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Fuzzy Taylor Formulae

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ABSTRACT

We produce Fuzzy Taylor formulae with integral remainder in the univariate and multivariate cases, analogs of the real setting.

RESUMEN

Se presentan versiones Fuzzy análogas a las reales de fórmulas de Taylor con resto integral en el caso univariado y multivariado.

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1 Background

We need the following

Definition A (see [10]). Let $\mu\colon \mathbb{R} \to [0,1]$ with the following properties.

(i) is *normal*, i.e., $\exists x_0 \in \mathbb{R}$; $\mu(x_0) = 1$.

(ii) $\mu(\lambda x + (1-\lambda)y) \geq \min\{\mu(x), \mu(y)\}$, $\forall x, y \in \mathbb{R}$, $\forall \lambda \in [0,1]$ (μ is called a convex fuzzy subset).

(iii) μ is *upper semicontinuous* on $\mathbb{R}$, i.e., $\forall x_0 \in \mathbb{R}$ and $\forall \varepsilon > 0$, $\exists$ neighborhood $V(x_0)$: $\mu(x) \leq \mu(x_0) + \varepsilon$, $\forall x \in V(x_0)$.

(iv) The set $\overline{\operatorname{supp}(\mu)}$ is compact in $\mathbb{R}$ (where $\operatorname{supp}(\mu) := \{x \in \mathbb{R};\ \mu(x) > 0\}$).

We call μ a *fuzzy real number*. Denote the set of all μ with $\mathbb{R}_{\mathcal{F}}$.

E.g., $\mathcal{X}_{\{x_0\}} \in \mathbb{R}_{\mathcal{F}}$, for any $x_0 \in \mathbb{R}$, where $\mathcal{X}_{\{x_0\}}$ is the characteristic function at x_0.

For $0 < r \leq 1$ and $\mu \in \mathbb{R}_{\mathcal{F}}$ define $[\mu]^r := \{x \in \mathbb{R}\colon\ \mu(x) \geq r\}$ and

$$[\mu]^0 := \overline{\{x \in \mathbb{R} : \mu(x) > 0\}}.$$

Then it is well known that for each $r \in [0,1]$, $[\mu]^r$ is a closed and bounded interval of $\mathbb{R}$. For $u, v \in \mathbb{R}_{\mathcal{F}}$ and $\lambda \in \mathbb{R}$, we define uniquely the sum $u \oplus v$ and the product $\lambda \odot u$ by

$$[u \oplus v]^r = [u]^r + [v]^r, \quad [\lambda \odot u]^r = \lambda [u]^r, \qquad \forall r \in [0,1],$$

where $[u]^r + [v]^r$ means the usual addition of two intervals (as subsets of $\mathbb{R}$) and $\lambda[u]^r$ means the usual product between a scalar and a subset of $\mathbb{R}$ (see, e.g., [10]). Notice $1 \odot u = u$ and it holds $u \oplus v = v \oplus u$, $\lambda \odot u = u \odot \lambda$. If $0 \leq r_1 \leq r_2 \leq 1$ then $[u]^{r_2} \subseteq [u]^{r_1}$. Actually $[u]^r = [u_-^{(r)}, u_+^{(r)}]$, where $u_-^{(r)} \leq u_+^{(r)}$, $u_-^{(r)}$, $u_+^{(r)} \in \mathbb{R}$, $\forall r \in [0,1]$. For $\lambda > 0$ one has $\lambda u_\pm^{(r)} = (\lambda \odot u)_\pm^{(r)}$, respectively.

Define

$$D\colon \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_+$$

by

$$D(u,v) := \sup_{r \in [0,1]} \max\{|u_-^{(r)} - v_-^{(r)}|, |u_+^{(r)} - v_+^{(r)}|\},$$

where $[v]^r = [v_-^{(r)}, v_+^{(r)}]$; $u, v \in \mathbb{R}_{\mathcal{F}}$. We have that D is a metric on $\mathbb{R}_{\mathcal{F}}$. Then $(\mathbb{R}_{\mathcal{F}}, D)$ is a complete metric space, see [10], with the properties

$$\begin{aligned} D(u \oplus w, v \oplus w) &= D(u,v), \quad \forall u, v, w \in \mathbb{R}_{\mathcal{F}}, \\ D(k \odot u, k \odot v) &= |k| D(u,v), \quad \forall u, v \in \mathbb{R}_{\mathcal{F}},\ \forall k \in \mathbb{R}, \\ D(u \oplus v, w \oplus e) &\leq D(u,w) + D(v,e), \quad \forall u, v, w, e \in \mathbb{R}_{\mathcal{F}}. \end{aligned}$$

Let $f, g\colon \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ be *fuzzy number valued functions*. The distance between f, g is defined by

$$D^*(f,g) := \sup_{x \in \mathbb{R}} D(f(x), g(x)).$$

On $\mathbb{R}_{\mathcal{F}}$ we define a *partial order by* "$\leq$": $u, v \in \mathbb{R}_{\mathcal{F}}$, $u \leq v$ iff $u_-^{(r)} \leq v_-^{(r)}$ and $u_+^{(r)} \leq v_+^{(r)}$, $\forall r \in [0,1]$.

