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Converse Fractional Opial Inequalities for Several Functions

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ABSTRACT

A variety of very general $L_p(0 < p < 1)$ form converse Opial type inequalities ([8]) is presented involving generalized fractional derivatives ([3],[6]) of several functions in different orders and powers. From the established results derive other particular results of special interest.

RESUMEN

Una variedad general de desigualdades inversas de tipo Opial en $L_p(0<p<1)$ son presentadas, las cuales envuelven deridadas fraccionarias generalidades ([3],[6]) de varias funciones con diferentes ordenes y potencias. Deducimos algunos casos particulares de especial interes.

Key Words and Phrases: *Opial type inequality, Fractional derivative.*

Math Subj. Class.: *26A33, 26D10, 26D15.*

1 Introduction

Opial inequalities appeared for the first time in [8] and then many authors dealt with them in different directions and for various cases. For a complete recent account on the activity of this field see [1], and still it remains a very active area of research. One of their main attractions to these inequalities is their applications, especially to proving uniqueness and upper bounds of solution of initial value problems in differential equations. The author was the first to present Opial inequalities involving fractional derivatives of functions in [2], [3] with applications to fractional differential equations.

Fractional derivatives come up naturally in a number of fields, especially in Physics, see the recent books [7], [9]. Here the author continues his study of fractional Opial inequalities now involving several different functions and produces a wide variety of converse results. To give an idea to the reader of the kind of inequalities we are dealing with, briefly we mention a specific one (see Corollary 15).

$$\int_{x_0}^{x}\left(\sum_{j=1}^{M}\left|\left(D_{x_0}^{\gamma_1}f_j\right)(w)\right|^{\lambda_\alpha}\left|\left(D_{x_0}^{v}f_j\right)(w)\right|^{\lambda_v}\right)dw \geq$$

$$C(x)\left[\int_{x_0}^{x}\left[\sum_{j=1}^{M}\left|\left(D_{x_0}^{v}f_j\right)(w)\right|^{p}\right]dw\right]^{\left(\frac{\lambda_\alpha+\lambda_v}{p}\right)}, \quad (*)$$

all $x_0 \leq x \leq b$.

In $(*)$, $C(x)$ is a constant that depends on x_0, x, and the involved parameters, $\gamma_1 \geq 0$, $1 \leq v-\gamma_1 < \frac{1}{p}$, $0<p<1$; $D_{x_0}^{v}f_j$ is of fixed sign on $[x_0,b]$, $j=1,\ldots,M \in \mathbb{N}$. Also $\lambda_\alpha \geq 0$, $\lambda_v > p$. Here $f_j^{(i)}(x_0)=0$, $i=0,1,\ldots,n-1, n:=[v]$ (integral part); $j=1,\ldots,M$.

And $D_{x_0}^{\gamma_1}f_j$, $D_{x_0}^{v}f_j$ are the generalized (of Canavati) type [6], [2] fractional derivatives of f_j of orders γ_1, v respectively.

2 Preliminaries

In the sequel we follow [6]. Let $g \in C([0,1])$. Let v be a positive number, $n:=[v]$ and $\alpha := v-n$ $(0<\alpha<1)$. Define

$$(J_v g)(x) := \frac{1}{\Gamma(v)}\int_0^x (x-t)^{v-1} g(t)\,dt, \quad 0 \leq x \leq 1, \tag{1}$$

the *Riemann-Liouville integral,* where Γ is the gamma function. We define the subspace $C^v([0,1])$ of $C^n([0,1])$ as follows:

$$C^v([0,1]) := \left\{g \in C^n([0,1]) : J_{1-\alpha}D^n g \in C^1([0,1])\right\},$$

where $D := \frac{d}{dx}$. So for $g \in C^v([0,1])$, we define the $v-$*fractional derivative* of g as

$$D^v g := DJ_{1-\alpha}D^n g. \tag{2}$$

When $v \geq 1$ we have the Taylor's formula

$$g(t) = g(0) + g'(0)t + g''(0)\frac{t^2}{2!} + \ldots + g^{(n-1)}(0)\frac{t^{n-1}}{(n-1)!}$$

$$+ (J_v D^v g)(t), \quad \text{for all } t \in [0,1]. \tag{3}$$

When $0 < v < 1$ we find

$$g(t) = (J_v D^v g)(t), \quad \text{for all } t \in [0,1]. \tag{4}$$

Next we carry above notions over to arbitrary $[a,b] \subseteq \mathbb{R}$ (see[3]). Let x, $x_0 \in [a,b]$ such that $x > x_0$, where $x_0 < b$ is fixed. Let $f \in C([a,b])$ and define

$$(J_v^{x_0} f)(x) := \frac{1}{\Gamma(v)} \int_{x_0}^{x} (x-t)^{v-1} f(t)\, dt, \quad x_0 \leq x \leq b, \tag{5}$$

the *generalized Riemann-Liouville integral.* We define the subspace $C_{x_0}^v([a,b])$ of $C^n([a,b])$:

$$C_{x_0}^v([a,b]) := \left\{f \in C^n([a,b]) : J_{1-\alpha}^{x_0} D^n f \in C^1([x_0,b])\right\},$$

clearly $C_{x_0}^0([a,b]) = C([a,b])$, also $C_{x_0}^n([a,b]) = C^n([a,b])$, $n \in \mathbb{N}$.

For $f \in C_{x_0}^v([a,b])$, we define the *generalized* $v-$*fractional derivative of* f *over* $[x_0,b]$ *as*

$$D_{x_0}^v f := DJ_{1-\alpha}^{x_0} f^{(n)} \quad \left(f^{(n)} := D^n f\right). \tag{6}$$

Notice that

$$\left(J_{1-\alpha}^{x_0} f^{(n)}\right)(x) = \frac{1}{\Gamma(1-\alpha)} \int_{x_0}^{x} (x-t)^{-\alpha} f^{(n)}(t)\, dt$$

exists for $f \in C_{x_0}^v([a,b])$.

We recall the following generalization of Taylor's formula (see [6], [3]).

Theorem 1. *Let $f \in C^v_{x_0}([a,b])$, $x_0 \in [a,b]$ fixed.*

(i) If $v \geq 1$ then

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + f''(x_0)\frac{(x-x_0)^2}{2} + \ldots +$$

$$f^{(n-1)}(x_0)\frac{(x-x_0)^{n-1}}{(n-1)!} + \left(J^{x_0}_v D^v_{x_0} f\right)(x), \quad \text{for all } x \in [a,b] : x \geq x_0. \tag{7}$$

(ii) If $0 < v < 1$ then

$$f(x) = \left(J^{x_0}_v D^v_{x_0} f\right)(x), \quad \text{for all } x \in [a,b] : x \geq x_0. \tag{8}$$

We make

Remark 2. 1) $\left(D^n_{x_0} f\right) = f^{(n)}$, $n \in \mathbb{N}$.

2) Let $f \in C^v_{x_0}([a,b])$, $v \geq 1$ and $f^{(i)}(x_0) = 0$, $i = 0, 1, \ldots, n-1$, $n := [v]$. Then by (7)

$$f(x) = \left(J^{x_0}_v D^v_{x_0} f\right)(x).$$

I.e.

$$f(x) = \frac{1}{\Gamma(v)} \int_{x_0}^x (x-t)^{v-1} \left(D^v_{x_0} f\right)(t)\, dt, \tag{9}$$

for all $x \in [a,b]$ with $x \geq x_0$. Notice that (9) is true, also when $0 < v < 1$.

We need from [3]

Lemma 3. Let $f \in C([a,b])$, $\mu, v > 0$. Then

$$J^{x_0}_\mu \left(J^{x_0}_v f\right) = J^{x_0}_{\mu+v}(f). \tag{10}$$

We also make

Remark 4. Let $v \geq \gamma + 1$, $\gamma \geq 0$, so that $\gamma < v$. Call $n := [v]$, $\alpha := v - n$; $m := [\gamma]$, $\rho := \gamma - m$. Note that $v - m \geq 1$ and $n - m \geq 1$. Let $f \in C^v_{x_0}([a,b])$ be such that $f^{(i)}(x_0) = 0$, $i = 0, 1, \ldots, n-1$. Hence by (7)

$$f(x) = \left(J^{x_0}_v D^v_{x_0} f\right)(x), \quad \text{for all } x \in [a,b] : x \geq x_0.$$

Therefore by Leibnitz's formula and $\Gamma(p+1) = p\Gamma(p)$, $p > 0$, we get that

$$f^{(m)}(x) = \left(J^{x_0}_{v-m} D^v_{x_0} f\right)(x), \quad \text{for all } x \geq x_0. \tag{11}$$

It follows that $f \in C^{\gamma}_{x_0}([a,b])$ and thus

$$\left(D^{\gamma}_{x_0} f\right)(x) := \left(D J^{x_0}_{1-\rho} f^{(m)}\right)(x) \quad \text{exists for all } x \geq x_0. \tag{12}$$

Really by the use of (11) we have on $[x_0, b]$

$$J^{x_0}_{1-\rho}\left(f^{(m)}\right) = J^{x_0}_{1-\rho}\left(J^{x_0}_{v-m} D^{v}_{x_0} f\right) = \left(J^{x_0}_{1-\rho} \circ J^{x_0}_{v-m}\right)\left(D^{v}_{x_0} f\right)$$

$$= J^{x_0}_{v-m+1-\rho}\left(D^{v}_{x_0} f\right) = J^{x_0}_{v-\gamma+1}\left(D^{v}_{x_0} f\right),$$

by (10). That is,

$$\left(J^{x_0}_{1-\rho} f^{(m)}\right)(x) = \frac{1}{\Gamma(v-\gamma+1)} \int_{x_0}^{x} (x-t)^{v-\gamma}\left(D^{v}_{x_0} f\right)(t)\, dt.$$

Therefore

$$\left(D^{\gamma}_{x_0} f\right)(x) = D\left(\left(J^{x_0}_{1-\rho} f^{(m)}\right)(x)\right) = \frac{1}{\Gamma(v-\gamma)} \cdot \int_{x_0}^{x} (x-t)^{(v-\gamma)-1}\left(D^{v}_{x_0} f\right)(t)\, dt; \tag{13}$$

hence

$$\left(D^{\gamma}_{x_0} f\right)(x) = \left(J^{x_0}_{v-\gamma}\left(D^{v}_{x_0} f\right)\right)(x) \quad \text{and is continuous in } x \text{ on } [x_0, b].$$

