$$x'(t) = f(t, x(t), x'(t)), \quad t \in [0, T], \tag{5}$$

investigated by Loris Faina [9] and other authors. This is the simplest form, in which x and f take scalar values, or values belonging to $\mathbb{R}^n$.

The initial value condition attached to (5) will be

$$x(0) = x^0 \in \mathbb{R}^n, \tag{6}$$

if one deals with the vector case.

Formally, let us denote

$$x'(t) = y(t), \tag{7}$$

which implies under rather general assumptions (see below)

$$x(t) = x^0 + \int_0^t y(s)ds. \tag{8}$$

The neutral equation (5) can be now written as

$$y(t) = f\left(t, x^0 + \int_0^t y(s)ds, y(t)\right). \tag{9}$$

Obviously, the right hand side of (9) engages only the values of y on the interval $0 \leq s \leq t$ $(t \leq T)$.

This means that equation (9) is an equation of the form (4), in which the right hand side is a causal operator (in y). There is, therefore, an equivalence between the initial value problem (5), (6), and the problem (8), (9). This equivalence will be further discussed when we precise the underlying function spaces. Since equation (9) contains only the unknown $y(t)$, we shall be able to investigate it in various function spaces (continuous or measurable functions), by using known results.

In L. Faina [9] there are more general neutral functional differential equations than (5). For instance, in (5) one assumes that f is a map from $R \times C(R) \times L_1(R)$ into $\mathbb{R}^n$, while the initial condition (6) is replaced by $x(t) = \varphi(t)$, $t \in (-\infty, t_0]$, $t_0 \in R$ fixed. In other terms, the infinite delay is dealt with. This case can be also covered by the scheme described above, though the procedure is more intricate.

In T.A. Burton [1], the following neutral functional differential equation is studied

$$x'(t) = f(t, x(t), x'(t - h(t)) + g(t, x(t), x(t - h(t)),$$

where $0 \leq h(t) \leq h_0$, $h_0 > 0$ being fixed. To the above equation one attaches the typical initial condition for delay equations, namely $x(t) = \varphi(t)$, $t \in [-h_0, 0]$.

We shall rewrite Burton's equation in the form

$$x'(t) = f(t, x(t), x'(\alpha(t))) + g(t, x(t), x(\alpha(t))), \tag{10}$$

where $\alpha(t)$ is such that $0 \leq \alpha(t) \leq t$ on some interval $[0, T]$. The right hand side in (10) can be regarded as a causal operator in x. Hence, (10) is also of the form (4). Moreover, one can use the initial condition (6) for determining a (unique) solution to (10), (6).

The literature on neutral functional equations is very rich, and a good amount of references can be found in our book [6]. In many more cases than those illustrated above, the method of reduction to equations of the form (1), with causal operator in the right hand side, can be successfully applied.

In what follows, we shall dwell on the equations (5) and (9), trying to apply the reduction procedure in these cases, as well as other methods.

3 The equation (9) in the space $C([0,T],\mathbb{R}^n)$

We shall consider in this section the equation (9) in the space $C([0,T],\mathbb{R}^n)$. The assumption to be made are of such a nature that the right hand side of this equation represent a compact operator on $C([0,T_1],\mathbb{R}^n)$, with $T_1 \le T$. Obviously, the Arzelà-Ascoli criterion of compactness in $C([0,T],\mathbb{R}^n)$ will be used.

The compactness of the operator

$$y(t) \longrightarrow f\left(t, x^0 + \int_0^t y(s)ds, y(t)\right), \tag{11}$$

on $C([0,T],\mathbb{R}^n)$ can be achieved under various sets of hypotheses. We shall describe such a set of hypeotheses, which will imply the existence of a local solution to the equation (9). Such a solution will generate a continuously differentiable solution to the problem (5), (6).