We mention

Lemma 2.2 ([5]). *For any* $a, b \in \mathbb{R}$: $a, b \geq 0$ *and any* $u \in \mathbb{R}_{\mathcal{F}}$ *we have*

$$D(a \odot u, b \odot u) \leq |a - b| \cdot D(u, \tilde{o}),$$

where $\tilde{o} \in \mathbb{R}_{\mathcal{F}}$ *is defined by* $\tilde{o} := \mathcal{X}_{\{0\}}$.

Lemma 4.1 ([5]).

(i) *If we denote* $\tilde{o} := \mathcal{X}_{\{0\}}$, *then* $\tilde{o} \in \mathbb{R}_{\mathcal{F}}$ *is the neutral element with respect to* $\oplus$, *i.e.,* $u \oplus \tilde{o} = \tilde{o} \oplus u = u$, $\forall u \in \mathbb{R}_{\mathcal{F}}$.

(ii) *With respect to* $\tilde{o}$, *none of* $u \in \mathbb{R}_{\mathcal{F}}$, $u \neq \tilde{o}$ *has opposite in* $\mathbb{R}_{\mathcal{F}}$.

(iii) *Let* $a, b \in \mathbb{R}$: $a \cdot b \geq 0$, *and any* $u \in \mathbb{R}_{\mathcal{F}}$, *we have* $(a + b) \odot u = a \odot u \oplus b \odot u$. *For general* $a, b \in \mathbb{R}$, *the above property is fale.*

(iv) *For any* $\lambda \in \mathbb{R}$ *and any* $u, v \in \mathbb{R}_{\mathcal{F}}$, *we have* $\lambda \odot (u \oplus v) = \lambda \odot u \oplus \lambda \odot v$.

(v) *For any* $\lambda, \mu \in \mathbb{R}$ *and* $u \in \mathbb{R}_{\mathcal{F}}$, *we have* $\lambda \odot (\mu \odot u) = (\lambda \cdot \mu) \odot u$.

(vi) *If we denote* $\|u\|_{\mathcal{F}} := D(u, \tilde{o})$, $\forall u \in \mathbb{R}_{\mathcal{F}}$, *then* $\|\cdot\|_{\mathcal{F}}$ *has the properties of a usual norm on* $\mathbb{R}_{\mathcal{F}}$, *i.e.,*

$$\begin{aligned} \|u\|_{\mathcal{F}} &= 0 \text{ iff } u = \tilde{o}, \|\lambda \odot u\|_{\mathcal{F}} = |\lambda| \cdot \|u\|_{\mathcal{F}}, \\ \|u \oplus v\|_{\mathcal{F}} &\leq \|u\|_{\mathcal{F}} + \|v\|_{\mathcal{F}}, \ \|u\|_{\mathcal{F}} - \|v\|_{\mathcal{F}} \leq D(u, v). \end{aligned}$$

Notice that $(\mathbb{R}_{\mathcal{F}}, \oplus, \odot)$ is *not* a linear space over $\mathbb{R}$, and consequently $(\mathbb{R}_{\mathcal{F}}, \|\cdot\|_{\mathcal{F}})$ is *not* a normed space.

We need

Definition B (see [10]). Let $x, y \in \mathbb{R}_{\mathcal{F}}$. If there exists a $z \in \mathbb{R}_{\mathcal{F}}$ such that $x = y + z$, then we call z the *H-difference* of x and y, denoted by $z := x - y$.

Definition 3.3 ([10]). Let $T := [x_0, x_0 + \beta] \subset \mathbb{R}$, with $\beta > 0$. A function $f : T \to \mathbb{R}_{\mathcal{F}}$ is *H-differentiable* at $x \in T$ if there exists a $f'(x) \in \mathbb{R}_{\mathcal{F}}$ such that the limits (with respect to metric D)

$$\lim_{h \to 0^+} \frac{f(x + h) - f(x)}{h}, \quad \lim_{h \to 0^+} \frac{f(x) - f(x - h)}{h}$$

exist and are equal to $f'(x)$. We call f' the *derivative* or *H-derivative* of f at x. If f is H-differentiable at any $x \in T$, we call f *differentiable* or *H-differentiable* and it has *H-derivative over* T the function f'.

The last definition was given first by M. Puri and D. Ralescu [9].

We need also a particular case of the *Fuzzy Henstock integral* ($\delta(x) = \frac{\delta}{2}$) introduced in [10], Definition 2.1.

That is,

Definition 13.14 ([6], p. 644). Let $f\colon [a,b] \to \mathbb{R}_{\mathcal{F}}$. We say that f is *Fuzzy-Riemann integrable* to $I \in \mathbb{R}_{\mathcal{F}}$ if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any division $P = \{[u,v];\xi\}$ of $[a,b]$ with the norms $\Delta(P) < \delta$, we have

$$D\left(\sum_{P}{}^{*}(v-u)\odot f(\xi), I\right) < \varepsilon,$$

where $\sum^{*}$ denotes the fuzzy summation. We choose to write

$$I := (FR)\int_a^b f(x)dx.$$

We also call an f as above (FR)-integrable.