In particular when $v \geq 2$ we have

$$\left(D^{v-1}_{x_0} f\right)(x) = \int_{x_0}^{x} \left(D^{v}_{x_0} f\right)(t)\, dt, \quad x \geq x_0. \tag{14}$$

That is

$$\left(D^{v-1}_{x_0} f\right)' = D^{v}_{x_0} f, \quad \left(D^{v-1}_{x_0} f\right)(x_0) = 0.$$

3 Main Results

3.1 Results involving two functions

We present our first main result

Theorem 5. *Let $\gamma_j \geq 0$, $1 \leq v - \gamma_j < 1/p$, $0 < p < 1$, $j = 1,2$, and $f_1, f_2 \in C^{v}_{x_0}([a,b])$ with $f_1^{(i)}(x_0) = f_2^{(i)}(x_0) = 0$, $i = 0, 1, \ldots, n-1$, $n := [v]$. Here $x, x_0 \in [a,b]$: $x \geq x_0$. We assume here that $D^{v}_{x_0} f_j$ is of fixed sign on $[x_0, b]$, $j = 1,2$. Consider also $p(t) > 0$ and $q(t) > 0$ continuous functions on $[x_0, b]$. Let $\lambda_v > 0$ and $\lambda_\alpha, \lambda_\beta \geq 0$ such that $\lambda_v > p$.*

Set

$$P_k(w) := \int_{x_0}^{w} (w-t)^{\frac{(v-\gamma_k-1)p}{p-1}} (p(t))^{-\frac{1}{p-1}} dt, \quad k=1,2, \ x_0 \le x \le b; \tag{15}$$

$$A(w) := \frac{q(w) \cdot (P_1(w))^{\lambda_\alpha\left(\frac{p-1}{p}\right)} \cdot (P_2(w))^{\lambda_\beta\left(\frac{p-1}{p}\right)} (p(w))^{-\frac{\lambda_v}{p}}}{(\Gamma(v-\gamma_1))^{\lambda_\alpha} (\Gamma(v-\gamma_2))^{\lambda_\beta}}, \tag{16}$$

$$A_0(x) := \left(\int_{x_0}^{x} (A(w))^{\frac{p}{p-\lambda_v}} dw\right)^{\frac{p-\lambda_v}{p}}, \tag{17}$$

and

$$\delta_1 := 2^{1-\left(\frac{\lambda_\alpha+\lambda_v}{p}\right)} \tag{18}$$

If $\lambda_\beta = 0$, *we obtain that,*

$$\int_{x_0}^{x} q(w) \left[\left|\left(D_{x_0}^{\gamma_1} f_1\right)(w)\right|^{\lambda_\alpha} \cdot \left|\left(D_{x_0}^{v} f_1\right)(w)\right|^{\lambda_v} + \left|\left(D_{x_0}^{\gamma_1} f_2\right)(w)\right|^{\lambda_\alpha} \cdot \left|\left(D_{x_0}^{v} f_2\right)(w)\right|^{\lambda_v}\right] dw$$

$$\ge \left(A_0(x)|_{\lambda_\beta=0}\right) \cdot \left(\frac{\lambda_v}{\lambda_\alpha+\lambda_v}\right)^{\frac{\lambda_v}{p}} \cdot \delta_1 \cdot \left[\int_{x_0}^{x} p(w) \left[\left|\left(D_{x_0}^{v} f_1\right)(w)\right|^p + \left|\left(D_{x_0}^{v} f_2\right)(w)\right|^p\right] dw\right]^{\left(\frac{\lambda_\alpha+\lambda_v}{p}\right)}. \tag{19}$$

Proof. From (13) and assumption we have

$$\left|\left(D_{x_0}^{\gamma_k} f_j\right)(w)\right| = \frac{1}{\Gamma(v-\gamma_k)} \int_{x_0}^{w} (w-t)^{v-\gamma_k-1} \left|\left(D_{x_0}^{v} f_j\right)(t)\right| dt,$$

for $k=1,2$, $j=1,2$ and for all $x_0 \le w \le b$.

Next applying Hölder's inequality with indices p, $\frac{p}{p-1}$ we get

$$\left|\left(D_{x_0}^{\gamma_k} f_j\right)(w)\right| = \frac{1}{\Gamma(v-\gamma_k)} \int_{x_0}^{w} (w-t)^{v-\gamma_k-1} (p(t))^{-\frac{1}{p}} (p(t))^{\frac{1}{p}} \left|\left(D_{x_0}^{v} f_j\right)(t)\right| dt$$

$$\ge \frac{1}{\Gamma(v-\gamma_k)} \left(\int_{x_0}^{w} \left((w-t)^{v-\gamma_k-1} (p(t))^{-\frac{1}{p}}\right)^{\frac{p}{p-1}} dt\right)^{\frac{p-1}{p}} \left(\int_{x_0}^{w} p(t) \left|\left(D_{x_0}^{v} f_j\right)(t)\right|^p dt\right)^{\frac{1}{p}}$$

$$= \frac{1}{\Gamma(v-\gamma_k)} (P_k(w))^{\frac{p-1}{p}} \left(\int_{x_0}^{w} p(t) \left| \left(D_{x_0}^{v} f_j \right)(t) \right|^p dt \right)^{\frac{1}{p}}.$$

I.e., it holds

$$\left| \left(D_{x_0}^{\gamma_k} f_j \right)(w) \right| \geq \frac{1}{\Gamma(v-\gamma_k)} (P_k(w))^{\frac{p-1}{p}} \left(\int_{x_0}^{w} p(t) \left| \left(D_{x_0}^{v} f_j \right)(t) \right|^p dt \right)^{\frac{1}{p}}. \quad (20)$$

Put

$$z_j(w) := \int_{x_0}^{w} p(t) \left| \left(D_{x_0}^{v} f_j \right)(t) \right|^p dt,$$

thus,

$$z_j'(w) = p(w) \left| \left(D_{x_0}^{v} f_j \right)(w) \right|^p, \ z_j(x_0) = 0; \ j = 1, 2.$$

Hence, we have

$$\left| \left(D_{x_0}^{\gamma_k} f_j \right)(w) \right| \geq \frac{1}{\Gamma(v-\gamma_k)} (P_k(w))^{\frac{p-1}{p}} (z_j(w))^{\frac{1}{p}},$$

and

$$\left| \left(D_{x_0}^{v} f_j \right)(w) \right|^{\lambda_v} = p(w)^{-\frac{\lambda_v}{p}} \left(z_j'(w) \right)^{\frac{\lambda_v}{p}}, \quad j = 1, 2.$$

Therefore we obtain

$$q(w) \left| \left(D_{x_0}^{\gamma_1} f_1 \right)(w) \right|^{\lambda_\alpha} \left| \left(D_{x_0}^{\gamma_2} f_2 \right)(w) \right|^{\lambda_\beta} \left| \left(D_{x_0}^{v} f_1 \right)(w) \right|^{\lambda_v}$$

$$\geq q(w) \frac{1}{(\Gamma(v-\gamma_1))^{\lambda_\alpha}} (P_1(w))^{\lambda_\alpha \left(\frac{p-1}{p}\right)} (z_1(w))^{\frac{\lambda_\alpha}{p}}$$

$$\frac{1}{(\Gamma(v-\gamma_2))^{\lambda_\beta}} (P_2(w))^{\lambda_\beta \left(\frac{p-1}{p}\right)} (z_2(w))^{\frac{\lambda_\beta}{p}} (p(w))^{-\frac{\lambda_v}{p}} \left(z_1'(w) \right)^{\frac{\lambda_v}{p}}$$

$$= A(w) (z_1(w))^{\frac{\lambda_\alpha}{p}} (z_2(w))^{\frac{\lambda_\beta}{p}} \left(z_1'(w) \right)^{\frac{\lambda_v}{p}}.$$

Consequently, by another Hölder's inequality application, we find (by $\frac{p}{\lambda_v} < 1$)

$$\int_{x_0}^{x} q(w) \left| \left(D_{x_0}^{\gamma_1} f_1 \right)(w) \right|^{\lambda_\alpha} \left| \left(D_{x_0}^{\gamma_2} f_2 \right)(w) \right|^{\lambda_\beta} \left| \left(D_{x_0}^{v} f_1 \right)(w) \right|^{\lambda_v} dw$$

$$\geq A_0(x) \left[\int_{x_0}^{x} (z_1(w))^{\frac{\lambda_\alpha}{\lambda_v}} (z_2(w))^{\frac{\lambda_\beta}{\lambda_v}} z_1'(w) \, dw \right]^{\frac{\lambda_v}{p}}. \quad (21)$$

Similary one finds

$$\int_{x_0}^{x} q(w) \left| \left(D_{x_0}^{\gamma_2} f_1 \right)(w) \right|^{\lambda_\beta} \left| \left(D_{x_0}^{\gamma_1} f_2 \right)(w) \right|^{\lambda_\alpha} \left| \left(D_{x_0}^{v} f_2 \right)(w) \right|^{\lambda_v} dw$$

$$\geq A_0(x)\left[\int_{x_0}^{x}(z_1(w))^{\frac{\lambda_\beta}{\lambda_v}}(z_2(w))^{\frac{\lambda_\alpha}{\lambda_v}}z_2'(w)\,dw\right]^{\frac{\lambda_v}{p}}. \tag{22}$$