Before we proceed with the statement of the hypotheses, it is instructive to look at a simple example for the equation (5). Namely, we will choose $f = 2x(t)x'(t) + 1$. This leads to the integral $x(t) = x^2(t) + t + c$, from which we derive $x(t) = \frac{1}{2}\left(1 + \sqrt{1-4c-4t}\right)$. Choosing the initial value $x^0 = 1/2$, we get $c = 1/4$, which means the solution is $x(t) = \frac{1}{2}\left(1 + 2\sqrt{-t}\right)$. This shows that we have no solutions of (5) on any $[0,T]$, $T > 0$, for $x^0 = 1/2$.

Therefore, a problem of the form (5), (6) may be deprived of (local) solutions, even though the right hand side of (5) is quite a usual function.

We shall return now to the general problem (5), (6), and provide some conditions which assure the existence of local solutions. But before getting into details, we shall modify somewhat the equation (5), in order to encompass a larger category of situations. Namely, let us rewrite (9) in the form

$$y(t) = f\left(t, x^0 + \int_0^t y(s)ds; y\right), \tag{9$'$}$$

and assume the right hand side in (9)$'$ is defined on the set $[0,T]\times\mathbb{R}^n\times C([0,T],\mathbb{R}^n)$. We shall assume continuity of the map $y \longrightarrow f$, but further hypotheses will be formulated. Compared to (9), the equation (9)$'$ involves now an operator in the right hand side, defined on the space $C([0,T],\mathbb{R}^n)$.

The following hypotheses will be made, in view of obtaining the existence of solutions to the equation (9)$'$:

H_1 The map

$$y \longrightarrow f\left(t, x^0 + \int_0^t y(s)ds; y\right) \tag{12}$$

is continuous from $[0,T]\times\mathbb{R}^n\times C([0,T],\mathbb{R}^n)$ into $\mathbb{R}^n$, and causal.

H_2 For each $\gamma > 0$, there exist two functions $\omega_1(r)$ and $\omega_2(r)$, continuous on $[0, \infty)$, $\omega_1(0) = \omega_2(0) = 0$, and positive for $r > 0$, such that

$$\begin{aligned} \left| f\left(t, x^0 + \int_0^t y(s)ds; y\right) - f\left(u, x^0 + \int_0^u y(s)ds; y\right)\right| \leq \\ \leq \omega_1(|t-u|) + \omega_2(\gamma|t-u|), \end{aligned} \tag{13}$$

for arbitrary $t, u \in [0, T]$, and all $y \in C$, with $|y|_C \leq \gamma$.

Let us notice the fact that choosing $\omega_1(r) = \alpha r$, $\omega_2(r) = \beta r$, $\alpha, \beta > 0$, the condition (13) becomes a Lipschitz type continuity condition.

H_3 The map (12) takes bounded sets in $C([0, T], \mathbb{R}^n)$, into bounded sets of $\mathbb{R}^n$.

We can now prove the following (local) existence theorem for the equation $(9)'$.

Theorem 1. *Consider the functional equation* $(9)'$ *in the space* $C([0, T], \mathbb{R}^n)$*. Assume that the map* (12) *satisfies the hypotheses* H_1*,* H_2 *and* H_3*. Then equation* $(9)'$ *has a local solution* (*i.e. defined on some interval* $[0, T_1]$, $T_1 \leq T$)*, provided* $f(0, x^0; y)$ *is independent of* y*.*

Moreover, the equation

$$x'(t) = f(t, x(t); x'), \tag{5$'$}$$

under initial condition (6)*, has a local solution, which is continuously differentiable.*

Remark 1. The localization is possible due to the causality of the operator (12) (hypothesis H_1).