We mention

Lemma 1 ([3]). *If $f,g\colon [a,b] \subseteq \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ are fuzzy continuous functions, then the function $F\colon [a,b] \to \mathbb{R}_+$ defined by $F(x) := D(f(x),g(x))$ is continuous on $[a,b]$, and*

$$D\left((FR)\int_a^b f(x)dx, (FR)\int_a^b g(x)dx\right) \le \int_a^b D(f(x),g(x))dx.$$

Lemma 2 ([3]). *Let $f\colon [a,b] \to \mathbb{R}_{\mathcal{F}}$ fuzzy continuous (with respect to metric D), then $D(f(x),\tilde{o}) \le M$, $\forall x \in [a,b]$, $M > 0$, that is f is fuzzy bounded. Equivalently we get $\chi_{-M} \le f(x) \le \chi_M$, $\forall x \in [a,b]$.*

Lemma 3 ([3]). *Let $f\colon [a,b] \subseteq \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ be fuzzy continuous. Then*

$$(FR)\int_a^x f(t)dt \quad \text{is a fuzzy continuous function in } x \in [a,b].$$

Lemma 5 ([4]). *Let $f\colon [a,b] \to \mathbb{R}_{\mathcal{F}}$ have an existing H-fuzzy derivative f' at $c \in [a,b]$. Then f is fuzzy continuous at c.*

We need

Theorem 3.2 ([7]). *Let $f\colon [a,b] \to \mathbb{R}_{\mathcal{F}}$ be fuzzy continuous. Then $(FR)\int_a^b f(x)dx$ exists and belongs to $\mathbb{R}_{\mathcal{F}}$, furthermore it holds*

$$\left[(FR)\int_a^b f(x)dx\right]^r = \left[\int_a^b (f)_-^{(r)}(x)dx, \int_a^b (f)_+^{(r)}(x)dx\right], \quad \forall r \in [0,1]. \tag{1}$$

Clearly $f_{\pm}^{(r)}\colon [a,b] \to \mathbb{R}$ are continuous functions.

We also need

Theorem 5.2 ([8]). *Let $f\colon [a,b] \subseteq \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ be H-fuzzy differentiable. Let $t \in [a,b]$, $0 \le r \le 1$. (Clearly*

$$[f(t)]^r = \left[(f(t))_-^{(r)}, (f(t))_+^{(r)}\right] \subseteq \mathbb{R}.) \tag{2}$$

Then $(f(t))_{\pm}^{(r)}$ are differentiable and

$$[f'(t)]^r = \left[((f(t))_-^{(r)})', ((f(t))_+^{(r)})'\right]. \tag{3}$$

The last can be used to find f'.

Here $C^n([a,b],\mathbb{R}_{\mathcal{F}})$, $n \geq 1$ denotes the space of n-times fuzzy continuously H-differentiable functions from $[a,b] \subseteq \mathbb{R}$ into $\mathbb{R}_{\mathcal{F}}$. By above Theorem 5.2 of [8] for $f \in C^n([a,b],\mathbb{R}_{\mathcal{F}})$ we obtain

$$[f^{(i)}(t)]^r = \left[((f(t))_-^{(r)})^{(i)}, ((f(t))_+^{(r)})^{(i)}\right], \tag{4}$$

for $i = 0, 1, 2, \ldots, n$ and in particular we have

$$(f_{\pm}^{(i)})^{(r)} = (f_{\pm}^{(r)})^{(i)}, \quad \forall r \in [0,1]. \tag{5}$$

Definition 1. Let $a_1, a_2, b_1, b_2 \in \mathbb{R}$ such that $a_1 \leq b_1$ and $a_2 \leq b_2$. Then we define

$$[a_1, b_1] + [a_2, b_2] = [a_1 + a_2, b_1 + b_2]. \tag{6}$$

Let $a, b \in \mathbb{R}$ such that $a \leq b$ and $k \in \mathbb{R}$, then we define,

$$\begin{aligned} &\text{if } k \geq 0, \quad k[a,b] = [ka, kb], \\ &\text{if } k < 0, \quad k[a,b] = [kb, ka]. \end{aligned} \tag{7}$$

Here we use

Lemma 1. *Let $f\colon [a,b] \to \mathbb{R}_{\mathcal{F}}$ be fuzzy continuous and let $g\colon [a,b] \to \mathbb{R}_+$ be continuous. Then $f(x) \odot g(x)$ is fuzzy continuous function $\forall x \in [a,b]$.*

Proof. The same as of Lemma 2 ([1]), using Lemma 2 of [3]. ∎

2 Main Results

We present the following fuzzy Taylor theorem in one dimension.