Taking $\lambda_\beta = 0$ and adding (21) and (22) we obtain

$$\int_{x_0}^{x} q(w)\Big[\left|\left(D_{x_0}^{\gamma_1}f_1\right)(w)\right|^{\lambda_\alpha}\cdot\left|\left(D_{x_0}^{v}f_1\right)(w)\right|^{\lambda_v}+$$
$$\left|\left(D_{x_0}^{\gamma_1}f_2\right)(w)\right|^{\lambda_\alpha}\cdot\left|\left(D_{x_0}^{v}f_2\right)(w)\right|^{\lambda_v}\Big]\,dw$$

$$\geq \left(A_0(x)\,|_{\lambda_\beta=0}\right)\left\{\left[\int_{x_0}^{x}(z_1(w))^{\frac{\lambda_\alpha}{\lambda_v}}z_1'(w)\,dw\right]^{\frac{\lambda_v}{p}}+\right.$$
$$\left.\left[\int_{x_0}^{x}(z_2(w))^{\frac{\lambda_\alpha}{\lambda_v}}z_2'(w)\,dw\right]^{\frac{\lambda_v}{p}}\right\}$$
$$=\left(A_0(x)\,|_{\lambda_\beta=0}\right)\left\{(z_1(x))^{\frac{(\lambda_\alpha+\lambda_v)}{p}}+\right.$$
$$\left.(z_2(x))^{\frac{(\lambda_\alpha+\lambda_v)}{p}}\right\}\left(\frac{\lambda v}{\lambda_\alpha+\lambda_v}\right)^{\frac{\lambda_v}{p}}$$
$$=\left(A_0(x)\,|_{\lambda_\beta=0}\right)\left(\frac{\lambda_v}{\lambda_\alpha+\lambda_v}\right)^{\frac{\lambda_v}{p}}\left\{\left(\int_{x_0}^{x}p(t)\left|\left(D_{x_0}^{v}f_1\right)(t)\right|^{p}dt\right)^{\frac{(\lambda_\alpha+\lambda_v)}{p}}\right.$$
$$\left.+\left(\int_{x_0}^{x}p(t)\left|\left(D_{x_0}^{v}f_2\right)(t)\right|^{p}dt\right)^{\frac{(\lambda_\alpha+\lambda_v)}{p}}\right\}=:(*).$$

In this article we are using frequently the basic inequalities

$$2^{r-1}(a^r+b^r)\leq(a+b)^r\leq a^r+b^r,\quad a,b\geq 0,\ 0\leq r\leq 1, \tag{23}$$

$$a^r+b^r\leq(a+b)^r\leq 2^{r-1}(a^r+b^r),\quad a,b\geq 0,\ r\geq 1. \tag{24}$$

Finally using (23), (24) and (18) we get

$$(*)\geq\left(A_0(x)\,|_{\lambda_\beta=0}\right)\cdot\left(\frac{\lambda_v}{\lambda_\alpha+\lambda_v}\right)^{\frac{\lambda_v}{p}}\cdot\delta_1$$

$$\left\{\int_{x_0}^{x} p(t)\left[\left|\left(D_{x_0}^{v} f_1\right)(t)\right|^{p}+\left|\left(D_{x_0}^{v} f_2\right)(t)\right|^{p}\right] dt\right\}^{\frac{(\lambda_\alpha+\lambda_v)}{p}}.$$

Inequality (19) has been established.

Here we see that $\left(\frac{p}{p-1}\right)(v-\gamma_i-1)+1>0$, $-\frac{1}{(p-1)}>0$ and $p(t)\in C([x_0,b])$, thus (see (15)) $P_i(w)\in\mathbb{R}$ for every $w\in[x_0,b]$, also $P_i(w)$ is continuous and bounded on $[x_0,b]$ for $i=1,2$.

By $\lambda_v>p>0$, we have $0<\frac{p}{\lambda_v}<1$, $\frac{p}{p-\lambda_v}<0$.

We observe that

$$\frac{1}{A(w)}=\frac{1}{q(w)}\left(\Gamma(v-\gamma_1)\right)^{\lambda_\alpha}\left(\Gamma(v-\gamma_2)\right)^{\lambda_\beta}(p(w))^{\lambda_v/p}$$

$$\left(P_1(w)\right)^{\lambda_\alpha\left(\frac{1-p}{p}\right)}\left(P_2(w)\right)^{\lambda_\beta\left(\frac{1-p}{p}\right)}\in C([x_0,b]),$$

and $\frac{1}{A(w)}>0$ on $(x_0,b]$, $\frac{1}{A(x_0)}=0$.

Therefore $0<A_0(x)<\infty$, and all we have done in this proof are valid. ☐

It follows the counterpart of the last theorem.

Theorem 6. *All here as in Theorem 5. Further assume $\lambda_\beta\geq\lambda_v$.*

Denote

$$\delta_2:=2^{1-(\lambda_\beta/\lambda_v)},\quad \delta_3:=(\delta_2-1)\,2^{-(\lambda_\beta/\lambda_v)}. \tag{25}$$

If $\lambda_\alpha=0$, then it holds

$$\int_{x_0}^{x} q(w)\left[\left|\left(D_{x_0}^{\gamma_2} f_2\right)(w)\right|^{\lambda_\beta}\cdot\left|\left(D_{x_0}^{v} f_1\right)(w)\right|^{\lambda_v}+\right.$$

$$\left.\left|\left(D_{x_0}^{\gamma_2} f_1\right)(w)\right|^{\lambda_\beta}\cdot\left|\left(D_{x_0}^{v} f_2\right)(w)\right|^{\lambda_v}\right] dw$$

$$\geq\left(A_0(x)\left|_{\lambda_\alpha=0}\right.\right)2^{\frac{p-\lambda_v}{p}}\left(\frac{\lambda_v}{\lambda_\beta+\lambda_v}\right)^{\frac{\lambda_v}{p}}\delta_3^{\frac{\lambda_v}{p}}$$

$$\cdot\left(\int_{x_0}^{x} p(w)\left[\left|\left(D_{x_0}^{v} f_1\right)(w)\right|^{p}+\left|\left(D_{x_0}^{v} f_2\right)(w)\right|^{p}\right] dw\right)^{\left(\frac{\lambda_v+\lambda_\beta}{p}\right)},\quad \text{all } x_0\leq x\leq b. \tag{26}$$

Proof. When $\lambda_\alpha=0$ from (21) and (22) we obtain

$$\int_{x_0}^{x} q(w)\left|\left(D_{x_0}^{\gamma_2} f_2\right)(w)\right|^{\lambda_\beta}\left|\left(D_{x_0}^{v} f_1\right)(w)\right|^{\lambda_v} dw$$

$$\geq \left(A_0(x)|_{\lambda_\alpha=0}\right)\left[\int_{x_0}^{x}(z_2(w))^{\frac{\lambda_\beta}{\lambda_v}}\, z_1^{'}(w)\,dw\right]^{\frac{\lambda_v}{p}}, \tag{27}$$

and

$$\int_{x_0}^{x} q(w)\left|\left(D_{x_0}^{\gamma_2}f_1\right)(w)\right|^{\lambda_\beta}\left|\left(D_{x_0}^{v}f_2\right)(w)\right|^{\lambda_v}dw$$

$$\geq \left(A_0(x)|_{\lambda_\alpha=0}\right)\left[\int_{x_0}^{x}(z_1(w))^{\frac{\lambda_\beta}{\lambda_v}}\, z_2^{'}(w)\,dw\right]^{\frac{\lambda_v}{p}}, \tag{28}$$

all $x_0 \leq x \leq b$. Adding (27) and (28) we get

$$\int_{x_0}^{x} q(w)\Big[\left|\left(D_{x_0}^{\gamma_2}f_2\right)(w)\right|^{\lambda_\beta}\cdot\left|\left(D_{x_0}^{v}f_1\right)(w)\right|^{\lambda_v}+$$

$$\left|\left(D_{x_0}^{\gamma_2}f_1\right)(w)\right|^{\lambda_\beta}\cdot\left|\left(D_{x_0}^{v}f_2\right)(w)\right|^{\lambda_v}\Big]dw$$

$$\geq \left(A_0(x)|_{\lambda_\alpha=0}\right)\left\{\left[\int_{x_0}^{x}(z_2(w))^{\frac{\lambda_\beta}{\lambda_v}}\, z_1^{'}(w)\,dw\right]^{\frac{\lambda_v}{p}}+\right.$$

$$\left.\left[\int_{x_0}^{x}(z_1(w))^{\frac{\lambda_\beta}{\lambda_v}}\, z_2^{'}(w)\,dw\right]^{\frac{\lambda_v}{p}}\right\}$$

$$\geq \left(A_0(x)|_{\lambda_\alpha=0}\right)\cdot 2^{\frac{p-\lambda_v}{p}}\cdot(M(x))^{\frac{\lambda_v}{p}} =: (*), \tag{29}$$

by $\frac{\lambda_v}{p} > 1$ and (24), where

$$M(x) := \int_{x_0}^{x}(z_2(w))^{\frac{\lambda_\beta}{\lambda_v}}\, z_1^{'}(w) + (z_1(w))^{\frac{\lambda_\beta}{\lambda_v}}\, z_2^{'}(w)\,dw. \tag{30}$$

Next we work on $M(x)$. We have that

$$M(x) = \int_{x_0}^{x}\left((z_1(w))^{\frac{\lambda_\beta}{\lambda_v}} + (z_2(w))^{\frac{\lambda_\beta}{\lambda_v}}\right)\left(z_1^{'}(w) + z_2^{'}(w)\right)dw$$

$$-\int_{x_0}^{x}\left[(z_1(w))^{\frac{\lambda_\beta}{\lambda_v}}\, z_1^{'}(w) + (z_2(w))^{\frac{\lambda_\beta}{\lambda_v}}\, z_2^{'}(w)\right]dw$$

$$\overset{(by\ (24))}{\geq} \delta_2\int_{x_0}^{x}(z_1(w)+z_2(w))^{\frac{\lambda_\beta}{\lambda_v}}\,(z_1(w)+z_2(w))'\,dw$$

$$-\left(\frac{\lambda_v}{\lambda_\beta+\lambda_v}\right)\left[(z_1(x))^{\left(\frac{\lambda_v+\lambda_\beta}{\lambda_v}\right)} + (z_2(x))^{\left(\frac{\lambda_v+\lambda_\beta}{\lambda_v}\right)}\right]$$

$$= \delta_2\,(z_1(x)+z_2(x))^{\left(\frac{\lambda_v+\lambda_\beta}{\lambda_v}\right)}\left(\frac{\lambda_v}{\lambda_v+\lambda_\beta}\right) - \left(\frac{\lambda_v}{\lambda_\beta+\lambda_v}\right)$$

$$\left[(z_1(x))^{\left(\frac{\lambda_v+\lambda_\beta}{\lambda_v}\right)}+(z_2(x))^{\left(\frac{\lambda_v+\lambda_\beta}{\lambda_v}\right)}\right]$$