Remark 2. In case f does not depend on the last argument, the equation $(5)'$ becomes $x'(t) = f(t, x(t))$, while hypotheses H_2 and H_3 are automatically satisfied in case of continuity. The result of Theorem 1 reduces to the classical Peano's existence theorem. The existence of the functions $\omega_1(r)$ and $\omega_2(r)$ is a simple consequence of the uniform continuity of $f(t, x)$ on a set of the form $[0, T] \times B$, with B compact in $\mathbb{R}^n$. The imposition of hypothesis H_2 is motivated by the fact that the last argument in $f\left(t, x^0 + \int_0^t y(s)ds; y\right)$ belongs to an infinite dimensional space, i.e. to $C([0, T], \mathbb{R}^n)$.

Proof of Theorem 1. The equation $(9)'$ is, according to our hypotheses, a functional equation with causal operaor of the form (4). The hypotheses H_1 and H_2 assure the continuity of the map (12) from $C([0, T], \mathbb{R}^n)$ into itself. Based on hypothesis H_3, the map (12) from $C([0, T], \mathbb{R}^n)$ into $C([0, T], \mathbb{R}^n)$ is also compact. Indeed, according to the hypothesis H_3, the image of the ball $|y|_C \leq \gamma$ is a bounded set in $C([0, T], \mathbb{R}^n)$. The inequality (13) in hypothesis H_2 tells us that the image of the ball $|x|_C \leq \gamma$ consists of equicontinuous functions on $[0, T]$, with values in $\mathbb{R}^n$. Since $\gamma > 0$ is an arbitrary number, we conclude that the operator (12) is compact (takes bounded sets into relatively compact sets). Hence Theorem 3.1 in [6] applies directly, keeping also in mind that the operator (12) enjoys the property of fixed initial value. Consequently, $(9)'$ has a local solution in some space $C([0,T_1],\mathbb{R}^n)$, with $T_1 \le T$.

This result leads immediately to the existence of a local solution for the problem $(5)'$, 6. This ends the proof of Theorem 1.

Remark 3. The case of measurable solutions to the equation $(9)'$, when the corresponding solutions to $(5)'$ will be absolutely continuous functions, can be treated in the same manner as in the continuous case. One has to use Theorem 3.3 in [6], instead of Theorem 3.1. We shall leave to the reader the task of formulating existence results.

4 Existence of solutions to equation (10)

If we denote again $x'(t) = y(t)$, and take into account the initial condition (6), then equation (10) becomes

$$\begin{aligned} y(t) = f\left(t, x^0 + \int_0^t y(s)ds, y(\alpha(t))\right) + \\ +g\left(t, x^0 + \int_0^t y(s)ds, y(\alpha(t))\right), \end{aligned} \tag{14}$$

which is precisely of the form (4), with causal operator in the right hand side. This is due to the assumption on $\alpha(t)$, namely $0 \le \alpha(t) \le t$, $t \in [0,T]$.

We shall consider now a particular case of equation (14), as far as the function g is concerned. Instead of the term f, we shall consider another operator-like term. More precisely, we shall deal with the functional equation

$$y(t) + g(y(\alpha(t))) = C + \int_0^t (Wy)(s)ds, \tag{15}$$

under the following assumptions:

1) $g : C([0,T],\mathbb{R}^n) \longrightarrow C([0,T],\mathbb{R}^n)$ is a contraction map on this space:
$$|g(x) - g(y)|_C \le \lambda |x-y|_C,\ \ \lambda \in [0,1);$$

2) $W : C([0,T],\mathbb{R}^n) \longrightarrow C([0,T],\mathbb{R}^n)$ is a continuous causal operator, taking bounded sets of $C([0,T],\mathbb{R}^n)$ into bounded sets;

3) $\alpha : [0,T] \longrightarrow [0,\infty)$ is continuous, and such that $\alpha(0) = 0$ and $0 \le \alpha(t) \le t$ for $t \in [0,T]$.

Remark 4. The vector $C \in \mathbb{R}^n$ is arbitrary, but it can be chosen in such a way to satisfy some kind of initial condition. For instance, if we assign to y the initial value y^0, and assume (without loss of generality) that $g(\theta) = \theta \in \mathbb{R}^n$, then one obtains $C = y^0$.