Theorem 1. *Let $f \in C^n([a,b],\mathbb{R}_{\mathcal{F}})$, $n \geq 1$, $[\alpha,\beta] \subseteq [a,b] \subseteq \mathbb{R}$. Then*

$$\begin{aligned} f(\beta) = f(\alpha) \quad &\oplus \quad f'(\alpha) \odot (\beta - \alpha) \oplus \cdots \oplus f^{(n-1)}(\alpha) \odot \frac{(\beta-\alpha)^{n-1}}{(n-1)!} \\ &\oplus \quad \frac{1}{(n-1)!} \odot (FR) \int_\alpha^\beta (\beta - t)^{n-1} \odot f^{(n)}(t)\, dt. \end{aligned} \tag{8}$$

The integral remainder is a fuzzy continuous function in β.

Proof. Let $r \in [0,1]$. We have here $[f(\beta)]^r = [f_-^{(r)}(\beta), f_+^{(r)}(\beta)]$, and by Theorem 5.2 ([8]) $f_{\pm}^{(r)}$ is n-times continuously differentiable on $[a,b]$. By (5) we get

$$(f_{\pm}^{(i)}(\alpha))^{(r)} = (f_{\pm}^{(r)}(\alpha))^{(i)}, \quad \text{all } \ i = 0, 1, \ldots, n, \tag{9}$$

and

$$[f^{(i)}(\alpha)]^r = \big[(f_-^{(r)}(\alpha))^{(i)}, (f_+^{(r)}(\alpha))^{(i)}\big].$$

Thus by Taylor's theorem we obtain

$$\begin{aligned} f_\pm^{(r)}(\beta) &= f_\pm^{(r)}(\alpha) + (f_\pm^{(r)}(\alpha))'(\beta-\alpha) \\ &+ \cdots + (f_\pm^{(r)}(\alpha))^{(n-1)}\frac{(\beta-\alpha)^{n-1}}{(n-1)!} + \frac{1}{(n-1)!}\int_\alpha^\beta (\beta-t)^{n-1}(f_\pm^{(r)})^{(n)}(t)dt. \end{aligned}$$

Furthermore by (9) we have

$$\begin{aligned} f_\pm^{(r)}(\beta) &= f_\pm^{(r)}(\alpha) + (f'_\pm(\alpha))^{(r)}(\beta-\alpha) \\ &+ \cdots + (f_\pm^{(n-1)}(\alpha)^{(r)}\frac{(\beta-\alpha)^{n-1}}{(n-1)!} + \frac{1}{(n-1)!}\int_\alpha^\beta (\beta-t)^{n-1}(f_\pm^{(n)})^{(r)}(t)dt. \end{aligned}$$

Here it holds $\beta-\alpha \geq 0$, $\beta - t \geq 0$ for $t \in [\alpha,\beta]$, and

$$(f_-^{(i)}(t))^{(r)} \leq (f_+^{(i)}(t))^{(r)}, \quad \forall t \in [a,b]$$

all $i = 0,1,\dots,n$, and any $r \in [0,1]$.

We see that

$$\begin{aligned} \Big[f_-^{(r)}(\beta), f_+^{(r)}(\beta)] &= [f_-^{(r)}(\alpha) + (f'_-(\alpha))^{(r)}(\beta-\alpha) + \cdots + (f_-^{(n-1)}(\alpha))^{(r)}\frac{(\beta-\alpha)^{n-1}}{(n-1)!} \\ &+ \frac{1}{(n-1)!}\int_\alpha^\beta (\beta-t)^{n-1}(f_-^{(n)})^{(r)}(t)dt,, f_+^{(r)}(\alpha) \\ &+ (f'_+(\alpha))^{(r)}(\beta-\alpha) + \cdots + (f_+^{(n-1)}(\alpha))^{(r)}\frac{(\beta-\alpha)^{n-1}}{(n-1)!} \\ &+ \frac{1}{(n-1)!}\int_\alpha^\beta (\beta-t)^{n-1}(f_+^{(n)})^{(r)}(t)\,dt\Big]. \end{aligned}$$

To split the above closed interval into a sum of smaller closed intervals is where we use $\beta-\alpha \geq 0$. So we get

$$\begin{aligned} [f(\beta)^r] = \ & [f_-^{(r)}(\beta), f_+^{(r)}(\beta)] = [f_-^{(r)}(\alpha), f_+^{(r)}(\alpha)] + [(f'_-(\alpha))^{(r)}, (f'_+(\alpha))^{(r)}](\beta-\alpha) \\ & + \cdots + [(f_-^{(n-1)}(\alpha))^{(r)}, (f_+^{(n-1)}(\alpha))^{(r)}]\tfrac{(\beta-\alpha)^{n-1}}{(n-1)!} \\ & + \tfrac{1}{(n-1)!}\left[\textstyle\int_\alpha^\beta (\beta-t)^{n-1}(f_-^{(n)})^{(r)}(t)dt, \int_\alpha^\beta (\beta-t)^{n-1}(f_+^{(n)})^{(r)}(t)dt\right] \\ = \ & [f(\alpha)]^r + [f'(\alpha)]^r(\beta-\alpha) + \cdots + [f^{(n-1)}(\alpha)]^r\tfrac{(\beta-\alpha)^{n-1}}{(n-1)!} \\ & + \tfrac{1}{(n-1)!}\left[\textstyle\int_\alpha^\beta ((\beta-t)^{n-1}\odot f^{(n)}(t))_-^{(r)}dt, \int_\alpha^\beta ((\beta-t)^{n-1}\odot f^{(n)}(t))_+^{(r)}dt\right]. \end{aligned}$$