$$=\left(\frac{\lambda_v}{\lambda_\beta+\lambda_v}\right)\left[\delta_2\left(z_1(x)+z_2(x)\right)^{\left(\frac{\lambda_v+\lambda_\beta}{\lambda_v}\right)}-\left((z_1(x))^{\left(\frac{\lambda_v+\lambda_\beta}{\lambda_v}\right)}+(z_2(x))^{\left(\frac{\lambda_v+\lambda_\beta}{\lambda_v}\right)}\right)\right]$$

$$\overset{(24)}{\geq}\left(\frac{\lambda_v}{\lambda_\beta+\lambda_v}\right)\left[\delta_2\left((z_1(x))^{\left(\frac{\lambda_v+\lambda_\beta}{\lambda_v}\right)}+(z_2(x))^{\left(\frac{\lambda_v+\lambda_\beta}{\lambda_v}\right)}\right)\right.$$
$$\left.-\left((z_1(x))^{\left(\frac{\lambda_v+\lambda_\beta}{\lambda_v}\right)}+(z_2(x))^{\left(\frac{\lambda_v+\lambda_\beta}{\lambda_v}\right)}\right)\right]$$
$$=\left(\frac{\lambda_v}{\lambda_\beta+\lambda_v}\right)(\delta_2-1)\left[(z_1(x))^{\left(\frac{\lambda_v+\lambda_\beta}{\lambda_v}\right)}+(z_2(x))^{\left(\frac{\lambda_v+\lambda_\beta}{\lambda_v}\right)}\right]$$
$$\overset{(24)}{\geq}\left(\frac{\lambda_v}{\lambda_\beta+\lambda_v}\right)\delta_3\left(z_1(x)+z_2(x)\right)^{\left(\frac{\lambda_v+\lambda_\beta}{\lambda_v}\right)}.$$

I.e. we present that

$$M(x)\geq\left(\frac{\lambda_v}{\lambda_\beta+\lambda_v}\right)\delta_3\left(z_1(x)+z_2(x)\right)^{\left(\frac{\lambda_v+\lambda_\beta}{\lambda_v}\right)}. \tag{31}$$

Consequently, by (29) and (31) we get

$$(*)\geq\left(A_0(x)\,|_{\lambda_\alpha=0}\right)2^{\frac{p-\lambda_v}{p}}\left(\frac{\lambda_v}{\lambda_\beta+\lambda_v}\right)^{\frac{\lambda_v}{p}}$$
$$\delta_3^{\frac{\lambda_v}{p}}\left(z_1(x)+z_2(x)\right)^{\left(\frac{\lambda_v+\lambda_\beta}{p}\right)}$$

$$=\left(A_0(x)\,|_{\lambda_\alpha=0}\right)2^{\frac{p-\lambda_v}{p}}\left(\frac{\lambda_v}{\lambda_\beta+\lambda_v}\right)^{\frac{\lambda_v}{p}}\delta_3^{\frac{\lambda_v}{p}}$$

$$\left(\int_{x_0}^{x}p(t)\left[\left|\left(D_{x_0}^{v}f_1\right)(t)\right|^p+\left|\left(D_{x_0}^{v}f_2\right)(t)\right|^p\right]dt\right)^{\left(\frac{\lambda_v+\lambda_\beta}{p}\right)}.$$

We have established (26). □

A special important case follows.

Theorem 7. *Let* $v\geq 2$ and $\gamma_1\geq 0$ such that $2\leq v-\gamma_1<\frac{1}{p}$, $0<p<1$. *Let* $f_1,f_2\in C_{x_0}^{v}([a,b])$ with $f_1^{(i)}(x_0)=f_2^{(i)}(x_0)=0$, $i=0,1,\ldots,n-1$, $n:=[v]$. *Here* $x,x_0\in[a,b]:x\geq x_0$. *We assume here that* $D_{x_0}^{v}f_j$ *is of fixed sign on* $[x_0,b]$, $j=1,2$. *Consider also* $p(t)>0$ *and* $q(t)>0$ *continuous functions on* $[x_0,b]$. *Let* $\lambda_\alpha\geq\lambda_{\alpha+1}>1$.

Denote

$$\theta_3 := \left(2^{1-(\lambda_\alpha/\lambda_{\alpha+1})} - 1\right) 2^{-\lambda_\alpha/\lambda_{\alpha+1}}, \tag{32}$$

$$L(x) := \left(2\int_{x_0}^{x} (q(w))^{\left(\frac{1}{1-\lambda_{\alpha+1}}\right)} dw\right)^{(1-\lambda_{\alpha+1})} \left(\frac{\theta_3 \lambda_{\alpha+1}}{\lambda_\alpha + \lambda_{\alpha+1}}\right)^{\lambda_{\alpha+1}}, \tag{33}$$

and

$$P_1(x) := \int_{x_0}^{x} (x-t)^{\frac{(v-\gamma_1-1)p}{p-1}} (p(t))^{-\frac{1}{p-1}} dt, \tag{34}$$

$$T(x) := L(x) \cdot \left(\frac{P_1(x)^{\left(\frac{p-1}{p}\right)}}{\Gamma(v-\gamma_1)}\right)^{(\lambda_\alpha+\lambda_{\alpha+1})},$$

$$\omega_1 := 2^{\left(\frac{p-1}{p}\right)(\lambda_\alpha+\lambda_{\alpha+1})}, \tag{35}$$

and

$$\Phi(x) := T(x)\,\omega_1. \tag{36}$$

Then

$$\int_{x_0}^{x} q(w)\Big[\left|\left(D_{x_0}^{\gamma_1} f_1\right)(w)\right|^{\lambda_\alpha} \left|\left(D_{x_0}^{\gamma_1+1} f_2\right)(w)\right|^{\lambda_{\alpha+1}}$$

$$+ \left|\left(D_{x_0}^{\gamma_1} f_2\right)(w)\right|^{\lambda_\alpha} \left|\left(D_{x_0}^{\gamma_1+1} f_1\right)(w)\right|^{\lambda_{\alpha+1}}\Big] dw \tag{37}$$

$$\geq \Phi(x) \left[\int_{x_0}^{x} p(w)\left(\left|\left(D_{x_0}^{v} f_1\right)(w)\right|^p + \left|\left(D_{x_0}^{v} f_2\right)(w)\right|^p\right) dw\right]^{\frac{(\lambda_\alpha+\lambda_{\alpha+1})}{p}}, \quad \text{all } x_0 \leq x \leq b.$$

Proof. For convenience we set $\gamma_2 := \gamma_1 + 1$. From (13) and assumption we obtain

$$\left|\left(D_{x_0}^{\gamma_k} f_j\right)(w)\right| = \frac{1}{\Gamma(v-\gamma_k)} \int_{x_0}^{w} (w-t)^{v-\gamma_k-1} \left|\left(D_{x_0}^{v} f_i\right)(t)\right| dt =: g_{j,\gamma_k}(w), \tag{38}$$

where $j = 1, 2$, $k = 1, 2$, all $x_0 \leq x \leq b$. We observe that

$$\left(\left(D_{x_0}^{\gamma_1} f_j\right)(x)\right)' = \left(D_{x_0}^{\gamma_1+1} f_j\right)(x) = \left(D_{x_0}^{\gamma_2} f_j\right)(x), \tag{39}$$

all $x_0 \le x \le b$. And also

$$(g_{j,\gamma_1}(w))' = g_{j,\gamma_2}(w)\,;\; g_{j,\gamma_k}(x_0) = 0. \tag{40}$$

Notice that if $v - \gamma_2 = 1$, then

$$g_{j,\gamma_2}(w) = \int_{x_0}^{w} \left|\left(D_{x_0}^{v} f_j\right)(t)\right| dt.$$

Next we apply Hölder's inequality with indices $\frac{1}{\lambda_{\alpha+1}} < 1$, $\frac{1}{(1-\lambda_{\alpha+1})} < 0$, we obtain

$$\int_{x_0}^{x} q(w) \left|\left(D_{x_0}^{\gamma_1} f_1\right)(w)\right|^{\lambda_\alpha} \left|\left(D_{x_0}^{\gamma_1+1} f_2\right)(w)\right|^{\lambda_{\alpha+1}} dw$$

$$= \int_{x_0}^{x} q(w) \left(g_{1,\gamma_1}(w)\right)^{\lambda_\alpha} \left(\left(g_{2,\gamma_1}(w)\right)'\right)^{\lambda_{\alpha+1}} dw \tag{41}$$

$$\ge \left(\int_{x_0}^{x} (q(w))^{\left(\frac{1}{1-\lambda_{\alpha+1}}\right)} dw\right)^{(1-\lambda_{\alpha+1})}$$

$$\left(\int_{x_0}^{x} \left(g_{1,\gamma_1}(w)\right)^{\frac{\lambda_\alpha}{\lambda_{\alpha+1}}} \left(g_{2,\gamma_1}(w)\right)' dw\right)^{\lambda_{\alpha+1}}.$$

Similarly we get

$$\int_{x_0}^{x} q(w) \left|\left(D_{x_0}^{\gamma_1} f_2\right)(w)\right|^{\lambda_\alpha} \left|\left(D_{x_0}^{\gamma_1+1} f_1\right)(w)\right|^{\lambda_{\alpha+1}} dw$$

$$\ge \left(\int_{x_0}^{x} (q(w))^{\left(\frac{1}{1-\lambda_{\alpha+1}}\right)} dw\right)^{(1-\lambda_{\alpha+1})}$$

$$\left(\int_{x_0}^{x} \left(g_{2,\gamma_1}(w)\right)^{\frac{\lambda_\alpha}{\lambda_{\alpha+1}}} \left(g_{1,\gamma_1}(w)\right)' dw\right)^{\lambda_{\alpha+1}}. \tag{42}$$