In regard to the equation (15), the following existence result can be stated:

Theorem 2. *Consider the functional equation* (15)*, under conditions* 1), 2), 3) *stated above. Then, there exists a solution* $y = y(t)$*, defined on some interval* $[0, T_1] \subset [0, T]$*, for each* $C \in \mathbb{R}^n$*. This solution is such that* $y(t) + g(y(\alpha(t)))$ *is continuously differentiable.*

Proof. The hypotheses accepted are of such a nature that allow the application of Theorem 6.1 in [6], which yields the existence result.

The idea of proof is based on the fact that the functional equation $y(t)+g(y(\alpha(t))) = f(t)$ is uniquely solvable in $C([0,T], \mathbb{R}^n)$. Moreover, $y(t)$ depends continuously of $f(t)$, which allows to deal with (15) by contraction mapping principle, or by another fixed point methods.

Details of this approach can be found in our paper [5], where further existence results are obtained.

5 Further considerations on equation (10)

The idea of proof mentioned above can be adapted to other functional equations. For an illustration we will consider the equation (10), as well as the auxiliary equation

$$x'(t) = g(t, x(t), x(\alpha(t))) + f(t), \tag{16}$$

with $f \in C([0,T], \mathbb{R}^n)$. The attached initial condition is (6). The functional integral equation equivalent to (16), (6) is

$$x(t) = x^0 + \int_0^t f(s)ds + \int_0^t g(s, x(s), x(\alpha(s)))ds. \tag{17}$$

Let us assume the following conditions on the data in equation (17):

1) $g : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is continuous, and satisfies the Lipschitz condition

$$|g(t,x,y) - g(t,\bar{x},\bar{y})| \leq L(|x - \bar{x}| + |y - \bar{y}|),$$

with $L > 0$;

2) $f \in C([0,T], \mathbb{R}^n)$;

3) $\alpha(t)$ is continuous on $[0,T]$, and $\alpha(0) = 0$, $0 \leq \alpha(t) \leq t$.

It is easy to see that the usual process of iteration leads to the following relationship:

$$x^{(k+1)}(t) - x^{(k)}(t) =$$
$$= \int_0^t \left[g\left(s, x^{(k)}(s), x^{(k)}(\alpha(s))\right) - g\left(s, x^{(k-1)}(s), x^{(k-1)}(\alpha(s))\right)\right] ds, \ k \geq 1,$$

with

$$x^{(0)}(t) = x^0 + \int_0^t f(s)ds.$$

We further derive on behalf of condition 1)

$$|x^{(k-1)}(t) - x^{(k)}(t)| \leq$$
$$\leq L \int_0^t \left[|x^{(k)}(s) - x^{(k-1)}(s)| + |x^{(k)}(\alpha(s)) - x^{(k-1)}(\alpha(s))| \right] ds,$$

which leads to

$$\sup_{0 \leq s \leq t} |x^{(k+1)}(s) - x^{(k)}(s)| \leq 2L \int_0^t \sup_{0 \leq u \leq s} |x^{(k)}(u) - x^{(k-1)}(u)| ds, \tag{18}$$

if we keep in mind that $0 \leq \alpha(t) \leq t$.

The inequality (18) can be processed in the usual manner, and one finds that $\lim x^{(k)}(t) = x(t)$ as $k \longrightarrow \infty$, uniformly on $[0,T]$, with $x(t)$ satisfying (16). The uniqueness can be also proven by the standard method, as well as the continuous dependence of the solution with respect to $f \in C([0,T], \mathbb{R}^n)$.

The auxiliary result established above enables us to make some progress in regard to the equation (10). We rewrite it for the reader's convenience,

$$x'(t) = g(t, x(t), x(\alpha(t))) + f(t, x(t), x'(\alpha(t))),$$

and regard it as a (nonlinear) perturbed equation associated to (16). Of course, we preserve the initial condition (6).