By Theorem 3.2 ([7]) we next get

$$\begin{aligned}[f(\beta)]^r &= [f(\alpha)]^r + [f'(\alpha)]^r(\beta-\alpha) + \cdots + [f^{(n-1)}(\alpha)]^r \frac{(\beta-\alpha)^{n-1}}{(n-1)!} \\ &\quad + \frac{1}{(n-1)!}\left[(FR)\int_\alpha^\beta (\beta-t)^{n-1} \odot f^{(n)}(t)dt\right]^r.\end{aligned}$$

Finally we obtain

$$\begin{aligned}[f(\beta)]^r &= \left[f(\alpha) \oplus f'(\alpha) \odot (\beta-\alpha) \oplus \cdots \oplus f^{(n-1)}(\alpha) \odot \frac{(\beta-\alpha)^{n-1}}{(n-1)!}\right. \\ &\quad \left.\oplus \frac{1}{(n-1)!} \odot (FR)\int_\alpha^\beta (\beta-t)^{n-1} \odot f^{(n)}(t)dt\right]^r, \quad \text{all } r \in [0,1].\end{aligned}$$

By Theorem 3.2 of [7] and Lemma 1 we get that the remainder of (8) is in $\mathbb{R}_{\mathcal{F}}$, and by Lemma 3 ([3]) is a fuzzy continuous function in β. The theorem has been proved. ∎

Next we present a multivariate fuzzy Taylor theorem.

We need the following multivariate fuzzy chain rule. Here the H-fuzzy partial derivatives are defined according to the Definition 3.3 of [10], see Section 1, and the analogous way to the real case.

Theorem 3 ([2]). *Let $\phi_i \colon [a,b] \subseteq \mathbb{R} \to \phi_i([a,b]) := I_i \subseteq \mathbb{R}$, $i = 1,\ldots,n$, $n \in \mathbb{N}$, are strictly increasing and differentiable functions. Denote $x_i := x_i(t) := \phi_i(t)$, $t \in [a,b]$, $i = 1,\ldots,n$. Consider U an open subset of $\mathbb{R}^n$ such that $\times_{i=1}^n I_i \subseteq U$. Consider $f \colon U \to \mathbb{R}_{\mathcal{F}}$ a fuzzy continuous function. Assume that $f_{x_i} \colon U \to \mathbb{R}_{\mathcal{F}}$, $i = 1,\ldots,n$, the H-fuzzy partial derivatives of f, exist and are fuzzy continuous. Call $z := z(t) := f(x_1,\ldots,x_n)$. Then $\frac{dz}{dt}$ exists and*

$$\frac{dz}{dt} = \sum_{i=1}^{n}{}^{*} \frac{dz}{dx_i} \odot \frac{dx_i}{dt}, \quad \forall t \in [a,b] \tag{10}$$

where $\frac{dz}{dt}$, $\frac{dz}{dx_i}$, $i = 1,\ldots,n$ are the H-fuzzy derivatives of f with respect to t, x_i, respectively.

The interchange of the order of H-fuzzy differentiation is needed too.

Theorem 4 ([2]). *Let U be an open subset of $\mathbb{R}^n$, $n \in \mathbb{N}$, and $f \colon U \to \mathbb{R}_{\mathcal{F}}$ be a fuzzy continuous function. Assume that all H-fuzzy partial derivatives of f up to order $m \in \mathbb{N}$ exist and are fuzzy continuous. Let $x := (x_1,\ldots,x_n) \in U$. Then the H-fuzzy mixed partial derivative of order k, $D_{x_{\ell_1},\ldots,x_{\ell_k}} f(x)$ is unchanged when the indices $\ell_1,\ldots,\ell_k$ are permuted. Each ℓ_i is a positive integer $\leq n$. Here some or all of ℓ_i's can be equal. Also $k = 2,\ldots,m$ and there are n^k partials of order k.*

We give

Theorem 2. *Let U be an open convex subset of $\mathbb{R}^n$, $n \in \mathbb{N}$ and $f\colon U \to \mathbb{R}_{\mathcal{F}}$ be a fuzzy continuous function. Assume that all H-fuzzy partial derivatives of f up to order $m \in \mathbb{N}$ exist and are fuzzy continuous. Let $z := (z_1, \ldots, z_n)$, $x_0 := (x_{01}, \ldots, x_{0n}) \in U$ such that $x_i \geq x_{0i}$, $i = 1, \ldots, n$. Let $0 \leq t \leq 1$, we define $x_i := x_{0i} + t(z_i - z_{0i})$, $i = 1, 2, \ldots, n$ and $g_z(t) := f(x_0 + t(z - x_0))$. (Clearly $x_0 + t(z - x_0) \in U$.) Then for $N = 1, \ldots, m$ we obtain*

$$g_z^{(N)}(t) = \left[\left(\sum_{i=1}^{n}{}^{*} (z_i - x_{0i}) \odot \frac{\partial}{\partial x_i}\right)^N f\right](x_1, x_2, \ldots, x_n). \tag{11}$$