Adding (41) and (42) we observe

$$\int_{x_0}^{x} q(w) \Big[\left|\left(D_{x_0}^{\gamma_1} f_1\right)(w)\right|^{\lambda_\alpha} \left|\left(D_{x_0}^{\gamma_1+1} f_2\right)(w)\right|^{\lambda_{\alpha+1}}$$

$$+ \left|\left(D_{x_0}^{\gamma_1} f_2\right)(w)\right|^{\lambda_\alpha} \left|\left(D_{x_0}^{\gamma_1+1} f_1\right)(w)\right|^{\lambda_{\alpha+1}}\Big] dw$$

$$\ge \left(\int_{x_0}^{x} (q(w))^{\left(\frac{1}{1-\lambda_{\alpha+1}}\right)} dw\right)^{(1-\lambda_{\alpha+1})}$$

$$\left[\left(\int_{x_0}^{x}(g_{1,\gamma_1}(w))^{\frac{\lambda_\alpha}{\lambda_{\alpha+1}}}(g_{2,\gamma_1}(w))'\,dw\right)^{\lambda_{\alpha+1}}\right.$$

$$\left.+\left(\int_{x_0}^{x}(g_{2,\gamma_1}(w))^{\frac{\lambda_\alpha}{\lambda_{\alpha+1}}}(g_{1,\gamma_1}(w))'\,dw\right)^{\lambda_{\alpha+1}}\right]$$

$$\overset{(24)}{\geq}\left(2\int_{x_0}^{x}(q(w))^{\left(\frac{1}{1-\lambda_{\alpha+1}}\right)}dw\right)^{(1-\lambda_{\alpha+1})}$$

$$\cdot\left[\int_{x_0}^{x}\left[(g_{1,\gamma_1}(w))^{\frac{\lambda_\alpha}{\lambda_{\alpha+1}}}(g_{2,\gamma_1}(w))'\right.\right.$$

$$\left.\left.+(g_{2,\gamma_1}(w))^{\frac{\lambda_\alpha}{\lambda_{\alpha+1}}}(g_{1,\gamma_1}(w))'\right]dw\right]^{\lambda_{\alpha+1}}$$

(notice (30) and the proof of (31), accordingly here we have)

$$\geq\left(2\int_{x_0}^{x}(q(w))^{\left(\frac{1}{1-\lambda_{\alpha+1}}\right)}dw\right)^{(1-\lambda_{\alpha+1})}$$

$$\left(\frac{\lambda_{\alpha+1}\theta_3}{\lambda_\alpha+\lambda_{\alpha+1}}\right)^{\lambda_{\alpha+1}}(g_{1,\gamma_1}(x)+g_{2,\gamma_1}(x))^{(\lambda_\alpha+\lambda_{\alpha+1})}$$

$$=L(x)\,(g_{1,\gamma_1}(x)+g_{2,\gamma_1}(x))^{(\lambda_\alpha+\lambda_{\alpha+1})}$$

$$=\frac{L(x)}{(\Gamma(v-\gamma_1))^{(\lambda_\alpha+\lambda_{\alpha+1})}}\left\{\int_{x_0}^{x}(x-t)^{v-\gamma_1-1}(p(t))^{-\frac{1}{p}}(p(t))^{\frac{1}{p}}\right.$$

$$\left.\left[\left|\left(D_{x_0}^{v}f_1\right)(t)\right|+\left|\left(D_{x_0}^{v}f_2\right)(t)\right|\right]dt\right\}^{(\lambda_\alpha+\lambda_{\alpha+1})}$$

(applying Hölder's inequality with indices $\frac{p}{p-1}$ and p we find)

$$\geq\frac{L(x)}{(\Gamma(v-\gamma_1))^{(\lambda_\alpha+\lambda_{\alpha+1})}}\cdot\left(\int_{x_0}^{x}(x-t)^{\frac{(v-\gamma_1-1)p}{p-1}}(p(t))^{-\frac{1}{p-1}}dt\right)^{\left(\frac{p-1}{p}\right)(\lambda_\alpha+\lambda_{\alpha+1})}$$

$$\left(\int_{x_0}^{x}p(t)\left[\left|\left(D_{x_0}^{v}f_1\right)(t)\right|+\left|\left(D_{x_0}^{v}f_2\right)(t)\right|\right]^{p}dt\right)^{\left(\frac{\lambda_\alpha+\lambda_{\alpha+1}}{p}\right)}$$

$$= T(x) \cdot \left[\int_{x_0}^{x} p(t) \left(\left| \left(D_{x_0}^{v} f_1 \right)(t) \right| + \left| \left(D_{x_0}^{v} f_2 \right)(t) \right| \right)^p dt \right]^{\left(\frac{\lambda_\alpha + \lambda_{\alpha+1}}{p} \right)}$$

$$\geq \Phi(x) \cdot \left[\int_{x_0}^{x} p(t) \left(\left| \left(D_{x_0}^{v} f_1 \right)(t) \right|^p + \left| \left(D_{x_0}^{v} f_2 \right)(t) \right|^p \right) dt \right]^{\left(\frac{\lambda_\alpha + \lambda_{\alpha+1}}{p} \right)} .$$

We have proved (37) . □

Next we treat the case of exponents $\lambda_\beta = \lambda_\alpha + \lambda_v$.

Theorem 8. *All here as in Theorem 5. Consider the special case of* $\lambda_\beta = \lambda_\alpha + \lambda_v$. *Assume here for* $j = 1, 2$ *that*

$$z_j(x) := \int_{x_0}^{x} p(t) \left| \left(D_{x_0}^{v} f_j \right)(t) \right|^p dt \in [H, \Psi], 0 < H < \Psi,$$

$$h := \frac{\Psi}{H} > 1, \quad M_h(1) := \frac{(h-1) h^{\frac{1}{h-1}}}{e \ln h}. \tag{43}$$

Denote

$$\tilde{T}(x) := A_0(x) \left(\frac{\lambda_v}{\lambda_\alpha + \lambda_v} \right)^{\frac{\lambda_v}{p}} 2^{\frac{p - 2\lambda_\alpha - 3\lambda_v}{p}} (M_h(1))^{-2(\lambda_\alpha + \lambda_v)/p} \tag{44}$$

Then

$$\int_{x_0}^{x} q(w) \Big[\left| \left(D_{x_0}^{\gamma_1} f_1 \right)(w) \right|^{\lambda_\alpha} \left| \left(D_{x_0}^{\gamma_2} f_2 \right)(w) \right|^{\lambda_\alpha + \lambda_v} \left| \left(D_{x_0}^{v} f_1 \right)(w) \right|^{\lambda_v}$$

$$+ \left| \left(D_{x_0}^{\gamma_2} f_1 \right)(w) \right|^{\lambda_\alpha + \lambda_v} \left| \left(D_{x_0}^{\gamma_1} f_2 \right)(w) \right|^{\lambda_\alpha} \left| \left(D_{x_0}^{v} f_2 \right)(w) \right|^{\lambda_v} \Big] dw$$

$$\geq \tilde{T}(x) \left(\int_{x_0}^{x} p(w) \left(\left| \left(D_{x_0}^{v} f_1 \right)(w) \right|^p + \left| \left(D_{x_0}^{v} f_2 \right)(w) \right|^p \right) dw \right)^{2\left(\frac{\lambda_\alpha + \lambda_v}{p} \right)}, \tag{45}$$

all $x_0 \leq x \leq b$.

Proof. We apply (21) and (22) for $\lambda_\beta = \lambda_\alpha + \lambda_v$ and add to get

$$\int_{x_0}^{x} q(w) \Big[\left| \left(D_{x_0}^{\gamma_1} f_1 \right)(w) \right|^{\lambda_\alpha} \left| \left(D_{x_0}^{\gamma_2} f_2 \right)(w) \right|^{\lambda_\alpha + \lambda_v} \left| \left(D_{x_0}^{v} f_1 \right)(w) \right|^{\lambda_v}$$

$$+\left|\left(D_{x_0}^{\gamma_2} f_1\right)(w)\right|^{\lambda_\alpha+\lambda_v}\left|\left(D_{x_0}^{\gamma_1} f_2\right)(w)\right|^{\lambda_\alpha}\left|\left(D_{x_0}^{v} f_2\right)(w)\right|^{\lambda_v}\Big] dw$$

$$\geq A_0(x)\left\{\left[\int_{x_0}^{x}\left(z_1(w)\right)^{\frac{\lambda_\alpha}{\lambda_v}}\left(z_2(w)\right)^{\frac{\lambda_\alpha}{\lambda_v}+1} z_1'(w)\, dw\right]^{\frac{\lambda_v}{p}}\right.$$

$$\left.+\left[\int_{x_0}^{x}\left(z_1(w)\right)^{\frac{\lambda_\alpha}{\lambda_v}+1}\left(z_2(w)\right)^{\frac{\lambda_\alpha}{\lambda_v}} z_2'(w)\, dw\right]^{\frac{\lambda_v}{p}}\right\}$$

$$\overset{(24)}{\geq} A_0(x)\, 2^{1-\frac{\lambda_v}{p}}\left\{\int_{x_0}^{x}\left[\left(z_1(w)\right)^{\frac{\lambda_\alpha}{\lambda_v}}\left(z_2(w)\right)^{\frac{\lambda_\alpha}{\lambda_v}+1} z_1'(w)\right.\right.$$

$$\left.\left.+\left(z_1(w)\right)^{\frac{\lambda_\alpha}{\lambda_v}+1}\left(z_2(w)\right)^{\frac{\lambda_\alpha}{\lambda_v}} z_2'(w)\right] dw\right\}^{\frac{\lambda_v}{p}}$$

$$= A_0(x)\, 2^{1-\frac{\lambda_v}{p}}\left\{\int_{x_0}^{x}\left(z_1(w)\, z_2(w)\right)^{\frac{\lambda_\alpha}{\lambda_v}}\right.$$

$$\left.\left[z_2(w)\, z_1'(w)+z_1(w)\, z_2'(w)\right] dw\right\}^{\frac{\lambda_v}{p}}$$

$$= A_0(x)\, 2^{1-\frac{\lambda_v}{p}}\left\{\int_{x_0}^{x}\left(z_1(w)\, z_2(w)\right)^{\frac{\lambda_\alpha}{\lambda_v}}\left(z_1(w)\, z_2(w)\right)' dw\right\}^{\frac{\lambda_v}{p}}$$

$$= A_0(x)\, 2^{1-\frac{\lambda_v}{p}}\left(\frac{\left(z_1(x)\, z_2(x)\right)^{\frac{\lambda_\alpha}{\lambda_v}+1}}{\frac{\lambda_\alpha}{\lambda_v}+1}\right)^{\frac{\lambda_v}{p}}$$

$$= A_0(x)\, 2^{\frac{p-\lambda_v}{p}}\left(\frac{\lambda_v}{\lambda_\alpha+\lambda_v}\right)^{\frac{\lambda_v}{p}}\left(z_1(x)\, z_2(x)\right)^{\frac{(\lambda_\alpha+\lambda_v)}{p}}$$

(see [10])

$$\geq A_0(x)\, 2^{\frac{p-\lambda_v}{p}}\left(\frac{\lambda_v}{\lambda_\alpha+\lambda_v}\right)^{\frac{\lambda_v}{p}}\left(\frac{z_1(x)+z_2(x)}{2M_h(1)}\right)^{\frac{2(\lambda_\alpha+\lambda_v)}{p}}$$

$$= \tilde{T}(x)\left(z_1(x)+z_2(x)\right)^{\frac{2(\lambda_\alpha+\lambda_v)}{p}}$$

$$= \tilde{T}(x)\left(\int_{x_0}^{x} p(w)\left(\left|\left(D_{x_0}^{v} f_1\right)(w)\right|^{p}+\left|\left(D_{x_0}^{v} f_2\right)(w)\right|^{p}\right) dw\right)^{2\left(\frac{\lambda_\alpha+\lambda_v}{p}\right)}.$$

We have established (45). $\square$

Next follow special cases of the above theorems.