The following fixed point scheme can be attached to the equation (19): for each continuously differntiable $u(t)$ on $[0,T]$, with values in$\mathbb{R}^n$, we shall attach the unique solution $x(t)$ of the equation like (16)

$$x'(t) = g(t, x(t), x(\alpha(t))) + f(t, u(t), u'(\alpha(t))) \tag{19}$$

with the initial condition (6). The existence and uniqueness of $x(t)$, under the above scheme, is guaranteed under the conditions 1), 2) and 3) specified above. Consequently, in the space $C([0,T], \mathbb{R}^n)$, or rather in the space $C^{(1)}([0,T], \mathbb{R}^n)$, we have defined an opeator $u \longrightarrow x$, where u and x are related by the equation (19), with x satisfying also (6).

Let us denote by V the operator defined above, i.e.

$$x(t) = (Vu)(t), \quad t \in [0,T], \tag{20}$$

with u and x as described above. The operator V appears as a compound operator: first, $u \longrightarrow f(t, u(t), u'(\alpha(t)))$, and second $f \longrightarrow x$, with x the solution of (19), (6).

As noticed earlier in this section, the second operator is continuous on $C([0,T], \mathbb{R}^n)$. The first operator involved, $u \longrightarrow f(t, u(t), u'(\alpha(t)))$ can be made continuous, under adequate hypotheses on the function f. It is obviously continuous from $C^{(1)}([0,T], \mathbb{R}^n)$ into $C([0,T], \mathbb{R}^n)$ when $f(t, u, v)$ is continuous on $[0,T] \times \mathbb{R}^n \times \mathbb{R}^n$.

Instead of pursuing the above scheme, which can certainly lead to results of existence for (10), we shall attempt to apply the contraction mapping principle to the equation (10), but modifying somewhat the scheme presented above. Namely, we will consider the scheme described by the following equation, attached to (10):

$$x'(t) = g(t, x(t), x(\alpha(t))) + f(t, x(t), u'(\alpha(t))). \tag{21}$$

By means of (21) and (6), we shall define the operator on $C^{(1)}([0,T], \mathbb{R}^n)$, say $x(t) = (Wu)(t)$, in the following manner. Given $u \in C^{(1)}([0,T], \mathbb{R}^n)$, the equation (21) can be solved in x under rather mild assumptions (as seen above, under Lipschitz condition). The unique solution of (21), (6) will be denoted by $x(t) = (Wu)(t)$. From (2) we derive the following relationship between $x = Wu$ and $y = Wv$, where $u, v \in C^{(1)}([0,T], \mathbb{R}^n)$:

$$\begin{aligned} x'(t) - y'(t) = g(t, x(t), x(\alpha(t))) - g(t, y(t), y(\alpha(t))) + \\ + f(t, x(t), u'(\alpha(t))) - f(t, y(t), v'(\alpha(t))). \end{aligned} \tag{22}$$

Assuming also a Lipschitz condition on $f(t, x, y)$, as we did already on $g(t, x, y)$, we obtain

$$\begin{aligned} |x'(t) = y'(t)| \leq L(|x(t) - y(t)| + |x(\alpha(t)) - y(\alpha(t))|) + \\ + M|x(t) - y(t)| + m|u'(\alpha(t)) - v'(\alpha(t))|, \end{aligned}$$

for any $t \in [0,T]$, where L, M and m are positive numbers. The above inequality yields

$$\begin{aligned} \sup |x'(t) - y'(t(| \leq (2L + M) \sup |x(t) - y(t)| + \\ + m \sup |u'(\alpha(t)) - v'(\alpha(t))|, \end{aligned} \tag{23}$$

with sup taken on $[0,T]$, or on any $[0,T_1]$, $T_1 \leq T$.