Furthermore it holds the following fuzzy multivariate Taylor formula

$$f(z) = f(x_0) \oplus \sum_{N=1}^{m-1}{}^{*} \frac{g_z^{(N)}(0)}{N!} \oplus \mathcal{R}_m(0, 1), \tag{12}$$

where

$$\mathcal{R}_m(0, 1) := \frac{1}{(m-1)!} \odot (FR) \int_0^1 (1 - s)^{m-1} \odot g_z^{(m)}(s) ds. \tag{13}$$

Comment. (Explaining formula (11)). When $N = n = 2$ we have ($z_i \geq x_{0i}$, $i = 1, 2$)

$$g_z(t) = f(x_{01} + t(z_1 - x_{01}), x_{02} + t(z_2 - x_{02})), \quad 0 \leq t \leq 1.$$

We apply Theorems 3 and 4 of [2] repeatedly, etc. Thus we find

$$g_z'(t) = (z_1 - x_{01}) \odot \frac{\partial f}{\partial x_1}(x_1, x_2) \oplus (z_2 - x_{02}) \odot \frac{\partial f}{\partial x_2}(x_1, x_2).$$

Furthermore it holds

$$\begin{aligned} g_z''(t) &= (z_1 - x_{01})^2 \odot \frac{\partial^2 f}{\partial x_1^2}(x_1, x_2) \oplus 2(z_1 - x_{01}) \cdot (z_2 - x_{02}) \\ &\quad \odot \frac{\partial^2 f(x_1, x_2)}{\partial x_1 \partial x_2} \oplus (z_2 - x_{02})^2 \odot \frac{\partial^2 f}{\partial x_2^2}(x_1, x_2). \end{aligned} \tag{14}$$

When $n = 2$ and $N = 3$ we obtain

$$\begin{aligned} g_z'''(t) &= (z_1 - x_{01})^3 \odot \frac{\partial^3 f}{\partial x_1^3}(x_1, x_2) \oplus 3(z_1 - x_{01})^2 (z_2 - x_{02}) \\ &\quad \odot \frac{\partial^3 f(x_1, x_2)}{\partial x_1^2 \partial x_2} \oplus 3(z_1 - x_{01})(z_2 - x_{02})^2 \cdot \frac{\partial^3 f(x_1, x_2)}{\partial x_1 \partial x_2^2} \\ &\quad \oplus (z_2 - x_{02})^3 \odot \frac{\partial^3 f}{\partial x_2^3}(x_1, x_2). \end{aligned} \tag{15}$$

When $n = 3$ and $N = 2$ we get ($z_i \geq x_{0i}$, $i = 1, 2, 3$)

$$\begin{aligned} g_z''(t) &= (z_1 - x_{01})^2 \odot \frac{\partial^2 f}{\partial x_1^2}(x_1, x_2, x_3) \oplus (z_2 - x_{02})^2 \odot \frac{\partial^2 f}{\partial x_2^2}(x_1, x_2, x_3) \\ &\oplus (z_3 - x_{03})^2 \odot \frac{\partial^2 f}{\partial x_3^2}(x_1, x_2, x_3) \oplus 2(z_1 - x_{01})(z_2 - x_{02}) \\ &\odot \frac{\partial^2 f(x_1, x_2, x_3)}{\partial x_1 \partial x_2} \oplus 2(z_2 - x_{02})(z_3 - x_{03}) \\ &\odot \frac{\partial^2 f(x_1, x_2, x_3)}{\partial x_2 \partial x_3} \oplus 2(z_3 - x_{03})(z_1 - x_{01}) \odot \frac{\partial^2 f}{\partial x_3 \partial x_1}(x_1, x_2, x_3), \quad (16) \end{aligned}$$

etc.

Proof of Theorem 2. Let $z := (z_1, \ldots, z_n)$, $x_0 := (x_{01}, \ldots, x_{0n}) \in U$, $n \in \mathbb{N}$, such that $z_i > x_{0i}$, $i = 1, 2, \ldots, n$. We define

$$x_i := \phi_i(t) := x_{0i} + t(z_i - x_{0i}), \quad 0 \leq t \leq 1; \quad i = 1, 2, \ldots, n.$$

Thus $\frac{dx_i}{dt} = z_i - x_{0i} > 0$. Consider

$$\begin{aligned} Z := g_z(t) := f(x_0 + t(z - x_0)) &= f(x_{01} + t(z_1 - x_{01}), \ldots, x_{0n} + t(z_n - x_{0n})) \\ &= f(\phi_1(t), \ldots, \phi_n(t)). \end{aligned}$$

Since by assumptions $f \colon U \to \mathbb{R}_{\mathcal{F}}$ is fuzzy continuous, also f_{x_i} exist and are fuzzy continuous, by Theorem 3 (10) of [2] we get

$$\begin{aligned} \frac{dZ(x_1, \ldots, x_n)}{dt} &= \sum_{i=1}^{n}{}^{*} \frac{\partial Z(x_1, \ldots, x_n)}{\partial x_i} \odot \frac{dx_i}{dt} \\ &= \sum_{i=1}^{n}{}^{*} \frac{\partial f(x_1, \ldots, x_n)}{\partial x_i} \odot (z_i - x_{0i}). \end{aligned}$$