Corollary 9. (to Theorem 5; $\lambda_\beta = 0,\ p(t) = q(t) = 1$). *Then*

$$\int_{x_0}^{x} \left[\left| \left(D_{x_0}^{\gamma_1} f_1 \right)(w) \right|^{\lambda_\alpha} \left| \left(D_{x_0}^{v} f_1 \right)(w) \right|^{\lambda_v} + \left| \left(D_{x_0}^{\gamma_1} f_2 \right)(w) \right|^{\lambda_\alpha} \left| \left(D_{x_0}^{v} f_2 \right)(w) \right|^{\lambda_v} \right] dw$$

$$\geq C_1(x) \cdot \left(\int_{x_0}^{x} \left(\left| \left(D_{x_0}^{v} f_1 \right)(w) \right|^{p} + \left| \left(D_{x_0}^{v} f_2 \right)(w) \right|^{p} \right) dw \right)^{\left(\frac{\lambda_\alpha + \lambda_v}{p} \right)}, \tag{46}$$

all $x_0 \leq x \leq b$, *where*

$$C_1(x) := \left(A_0(x)\,|_{\lambda_\beta = 0} \right) \cdot \left(\frac{\lambda_v}{\lambda_\alpha + \lambda_v} \right)^{\frac{\lambda_v}{p}} \cdot \delta_1, \tag{47}$$

$$\delta_1 := 2^{1 - \left(\frac{\lambda_\alpha + \lambda_v}{p} \right)} \tag{48}$$

We have that

$$\left(A_0(x)\,|_{\lambda_\beta = 0} \right) = \left\{ \left(\frac{(p-1)^{\left(\frac{\lambda_\alpha p - \lambda_\alpha}{p} \right)}}{\left(\Gamma(v - \gamma_1) \right)^{\lambda_\alpha} (vp - \gamma_1 p - 1)^{\left(\frac{\lambda_\alpha p - \lambda_\alpha}{p} \right)}} \right) \right. \tag{49}$$

$$\left. \cdot \left(\frac{(p - \lambda_v)^{\left(\frac{p - \lambda_v}{p} \right)}}{(\lambda_\alpha v p - \lambda_\alpha \gamma_1 p - \lambda_\alpha + p - \lambda_v)^{\left(\frac{p - \lambda_v}{p} \right)}} \right) \right\} \cdot$$

$$(x - x_0)^{\left(\frac{\lambda_\alpha v p - \lambda_\alpha \gamma_1 p - \lambda_\alpha + p - \lambda_v}{p} \right)}.$$

Proof. By Theorem 5. The constant $\left(A_0(x)\,|_{\lambda_\beta = 0} \right)$ was calculated in [4]. $\square$

Corollary 10. (to Theorem 6; $\lambda_\alpha = 0$, $p(t) = q(t) = 1$, $\lambda_\beta \geq \lambda_v$). *Then*

$$\int_{x_0}^{x} \left[\left| \left(D_{x_0}^{\gamma_2} f_2 \right)(w) \right|^{\lambda_\beta} \left| \left(D_{x_0}^{v} f_1 \right)(w) \right|^{\lambda_v} + \left| \left(D_{x_0}^{\gamma_2} f_1 \right)(w) \right|^{\lambda_\beta} \left| \left(D_{x_0}^{v} f_2 \right)(w) \right|^{\lambda_v} \right] dw$$

$$\geq C_2(x) \left(\int_{x_0}^{x} \left[\left| \left(D_{x_0}^{v} f_1 \right)(w) \right|^{p} + \left| \left(D_{x_0}^{v} f_2 \right)(w) \right|^{p} \right] dw \right)^{\left(\frac{\lambda_v + \lambda_\beta}{p} \right)}, \tag{50}$$

all $x_0 \leq x \leq b$, *where*

$$C_2(x) := (A_0(x)|_{\lambda_\alpha=0})\, 2^{\frac{p-\lambda_v}{p}} \left(\frac{\lambda_v}{\lambda_\beta+\lambda_v}\right)^{\frac{\lambda_v}{p}} \delta_3^{\frac{\lambda_v}{p}}. \tag{51}$$

We have that

$$(A_0(x)|_{\lambda_\alpha=0}) = \left\{\left(\frac{(p-1)^{\left(\frac{\lambda_\beta p-\lambda_\beta}{p}\right)}}{(\Gamma(v-\gamma_2))^{\lambda_\beta}(vp-\gamma_2 p-1)^{\left(\frac{\lambda_\beta p-\lambda_\beta}{p}\right)}}\right)\right. \tag{52}$$

$$\left.\cdot\left(\frac{(p-\lambda_v)^{\left(\frac{p-\lambda_v}{p}\right)}}{(\lambda_\beta vp-\lambda_\beta\gamma_2 p-\lambda_\beta+p-\lambda_v)^{\left(\frac{p-\lambda_v}{p}\right)}}\right)\right\}\cdot$$

$$(x-x_0)^{\left(\frac{\lambda_\beta vp-\lambda_\beta\gamma_2 p-\lambda_\beta+p-\lambda_v}{p}\right)}.$$

Proof. By Theorem 6. The constant $(A_0(x)|_{\lambda_\alpha=0})$ was calculated in [4]. □

3.2 Results involving several functions

Here we use the following basic inequality. Let $\alpha_1,...,\alpha_n \geq 0$, $n \in \mathbb{N}$, then

$$a_1^r+\ldots+a_n^r \leq (a_1+\ldots+a_n)^r \leq n^{r-1}\left(\sum_{i=1}^n a_i^r\right), r \geq 1, \tag{53}$$

We present

Theorem 11. *Let* $\gamma_1,\gamma_2 \geq 0$ such that $1 \leq v-\gamma_i < \frac{1}{p}$, $0 < p < 1$, $i=1,2$, *and* $f_j \in C_{x_0}^v([a,b])$ with $f_j^{(i)}(x_0)=0$, $i=0,1,\ldots,n-1$, $n:=[v]$, $j=1,\ldots,M \in \mathbb{N}$. *Here* $x,x_0 \in [a,b] : x \geq x_0$. *We assume that* $D_{x_0}^v f_j$ *is of fixed sign on* $[x_0,b]$, $j=1,\ldots,M$. *Consider also* $p(t)>0$, *and* $q(t)>0$ *continuous functions on* $[x_0,b]$. *Let* $\lambda_v>0$ *and* $\lambda_\alpha,\lambda_\beta \geq 0$ *such that* $\lambda_v > p$.

Set

$$P_k(w) := \int_{x_0}^w (w-t)^{\frac{(v-\gamma_k-1)p}{p-1}}(p(t))^{-\frac{1}{p-1}}dt,\ k=1,2;\ x_0 \leq w \leq b; \tag{54}$$

$$A(w) := \frac{q(w)(P_1(w))^{\lambda_\alpha\left(\frac{p-1}{p}\right)}(P_2(w))^{\lambda_\beta\left(\frac{p-1}{p}\right)}(p(w))^{-\frac{\lambda_v}{p}}}{(\Gamma(v-\gamma_1))^{\lambda_\alpha}(\Gamma(v-\gamma_2))^{\lambda_\beta}}; \tag{55}$$

$$A_0(x) := \left(\int_{x_0}^{x} (A(w))^{\frac{p}{p-\lambda_v}} dw\right)^{\frac{p-\lambda_v}{p}}. \tag{56}$$

Call

$$\varphi_1(x) := \left(A_0(x)|_{\lambda_\beta=0}\right) \cdot \left(\frac{\lambda_v}{\lambda_\alpha+\lambda_v}\right)^{\frac{\lambda_v}{p}}, \tag{57}$$

$$\delta_1^* := M^{1-\left(\frac{\lambda_\alpha+\lambda_v}{p}\right)}. \tag{58}$$

If $\lambda_\beta = 0$, *we obtain that*

$$\int_{x_0}^{x} q(w) \left(\sum_{j=1}^{M} \left|\left(D_{x_0}^{\gamma_1} f_j\right)(w)\right|^{\lambda_\alpha} \left|\left(D_{x_0}^{v} f_j\right)(w)\right|^{\lambda_v}\right) dw$$

$$\geq \delta_1^* \cdot \varphi_1(x) \cdot \left[\int_{x_0}^{x} p(w) \left(\sum_{j=1}^{M} \left|\left(D_{x_0}^{v} f_j\right)(w)\right|^{p}\right) dw\right]^{\left(\frac{\lambda_\alpha+\lambda_v}{p}\right)}, \tag{59}$$

all $x_0 \leq x \leq b$.

Proof. By Theorem 5 we get

$$\int_{x_0}^{x} q(w) \Big[\left|\left(D_{x_0}^{\gamma_1} f_j\right)(w)\right|^{\lambda_\alpha} \left|\left(D_{x_0}^{v} f_j\right)(w)\right|^{\lambda_v}$$

$$+ \left|\left(D_{x_0}^{\gamma_1} f_{j+1}\right)(w)\right|^{\lambda_\alpha} \left|\left(D_{x_0}^{v} f_{j+1}\right)(w)\right|^{\lambda_v}\Big] dw \tag{60}$$

$$\geq \delta_1 \varphi_1(x) \left[\int_{x_0}^{x} p(w) \left[\left|\left(D_{x_0}^{v} f_j\right)(w)\right|^{p} + \left|\left(D_{x_0}^{v} f_{j+1}\right)(w)\right|^{p}\right] dw\right]^{\left(\frac{\lambda_\alpha+\lambda_v}{p}\right)},$$

$j = 1, 2, \ldots, M-1$.