But

$$x(t) - y(t) = \int_0^t [x'(s) - y'(s)] ds, \tag{24}$$

because $x(0) = y(0) = x^0$, according to (6). From (23) we derive

$$\sup |x(t) - y(t)| \leq T \sup |x'(s) - y'(s)|, \tag{25}$$

with sup taken on $[0,T]$. Taking into account (23), (24) and (25) we obtain

$$\begin{aligned} \sup |x'(t) - y'(t)| \leq (2L + M)T \sup |x'(t) - y'(t)| + \\ + m \sup |u'(t) - v'(t)|. \end{aligned} \tag{26}$$

Since we want (26) to be a relation showing the fact that the operator W is a contraction on $C^{(1)}([0,T], \mathbb{R}^n)$, we see from (26) that a first condition to be imposed is

$$(2L + M)T < 1. \tag{27}$$

If we admit (27), then (26) allows us to write

$$\sup |x'(t) - y'(t)| \leq m[1 - (2L + M)T]^{-1} \sup |u'(t) - v'(t)|,$$

which really represents a contraction condition for W, as soon as

$$\lambda = m[1 - (2L + M)T]^{-1} < 1. \tag{28}$$

It is appropriate to notice the fact that the norm in $C^{(1)}([0, T], \mathbb{R}^n)$ is (by our choice)

$$|x^0| + \sup |x'(t)|. \tag{29}$$

Accordingly, the norm for $u - v$ should be $|u^0 - v^0| + \sup |u'(t) - v'(t)|$, which is in advantage of the contraction inequality

$$|Wu - Wv|_{C^{(1)}} \leq \lambda |u - v|_{C^{(1)}}, \tag{30}$$

as derived from above.

Therefore, we can now state the following (global) existence result for the problem (10), (6):

Theorem 3. *Consider the problem* (10), (6)*, and assume the following conditions are verified by the functions f and g:*

1) $f, g : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ *are continuous maps;*

2) *f and g satisfy the Lipschitz type conditions*

$$|f(t, x, y) - f(t, \bar{x}, \bar{y})| \leq L(|x - \bar{x}| + |y - \bar{y}|),$$
$$|g(t, x, y) - g(t, \bar{x}, \bar{y})| \leq M|x - \bar{x}| + m|y - \bar{y}|,$$

with positive constants L, M and m;

3) *the inequalities* (27) *and* (28) *are satisfied.*

Then, there exists a unique solution $x(t) \in C^{(1)}([0, T], \mathbb{R}^n)$, which can be approximated by the scheme described by the equation (21).

The proof of Theorem 3 has been carried out above, before its statement.

Remark 5. It is obvious from the inequalities (27) and (28) that severe restrictions must be imposed to the constants L, M, m and T.

First, if L, M are fixed, it is obvious that the inequality (27) can be satisfied provided we choose T small enough: $T < (2L + M)^{-1}$. This restriction suggests that we need to confine our investigation, possibly, to a smaller interval than the original interval $[0, T]$. But this kind of restriction is in accordance with the fact we are looking for local solutions to our problem.

Second, once we choose T such that (27) takes place, there remains the inequality (28) to be satisfied. If the constants L, M and T are fixed, then the only way to satisfy (28) is to choose m small enough.

In conclusion, local existence for the problem (10), (6) is always assured by choosing the constant m sufficiently small.

Other neutral equations can be investigated in regard to the existence of their solutions, using approaches described above. We suggest to the reader to try such procedures on equations of the form

$$x'(t) = f(t, x(t), x(t-h)) + g(t, x(t), x'(t-h)),$$

under an initial condition of the form $x(t) = \varphi(t)$, $t \in [-h, 0]$. Also, similar to the equation (15), is

$$\frac{d}{dt}[x(t) + g(x(t-h))] = (Wx)(t),$$

with initial datum $x(t) = \varphi(t)$, $t \in [-h, 0]$.

See our paper [4] for details.

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