Thus

$$g_z'(t) = \sum_{i=1}^{n}{}^{*} \frac{\partial f(x_1, \ldots, x_n)}{\partial x_i} \odot (z_i - x_{0i}).$$

Next we observe that

$$\begin{aligned} \frac{d^2 Z}{dt^2} &= g_z''(t) = \frac{d}{dt}\left(\sum_{i=1}^{n}{}^{*} \frac{\partial f(x_1, \ldots, x_n)}{\partial x_i} \odot (z_i - x_{0i})\right) \\ &= \sum_{i=1}^{n}{}^{*} (z_i - x_{0i}) \odot \frac{d}{dt}\left(\frac{\partial f(x_1, \ldots, x_n)}{\partial x_i}\right) \\ &= \sum_{i=1}^{n}{}^{*} (z_i - x_{0i}) \odot \left[\sum_{j=1}^{n}{}^{*} \frac{\partial^2 f(x_1, \ldots, x_n)}{\partial x_j \partial x_i} \odot (z_j - x_{0j})\right] \\ &= \sum_{i=1}^{n}{}^{*} \sum_{j=1}^{n}{}^{*} \frac{\partial^2 f(x_1, \ldots, x_n)}{\partial x_j \partial x_i} \odot (z_i - x_{0i}) \cdot (z_j - x_{0j}). \end{aligned}$$

That is

$$g_z''(t) = \sum_{i=1}^{n}{}^{*} \sum_{j=1}^{n}{}^{*} \frac{\partial^2 f(x_1, \ldots, x_n)}{\partial x_j \partial x_i} \odot (z_i - x_{0i}) \cdot (z_j - x_{0j}).$$

The last is true by Theorem 3 (10) of [2] under the additional assumptions that f_{x_i}; $\frac{\partial^2 f}{\partial x_j \partial x_i}$, $i, j = 1, 2, \ldots, n$ exist and are fuzzy continuous.

Working the same way we find

$$\begin{aligned} \frac{d^3 Z}{dt^3} &= g_z'''(t) = \frac{d}{dt}\left(\sum_{i=1}^{n}{}^{*} \sum_{j=1}^{n}{}^{*} \frac{\partial^2 f(x_1, \ldots, x_n)}{\partial x_j \partial x_i} \odot (z_i - x_{0i}) \cdot (z_j - x_{0j})\right) \\ &= \sum_{i=1}^{n}{}^{*} \sum_{j=1}^{n}{}^{*} (z_i - x_{0i}) \cdot (z_j - x_{0j}) \frac{d}{dt}\left(\frac{\partial^2 f(x_1, \ldots, x_n)}{\partial x_j \partial x_i}\right) \\ &= \sum_{i=1}^{n}{}^{*} \sum_{j=1}^{n}{}^{*} (z_i - x_{0i}) \cdot (z_j - x_{0j}) \left[\sum_{k=1}^{n}{}^{*} \frac{\partial^3 f(x_1, \ldots, x_n)}{\partial x_k \partial x_j \partial x_i} \odot (z_k - x_{0k})\right] \\ &= \sum_{i=1}^{n}{}^{*} \sum_{j=1}^{n}{}^{*} \sum_{k=1}^{n}{}^{*} \frac{\partial^3 f(x_1, \ldots, x_n)}{\partial x_k \partial x_j \partial x_i} \odot (z_i - x_{0i}) \cdot (z_j - x_{0j}) \cdot (z_k - x_{0k}). \end{aligned}$$

Therefore,

$$g_z'''(t) = \sum_{i=1}^{n}{}^{*} \sum_{j=1}^{n}{}^{*} \sum_{k=1}^{n}{}^{*} \frac{\partial^3 f(x_1, \ldots, x_n)}{\partial x_k \partial x_j \partial x_i} \odot (z_i - x_{0i}) \cdot (z_j - x_{0j}) \cdot (z_k - x_{0k}).$$

That last is true by Theorem 3 (10) of [2] under the additional assumptions that

$$\frac{\partial^3 f(x_1, \ldots, x_n)}{\partial x_k \partial x_j \partial x_i}, \quad i, j, k = 1, \ldots, n$$

do exist and are fuzzy continuous. Etc. In general one obtains that for $N = 1, \ldots, m \in \mathbb{N}$,

$$g_z^{(N)}(t) = \sum_{i_1=1}^{n}{}^{*} \sum_{i_2=1}^{n}{}^{*} \cdots \sum_{i_N=1}^{n}{}^{*} \frac{\partial^N f(x_1, \ldots, x_n)}{\partial x_{i_N} \partial x_{i_{N-1}} \cdots \partial x_{i_1}} \odot \prod_{r=1}^{N} (z_{i_r} - x_{0i_r}),$$

which by Theorem 4 of [2] is the same as (11) for the case $z_i > x_{0i}$, see also (14), (15), and (16). The last is true by Theorem 3 (10) of [2] under the assumptions that all H-partial derivatives of f up to order m exist and they are all fuzzy continuous including f itself.