Hence by adding all the above we find

$$\int_{x_0}^{x} q(w) \left(\sum_{j=1}^{M-1} \Big[\left|\left(D_{x_0}^{\gamma_1} f_j\right)(w)\right|^{\lambda_\alpha} \left|\left(D_{x_0}^{v} f_j\right)(w)\right|^{\lambda_v}\right.$$

$$\left. + \left|\left(D_{x_0}^{\gamma_1} f_{j+1}\right)(w)\right|^{\lambda_\alpha} \left|\left(D_{x_0}^{v} f_{j+1}\right)(w)\right|^{\lambda_v}\Big]\right) dw \tag{61}$$

$$\geq \delta_1 \varphi_1(x) \cdot \left(\sum_{j=1}^{M-1} \left[\int_{x_0}^{x} p(w) \left[\left| \left(D_{x_0}^{v} f_j \right)(w) \right|^p + \left| \left(D_{x_0}^{v} f_{j+1} \right)(w) \right|^p \right] dw \right]^{\left(\frac{\lambda_\alpha + \lambda_v}{p}\right)} \right).$$

Also it holds

$$\int_{x_0}^{x} q(w) \left[\left| \left(D_{x_0}^{\gamma_1} f_1 \right)(w) \right|^{\lambda_\alpha} \left| \left(D_{x_0}^{v} f_1 \right)(w) \right|^{\lambda_v} + \left| \left(D_{x_0}^{\gamma_1} f_M \right)(w) \right|^{\lambda_\alpha} \left| \left(D_{x_0}^{v} f_M \right)(w) \right|^{\lambda_v} \right] dw \tag{62}$$

$$\geq \delta_1 \varphi_1(x) \left[\int_{x_0}^{x} p(w) \left[\left| \left(D_{x_0}^{v} f_1 \right)(w) \right|^p + \left| \left(D_{x_0}^{v} f_M \right)(w) \right|^p \right] dw \right]^{\left(\frac{\lambda_\alpha + \lambda_v}{p}\right)}.$$

Adding (61) and (62), and using (53) we have

$$2 \int_{x_0}^{x} q(w) \left(\sum_{j=1}^{M} \left| \left(D_{x_0}^{\gamma_1} f_j \right)(w) \right|^{\lambda_\alpha} \left| \left(D_{x_0}^{v} f_j \right)(w) \right|^{\lambda_v} \right) dw$$

$$\geq \delta_1 \varphi_1(x) \left\{ \left\{ \sum_{j=1}^{M-1} \left[\int_{x_0}^{x} p(w) \left[\left| \left(D_{x_0}^{v} f_j \right)(w) \right|^p + \left| \left(D_{x_0}^{v} f_{j+1} \right)(w) \right|^p \right] dw \right]^{\left(\frac{\lambda_\alpha + \lambda_v}{p}\right)} \right\} \right. \tag{63}$$

$$\left. + \left\{ \int_{x_0}^{x} p(w) \left[\left| \left(D_{x_0}^{v} f_1 \right)(w) \right|^p + \left| \left(D_{x_0}^{v} f_M \right)(w) \right|^p \right] dw \right\}^{\left(\frac{\lambda_\alpha + \lambda_v}{p}\right)} \right\} \geq$$

$$M^{1-\left(\frac{\lambda_\alpha + \lambda_v}{p}\right)} \delta_1 \varphi_1(x) \left\{ \int_{x_0}^{x} p(w) \left[2 \sum_{j=1}^{M} \left| \left(D_{x_0}^{v} f_j \right)(w) \right|^p \right] dw \right\}^{\left(\frac{\lambda_\alpha + \lambda_v}{p}\right)}. \tag{64}$$

We have proved

$$\int_{x_0}^{x} q(w) \left(\sum_{j=1}^{M} \left| \left(D_{x_0}^{\gamma_1} f_j \right)(w) \right|^{\lambda_\alpha} \left| \left(D_{x_0}^{v} f_j \right)(w) \right|^{\lambda_v} \right) dw \geq \tag{65}$$

$$M^{1-\left(\frac{\lambda_\alpha + \lambda_v}{p}\right)} \delta_1 \left(2^{\left(\frac{\lambda_\alpha + \lambda_v}{p}\right) - 1} \right) \varphi_1(x)$$

$$\cdot \left\{ \int_{x_0}^{x} p(w) \left[\sum_{j=1}^{M} \left| \left(D_{x_0}^{v} f_j \right)(w) \right|^p \right] dw \right\}^{\left(\frac{\lambda_\alpha + \lambda_v}{p}\right)}.$$

That is proving (59). $\square$

Next we give

Theorem 12. *All here as in Theorem 11. Assume $\lambda_\beta \geq \lambda_v$.*

Denote

$$\varphi_2(x) := (A_0(x)|_{\lambda_\alpha=0})\, 2^{\frac{(p-\lambda_v)}{p}} \left(\frac{\lambda_v}{\lambda_\beta+\lambda_v}\right)^{\frac{\lambda_v}{p}} \delta_3^{\frac{\lambda_v}{p}}. \tag{66}$$

If $\lambda_\alpha = 0$, then

$$\int_{x_0}^{x} q(w) \Bigg\{ \Bigg\{ \sum_{j=1}^{M-1} \Big[\big|\big(D_{x_0}^{\gamma_2} f_{j+1}\big)(w)\big|^{\lambda_\beta} \big|\big(D_{x_0}^{v} f_j\big)(w)\big|^{\lambda_v}$$

$$+ \big|\big(D_{x_0}^{\gamma_2} f_j\big)(w)\big|^{\lambda_\beta} \big|\big(D_{x_0}^{v} f_{j+1}\big)(w)\big|^{\lambda_v} \Big] \Bigg\}$$

$$+ \Big[\big|\big(D_{x_0}^{\gamma_2} f_M\big)(w)\big|^{\lambda_\beta} \big|\big(D_{x_0}^{v} f_1\big)(w)\big|^{\lambda_v}$$

$$+ \big|\big(D_{x_0}^{\gamma_2} f_1\big)(w)\big|^{\lambda_\beta} \big|\big(D_{x_0}^{v} f_M\big)(w)\big|^{\lambda_v} \Big] \Bigg\} dw \geq$$

$$M^{1-\left(\frac{\lambda_v+\lambda_\beta}{p}\right)} 2^{\left(\frac{\lambda_v+\lambda_\beta}{p}\right)} \varphi_2(x) \cdot \Bigg\{ \int_{x_0}^{x} p(w)$$

$$\left[\sum_{j=1}^{M} \big|\big(D_{x_0}^{v} f_j\big)(w)\big|^{p} \right] dw \Bigg\}^{\left(\frac{\lambda_v+\lambda_\beta}{p}\right)}, \quad x \geq x_0. \tag{67}$$

Proof. From Theorem 6 we have

$$\int_{x_0}^{x} q(w) \Big[\big|\big(D_{x_0}^{\gamma_2} f_{j+1}\big)(w)\big|^{\lambda_\beta} \big|\big(D_{x_0}^{v} f_j\big)(w)\big|^{\lambda_v}$$

$$+ \big|\big(D_{x_0}^{\gamma_2} f_j\big)(w)\big|^{\lambda_\beta} \big|\big(D_{x_0}^{v} f_{j+1}\big)(w)\big|^{\lambda_v} \Big] dw$$

$$\geq \varphi_2(x) \left(\int_{x_0}^{x} p(w) \left[\big|\big(D_{x_0}^{v} f_j\big)(w)\big|^{p} + \big|\big(D_{x_0}^{v} f_{j+1}\big)(w)\big|^{p} \right] dw \right)^{\left(\frac{\lambda_v+\lambda_\beta}{p}\right)}, \tag{68}$$

for $j = 1, \ldots, M-1$. Hence by adding all of the above we get

$$\int_{x_0}^{x} q(w)\left(\sum_{j=1}^{M-1}\left[\left|\left(D_{x_0}^{\gamma_2}f_{j+1}\right)(w)\right|^{\lambda_\beta}\left|\left(D_{x_0}^{v}f_j\right)(w)\right|^{\lambda_v}\right.\right.$$

$$\left.\left.+\left|\left(D_{x_0}^{\gamma_2}f_j\right)(w)\right|^{\lambda_\beta}\left|\left(D_{x_0}^{v}f_{j+1}\right)(w)\right|^{\lambda_v}\right]\right)dw$$

$$\geq \varphi_2(x)\left\{\sum_{j=1}^{M-1}\left(\int_{x_0}^{x} p(w)\left[\left|\left(D_{x_0}^{v}f_j\right)(w)\right|^{p}\right.\right.\right.$$

$$\left.\left.\left.+\left|\left(D_{x_0}^{v}f_{j+1}\right)(w)\right|^{p}\right]dw\right)^{\left(\frac{\lambda_v+\lambda_\beta}{p}\right)}\right\}. \tag{69}$$

Similarly it holds

$$\int_{x_0}^{x} q(w)\left[\left|\left(D_{x_0}^{\gamma_2}f_M\right)(w)\right|^{\lambda_\beta}\left|\left(D_{x_0}^{v}f_1\right)(w)\right|^{\lambda_v}\right.$$

$$\left.+\left|\left(D_{x_0}^{\gamma_2}f_1\right)(w)\right|^{\lambda_\beta}\left|\left(D_{x_0}^{v}f_M\right)(w)\right|^{\lambda_v}\right]dw$$

$$\geq \varphi_2(x)\left(\int_{x_0}^{x} p(w)\left[\left|\left(D_{x_0}^{v}f_1\right)(w)\right|^{p}+\left|\left(D_{x_0}^{v}f_M\right)(w)\right|^{p}\right]dw\right)^{\left(\frac{\lambda_v+\lambda_\beta}{p}\right)}. \tag{70}$$

Adding (69), (70) and using (53) we derive (67). □

We continue with

Theorem 13. *Let $v \geq 2$ and $\gamma_1 \geq 0$ such that $2 \leq v-\gamma_1 < 1/p$, $0 < p < 1$. Let $f_j \in C_{x_0}^{v}([a,b])$ with $f_j^{(i)}(x_0) = 0$, $i = 0,1,\ldots,n-1$, $n := [v]$, $j = 1,\ldots,M \in \mathbb{N}$. Here $x, x_0 \in [a,b] : x \geq x_0$. Assume that $D_{x_0}^{v}f_j$ is of fixed sign on $[x_0,b]$, $j = 1,\ldots,M$. Consider also $p(t) > 0$, and $q(t) > 0$ continuous functions on $[x_0,b]$. Let $\lambda_\alpha \geq \lambda_{\alpha+1} > 1$, Φ is as in Theorem 7.*

Then

$$\int_{x_0}^{x} q(w)\left\{\left\{\sum_{j=1}^{M-1}\left[\left|\left(D_{x_0}^{\gamma_1}f_j\right)(w)\right|^{\lambda_\alpha}\left|D_{x_0}^{\gamma_1+1}f_{j+1}(w)\right|^{\lambda_{\alpha+1}}\right.\right.\right.$$

$$\left.\left.+\left|\left(D_{x_0}^{\gamma_1}f_{j+1}\right)(w)\right|^{\lambda_\alpha}\left|D_{x_0}^{\gamma_1+1}f_j(w)\right|^{\lambda_{\alpha+1}}\right]\right\}$$

$$+\left[\left|\left(D_{x_0}^{\gamma_1}f_1\right)(w)\right|^{\lambda_\alpha}\left|D_{x_0}^{\gamma_1+1}f_M(w)\right|^{\lambda_{\alpha+1}}\right.$$

$$\left.\left.+\left|\left(D_{x_0}^{\gamma_1}f_M\right)(w)\right|^{\lambda_\alpha}\left|D_{x_0}^{\gamma_1+1}f_1(w)\right|^{\lambda_{\alpha+1}}\right]\right\}dw \geq$$

$$M^{1-\left(\frac{\lambda_\alpha+\lambda_{\alpha+1}}{p}\right)}2^{\left(\frac{\lambda_\alpha+\lambda_{\alpha+1}}{p}\right)}\Phi(x)\left[\int_{x_0}^{x}p(w)\left(\sum_{j=1}^{M}\left|\left(D_{x_0}^{v}f_j\right)(w)\right|^p\right)dw\right]^{\left(\frac{\lambda_\alpha+\lambda_{\alpha+1}}{p}\right)}, \quad (71)$$

all $x_0 \le x \le b$.

Proof. From Theorem 7 we get

$$\int_{x_0}^{x}q(w)\sum_{j=1}^{M-1}\left[\left|\left(D_{x_0}^{\gamma_1}f_j\right)(w)\right|^{\lambda_\alpha}\left|D_{x_0}^{\gamma_1+1}f_{j+1}(w)\right|^{\lambda_{\alpha+1}}\right.$$

$$\left.+\left|\left(D_{x_0}^{\gamma_1}f_{j+1}\right)(w)\right|^{\lambda_\alpha}\left|D_{x_0}^{\gamma_1+1}f_j(w)\right|^{\lambda_{\alpha+1}}\right]dw$$

$$\ge\Phi(x)\sum_{j=1}^{M-1}\left[\int_{x_0}^{x}p(w)\left(\left|\left(D_{x_0}^{v}f_j\right)(w)\right|^p+\left|\left(D_{x_0}^{v}f_{j+1}\right)(w)\right|^p\right)dw\right]^{\left(\frac{\lambda_\alpha+\lambda_{\alpha+1}}{p}\right)} \quad (72)$$

all $x_0 \le x \le b$.

Also it holds

$$\int_{x_0}^{x}q(w)\left[\left|\left(D_{x_0}^{\gamma_1}f_1\right)(w)\right|^{\lambda_\alpha}\left|D_{x_0}^{\gamma_1+1}f_M(w)\right|^{\lambda_{\alpha+1}}\right.$$

$$\left.+\left|\left(D_{x_0}^{\gamma_1}f_M\right)(w)\right|^{\lambda_\alpha}\left|D_{x_0}^{\gamma_1+1}f_1(w)\right|^{\lambda_{\alpha+1}}\right]dw$$

$$\ge\Phi(x)\left[\int_{x_0}^{x}p(w)\left(\left|\left(D_{x_0}^{v}f_1\right)(w)\right|^p+\left|\left(D_{x_0}^{v}f_M\right)(w)\right|^p\right)dw\right]^{\left(\frac{\lambda_\alpha+\lambda_{\alpha+1}}{p}\right)}, \quad (73)$$

all $x_0 \le x \le b$. Adding (72) and (73), along with (53) we derive (71). □

Next it comes

Theorem 14. *All here as in Theorem 11. Consider the special case of* $\lambda_\beta = \lambda_\alpha + \lambda_v$. *Here* $\tilde{T}(x)$ *as in* (44).

Assume here for $j = 1, \ldots, M$ *that*

$$z_j(x) := \int_{x_0}^{x}p(t)\left|D_{x_0}^{v}f_j(t)\right|^p dt \in [H,\Psi],\ 0 < H < \Psi.$$

Then

$$\int_{x_0}^{x} q(w) \left\{ \left\{ \sum_{j=1}^{M-1} \left[\left| \left(D_{x_0}^{\gamma_1} f_j \right)(w) \right|^{\lambda_\alpha} \left| \left(D_{x_0}^{\gamma_2} f_{j+1} \right)(w) \right|^{\lambda_\alpha + \lambda_v} \left| \left(D_{x_0}^{v} f_j \right)(w) \right|^{\lambda_v} \right. \right. \right.$$

$$\left. \left. + \left| \left(D_{x_0}^{\gamma_2} f_j \right)(w) \right|^{\lambda_\alpha + \lambda_v} \left| \left(D_{x_0}^{\gamma_1} f_{j+1} \right)(w) \right|^{\lambda_\alpha} \left| \left(D_{x_0}^{v} f_{j+1} \right)(w) \right|^{\lambda_v} \right] \right\}$$

$$+ \left[\left| \left(D_{x_0}^{\gamma_1} f_1 \right)(w) \right|^{\lambda_\alpha} \left| \left(D_{x_0}^{\gamma_2} f_M \right)(w) \right|^{\lambda_\alpha + \lambda_v} \left| \left(D_{x_0}^{v} f_1 \right)(w) \right|^{\lambda_v} \right.$$

$$\left. \left. + \left| \left(D_{x_0}^{\gamma_2} f_1 \right)(w) \right|^{\lambda_\alpha + \lambda_v} \left| \left(D_{x_0}^{\gamma_1} f_M \right)(w) \right|^{\lambda_\alpha} \left| \left(D_{x_0}^{v} f_M \right)(w) \right|^{\lambda_v} \right] \right\} dw \geq \tag{74}$$

$$M^{\left(1 - \frac{2(\lambda_\alpha + \lambda_v)}{p}\right)} 2^{2\left(\frac{\lambda_\alpha + \lambda_v}{p}\right)} \tilde{T}(x) \left[\int_{x_0}^{x} p(w) \left(\sum_{j=1}^{M} \left| \left(D_{x_0}^{v} f_j \right)(w) \right|^p \right) dw \right]^{\left(2\left(\frac{\lambda_\alpha + \lambda_v}{p}\right)\right)},$$

all $x_0 \leq x \leq b$.

Proof. Based on Theorem 8. The rest as in the proof of Theorem 13. □

We continue with

Corollary 15. (to Theorem 11, $\lambda_\beta = 0$, $p(t) = q(t) = 1$). *Then*

$$\int_{x_0}^{x} \left(\sum_{j=1}^{M} \left| \left(D_{x_0}^{\gamma_1} f_j \right)(w) \right|^{\lambda_\alpha} \left| \left(D_{x_0}^{v} f_j \right)(w) \right|^{\lambda_v} \right) dw$$

$$\geq \delta_1^* \varphi_1(x) \left[\int_{x_0}^{x} \left[\sum_{j=1}^{M} \left| \left(D_{x_0}^{v} f_j \right)(w) \right|^p \right] dw \right]^{\left(\frac{\lambda_\alpha + \lambda_v}{p}\right)} \tag{75}$$

all $x_0 \leq x \leq b$.

In (75), $\left(A_0(x)\,|_{\lambda_\beta = 0}\right)$ *of* $\varphi_1(x)$ *is given by* (49).

Proof. Based on Theorem 11. □

Corollary 16. (to Theorem 12, $\lambda_\alpha = 0$, $p(t) = q(t) = 1$). *It holds*

$$\int_{x_0}^{x}\left\{\left\{\sum_{j=1}^{M-1}\left[\left|\left(D_{x_0}^{\gamma_2}f_{j+1}\right)(w)\right|^{\lambda_\beta}\left|\left(D_{x_0}^{v}f_j\right)(w)\right|^{\lambda_v}\right.\right.\right.$$
$$\left.\left.+\left|\left(D_{x_0}^{\gamma_2}f_j\right)(w)\right|^{\lambda_\beta}\left|\left(D_{x_0}^{v}f_{j+1}\right)(w)\right|^{\lambda_v}\right]\right\}$$
$$+\left[\left|\left(D_{x_0}^{\gamma_2}f_M\right)(w)\right|^{\lambda_\beta}\left|\left(D_{x_0}^{v}f_1\right)(w)\right|^{\lambda_v}\right.$$
$$\left.\left.+\left|\left(D_{x_0}^{\gamma_2}f_1\right)(w)\right|^{\lambda_\beta}\left|\left(D_{x_0}^{v}f_M\right)(w)\right|^{\lambda_v}\right]\right\}dw\geq$$
$$\left(M^{1-\left(\frac{\lambda_v+\lambda_\beta}{p}\right)}\right)2^{\left(\frac{\lambda_v+\lambda_\beta}{p}\right)}\varphi_2(x)\left\{\int_{x_0}^{x}\left[\sum_{j=1}^{M}\left|\left(D_{x_0}^{v}f_j\right)(w)\right|^{p}\right]dw\right\}^{\left(\frac{\lambda_v+\lambda_\beta}{p}\right)}, \tag{76}$$

all $x_0 \leq x \leq b$.

In $(76), (A_0(x)|_{\lambda_\alpha=0})$ *of* $\varphi_2(x)$ *is given by* (52).

Proof. Based on Theorem 12. □

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