Next let $t_{\tilde{m}} \to \tilde{t}$, as $\tilde{m} \to +\infty$, $t_{\tilde{m}}$, $\tilde{t} \in [0, 1]$. Consider

$$x_{i\tilde{m}} := x_{0i} + t_{\tilde{m}}(z_i - x_{0i})$$

and

$$\tilde{x}_i := x_{0i} + \tilde{t}(z_i - x_{0i}), \quad i = 1, 2, \ldots, n.$$

That is

$$x_{\tilde{m}} = (x_{1\tilde{m}}, x_{2\tilde{m}}, \dots, x_{n\tilde{m}}) \text{ and } \tilde{x} = (\tilde{x}_1, \dots, \tilde{x}_n) \text{ in } U.$$

Then $x_{\tilde{m}} \to \tilde{x}$, as $\tilde{m} \to +\infty$. Clearly using the properties of D-metric and under the theorem's assumptions, we obtain that

$$g_z^{(N)}(t) \text{ is fuzzy continuous for } N = 0, 1, \dots, m.$$

Then by Theorem 1, from the univariate fuzzy Taylor formula (8), we find

$$g_z(1) = g_z(0) \oplus g_z'(0) \oplus \frac{g_z''(0)}{2!} \oplus \cdots \oplus \frac{g_z^{(m-1)}(0)}{(m-1)!} \oplus \mathcal{R}_m(0,1),$$

where $\mathcal{R}_m(0,1)$ comes from (13).

By Theorem 3.2 of [7] and Lemma 1 we get that $\mathcal{R}_m(0,1) \in \mathbb{R}_{\mathcal{F}}$. That is we get the multivariate fuzzy Taylor formula

$$f(z) = f(x_0) \oplus g_z'(0) \oplus \frac{g_z''(0)}{2!} \oplus \cdots \oplus \frac{g_z^{(m-1)}(0)}{(m-1)!} \oplus \mathcal{R}_m(0,1),$$

when $z_i > x_{0i}$, $i = 1, 2, \dots, n$.

Finally we would like to take care of the case that some $x_{0i} = z_i$. Without loss of generality we may assume that $x_{01} = z_1$, and $z_i > x_{0i}$, $i = 2, \dots, n$. In this case we define

$$\tilde{Z} := \tilde{g}_z(t) := f(x_{01}, x_{02} + t(z_2 - x_{02}), \dots, x_{0n} + t(z_n - x_{0n})).$$

Therefore one has

$$\tilde{g}_z'(t) = \sum_{i=2}^{n}{}^{*} \frac{\partial f(x_{01}, x_2, \dots, x_n)}{\partial x_i} \odot (z_i - x_{0i}),$$

and in general we find

$$\tilde{g}_z^{(N)}(t) = \sum_{i_2=2,\dots,i_N=2}^{n}{}^{*} \frac{\partial^N f(x_{01}, x_2, \dots, x_n)}{\partial x_{i_N} \partial x_{N-1} \cdots \partial x_{i_2}} \odot \prod_{r=2}^{N} (z_{i_r} - x_{0i_r}),$$

for $N = 1, \dots, m \in \mathbb{N}$. Notice that all $\tilde{g}_z^{(N)}$, $N = 0, 1, \dots, m$ are fuzzy continuous and

$$\tilde{g}_z(0) = f(x_{01}, x_{02}, \dots, x_{0n}), \quad \tilde{g}_z(1) = f(x_{01}, z_2, z_3, \dots, z_n).$$

Then one can write down a fuzzy Taylor formula, as above, for $\tilde{g}_z$. But $\tilde{g}_z^{(N)}(t)$ coincides with $g_z^{(N)}(t)$ formula at $z_1 = x_{01} = x_1$. That is both Taylor formulae in that case coincide.

At last we remark that if $z = x_0$, then we define $Z^* := g_z^*(t) := f(x_0) =: c \in \mathbb{R}_{\mathcal{F}}$ a constant. Since $c = c + \tilde{o}$, that is $c - c = \tilde{o}$, we obtain the H-fuzzy derivative $(c)' = \tilde{o}$. Consequently we have that

$$g_z^{*(N)}(t) = \tilde{o}, \quad N = 1, \dots, m.$$

The last coincide with the $g_z^{(N)}$ formula, established earlier, if we apply there $z = x_0$. And, of course, the fuzzy Taylor formula now can be applied trivially for g_z^*. Furthermore in that case it coincides with the Taylor formula proved earlier for g_z. We have established a multivariate fuzzy Taylor formula for the case of $z_i \geq x_{0i}$, $i = 1, 2, \ldots, n$. That is (11)–(13) are true. ∎

Note. Theorem 2 is still valid when U is a compact convex subset of $\mathbb{R}^n$ such that $U \subseteq W$, where W is an open subset of $\mathbb{R}^n$. Now $f \colon W \to \mathbb{R}_{\mathcal{F}}$ and it has all the properties of f as in Theorem 2. Clearly here we take $x_0, z \in U$.

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